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EXISTENCE OF POSITIVE SOLUTIONS FOR COUPLED FRACTIONAL DIFFERENTIAL SYSTEM WITH IMPROPER INTEGRAL BOUNDARY CONDITIONS ON THE HALF-LINE

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Abstract This article is devoted to proving the existence of positive solutions for a class of coupled fractional boundary value problems involving an improper integral and the infinite-point on the half-line. By making use of the monotone iterative technique along with Banach's contraction mapping principle, some explicit monotone iterative sequences for approximating the extreme positive solutions and the unique positive solution for the problem are constructed, an error estimate formula of the positive solution is also given. In the end, a numerical simulation is given to illustrate the main results.

Keywords Monotone iterative technique, fractional differential equation, improper integral, infinite interval, numerical simulation.

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1. Introduction

Fractional differential equations are utilized to explain a wide range of phenomena across numerous natural science fields. These include physics, mechanics, epidemiology, biology, and other disciplines, see [1, 14, 22, 30]. Fractional-order differential equations are a natural extension of the integer-order case. For example, in fluid dynamics, the authors in [4, 9] show that solutes moving through a highly heterogeneous aquifer do not abide by Fick's first law. Thus, the fractional-order advection-dispersion equation can be utilized to characterize the convection-diffusion process to improve the model's accuracy. Viscoelastic media demonstrate intricate nonlinear behaviors in their internal structures, which can no longer be accurately described by classical integer-order mechanical constitutive models. Fractional differential systems are more suitable for precisely capture and predict the viscoelastic behavior of systems with their flexible non-local properties, see [6, 28]. Due to their extensive use in mathematical models and applied sciences, they have attracted significant attention, especially when dealing with a wide range of boundary conditions, some latest results on the topic can see for example [2, 3, 7, 8, 10, 12, 13, 17–19, 21, 23, 25, 26].

In recent years, boundary value problems for fractional differential equation have become a hot research topic, and lots of excellent results have been obtained by means of fixed point theorems, such as Guo-Krasnosel'skii fixed point theorem [24, 30, 33], Avery-Henderson fixed point theorem [31], Leggett-Williams fixed point theorem [24, 33], Avery-Peterson fixed point theorem [15], topological degree methods [27], upper and lower solutions technique [32], and so forth. In particular, the monotone iterative technique is regarded as an effective and significant

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approach for handling sequences of monotone solutions in initial and boundary value problems, see [11, 16, 20, 29].

In [20], the authors studied the following Hadamard fractional integro-differential equation on an infinite interval

$$\begin{cases} {}^H\mathcal{D}^{\alpha}_{1+}u(t) + f(t, u(t), {}^H I^r u(t), {}^H \mathcal{D}^{\alpha-1}_{1+}u(t)) = 0, \quad 1 < \alpha < 2, \quad t \in (1, \infty), \\ u(1) = 0, \quad {}^H\mathcal{D}^{\alpha-1}_{1+}u(\infty) = \sum_{i=1}^{m} \lambda_i {}^H I^{\beta_i}_{1+}u(\eta), \end{cases}$$

where ${}^H\mathcal{D}^{\alpha}_{1+}$ denotes Hadamard fractional derivative of order α and ${}^H I^{(\cdot)}_{1+}$ is a Hadamard type fractional order integral, $r, \eta, \beta_i, \lambda_i (i = 1, 2, 3, \cdots, m)$ are some given constants satisfy $\Gamma(\alpha) > \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\alpha)}{\Gamma(\alpha+\beta_i)}(\log \eta)^{\alpha+\beta_i-1}$. By using the monotone iterative technique, the existence of positive solutions was established.

In [16], the authors discussed the following Hadamard type fractional-order differential system

$$\begin{cases} {}^H\mathcal{D}^{p}_{1+}x(t) + f_1(t, x(t), y(t), {}^H \mathcal{D}^{p-1}_{1+}x(t), {}^H \mathcal{D}^{q-1}_{1+}y(t)) = 0, \quad 1 < p \leq 2, t \in [1, \infty), \\ {}^H\mathcal{D}^{q}_{1+}x(t) + f_2(t, x(t), y(t), {}^H \mathcal{D}^{p-1}_{1+}x(t), {}^H \mathcal{D}^{q-1}_{1+}y(t)) = 0, \quad 1 < q \leq 2, t \in [1, \infty), \\ x(1) = 0, \quad {}^H\mathcal{D}^{p-1}_{1+}x(\infty) = \sum_{i=1}^{m} \lambda_i {}^H I^{\alpha_i}_{1+}y(\eta), \quad \eta \in [1, \infty), \\ y(1) = 0, \quad {}^H\mathcal{D}^{q-1}_{1+}y(\infty) = \sum_{j=1}^{n} \sigma_j {}^H I^{\beta_j}_{1+}x(\xi), \quad \xi \in [1, \infty), \end{cases}$$

where ${}^H\mathcal{D}^{\phi}_{1+}$ are Hadamard fractional derivatives of $\phi \in \{p, q\}$ and ${}^H I^{\psi}_{1+}$ are Hadamard fractional integrals of $\psi \in \{\alpha_i, \beta_j\}$, $\lambda_i, \sigma_j > 0 (i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$. By using the monotone iterative technique, iterative sequences of the positive extremal solutions were acquired.

On the other hand, an improper integral and infinite-point boundary value conditions is more general than those of multipoint boundary value conditions in the known papers, see [5, 26].

In [26], the authors investigated the following fractional boundary value problems involving an improper integral and the infinite-point on the half-line

$$\begin{cases} \mathcal{D}^{\beta}_{0+}x(t) + a(t) f_1(t, x(t), x'(t)) = 0, \quad t \in [0, \infty), \\ x(0) = x'(0) = 0, \quad \lim_{t\to\infty} \mathcal{D}^{\beta-1}_{0+}x(t) = \int_0^{\infty} h(t)x'(t)dt + \sum_{i=1}^{\infty} \eta_i \mathcal{D}^{\gamma}_{0+}x(\xi_i), \end{cases}$$

where $2 < \beta \leq 3$, $0 \leq \gamma \leq \beta - 1$, and $\mathcal{D}^{\beta}_{0+}$ denotes Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \xi_{i+1} < \cdots < \infty$, $\eta_i > 0 (i = 1, 2, \cdots)$. By utilizing the Green function and Avery-Peterson fixed point theorem, the existence of multiple positive solutions were obtained.

Motivated by the excellent results above, in this paper, we consider the following coupled fractional differential systems on the half-line

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}u(t) + a(t) f_1(t, u(t), v(t), \mathcal{D}^{\alpha-1}_{0+}u(t), \mathcal{D}^{\beta-1}_{0+}v(t)) = 0, \quad t \in \mathbb{R}^+_0, \\ \mathcal{D}^{\beta}_{0+}v(t) + b(t) f_2(t, u(t), v(t), \mathcal{D}^{\alpha-1}_{0+}u(t), \mathcal{D}^{\beta-1}_{0+}v(t)) = 0, \quad t \in \mathbb{R}^+_0, \end{cases} \tag{1.1}$$

subject to an improper integral and the infinite-point boundary conditions

$$\begin{cases} u(0)=u'(0)=0,\ \lim\limits_{t\to\infty}\mathcal{D}_{0^+}^{\alpha-1}u(t)=\displaystyle\int_0^\infty h_1(t)v'(t)dt+\sum_{i=1}^{\infty}\eta_i\mathcal{D}_{0^+}^{\gamma_1}v(\sigma_i), \\ v(0)=v'(0)=0,\ \lim\limits_{t\to\infty}\mathcal{D}_{0^+}^{\beta-1}v(t)=\displaystyle\int_0^\infty h_2(t)u'(t)dt+\sum_{j=1}^{\infty}\lambda_j\mathcal{D}_{0^+}^{\gamma_2}u(\xi_j), \end{cases} \tag{1.2}$$

where $\mathcal{D}_{0^+}^{\phi}$ are Riemann-Liouville fractional derivatives of $\phi\in\{\alpha,\beta,\gamma_1,\gamma_2\}$, $2<\alpha\le 3$, $2<\beta\le 3$, $0\le\gamma_1\le\beta-1$, $0\le\gamma_2\le\alpha-1$, $0<\sigma_1<\sigma_2<\cdots<\sigma_i<\sigma_{i+1}<\cdots<\infty$, $0<\xi_1<\xi_2<\cdots<\xi_j<\xi_{j+1}<\cdots<\infty$, $\eta_i>0$, $\lambda_i>0(i=1,2,\cdots,j=1,2,\cdots)$, $f_1,f_2:\mathbb{R}^+\times\mathbb{R}^4\to\mathbb{R}^+$ are continuous, in which $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^+=[0,\infty)$ and $\mathbb{R}_0^+=(0,\infty)$.

The purpose of this paper is to investigate the existence of positive solutions for problem (1.1)-(1.2) on an infinite interval. Compared with [16, 20], we investigate the problem which involves an improper integral and infinite-point boundary value conditions on the half-line. For any given initial value, we use numerical algorithms to generate a series of iterative process. Compared with [26], through the application of the monotone iterative technique, we have obtained some explicit monotone iterative sequences for approximating the extreme positive solutions as well as the unique positive solution, an error estimate formula of the positive solution is also given. This endeavor is of greater value and interest compared to merely verifying the existence of solutions.

The structure of this paper is as follows. Some necessary preliminaries from fractional calculus are recalled in Section 2. Next, we obtain the Green functions corresponding to the problem. Section 3 is devoted to the existence and uniqueness of positive solutions for boundary value problems (1.1)-(1.2). Also, two example are prepared to validate the theoretical results in Section 4. Finally, we propose some conclusions.

2. Preliminaries and lemmas

In this section, for the convenience of reader, we present some notations and lemmas that will be used in the proof of our main results.

Definition 2.1. [14] The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:\mathbb{R}_0^+\to\mathbb{R}$ is given by

$$I_{0^+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds,$$

provided that the right-hand side is pointwise defined on $\mathbb{R}_0^+$.

Definition 2.2. [14] The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:\mathbb{R}_0^+\to\mathbb{R}$ is given by

$$\mathcal{D}_{0^+}^{\alpha}y(t)=\left(\frac{d}{dt}\right)^n I_{0^+}^{n-\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t\frac{y(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha\le n$, provided that the right-hand side is pointwise defined on $\mathbb{R}_0^+$.

Lemma 2.1. [14] *Assume that $y\in C(0,+\infty)\cap L^1(0,+\infty)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,+\infty)\cap L^1(0,+\infty)$. Then*

$$I_{0^+}^{\alpha}\mathcal{D}_{0^+}^{\alpha}y(t)=y(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

for some $C_i \in \mathbb{R}$ $(i = 1, 2, \cdots, n)$, *where* $n-1 < \alpha \le n$.

Lemma 2.2. *Let* $g_1, g_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$, *then the solution of boundary value problem*

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} u(t) + g_1(t) = 0, \quad t \in \mathbb{R}_0^+, \quad 2 < \alpha \le 3, \\ \mathcal{D}_{0^+}^{\beta} v(t) + g_2(t) = 0, \quad t \in \mathbb{R}_0^+, \quad 2 < \beta \le 3 \end{cases} \tag{2.1}$$

with boundary conditions (1.2) *is equivalent to the integral system*

$$\begin{cases} u(t) = \int_0^{\infty} \mathcal{G}_1(t,s) g_1(s) ds + \int_0^{\infty} \mathcal{G}_2(t,s) g_2(s) ds, \\ v(t) = \int_0^{\infty} \mathcal{G}_3(t,s) g_2(s) ds + \int_0^{\infty} \mathcal{G}_4(t,s) g_1(s) ds, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{G}_1(t,s) &= g_\alpha(t,s) + \frac{\Omega_2 t^{\alpha-1}}{\Delta} \sum_{j=1}^{\infty} \frac{\lambda_j g_{\gamma_2}^{\alpha}(\xi_j, s)}{\Gamma(\alpha-\gamma_2)} + \frac{\Omega_2 \delta_1(s) t^{\alpha-1}}{\Gamma(\alpha-1)\Delta}, \\ \mathcal{G}_2(t,s) &= \frac{\Gamma(\beta) t^{\alpha-1}}{\Delta} \sum_{i=1}^{\infty} \frac{\eta_i g_{\gamma_1}^{\beta}(\sigma_i, s)}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)\delta_2(s) t^{\alpha-1}}{\Delta}, \\ \mathcal{G}_3(t,s) &= g_\beta(t,s) + \frac{\Omega_1 t^{\beta-1}}{\Delta} \sum_{i=1}^{\infty} \frac{\eta_i g_{\gamma_1}^{\beta}(\sigma_i, s)}{\Gamma(\beta-\gamma_1)} + \frac{\Omega_1 \delta_2(s) t^{\beta-1}}{\Gamma(\beta-1)\Delta}, \\ \mathcal{G}_4(t,s) &= \frac{\Gamma(\alpha) t^{\beta-1}}{\Delta} \sum_{j=1}^{\infty} \frac{\lambda_j g_{\gamma_2}^{\alpha}(\xi_j, s)}{\Gamma(\alpha-\gamma_2)} + \frac{(\alpha-1)\delta_1(s) t^{\beta-1}}{\Delta}, \end{aligned}$$

with

$$\begin{aligned} & Z_1(s) = \int_s^{\infty} h_2(\tau)(\tau-s)^{\alpha-2} d\tau, \quad Z_2(s) = \int_s^{\infty} h_1(\tau)(\tau-s)^{\beta-2} d\tau, \\ & \delta_1(s) = \int_0^{\infty} h_2(\tau)\tau^{\alpha-2} d\tau - \int_s^{\infty} h_2(\tau)(\tau-s)^{\alpha-2} d\tau = Z_1(0) - Z_1(s), \\ & \delta_2(s) = \int_0^{\infty} h_1(\tau)\tau^{\beta-2} d\tau - \int_s^{\infty} h_1(\tau)(\tau-s)^{\beta-2} d\tau = Z_2(0) - Z_2(s), \\ & \Omega_1 = \int_0^{\infty} (\alpha-1) h_2(\tau)\tau^{\alpha-2} d\tau + \Gamma(\alpha) \sum_{j=1}^{\infty} \frac{\lambda_j \xi_j^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)}, \\ & \Omega_2 = \int_0^{\infty} (\beta-1) h_1(\tau)\tau^{\beta-2} d\tau + \Gamma(\beta) \sum_{i=1}^{\infty} \frac{\eta_i \sigma_i^{\beta-\gamma_1-1}}{\Gamma(\beta-\gamma_1)}, \\ & \Delta = \Gamma(\alpha)\Gamma(\beta) - \Omega_1\Omega_2 > 0 \end{aligned} \tag{2.3}$$

and

$$g_\varphi(t,s) = \frac{1}{\Gamma(\varphi)} \begin{cases} t^{\varphi-1} - (t-s)^{\varphi-1}, & 0 \le s \le t \le \infty, \\ t^{\varphi-1}, & 0 \le t \le s \le \infty, \end{cases}$$

$$g_\psi^\varphi(\rho,s) = \begin{cases} \rho^{\varphi-\psi-1} - (\rho-s)^{\varphi-\psi-1}, & 0 \le s \le \rho \le \infty, \\ \rho^{\varphi-\psi-1}, & 0 \le \rho \le s \le \infty, \end{cases}$$

with $\varphi \in \{\alpha, \beta\}$, $\psi \in \{\gamma_1, \gamma_2\}$, $\rho \in \{\sigma_i, \xi_j\} (i, j = 1, 2, \cdots)$.

Proof. By means of Lemma 2.1, applying the Riemann-Liouville fractional integral of orders α and β to both sides of the two equations of system (2.1), respectively, we obtain

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_1(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$
$$v(t) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g_2(s)ds + d_1 t^{\beta-1} + d_2 t^{\beta-2} + d_3 t^{\beta-3},$$

where $c_i, d_i \in \mathbb{R}(i = 1, 2, 3)$ are constants to be determined. Since $u(0) = u'(0) = 0$, $v(0) = v'(0) = 0$, we can get $c_2 = c_3 = 0$, $d_2 = d_3 = 0$. Hence

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_1(s)ds + c_1 t^{\alpha-1}, \\ v(t) &= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g_2(s)ds + d_1 t^{\beta-1}. \end{aligned} \tag{2.4}$$

By (2.4), we obtain

$$u'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} g_1(s)ds + c_1(\alpha-1) t^{\alpha-2},$$
$$v'(t) = -\frac{1}{\Gamma(\beta-1)}\int_0^t (t-s)^{\beta-2} g_2(s)ds + d_1(\beta-1) t^{\beta-2},$$

and

$$\mathcal{D}_{0^+}^{\alpha-1} u(t) = -\int_0^t g_1(s)ds + c_1\Gamma(\alpha),$$
$$\mathcal{D}_{0^+}^{\beta-1} v(t) = -\int_0^t g_2(s)ds + d_1\Gamma(\beta).$$

By applying the Riemann-Liouville fractional derivative of orders γ_2 and γ_1 to (2.4), and also substitution $t = \xi_j$ and $t = \sigma_i$ respectively, we get

$$\mathcal{D}_{0^+}^{\gamma_2} u(\xi_j) = -\frac{1}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j} (\xi_j - s)^{\alpha-\gamma_2-1} g_1(s)ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_2)}\xi_j^{\alpha-\gamma_2-1},$$
$$\mathcal{D}_{0^+}^{\gamma_1} v(\sigma_i) = -\frac{1}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i} (\sigma_i - s)^{\beta-\gamma_1-1} g_2(s)ds + d_1 \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_1)}\sigma_i^{\beta-\gamma_1-1}.$$

Using the boundary conditions (1.2), by Cramer's rule and simple calculations, it follows that

$$\begin{aligned} c_1 = &\frac{\Gamma(\beta)}{\Delta}\int_0^\infty g_1(s)ds - \frac{\beta-1}{\Delta}\int_0^\infty h_1(\tau)\left(\int_0^\tau (\tau-s)^{\beta-2} g_2(s)ds\right)d\tau \\ &- \frac{\Gamma(\beta)}{\Delta}\sum_{i=1}^\infty \frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i} (\sigma_i - s)^{\beta-\gamma_1-1} g_2(s)ds + \frac{\Omega_2}{\Delta}\int_0^\infty g_2(s)ds \\ &- \frac{\Omega_2}{\Gamma(\alpha-1)\Delta}\int_0^\infty h_2(\tau)\left(\int_0^\tau (\tau-s)^{\alpha-2} g_1(s)ds\right)d\tau \\ &- \frac{\Omega_2}{\Delta}\sum_{j=1}^\infty \frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j} (\xi_j - s)^{\alpha-\gamma_2-1} g_1(s)ds, \end{aligned}$$

$$
\begin{aligned}
d_1 = &\frac{\Gamma(\alpha)}{\Delta}\int_0^\infty g_2(s)ds - \frac{\alpha-1}{\Delta}\int_0^\infty h_2(\tau)\left(\int_0^\tau (\tau-s)^{\alpha-2}g_1(s)ds\right)d\tau \\
&- \frac{\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j}(\xi_i-s)^{\alpha-\gamma_2-1}g_1(s)ds + \frac{\Omega_1}{\Delta}\int_0^\infty g_1(s)ds \\
&- \frac{\Omega_1}{\Gamma(\beta-1)\Delta}\int_0^\infty h_1(\tau)\left(\int_0^\tau (\tau-s)^{\beta-2}g_2(s)ds\right)d\tau \\
&- \frac{\Omega_1}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i}(\sigma_i-s)^{\beta-\gamma_1-1}g_2(s)ds.
\end{aligned}
$$

Substituting the values of c_1 and d_1 in (2.4), one has

$$
\begin{aligned}
u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g_1(s)ds + t^{\alpha-1}\Bigg[\frac{\Gamma(\beta)}{\Delta}\int_0^\infty g_1(s)ds \\
&- \frac{\beta-1}{\Delta}\int_0^\infty h_1(\tau)\left(\int_0^\tau (\tau-s)^{\beta-2}g_2(s)ds\right)d\tau \\
&- \frac{\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i}(\sigma_i-s)^{\beta-\gamma_1-1}g_2(s)ds \\
&- \frac{\Omega_2}{\Gamma(\alpha-1)\Delta}\int_0^\infty h_2(\tau)\left(\int_0^\tau (\tau-s)^{\alpha-2}g_1(s)ds\right)d\tau + \frac{\Omega_2}{\Delta}\int_0^\infty g_2(s)ds \\
&- \frac{\Omega_2}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma_2-1}g_1(s)ds\Bigg] + \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}g_1(s)ds \\
&- \frac{\Gamma(\alpha)\Gamma(\beta)-\Omega_1\Omega_2}{\Gamma(a)\Delta}\int_0^\infty t^{\alpha-1}g_1(s)ds \\
= &\int_0^\infty g_\alpha(t,s)g_1(s)ds + \frac{\Gamma(\beta)t^{\alpha-1}}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^\infty g^{\beta}_{\gamma_1}(\sigma_i,s)g_2(s)ds \\
&+ \frac{\Omega_2 t^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^\infty g^{\alpha}_{\gamma_2}(\xi_j,s)g_1(s)ds + \int_0^\infty \frac{(\beta-1)\delta_2(s)t^{\alpha-1}}{\Delta}g_2(s) \\
&+ \int_0^\infty \frac{\Omega_2\delta_1(s)t^{\alpha-1}}{\Gamma(\alpha-1)\Delta}g_1(s) \\
= &\int_0^\infty \mathcal{G}_1(t,s)g_1(s)ds + \int_0^\infty \mathcal{G}_2(t,s)g_2(s)ds.
\end{aligned}
$$

In a similar way, we have

$$
\begin{aligned}
v(t) = &-\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g_2(s)ds + t^{\beta-1}\Bigg[\frac{\Gamma(\alpha)}{\Delta}\int_0^\infty g_2(s)ds \\
&- \frac{\alpha-1}{\Delta}\int_0^\infty h_2(\tau)\left(\int_0^\tau (\tau-s)^{\alpha-2}g_1(s)ds\right)d\tau \\
&- \frac{\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma_2-1}g_1(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\Omega_1}{\Gamma(\beta-1)\Delta}\int_0^\infty h_1(\tau)\left(\int_0^\tau(\tau-s)^{\beta-2}g_2(s)ds\right)d\tau + \frac{\Omega_1}{\Delta}\int_0^\infty g_1(s)ds \\
&- \frac{\Omega_1}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i}(\sigma_i-s)^{\beta-\gamma_1-1}g_2(s)ds\Bigg] + \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}g_2(s)ds \\
&- \frac{\Gamma(\alpha)\Gamma(\beta)-\Omega_1\Omega_2}{\Gamma(\beta)\Delta}\int_0^\infty t^{\beta-1}g_2(s)ds \\
=& \int_0^\infty g_\beta(t,s)g_2(s)ds + \frac{\Gamma(\alpha)t^{\beta-1}}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^\infty g_{\gamma_2}^{\alpha}(\xi_j,s)g_1(s)ds \\
&+ \frac{\Omega_1 t^{\beta-1}}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^\infty g_{\gamma_1}^{\beta}(\sigma_i,s)g_2(s)ds + \int_0^\infty \frac{(\alpha-1)\delta_1(s)t^{\beta-1}}{\Delta}g_1(s) \\
&+ \int_0^\infty \frac{\Omega_1\delta_2(s)t^{\beta-1}}{\Gamma(\beta-1)\Delta}g_2(s) \\
=& \int_0^\infty \mathcal{G}_3(t,s)g_2(s)ds + \int_0^\infty \mathcal{G}_4(t,s)g_1(s)ds,
\end{aligned}
$$

which proves that (2.2) holds. This completes the proof. □

Lemma 2.3. *Let $g_1, g_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$, then the subsequent expression can be derived from the integral equations* (2.2)

$$
\begin{cases}
\mathcal{D}_{0^+}^{\alpha-1}u(t) = \int_0^\infty \mathcal{G}_1^*(t,s)g_1(s)ds + \int_0^\infty \mathcal{G}_2^*(t,s)g_2(s)ds, \\
\mathcal{D}_{0^+}^{\beta-1}v(t) = \int_0^\infty \mathcal{G}_3^*(t,s)g_2(s)ds + \int_0^\infty \mathcal{G}_4^*(t,s)g_1(s)ds,
\end{cases}
\tag{2.5}
$$

where

$$
\begin{aligned}
\mathcal{G}_1^*(t,s) &= \mathcal{G}_0(t,s) + \frac{\Omega_2\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j g_{\gamma_2}^{\alpha}(\xi_j,s)}{\Gamma(\alpha-\gamma_2)} + \frac{(\alpha-1)\Omega_2\delta_1(s)}{\Delta}, \\
\mathcal{G}_2^*(t,s) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i g_{\gamma_1}^{\beta}(\sigma_i,s)}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)\Gamma(\alpha)\delta_2(s)}{\Delta}, \\
\mathcal{G}_3^*(t,s) &= \mathcal{G}_0(t,s) + \frac{\Omega_1\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i g_{\gamma_1}^{\beta}(\sigma_i,s)}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)\Omega_1\delta_2(s)}{\Delta}, \\
\mathcal{G}_4^*(t,s) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j g_{\gamma_2}^{\alpha}(\xi_j,s)}{\Gamma(\alpha-\gamma_2)} + \frac{(\alpha-1)\Gamma(\beta)\delta_1(s)}{\Delta},
\end{aligned}
\tag{2.6}
$$

and

$$
\mathcal{G}_0(t,s) = \begin{cases} 0, & 0 \le s \le t \le \infty, \\ 1, & 0 \le t \le s \le \infty. \end{cases}
$$

Proof. Using Lemma 2.2, we can obtain

$$
\mathcal{D}_{0^+}^{\alpha-1}u(t) = -\int_0^t g_1(s)ds + \Gamma(\alpha)\Bigg[\frac{\Gamma(\beta)}{\Delta}\int_0^\infty g_1(s)ds
$$

$$
\begin{aligned}
&- \frac{\beta-1}{\Delta}\int_0^\infty h_1(\tau)\left(\int_0^\tau (\tau-s)^{\beta-2}g_2(s)ds\right)d\tau \\
&- \frac{\Gamma(\beta)}{\Delta}\sum_{i=1}^\infty \frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i}(\sigma_i-s)^{\beta-\gamma_1-1}g_2(s)ds \\
&- \frac{\Omega_2}{\Gamma(\alpha-1)\Delta}\int_0^\infty h_2(\tau)\left(\int_0^\tau (\tau-s)^{\alpha-2}g_1(s)ds\right)d\tau \\
&+ \frac{\Omega_2}{\Delta}\int_0^\infty g_2(s)ds - \frac{\Omega_2}{\Delta}\sum_{j=1}^\infty \frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma_2-1}g_1(s)ds\Bigg] \\
&+ \int_0^\infty g_1(s)ds - \int_0^\infty g_1(s)ds \\
=& \int_0^\infty \mathcal{G}_0(t,s)g_1(s)ds + \frac{\Omega_2\Gamma(\alpha)}{\Delta}\sum_{j=1}^\infty \frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^\infty g^\alpha_{\gamma_2}(\xi_j,s)g_1(s)ds \\
&+ \frac{(\alpha-1)\Omega_2}{\Delta}\int_0^\infty \delta_1(s)g_1(s)ds \\
&+ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta}\sum_{i=1}^\infty \frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^\infty g^\beta_{\gamma_1}(\sigma_i,s)g_2(s)ds + \frac{(\beta-1)\Gamma(\alpha)}{\Delta}\int_0^\infty \delta_2(s)g_2(s)ds \\
=& \int_0^\infty \mathcal{G}_1^*(t,s)g_1(s)ds + \int_0^\infty \mathcal{G}_2^*(t,s)g_2(s)ds.
\end{aligned}
$$

In an analogous way, we have

$$
\begin{aligned}
\mathcal{D}^{\beta-1}_{0^+}u(t) =& - \int_0^t g_2(s)ds + \Gamma(\beta)\Bigg[\frac{\Gamma(\alpha)}{\Delta}\int_0^\infty g_2(s)ds \\
&- \frac{\alpha-1}{\Delta}\int_0^\infty h_2(\tau)\left(\int_0^\tau (\tau-s)^{\alpha-2}g_1(s)ds\right)d\tau \\
&- \frac{\Gamma(\alpha)}{\Delta}\sum_{j=1}^\infty \frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^{\xi_j}(\xi_j-s)^{\alpha-\gamma_2-1}g_1(s)ds \\
&- \frac{\Omega_1}{\Gamma(\beta-1)\Delta}\int_0^\infty h_1(\tau)\left(\int_0^\tau (\tau-s)^{\beta-2}g_2(s)ds\right)d\tau \\
&+ \frac{\Omega_1}{\Delta}\int_0^\infty g_1(s)ds - \frac{\Omega_1}{\Delta}\sum_{i=1}^\infty \frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^{\sigma_i}(\sigma_i-s)^{\beta-\gamma_1-1}g_2(s)ds\Bigg] \\
&+ \int_0^\infty g_2(s)ds - \int_0^\infty g_2(s)ds \\
=& \int_0^\infty \mathcal{G}_0(t,s)g_2(s)ds + \frac{\Omega_1\Gamma(\beta)}{\Delta}\sum_{i=1}^\infty \frac{\eta_i}{\Gamma(\beta-\gamma_1)}\int_0^\infty g^\beta_{\gamma_1}(\sigma_i,s)g_2(s)ds \\
&+ \frac{(\beta-1)\Omega_1}{\Delta}\int_0^\infty \delta_2(s)g_2(s)ds \\
&+ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta}\sum_{j=1}^\infty \frac{\lambda_j}{\Gamma(\alpha-\gamma_2)}\int_0^\infty g^\alpha_{\gamma_2}(\xi_j,s)g_1(s)ds
\end{aligned}
$$

$$+\frac{(\alpha-1)\Gamma(\beta)}{\Delta}\int_0^\infty \delta_1(s)g_1(s)ds$$
$$=\int_0^\infty \mathcal{G}_3^*(t,s)g_2(s)ds+\int_0^\infty \mathcal{G}_4^*(t,s)g_1(s)ds,$$

which proves that (2.5) holds. This completes the proof. □

In the rest of the paper, we make the following assumptions:

(H_1) The functions $f_1, f_2 \in C(\mathbb{R}^+ \times \mathbb{R}^4, \mathbb{R}^+)$ and $\Delta = \Gamma(\alpha)\Gamma(\beta) - \Omega_1\Omega_2 > 0$.

(H_2) The functions $a, b, h_1, h_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ are not identical zero on any closed subinterval of $\mathbb{R}^+$.

(H_3) There exist nonnegative functions $a_i(t), b_i(t) \in L(\mathbb{R}^+)(i = 0, 1, 2, 3, 4)$ and the nonnegative constants $0 \le \iota_k, \tau_k < 1(k = 1, 2, 3, 4)$ satisfy

$$|f_1(t,x,y,w,z)| \le a_0(t)+a_1(t)|x|^{\iota_1}+a_2(t)|y|^{\iota_2}+a_3(t)|w|^{\iota_3}+a_4(t)|z|^{\iota_4}, \text{ a.e. } t\in\mathbb{R}^+,$$
$$|f_2(t,x,y,w,z)| \le b_0(t)+b_1(t)|x|^{\tau_1}+b_2(t)|y|^{\tau_2}+b_3(t)|w|^{\tau_3}+b_4(t)|z|^{\tau_4}, \text{ a.e. } t\in\mathbb{R}^+,$$

where $x, y, w, z \in \mathbb{R}$ and

$$\int_0^\infty a(t)a_0(t)dt = a_0^* < \infty, \quad \int_0^\infty a(t)a_1(t)(1+t^{\alpha+\beta-1})^{\iota_1}dt = a_1^* < \infty,$$
$$\int_0^\infty a(t)a_2(t)(1+t^{\alpha+\beta-1})^{\iota_2}dt = a_2^* < \infty, \quad \int_0^\infty a(t)a_3(t)dt = a_3^* < \infty,$$
$$\int_0^\infty a(t)a_4(t)dt = a_4^* < \infty,$$
$$\int_0^\infty b(t)b_0(t)dt = b_0^* < \infty, \quad \int_0^\infty b(t)b_1(t)(1+t^{\alpha+\beta-1})^{\tau_1}dt = b_1^* < \infty,$$
$$\int_0^\infty b(t)b_2(t)(1+t^{\alpha+\beta-1})^{\tau_2}dt = b_2^* < \infty, \quad \int_0^\infty b(t)b_3(t)dt = b_3^* < \infty,$$
$$\int_0^\infty b(t)b_4(t)dt = b_4^* < \infty.$$

(H_4) The functions $f_1(t, u_1, u_2, u_3, u_4)$ and $f_2(t, u_1, u_2, u_3, u_4)$ are increasing with respect to the variables u_1, u_2, u_3, u_4, and $\forall\ t \in \mathbb{R}^+,\ f_1(t, 0, 0, 0, 0) \not\equiv 0,\ f_2(t, 0, 0, 0, 0) \not\equiv 0$.

(H_5) There exist nonnegative functions $c_{jk}(t) \in L(\mathbb{R}^+)(j = 1, 2, k = 1, 2, 3, 4)$ such that

$$|f_j(t,u_1,u_2,u_3,u_4)-f_j(t,\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)| \le \sum_{k=1}^4 c_{jk}(t)|u_k-\overline{u}_k|, \text{ a.e. } t\in\mathbb{R}^+,$$
$$j=1,2, u_k, \overline{u}_k \in \mathbb{R}(k=1,2,3,4)$$

and

$$\int_0^\infty a(t)c_{11}(t)(1+t^{\alpha+\beta-1})dt = c_{11}^* < \infty, \quad \int_0^\infty a(t)c_{13}(t)dt = c_{13}^* < \infty,$$
$$\int_0^\infty a(t)c_{12}(t)(1+t^{\alpha+\beta-1})dt = c_{12}^* < \infty, \quad \int_0^\infty a(t)c_{14}(t)dt = c_{14}^* < \infty,$$

$$\int_0^\infty b(t)c_{21}(t)(1+t^{\alpha+\beta-1})dt = c_{21}^* < \infty, \quad \int_0^\infty b(t)c_{23}(t)dt = c_{23}^* < \infty,$$
$$\int_0^\infty b(t)c_{22}(t)(1+t^{\alpha+\beta-1})dt = c_{22}^* < \infty, \quad \int_0^\infty b(t)c_{24}(t)dt = c_{24}^* < \infty,$$
$$\int_0^\infty |f_1(t,0,0,0,0)|dt = m_1 < \infty, \quad \int_0^\infty |f_2(t,0,0,0,0)|dt = m_2 < \infty.$$

Before establishing some properties of the Green's functions, we set

$$K_1 = \frac{1}{\Gamma(\alpha)} + \frac{\Omega_2}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j\xi_j^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)} + \frac{\Omega_2 Z_1(0)}{\Gamma(\alpha-1)\Delta} = \frac{\Gamma(\beta)}{\Delta},$$
$$K_2 = \frac{\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i\sigma_i^{\beta-\gamma_1-1}}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)Z_2(0)}{\Delta} = \frac{\Omega_2}{\Delta},$$
$$K_3 = \frac{1}{\Gamma(\beta)} + \frac{\Omega_1}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i\sigma_i^{\beta-\gamma_1-1}}{\Gamma(\beta-\gamma_1)} + \frac{\Omega_1 Z_2(0)}{\Gamma(\beta-1)\Delta} = \frac{\Gamma(\alpha)}{\Delta},$$
$$K_4 = \frac{\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j\xi_j^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)} + \frac{(\alpha-1)Z_1(0)}{\Delta} = \frac{\Omega_1}{\Delta},$$
$$Q_1 = 1 + \frac{\Omega_2\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j\xi_j^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)} + \frac{(\alpha-1)\Omega_2 Z_1(0)}{\Delta} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta},$$
$$Q_2 = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i\sigma_i^{\beta-\gamma_1-1}}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)\Gamma(\alpha)Z_2(0)}{\Delta} = \frac{\Gamma(\alpha)\Omega_2}{\Delta},$$
$$Q_3 = 1 + \frac{\Omega_1\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i\sigma_i^{\beta-\gamma_1-1}}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)\Omega_1 Z_2(0)}{\Delta} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta},$$
$$Q_4 = \frac{\Gamma(\beta)\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j\xi_j^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)} + \frac{(\alpha-1)\Gamma(\beta)Z_1(0)}{\Delta} = \frac{\Gamma(\beta)\Omega_1}{\Delta}.$$

Lemma 2.4. *If (H_2) holds, then the function $0 \le Z_i(s) \le Z_i(0), s \in \mathbb{R}^+$ and $Z_i(s)$ is nonincreasing on $\mathbb{R}^+$, $i = 1, 2$.*

Proof. From hypothesis $(H_1), (H_2)$ and (2.3), we have

$$Z_1'(s) = -(\alpha-2)\int_s^\infty (\tau-s)^{\alpha-2}h_2(\tau)d\tau,$$
$$Z_2'(s) = -(\beta-2)\int_s^\infty (\tau-s)^{\beta-2}h_1(\tau)d\tau.$$

Consequently, $Z_i(s)$ is nonincreasing on $\mathbb{R}^+$ and $0 \le Z_i(s) \le Z_i(0)$, $i = 1, 2$. □

Lemma 2.5. *The Green functions $\mathcal{G}_i(t,s)$ and $\mathcal{G}_i^*(t,s)(i = 1,2,3,4)$ defined in (2.2) and (2.6) have the following properties:*

(1) $\mathcal{G}_i(t,s)$ are continuous and $\mathcal{G}_i(t,s) \ge 0$, for all $t, s \in \mathbb{R}^+$, $i = 1,2,3,4$;

(2) $\mathcal{G}_i(t,s) \le K_i t^{\alpha-1}$, $\mathcal{G}_j(t,s) \le K_j t^{\beta-1}$, for all $t, s \in \mathbb{R}^+$, $i = 1,2$, $j = 3,4$;

(3) $\frac{\mathcal{G}_i(t,s)}{1+t^{\alpha+\beta-1}} \le K_i$, *for all* $t, s \in \mathbb{R}^+$, $i = 1, 2, 3, 4$*;*

(4) $0 \le \mathcal{G}_i^*(t,s) \le Q_i$, *for all* $t, s \in \mathbb{R}^+$, $i = 1, 2, 3, 4$.

Proof. It is easy to prove that (1) hold, and we omit it.

To prove (2) and (3), for $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$, it is obvious that

$$g_\varphi(t,s) \le \frac{t^{\varphi-1}}{\Gamma(\varphi)},\ g^\alpha_{\gamma_2}(\xi_j, s) \le \xi_j^{\alpha-\gamma_2-1},\ g^\beta_{\gamma_1}(\sigma_i, s) \le \sigma_i^{\beta-\gamma_1-1},\ \sigma_i^{\beta-\gamma_1-1}\ \delta_i(s) \le Z_i(0),$$

and then for any $(t, s) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have

$$\mathcal{G}_1(t,s) \le \left[\frac{1}{\Gamma(\alpha)} + \frac{\Omega_2}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j \xi_j^{\alpha-\gamma_2-1}}{\Gamma(\alpha-\gamma_2)} + \frac{\Omega_2 Z_1(0)}{\Gamma(\alpha-1)\Delta}\right] t^{\alpha-1} = \frac{\Gamma(\beta)}{\Delta} t^{\alpha-1} = K_1 t^{\alpha-1},$$

$$\mathcal{G}_2(t,s) \le \left[\frac{\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i \sigma_i^{\beta-\gamma_1-1}}{\Gamma(\beta-\gamma_1)} + \frac{(\beta-1)Z_2(0)}{\Delta}\right] t^{\alpha-1} = \frac{\Omega_2}{\Delta} t^{\alpha-1} = K_2 t^{\alpha-1}.$$

Furthermore,

$$0 \le \frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}} \le K_1, \quad 0 \le \frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}} \le K_2, \quad (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

By a similar calculation, we can prove other inequality results about $\mathcal{G}_3(t, s)$ and $\mathcal{G}_4(t, s)$. So properties (2) and (3) hold. From the Green functions $\mathcal{G}_i^*(t, s) (i = 1, 2, 3, 4)$ in Lemma 2.3, it is easy to observe that property (4) holds. □

Define two spaces

$$X = \left\{u \in C(\mathbb{R}^+), \mathcal{D}^{\alpha-1}_{0^+} u \in C(\mathbb{R}^+) : \sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^{\alpha+\beta-1}} < \infty, \sup_{t\in\mathbb{R}^+} |\mathcal{D}^{\alpha-1}_{0^+} u(t)| < \infty\right\},$$

$$Y = \left\{v \in C(\mathbb{R}^+), \mathcal{D}^{\beta-1}_{0^+} v \in C(\mathbb{R}^+) : \sup_{t\in\mathbb{R}^+} \frac{|v(t)|}{1+t^{\alpha+\beta-1}} < \infty, \sup_{t\in\mathbb{R}^+} |\mathcal{D}^{\beta-1}_{0^+} v(t)| < \infty\right\},$$

equipped with the norms

$$\|u\|_X = \max\left\{\sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t^{\alpha+\beta-1}}, \sup_{t\in\mathbb{R}^+} |\mathcal{D}^{\alpha-1}_{0^+} u(t)|\right\},$$

$$\|v\|_Y = \max\left\{\sup_{t\in\mathbb{R}^+} \frac{|v(t)|}{1+t^{\alpha+\beta-1}}, \sup_{t\in\mathbb{R}^+} |\mathcal{D}^{\beta-1}_{0^+} v(t)|\right\}.$$

We can obtain $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces. Moreover, it is straightforward to observe that the product space $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space with the norm

$$\|\cdot\|_{X\times Y} = \max\{\|u\|_X, \|v\|_Y\}.$$

Lemma 2.6. [23] *Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following hold:*

(i) *For any $u \in U$, $\frac{u(t)}{1+t^{\alpha+\beta-1}}$ and $\mathcal{D}^{\alpha-1}_{0^+} u(t)$ are equicontinuous on any compact interval of $\mathbb{R}^+$;*

(ii) *For any $\epsilon > 0$, there is a constant $C = C(\epsilon) > 0$ such that $\left|\frac{u(t_1)}{1+t_1^{\alpha+\beta-1}} - \frac{u(t_2)}{1+t_2^{\alpha+\beta-1}}\right| < \epsilon$ and $|\mathcal{D}^{\alpha-1}_{0^+} u(t_1) - \mathcal{D}^{\alpha-1}_{0^+} u(t_2)| < \epsilon$ for any $t_1, t_2 \ge C$ and $u \in U$.*

3. Main results

We define the cone $P \subset X \times Y$ as $P = \{(u,v) \in X \times Y : u(t) \geq 0, v(t) \geq 0, \mathcal{D}_{0^+}^{\alpha-1}u(t) \geq 0, \mathcal{D}_{0^+}^{\beta-1}v(t) \geq 0, t \in \mathbb{R}^+\}$, and the operator

$$\mathcal{F} : P \to X \times Y$$

as

$$\mathcal{F}(u,v)(t) = (\mathcal{F}_1(u,v)(t), \mathcal{F}_2(u,v)(t))$$

for all $t \in \mathbb{R}^+$, where the operators $\mathcal{F}_1 : P \to X$ and $\mathcal{F}_2 : P \to Y$ are given by

$$\begin{pmatrix} \mathcal{F}_1(u,v)(t) \\ \mathcal{F}_2(u,v)(t) \end{pmatrix} = \begin{pmatrix} \int_0^\infty \mathcal{G}_1(t,s) f_{1(u,v)}(s)ds + \int_0^\infty \mathcal{G}_2(t,s) f_{2(u,v)}(s)ds \\ \int_0^\infty \mathcal{G}_3(t,s) f_{2(u,v)}(s)ds + \int_0^\infty \mathcal{G}_4(t,s) f_{1(u,v)}(s)ds \end{pmatrix}, \tag{3.1}$$

for $(u,v) \in P$, $t \in \mathbb{R}^+$, where

$$\begin{cases} f_{1(u,v)}(s) = a(s) f_1(s, u(s), v(s), \mathcal{D}_{0^+}^{\alpha-1}u(s), \mathcal{D}_{0^+}^{\beta-1}v(s)), \\ f_{2(u,v)}(s) = b(s) f_2(s, u(s), v(s), \mathcal{D}_{0^+}^{\alpha-1}u(s), \mathcal{D}_{0^+}^{\beta-1}v(s)). \end{cases}$$

From Lemma 2.3 and (3.1), for $(u,v) \in P$, $t \in \mathbb{R}^+$, we have

$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(u,v)(t) \end{pmatrix} = \begin{pmatrix} \int_0^\infty \mathcal{G}_1^*(t,s) f_{1(u,v)}(s)ds + \int_0^\infty \mathcal{G}_2^*(t,s) f_{2(u,v)}(s)ds \\ \int_0^\infty \mathcal{G}_3^*(t,s) f_{2(u,v)}(s)ds + \int_0^\infty \mathcal{G}_4^*(t,s) f_{1(u,v)}(s)ds \end{pmatrix}. \tag{3.2}$$

Clearly, if $(u,v) \in P$ is a fixed point of $\mathcal{F}$, then (u,v) is a solution of system (1.1)-(1.2).

Lemma 3.1. *If $(H_1)$and (H_3) hold, then*

$$\int_0^\infty f_{1(u,v)}(s)ds \leq a_0^* + \sum_{i=1}^4 a_i^* \|(u,v)\|_{X\times Y}^{\iota_i}, \ \forall\ (u,v) \in X \times Y,$$

$$\int_0^\infty f_{2(u,v)}(s)ds \leq b_0^* + \sum_{i=1}^4 b_i^* \|(u,v)\|_{X\times Y}^{\tau_i}, \ \forall\ (u,v) \in X \times Y.$$

Proof. By hypotheses (H_1) and (H_3), for all $(u,v) \in X \times Y$, we have

$$\begin{aligned} \int_0^\infty f_{1(u,v)}(s)ds \leq & \int_0^\infty a(s)\Big(a_0(s) + a_1(s)|u(s)|^{\iota_1} \\ & + a_2(s)|v(s)|^{\iota_2} + a_3(s)|D_{0^+}^{\alpha-1}u(s)|^{\iota_3} + a_4(s)|D_{0^+}^{\beta-1}v(s)|^{\iota_4}\Big)ds \\ \leq & a_0^* + \int_0^\infty a(s)a_1(s)(1+t^{\alpha+\beta-1})^{\iota_1} \frac{|u(s)|^{\iota_1}}{(1+t^{\alpha+\beta-1})^{\iota_1}} ds \\ & + \int_0^\infty a(s)a_2(s)(1+t^{\alpha+\beta-1})^{\iota_2} \frac{|v(s)|^{\iota_2}}{(1+t^{\alpha+\beta-1})^{\iota_2}} ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty a(s)a_3(s)|D_{0^+}^{\alpha-1}u(s)|^{\iota_3}ds+\int_0^\infty a(s)a_4(s)|D_{0^+}^{\beta-1}v(s)|^{\iota_4}ds\\
&\le a_0^*+a_1^*\|u\|_X^{\iota_1}+a_2^*\|v\|_Y^{\iota_2}+a_3^*\|u\|_X^{\iota_3}+a_4^*\|v\|_Y^{\iota_4}\\
&\le a_0^*+\sum_{i=1}^4 a_i^*\|(u,v)\|_{X\times Y}^{\iota_i}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_0^\infty f_{2(u,v)}(s)ds &\le b_0^*+b_1^*\|u\|_X^{\tau_1}+b_2^*\|v\|_Y^{\tau_2}+b_3^*\|u\|_X^{\tau_3}+b_4^*\|v\|_Y^{\tau_4}\\
&\le b_0^*+\sum_{i=1}^4 b_i^*\|(u,v)\|_{X\times Y}^{\tau_i}.
\end{aligned}
$$

This completes the proof. □

Lemma 3.2. *If $(H_1),(H_2)$ and (H_3) hold, then the operator $\mathcal{F}:P\to P$ is completely continuous.*

Proof. To complete the proof, we divide it into the following five steps:

Step 1. We will show that $\mathcal{F}:P\to P$. Since $\mathcal{G}_i(t,s)\ge 0$, $\mathcal{G}_i^*(t,s)\ge 0 (i=1,2,3,4)$ and $f_1,f_2\ge 0$, it follows (3.1), (3.2) that $\mathcal{F}_1(u,v)\ge 0$, $\mathcal{F}_2(u,v)\ge 0$, $D_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)\ge 0$, $D_{0^+}^{\beta-1}\mathcal{F}_2(u,v)(t)\ge 0$, for any $(u,v)\in P$, $t\in\mathbb{R}^+$. Consequently, $\mathcal{F}:P\to P$.

Step 2. We will prove that $\mathcal{F}U$ is uniformly bounded.

Let $U=\{(u,v):(u,v)\in P,\|(u,v)\|_{X\times Y}\le L\}$ for some $L>0$. For all $(u,v)\in U$, from (3.1), Lemma 2.5 and Lemma 3.1, we have

$$
\begin{aligned}
\sup_{t\in\mathbb{R}^+}\frac{|\mathcal{F}_1(u,v)(t)|}{1+t^{\alpha+\beta-1}} &\le \sup_{t\in\mathbb{R}^+}\left|\int_0^\infty \frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}}f_{1(u,v)}(s)ds\right|\\
&\quad+\sup_{t\in\mathbb{R}^+}\left|\int_0^\infty \frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}}f_{2(u,v)}(s)ds\right|\\
&\le K_1\int_0^\infty |f_{1(u,v)}(s)|ds+K_2\int_0^\infty |f_{2(u,v)}(s)|ds\\
&\le (K_1+K_2)\left[a_0^*+\sum_{i=1}^4 a_i^*\|(u,v)\|_{X\times Y}^{\iota_i}+b_0^*+\sum_{i=1}^4 b_i^*\|(u,v)\|_{X\times Y}^{\tau_i}\right]\\
&\le (K_1+K_2)\left[a_0^*+b_0^*+\sum_{i=1}^4 (a_i^*L^{\iota_i}+b_i^*L^{\tau_i})\right],
\end{aligned}
\tag{3.3}
$$

and from (3.2), Lemma 2.5 and Lemma 3.1, we have

$$
\begin{aligned}
\sup_{t\in\mathbb{R}^+}|\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)| &\le \sup_{t\in\mathbb{R}^+}\left|\int_0^\infty \mathcal{G}_1^*(t,s)f_{1(u,v)}(s)ds\right|\\
&\quad+\sup_{t\in\mathbb{R}^+}\left|\int_0^\infty \mathcal{G}_2^*(t,s)f_{2(u,v)}(s)ds\right|\\
&\le Q_1\int_0^\infty |f_{1(u,v)}(s)|ds+Q_2\int_0^\infty |f_{2(u,v)}(s)|ds
\end{aligned}
$$

$$\le (Q_1+Q_2)\left[a_0^* + \sum_{i=1}^{4} a_i^* \|(u,v)\|_{X\times Y}^{\iota_i} + b_0^* + \sum_{i=1}^{4} b_i^* \|(u,v)\|_{X\times Y}^{\tau_i}\right]$$
$$\le (Q_1+Q_2)\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* L^{\iota_i} + b_i^* L^{\tau_i})\right]. \tag{3.4}$$

Thus

$$\begin{aligned}\|\mathcal{F}_1(u,v)\|_X &= \max\left\{\sup_{t\in\mathbb{R}^+}\frac{|\mathcal{F}_1(u,v)(t)|}{1+t^{\alpha+\beta-1}}, \sup_{t\in\mathbb{R}^+}|D_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)|\right\}\\ &\le \max\{K_1+K_2, Q_1+Q_2\}\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* L^{\iota_i} + b_i^* L^{\tau_i})\right],\end{aligned}$$

and similarly

$$\|\mathcal{F}_2(u,v)\|_Y \le \max\{K_3+K_4, Q_3+Q_4\}\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* L^{\iota_i} + b_i^* L^{\tau_i})\right].$$

Therefore

$$\begin{aligned}&\|\mathcal{F}(u,v)\|_{X\times Y}\\ =&\max\{\|\mathcal{F}_1(u,v)\|_X, \|\mathcal{F}_2(u,v)\|_Y\}\\ \le&\max\{K_1+K_2, Q_1+Q_2, K_3+K_4, Q_3+Q_4\}\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* L^{\iota_i} + b_i^* L^{\tau_i})\right],\end{aligned}$$

which implies that $\mathcal{F}U$ is uniformly bounded.

Step 3. Let $I\subset\mathbb{R}_+$ be any compact interval. For all $t_1, t_2\in I$, $t_2>t_1$ and $(u,v)\in U$, we have

$$\begin{aligned}\left|\frac{|\mathcal{F}_1(u,v)(t_2)|}{1+t_2^{\alpha+\beta-1}} - \frac{|\mathcal{F}_1(u,v)(t_1)|}{1+t_1^{\alpha+\beta-1}}\right| \le& \int_0^\infty \left|\frac{\mathcal{G}_1(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+t_1^{\alpha+\beta-1}}\right| \left|f_{1(u,v)}(s)\right| ds\\ &+\int_0^\infty \left|\frac{\mathcal{G}_2(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+t_1^{\alpha+\beta-1}}\right| \left|f_{2(u,v)}(s)\right| ds.\end{aligned} \tag{3.5}$$

In fact, for all $(u,v)\in U$, we have

$$\begin{aligned}&\int_0^\infty \left|\frac{\mathcal{G}_1(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+t_1^{\alpha+\beta-1}}\right| \left|f_{1(u,v)}(s)\right| ds\\ \le&\int_0^\infty \left|\frac{\mathcal{G}_1(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+t_2^{\alpha+\beta-1}}\right| \left|f_{1(u,v)}(s)\right| ds\\ &+\int_0^\infty \left|\frac{\mathcal{G}_1(t_1,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+t_1^{\alpha+\beta-1}}\right| \left|f_{1(u,v)}(s)\right| ds\\ =&\int_0^\infty \left|\frac{\mathcal{G}_1(t_2,s)-\mathcal{G}_1(t_1,s)}{1+t_2^{\alpha+\beta-1}}\right| \left|f_{1(u,v)}(s)\right| ds\\ &+\int_0^\infty \frac{\mathcal{G}_1(t_1,s)|t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}|}{(1+t_1^{\alpha+\beta-1})(1+t_2^{\alpha+\beta-1})} \left|f_{1(u,v)}(s)\right| ds\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_1} \left| \frac{\mathcal{G}_1(t_2,s)-\mathcal{G}_1(t_1,s)}{1+t_2^{\alpha+\beta-1}} \right| \left| f_{1(u,v)}(s) \right| ds + \int_{t_1}^{t_2} \left| \frac{\mathcal{G}_1(t_2,s)-\mathcal{G}_1(t_1,s)}{1+t_2^{\alpha+\beta-1}} \right| \left| f_{1(u,v)}(s) \right| ds \\
&\quad + \int_{t_2}^{\infty} \left| \frac{\mathcal{G}_1(t_2,s)-\mathcal{G}_1(t_1,s)}{1+t_2^{\alpha+\beta-1}} \right| \left| f_{1(u,v)}(s) \right| ds \\
&\quad + K_1 \int_0^{\infty} \frac{t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}}{1+t_2^{\alpha+\beta-1}} \left| f_{1(u,v)}(s) \right| ds \to 0 \quad t_1 \to t_2,
\end{aligned}
$$

and similarly

$$
\int_0^{\infty} \left| \frac{\mathcal{G}_2(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+t_1^{\alpha+\beta-1}} \right| \left| f_{2(u,v)}(s) \right| ds \to 0, \quad t_1 \to t_2.
$$

So the function $\frac{\mathcal{F}_1(u,v)(t)}{1+t^{\alpha+\beta-1}}$ is equicontinuous on I.

Note that $\mathcal{G}_1^*(t,s), \mathcal{G}_2^*(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+)$ and by Lemma 2.5, for all $t_1, t_2 \in I$, $t_2 > t_1$ and $(u,v) \in U$, we have

$$
\begin{aligned}
\left| D_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t_2) - D_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t_1) \right| &\leq \int_0^{+\infty} \left| \mathcal{G}_1^*(t_2,s) - \mathcal{G}_1^*(t_1,s) \right| \left| f_{1(u,v)}(s) \right| ds \\
&\quad + \int_0^{+\infty} \left| \mathcal{G}_2^*(t_2,s) - \mathcal{G}_2^*(t_1,s) \right| \left| f_{2(u,v)}(s) \right| ds.
\end{aligned} \tag{3.6}
$$

In fact, for all $(u,v) \in P$, we have

$$
\begin{aligned}
&\int_0^{+\infty} \left| \mathcal{G}_1^*(t_2,s) - \mathcal{G}_1^*(t_1,s) \right| \left| f_{1(u,v)}(s) \right| ds \\
&\leq \int_0^{t_1} \left| \mathcal{G}_1^*(t_2,s) - \mathcal{G}_1^*(t_1,s) \right| \left| f_{1(u,v)}(s) \right| ds + \int_{t_1}^{t_2} \left| \mathcal{G}_1^*(t_2,s) - \mathcal{G}_1^*(t_1,s) \right| \left| f_{1(u,v)}(s) \right| ds \\
&\quad + \int_{t_2}^{+\infty} \left| \mathcal{G}_1^*(t_2,s) - \mathcal{G}_1^*(t_1,s) \right| \left| f_{1(u,v)}(s) \right| ds \\
&\to 0, \quad t_1 \to t_2,
\end{aligned}
$$

and similarly

$$
\int_0^{+\infty} \left| \mathcal{G}_2^*(t_2,s) - \mathcal{G}_2^*(t_1,s) \right| \left| f_{2(u,v)}(s) \right| ds \to 0 \quad t_1 \to t_2.
$$

So $D_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)$ is equicontinuous on I. In the same way, we can show that $\frac{\mathcal{F}_2(u,v)(t)}{1+t^{\alpha+\beta-1}}$ and $D_{0^+}^{\beta-1}\mathcal{F}_2(u,v)(t)$ are equicontinuous. Thus condition (i) of Lemma 2.6 is satisfied.

Step 4. We will show that operators $\mathcal{F}_1, \mathcal{F}_2$ are equiconvergent at ∞. Since

$$
\lim_{t\to\infty} \frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}} = 0, \quad \lim_{t\to\infty} \frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}} = 0.
$$

We can infer that, for any $\epsilon > 0$, there exists a sufficiently large constant $C = C(\epsilon) > 0$, such that for any $t_1, t_2 \geq C$ and $s \in \mathbb{R}^+$, we have

$$
\left| \frac{\mathcal{G}_1(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+t_1^{\alpha+\beta-1}} \right| < \epsilon, \quad \left| \frac{\mathcal{G}_2(t_2,s)}{1+t_2^{\alpha+\beta-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+t_1^{\alpha+\beta-1}} \right| < \epsilon.
$$

Similarly,

$$\lim_{t\to+\infty}\mathcal{G}_1^*(t,s)\leq 1+\frac{\Omega_2\Gamma(\alpha)}{\Delta}\sum_{j=1}^{\infty}\frac{\lambda_j g_{\gamma_2}^{\alpha}(\xi_j,s)}{\Gamma(\alpha-\gamma_2)}+\frac{(\alpha-1)\Omega_2\delta_1(s)}{\Delta}<+\infty,$$
$$\lim_{t\to+\infty}\mathcal{G}_2^*(t,s)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Delta}\sum_{i=1}^{\infty}\frac{\eta_i g_{\gamma_1}^{\beta}(\sigma_i,s)}{\Gamma(\beta-\gamma_1)}+\frac{(\beta-1)\Gamma(\alpha)\delta_2(s)}{\Delta}<+\infty.$$

Also we can infer that, for any $\epsilon>0$, there exists a sufficiently large constant $C=C(\epsilon)>0$, such that for any $t_1,t_2\geq C$ and $s\in\mathbb{R}^+$, we have

$$|\mathcal{G}_1^*(t_2,s)-\mathcal{G}_1^*(t_1,s)|<\epsilon,\quad |\mathcal{G}_2^*(t_2,s)-\mathcal{G}_2^*(t_1,s)|<\epsilon.$$

By Lemma 3.1, (3.5), (3.6), we obtain that $\frac{\mathcal{F}_1(u,v)(t)}{1+t_1^{\alpha+\beta-1}}$, $D_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)$ are equiconvergent at $+\infty$. In the same way, we can show that $\frac{\mathcal{F}_2(u,v)(t)}{1+t^{\alpha+\beta-1}}$ and $D_{0^+}^{\beta-1}\mathcal{F}_2(u,v)(t)$ are equiconvergent at $+\infty$. Thus condition (ii) of Lemma 2.6 is satisfied.

Step 5. We will check the continuity of the operator $\mathcal{F}$.

Let $(u_n,v_n),(u,v)\in P$, such that $(u_n,v_n)\to(u,v)(n\to\infty)$. Then $\|(u_n,v_n)\|_{X\times Y}<\infty,\|(u,v)\|_{X\times Y}<\infty$. Similar to (3.3) and (3.4), we have

$$\begin{aligned}\sup_{t\in\mathbb{R}^+}\frac{|\mathcal{F}_1(u_n,v_n)(t)|}{1+t^{\alpha+\beta-1}}\leq&(K_1+K_2)\Bigg[a_0^*+b_0^*\\&+\sum_{i=1}^{4}(a_i^*\|(u_n,v_n)\|_{X\times Y}^{\iota_i}+b_i^*\|(u_n,v_n)\|_{X\times Y}^{\tau_i})\Bigg]\\<&\infty,\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in\mathbb{R}^+}|\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_n,v_n)(t)|\leq&(Q_1+Q_2)\Bigg[a_0^*+b_0^*\\&+\sum_{i=1}^{4}(a_i^*\|(u_n,v_n)\|_{X\times Y}^{\iota_i}+b_i^*\|(u_n,v_n)\|_{X\times Y}^{\tau_i})\Bigg]\\<&\infty.\end{aligned}$$

By the Lebesgue dominated convergence theorem and continuity of f_1,f_2, we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{\mathcal{F}_1(u_n,v_n)(t)}{1+t^{\alpha+\beta-1}}\\&=\lim_{n\to\infty}\left[\int_0^\infty\frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}}f_{1(u_n,v_n)}(s)ds+\int_0^\infty\frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}}f_{2(u_n,v_n)}(s)ds\right]\\&=\int_0^\infty\frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}}f_{1(u,v)}(s)ds+\int_0^\infty\frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}}f_{2(u,v)}(s)ds=\frac{\mathcal{F}_1(u,v)(t)}{1+t^{\alpha+\beta-1}},\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_n,v_n)(t)\\&=\lim_{n\to\infty}\left[\int_0^\infty\mathcal{G}_1^*(t,s)f_{1(u_n,v_n)}(s)ds+\int_0^\infty\mathcal{G}_2^*(t,s)f_{2(u_n,v_n)}(s)ds\right]\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mathcal{G}_1^*(t,s) f_{1(u,v)}(s)ds + \int_0^\infty \mathcal{G}_2^*(t,s) f_{2(u,v)}(s)ds \\
&= \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t).
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \frac{|\mathcal{F}_1(u_n,v_n)(t)-\mathcal{F}_1(u,v)(t)|}{1+t^{\alpha+\beta-1}} \\
\le &\sup_{t\in\mathbb{R}^+} \int_0^\infty \frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}} \left| f_{1(u_n,v_n)}(s) - f_{1(u,v)}(s)\right| ds \\
&+ \sup_{t\in\mathbb{R}^+} \int_0^\infty \frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}} \left| f_{2(u_n,v_n)}(s) - f_{2(u,v)}(s)\right| ds \\
\le &(K_1+K_2)\left[\int_0^\infty \left| f_{1(u_n,v_n)}(s) - f_{1(u,v)}(s)\right| ds + \int_0^\infty \left| f_{2(u_n,v_n)}(s) - f_{2(u,v)}(s)\right| ds\right] \\
\to &0, \quad n\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}^+} \left|\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_n,v_n)(t) - \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)\right| \\
\le &\sup_{t\in\mathbb{R}^+} \int_0^\infty \frac{\mathcal{G}_1^*(t,s)}{1+t^{\alpha+\beta-1}} \left| f_{1(u_n,v_n)}(s) - f_{1(u,v)}(s)\right| ds \\
&+ \sup_{t\in\mathbb{R}^+} \int_0^\infty \frac{\mathcal{G}_2^*(t,s)}{1+t^{\alpha+\beta-1}} \left| f_{2(u_n,v_n)}(s) - f_{2(u,v)}(s)\right| ds \\
\le &(Q_1+Q_2)\left[\int_0^\infty \left| f_{1(u_n,v_n)}(s) - f_{1(u,v)}(s)\right| ds + \int_0^\infty \left| f_{2(u_n,v_n)}(s) - f_{2(u,v)}(s)\right| ds\right] \\
\to &0, \quad n\to\infty.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|\mathcal{F}_1(u_n,v_n) - \mathcal{F}_1(u,v)\|_X \\
= &\max\left\{\sup_{t\in\mathbb{R}^+} \frac{|\mathcal{F}_1(u_n,v_n)-\mathcal{F}_1(u,v)(t)|}{1+t^{\alpha+\beta-1}}, \sup_{t\in\mathbb{R}^+}\left|\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_n,v_n)(t) - \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)\right|\right\} \\
\to &0, \quad n\to\infty.
\end{aligned}
$$

In consequence, $\mathcal{F}_1$ is continuous. In a similar way we can show that $\mathcal{F}_2$ is continuous. Thus $\mathcal{F}$ is continuous.

From the above steps, the operator $\mathcal{F}: P \to P$ is completely continuous. □

Define a partial order on the product space:

$$
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \ge \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}
$$

if $u_1(t)\ge u_2(t)$, $v_1(t)\ge v_2(t)$, $D_{0^+}^{\alpha-1}u_1(t) \ge D_{0^+}^{\alpha-1}u_2(t)$, $D_{0^+}^{\beta-1}v_1(t)\ge D_{0^+}^{\beta-1}v_2(t), t\in\mathbb{R}^+$.

Theorem 3.1. *If* $(H_1)-(H_4)$ *hold, then the problem* (1.1)-(1.2) *has positive maximal and minimal solutions* (u^*,v^*) *and* (x^*,y^*) *satisfying* $0<\|(u^*,v^*)\|_{X\times Y}\le R$ *and* $0<\|(x^*,y^*)\|_{X\times Y}\le R$ *with* $\lim_{n\to+\infty}(u_n,v_n)=(u^*,v^*)$ *and* $\lim_{n\to+\infty}(x_n,y_n)=(x^*,y^*)$, *where* R *is a given real constant,* (u_n,v_n) *and* (x_n,y_n) *can be given by the following monotone iterative sequences:*

$$\begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(u_{n-1},v_{n-1})(t) \\ \mathcal{F}_2(u_{n-1},v_{n-1})(t) \end{pmatrix}, n=1,2,\cdots, \text{with} \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} = \begin{pmatrix} Rt^{\alpha-1} \\ Rt^{\beta-1} \end{pmatrix}, \tag{3.7}$$

and

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(x_{n-1},y_{n-1})(t) \\ \mathcal{F}_2(x_{n-1},y_{n-1})(t) \end{pmatrix}, n=1,2,\cdots, \text{with} \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

In addition

$$\begin{aligned} &\begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} \le \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} u^*(t) \\ v^*(t) \end{pmatrix} \\ &\le \cdots \le \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} \le \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} \le \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_0(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_0(t) \end{pmatrix} \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_1(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_n(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_n(t) \end{pmatrix} \le \cdots \\ &\le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x^*(t) \\ \mathcal{D}_{0^+}^{\beta-1}y^*(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u^*(t) \\ \mathcal{D}_{0^+}^{\beta-1}v^*(t) \end{pmatrix} \le \cdots \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_n(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_n(t) \end{pmatrix} \\ &\le \cdots \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_2(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_2(t) \end{pmatrix} \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_1(t) \end{pmatrix} \le \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_0(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_0(t) \end{pmatrix}. \end{aligned} \tag{3.10}$$

Proof. Recall $\mathcal{F}P\subset P$. Let

$$R\ge \max\left\{10\Theta a_0^*, 10\Theta b_0^*, (10\Theta a_i^*)^{\frac{1}{1-\iota_i}}, (10\Theta b_i^*)^{\frac{1}{1-\tau_i}}, i=1,2,3,4\right\},$$

where

$$\Theta=\max\{K_1+K_2, Q_1+Q_2, K_3+K_4, Q_3+Q_4\}.$$

Set $U_R=\{(u,v)\in P:\|(u,v)\|_{X\times Y}\|\le R\}$. For any $(u,v)\in U_R$, similar to (3.3) and (3.4), we have

$$\sup_{t\in\mathbb{R}^+}\frac{|\mathcal{F}_1(u,v)(t)|}{1+t^{\alpha+\beta-1}}\le\Theta\left[a_0^*+b_0^*+\sum_{i=1}^4(a_i^*R^{\iota_i}+b_i^*R^{\tau_i})\right]\le R,$$

and

$$\sup_{t\in\mathbb{R}^+}\left|\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)\right|\le\Theta\left[a_0^*+b_0^*+\sum_{i=1}^4(a_i^*R^{\iota_i}+b_i^*R^{\tau_i})\right]\le R,$$

which infers that $\|\mathcal{F}_1(u,v)\|_X \le R$. In a similar way, $\|\mathcal{F}_2(u,v)\|_Y \le R$ for all $(u,v)\in U_R$. Then we know for all $(u,v)\in U_R$,

$$\|\mathcal{F}(u,v)\|_{X\times Y} = \max\{\|\mathcal{F}_1(u,v)\|_X, \|\mathcal{F}_2(u,v)\|_Y\} \le R.$$

That is $\mathcal{F}U_R \subset U_R$. From (3.7) and (3.8), we obtain that $(u_0(t),v_0(t)),(x_0(t),y_0(t)) \in R$. By the continuity of the operator $\mathcal{F}$, we define the sequences (u_n,v_n) and (x_n,y_n) as $(u_n,v_n) = \mathcal{F}(u_{n-1},v_{n-1}),(x_n,y_n) = \mathcal{F}(x_{n-1},y_{n-1})$ for $n = 1,2,\cdots$. Since $\mathcal{F}U_R \subset U_R$, we see that $(u_n,v_n),(x_n,y_n) \in \mathcal{F}U_R$, for $n = 1,2,\cdots$. For $t\in\mathbb{R}^+$, via Lemma 2.5, (3.1) and (3.7), we have

$$u_1(t) = \mathcal{F}_1(u_0,v_0)(t) \le \Theta\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* R^{\iota_i} + b_i^* R^{\tau_i})\right]t^{\alpha-1} \le Rt^{\alpha-1} = u_0(t),$$

and

$$v_1(t) = \mathcal{F}_2(u_0,v_0)(t) \le \Theta\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* R^{\iota_i} + b_i^* R^{\tau_i})\right]t^{\beta-1} \le Rt^{\beta-1} = v_0(t),$$

that is

$$\begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(u_0,v_0)(t) \\ \mathcal{F}_2(u_0,v_0)(t) \end{pmatrix} \le \begin{pmatrix} Rt^{\alpha-1} \\ Rt^{\beta-1} \end{pmatrix} = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}. \tag{3.11}$$

For $t\in\mathbb{R}^+$, via Lemma 2.5 and (3.11), we have

$$\begin{aligned} \mathcal{D}_{0^+}^{\alpha-1}u_1(t) &= \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_0,v_0)(t) \\ &\le \Gamma(\alpha)\Theta\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* R^{\iota_i} + b_i^* R^{\tau_i})\right] \\ &\le \Gamma(\alpha)R \\ &= \mathcal{D}_{0^+}^{\alpha-1}u_0(t), \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}_{0^+}^{\beta-1}v_1(t) &= \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(u_0,v_0)(t) \\ &\le \Gamma(\beta)\Theta\left[a_0^* + b_0^* + \sum_{i=1}^{4}(a_i^* R^{\iota_i} + b_i^* R^{\tau_i})\right] \\ &\le \Gamma(\beta)R \\ &= \mathcal{D}_{0^+}^{\beta-1}v_0(t), \end{aligned}$$

that is

$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_1(t) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_0,v_0)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(u_0,v_0)(t) \end{pmatrix} \le \begin{pmatrix} \Gamma(\alpha)R \\ \Gamma(\beta)R \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_0(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_0(t) \end{pmatrix}. \tag{3.12}$$

For $t\in\mathbb{R}^+$, via (3.11), (3.12) and (H_4), we get

$$\begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(u_1,v_1)(t) \\ \mathcal{F}_2(u_1,v_1)(t) \end{pmatrix} \le \begin{pmatrix} \mathcal{F}_1(u_0,v_0)(t) \\ \mathcal{F}_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_2(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_1,v_1)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(u_1,v_1)(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u_0,v_0)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_(t) \end{pmatrix}.$$

By recursion, for $t \in \mathbb{R}^+$, and $n = 0, 1, 2, \cdots$, we have

$$\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} \leq \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \quad \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_{n+1}(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_{n+1}(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_n(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_n(t) \end{pmatrix}. \tag{3.13}$$

Applying the complete continuity of $\mathcal{F}$ and $(u_{n+1}, v_{n+1}) = \mathcal{F}(u_n, v_n)$, there exists a $(u^*, v^*) \in U_R$ such $(u_{n_k}, v_{n_k}) \to (u^*, v^*)$ as $k \to \infty$, which can be obtained through that $\{(u_n, v_n)\}_{n=1}^{\infty}$ has a convergent subsequence $\{(u_{n_k}, v_{n_k})\}_{k=1}^{\infty}$. From (3.13), we have $(u_n, v_n) \to (u^*, v^*)$ as $n \to \infty$. We can also have $\mathcal{F}(u^*, v^*) = (u^*, v^*)$ by $\mathcal{F}$ is continuous, that is (u^*, v^*) is a fixed point of $\mathcal{F}$.

For the sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$, we take a similar discussion. For $t \in \mathbb{R}^+$, and $n = 0, 1, 2, \cdots$, we have

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{F}_1(x_0,y_0)(t) \\ \mathcal{F}_2(x_0,y_0)(t) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^\infty \mathcal{G}_1(t,s) f_{1(x_0,y_0)}(s)ds + \int_0^\infty \mathcal{G}_2(t,s) f_{2(x_0,y_0)}(s)ds \\ \int_0^\infty \mathcal{G}_3(t,s) f_{2(x_0,y_0)}(s)ds + \int_0^\infty \mathcal{G}_4(t,s) f_{1(x_0,y_0)}(s)ds \end{pmatrix} \\ &\geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_1(t) \end{pmatrix} &= \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(x_0,y_0)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(x_0,y_0)(t) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^\infty \mathcal{G}_1^*(t,s) f_{1(x_0,y_0)}(s)ds + \int_0^\infty \mathcal{G}_2^*(t,s) f_{2(x_0,y_0)}(s)ds \\ \int_0^\infty \mathcal{G}_3^*(t,s) f_{2(x_0,y_0)}(s)ds + \int_0^\infty \mathcal{G}_4^*(t,s) f_{1(x_0,y_0)}(s)ds \end{pmatrix} \\ &\geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_0(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_0(t) \end{pmatrix}. \end{aligned}$$

From (H_4) we have

$$\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(x_1,y_1)(t) \\ \mathcal{F}_2(x_1,y_1)(t) \end{pmatrix} \geq \begin{pmatrix} \mathcal{F}_1(x_0,y_0)(t) \\ \mathcal{F}_2(x_0,y_0)(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_2(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_2(t) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(x_1,y_1)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(x_1,y_1)(t) \end{pmatrix} \geq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(x_0,y_0)(t) \\ \mathcal{D}_{0^+}^{\beta-1}\mathcal{F}_2(x_0,y_0)(t) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_1(t) \end{pmatrix}.$$

Analogously, for $n = 0, 1, 2, \cdots$ and $t \in \mathbb{R}^+$, we get

$$\begin{pmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix}, \quad \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_{n+1}(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_{n+1}(t) \end{pmatrix} \geq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_n(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_n(t) \end{pmatrix}.$$

Applying the complete continuity of $\mathcal{F}$ and $(x_{n+1}, y_{n+1}) = \mathcal{F}(x_n, y_n)$, we have $(x_n, y_n) \to (x^*, y^*)$ as $n \to \infty$ and $\mathcal{F}(x^*, y^*) = (x^*, y^*)$, that is (x^*, y^*) is a fixed point of $\mathcal{F}$.

Finally, we will show that (u^*, v^*) and (x^*, y^*) are the positive maximal and minimal solutions for the system (1.1)-(1.2). Suppose that $(\zeta(t), \eta(t))$ is any positive solution of system (1.1)-(1.2), then $\mathcal{F}(\zeta(t), \eta(t)) = (\zeta(t), \eta(t))$ and

$$\begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} Rt^{\alpha-1} \\ Rt^{\alpha-1} \end{pmatrix} = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix},$$
$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_0(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_0(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\zeta(t) \\ \mathcal{D}_{0^+}^{\beta-1}\eta(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_0(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_0(t) \end{pmatrix}.$$

Applying the monotone property of the operator $\mathcal{F}$, we get

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(x_0,y_0)(t) \\ \mathcal{F}_2(x_0,y_0)(t) \end{pmatrix} \leq \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{F}_1(u_0,v_0)(t) \\ \mathcal{F}_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix},$$
$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_1(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\zeta(t) \\ \mathcal{D}_{0^+}^{\beta-1}\eta(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_1(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_1(t) \end{pmatrix}.$$

Repeating the above process, we have

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} \leq \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix} \leq \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix},$$
$$\begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}x_n(t) \\ \mathcal{D}_{0^+}^{\beta-1}y_n(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}\zeta(t) \\ \mathcal{D}_{0^+}^{\beta-1}\eta(t) \end{pmatrix} \leq \begin{pmatrix} \mathcal{D}_{0^+}^{\alpha-1}u_n(t) \\ \mathcal{D}_{0^+}^{\beta-1}v_n(t) \end{pmatrix}.$$

Combining $\lim_{n\to\infty}(u_n, v_n) = (u^*, v^*)$ and $\lim_{n\to\infty}(x_n, y_n) = (x^*, y^*)$, the results (3.9) and (3.10) hold.

Since $f_1(t, 0, 0, 0, 0) \not\equiv 0, f_2(t, 0, 0, 0, 0) \not\equiv 0, \forall\ t \in [0, \infty)$, then the system (1.1)-(1.2) have no zero solution, it is follows that $(u^*, v^*), (x^*, y^*) > (0, 0)$ are the extreme positive solutions of system(1.1)-(1.2), which are given by two monotone iterative sequences in (3.7) and (3.8). □

Theorem 3.2. *If* (H_1), (H_2), (H_5) *hold, and*

$$\mathcal{B} = \Theta \max\left\{\sum_{k=1}^{4} c_{1k}^*, \sum_{k=1}^{4} c_{2k}^*\right\} < 1. \tag{3.14}$$

Then (1.1)-(1.2) *has a unique positive solution* $(\varkappa^*, \varrho^*)$, *which can be obtained by the following iterative sequence:*

$$\begin{pmatrix} \varkappa_n(t) \\ \varrho_n(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(u_{n-1}, v_{n-1})(t) \\ \mathcal{F}_2(u_{n-1}, v_{n-1})(t) \end{pmatrix}, \quad n = 1, 2, \cdots,$$

for any $(\varkappa_0, \varrho_0) \in P$. *Furthermore, the following error estimate formula holds:*

$$\left\|(\varkappa^*, \varrho^*) - (\varkappa_n, \varrho_n)\right\|_{X\times Y} \le \frac{\mathcal{B}^n}{1-\mathcal{B}}\left\|(\varkappa_1, \varrho_1) - (\varkappa_0, \varrho_0)\right\|_{X\times Y}, \quad n = 1, 2, \cdots. \tag{3.15}$$

Proof. For any $(u, v) \in X \times Y$, by (H_5), we have

$$\begin{aligned}
&\left|f_j(t, u(t), v(t), \mathcal{D}_{0^+}^{\alpha-1}u(t), \mathcal{D}_{0^+}^{\beta-1}v(t))\right| \\
\le&\left|f_j(t, u(t), v(t), \mathcal{D}_{0^+}^{\alpha-1}u(t), \mathcal{D}_{0^+}^{\beta-1}v(t)) - f_j(t, 0, 0, 0, 0)\right| + |f_j(t, 0, 0, 0, 0)| \\
\le& c_{j1}(t)(1 + t^{\alpha+\beta-1})\frac{|u(t)|}{1 + t^{\alpha+\beta-1}} + c_{j2}(t)(1 + t^{\alpha+\beta-1})\frac{|v(t)|}{1 + t^{\alpha+\beta-1}} + c_{j3}(t)\left|\mathcal{D}_{0^+}^{\alpha-1}u(t)\right| \\
&+ c_{j4}(t)\left|\mathcal{D}_{0^+}^{\beta-1}v(t)\right| + |f_j(t, 0, 0, 0, 0)| \\
\le& c_{j1}(t)(1 + t^{\alpha+\beta-1})\|u\|_X + c_{j2}(t)(1 + t^{\alpha+\beta-1})\|v\|_Y \\
&+ c_{j3}(t)\|u\|_X + c_{j4}(t)\|v\|_Y + |f_j(t, 0, 0, 0, 0)|, \quad j = 1, 2.
\end{aligned}$$

So we have

$$\begin{aligned}
\int_0^\infty f_{j(u,v)}(s)ds &\le c_{j1}^*(t)\|u\|_X + c_{j2}^*(t)\|v\|_Y + c_{j3}^*(t)\|u\|_X + c_{j4}^*(t)\|v\|_Y + m_j \\
&\le \sum_{k=1}^4 c_{jk}^*\|(u, v)\|_{X\times Y} + m_j, \quad j = 1, 2.
\end{aligned}$$

Thus take $r \ge \frac{\Theta\widetilde{m}}{1-\mathcal{B}}$, where $\widetilde{m} = \max\{m_1, m_2\}$. Next we show that $\mathcal{F}U_r \subset U_r$, where $U_r = \{(u, v) \in P : \|(u, v)\|_{X\times Y} \le r\}$. For any $(u, v) \in U_r$, we have

$$\sup_{t\in\mathbb{R}^+} \frac{|\mathcal{F}_1(u, v)(t)|}{1 + t^{\alpha+\beta-1}} \le \Theta\left[\max\left\{\sum_{k=1}^4 c_{1k}^* r, \sum_{k=1}^4 c_{2k}^* r\right\} + \widetilde{m}\right] \le r,$$

and

$$\sup_{t\in\mathbb{R}^+} |\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u, v)(t)| \le \Theta\left[\max\left\{\sum_{k=1}^4 c_{1k}^* r, \sum_{k=1}^4 c_{2k}^* r\right\} + \widetilde{m}\right] \le r,$$

which infers that $\|\mathcal{F}_1(u, v)\|_X \le r$. In a similar way, $\|\mathcal{F}_2(u, v)\|_Y \le r$ for all $(u, v) \in U_r$. Then we know for all $(u, v) \in U_r$,

$$\|\mathcal{F}(u, v)\|_{X\times Y} = \max\{\|\mathcal{F}_1(u, v)\|_X, \|\mathcal{F}_2(u, v)\|_Y\} \le r.$$

That is $\mathcal{F}U_r \subset U_r$.

Now we show that operator $\mathcal{F}$ is a contraction. In view of (H_5), for any $(x,y),(u,v)\in U_r$, we can get

$$\begin{aligned}
&\left|\frac{\mathcal{F}_1(x,y)(t)}{1+t^{\alpha+\beta-1}}-\frac{\mathcal{F}_1(u,v)(t)}{1+t^{\alpha+\beta-1}}\right|\\
\leq&\int_0^\infty \frac{\mathcal{G}_1(t,s)}{1+t^{\alpha+\beta-1}}\left|f_{1(x,y)}(s)-f_{1(u,v)}(s)\right|ds\\
&+\int_0^\infty \frac{\mathcal{G}_2(t,s)}{1+t^{\alpha+\beta-1}}\left|f_{2(x,y)}(s)-f_{2(u,v)}(s)\right|ds\\
\leq& K_1\int_0^\infty a(s)\bigg[c_{11}(s)(1+s^{\alpha+\beta-1})\frac{|x(s)-u(s)|}{1+s^{\alpha+\beta-1}}+c_{12}(s)(1+s^{\alpha+\beta-1})\frac{|y(s)-v(s)|}{1+s^{\alpha+\beta-1}}\\
&+c_{13}(s)|\mathcal{D}_{0^+}^{\alpha-1}x(s)-\mathcal{D}_{0^+}^{\alpha-1}u(s)|+c_{14}(s)|\mathcal{D}_{0^+}^{\beta-1}y(s)-\mathcal{D}_{0^+}^{\beta-1}v(s)|\bigg]ds\\
&+K_2\int_0^\infty b(s)\bigg[c_{21}(s)(1+s^{\alpha+\beta-1})\frac{|x(s)-u(s)|}{1+s^{\alpha+\beta-1}}+c_{22}(s)(1+s^{\alpha+\beta-1})\frac{|y(s)-v(s)|}{1+s^{\alpha+\beta-1}}\\
&+c_{23}(s)|\mathcal{D}_{0^+}^{\alpha-1}x(s)-\mathcal{D}_{0^+}^{\alpha-1}u(s)|+c_{24}(s)|\mathcal{D}_{0^+}^{\beta-1}y(s)-\mathcal{D}_{0^+}^{\beta-1}v(s)|\bigg]ds\\
\leq&\left(K_1\sum_{k=1}^4 c_{1k}^*+K_2\sum_{k=1}^4 c_{2k}^*\right)\|(x,y)-(u,v)\|_{X\times Y},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(x,y)(t)-\mathcal{D}_{0^+}^{\alpha-1}\mathcal{F}_1(u,v)(t)\right|\\
\leq&\int_0^\infty \mathcal{G}_1^*(t,s)\left|f_{1(x,y)}(s)-f_{1(u,v)}(s)\right|ds+\int_0^\infty \mathcal{G}_2^*(t,s)\left|f_{2(x,y)}(s)-f_{2(u,v)}(s)\right|ds\\
\leq&\left(Q_1\sum_{k=1}^4 c_{1k}^*+Q_2\sum_{k=1}^4 c_{2k}^*\right)\|(x,y)-(u,v)\|_{X\times Y},
\end{aligned}$$

which implies

$$\|\mathcal{F}_1(x,y)-\mathcal{F}_1(u,v)\|_X\leq\Theta\max\left\{\sum_{k=1}^4 c_{1k}^*,\sum_{k=1}^4 c_{2k}^*\right\}\|(x,y)-(u,v)\|_{X\times Y}.$$

In the same way, we can get

$$\|\mathcal{F}_2(x,y)-\mathcal{F}_2(u,v)\|_X\leq\Theta\max\left\{\sum_{k=1}^4 c_{1k}^*,\sum_{k=1}^4 c_{2k}^*\right\}\|(x,y)-(u,v)\|_{X\times Y}.$$

So we have

$$\|\mathcal{F}(x,y)-\mathcal{F}(u,v)\|_{X\times Y}\leq\mathcal{B}\|(x,y)-(u,v)\|_{X\times Y},\quad\forall\ (x,y),\ (u,v)\in U_r.$$

Due to $\mathcal{B}<1$ as shown in (3.14), then operator $\mathcal{F}$ is a contraction map. With the help of the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point $(\varkappa^*,\varrho^*)$ in U_r. And the error estimate formula (3.15) holds. □

Theorem 3.3. *If* $(H_1)-(H_5)$ *hold, then* (1.1)-(1.2) *has a unique positive solution* $(\varkappa^*, \varrho^*)$ *such that* $\|(\varkappa^*, \varrho^*)\| \in (0, R]$, *which can be obtained by the following iterative sequence:*

$$\begin{pmatrix} \varkappa_n(t) \\ \varrho_n(t) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(u_{n-1}, v_{n-1})(t) \\ \mathcal{F}_2(u_{n-1}, v_{n-1})(t) \end{pmatrix}, \quad n = 1, 2, \cdots,$$

with the initial value $(\varkappa_0(t), \varrho_0(t)) = (0, 0)$ *or* $(\varkappa_0(t), \varrho_0(t)) = (Rt^{\alpha-1}, Rt^{\beta-1})$.

Proof. By using Theorems 3.1 and 3.2, the conclusions are obvious. □

4. Applications

Example 4.1. Consider the following coupled fractional differential system

$$\begin{cases} \mathcal{D}_{0^+}^{2.5}u(t) + e^{-t}\Big[\dfrac{2}{(9+t)^2} + \dfrac{te^{-t}|u(t)|^{0.1}}{(1+t^{3.6})^{0.1}} + \dfrac{te^{-2t}|v(t)|^{0.3}}{(1+t^{3.6})^{0.3}} \\ \quad + te^{-10t}|\mathcal{D}_{0^+}^{1.5}u(t)|^{0.4} + \dfrac{|\mathcal{D}_{0^+}^{1.1}v(t)|^{0.1}}{1+t^2}\Big] = 0, \\ \mathcal{D}_{0^+}^{2.1}v(t) + e^{-2t}\Big[\dfrac{1}{(1+t)^2} + \dfrac{te^{-3t}|u(t)|^{0.2}}{(1+t^{3.6})^{0.2}} + \dfrac{te^{-4t}|v(t)|^{0.4}}{(1+t^{3.6})^{0.4}} \\ \quad + \dfrac{t^3|\mathcal{D}_{0^+}^{1.5}u(t)|^{0.2}}{(1+t^3)^2} + \dfrac{|\mathcal{D}_{0^+}^{1.1}v(t)|^{0.3}}{5(1+t^2)}\Big] = 0, \end{cases} \tag{4.1}$$

with an improper integral and the infinite-point boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \ \lim\limits_{t\to\infty} \mathcal{D}_{0^+}^{1.5}u(t) = \displaystyle\int_0^\infty h_1(t)v'(t)dt + \sum_{i=1}^\infty \frac{1}{4^i}\mathcal{D}_{0^+}^{0.9}v(1 - \frac{1}{i+2}), \\ v(0) = v'(0) = 0, \ \lim\limits_{t\to\infty} \mathcal{D}_{0^+}^{1.1}v(t) = \displaystyle\int_0^\infty h_2(t)u'(t)dt + \sum_{j=1}^\infty \frac{1}{2^j}\mathcal{D}_{0^+}^{1}u(1 - \frac{1}{j+1}), \end{cases} \tag{4.2}$$

where $h_1(t) = \frac{1}{4}e^{-t}$, $h_2(t) = \frac{1}{4}e^{-2t}$, $a(t) = e^{-t}$, $b(t) = e^{-2t}$ and

$$f_1(t, u_1, u_2, u_3, u_4) = \frac{2}{(9+t)^2} + \frac{te^{-t}|u_1|^{0.1}}{(1+t^{3.6})^{0.1}} + \frac{te^{-2t}|u_2|^{0.3}}{(1+t^{3.6})^{0.3}} + te^{-10t}|u_3|^{0.4} + \frac{|u_4|^{0.1}}{1+t^2},$$

$$f_2(t, u_1, u_2, u_3, u_4) = \frac{1}{(1+t)^2} + \frac{te^{-3t}|u_1|^{0.2}}{(1+t^{3.6})^{0.2}} + \frac{te^{-4t}|u_2|^{0.4}}{(1+t^{3.6})^{0.4}} + \frac{t^3|u_3|^{0.2}}{(1+t^3)^2} + \frac{|u_4|^{0.3}}{5(1+t^2)},$$

and $\iota_1 = 0.1$, $\iota_2 = 0.3$, $\iota_3 = 0.4$, $\iota_4 = 0.1$, $\tau_1 = 0.2$, $\tau_2 = 0.4$, $\tau_3 = 0.2$, $\tau_4 = 0.3$.

By calculating, we can obtain $\Omega_1 \approx 1.286806$, $\Omega_2 \approx 0.614382$, $\Delta = \Gamma(2.5)\Gamma(2.1) - \Omega_1\Omega_2 \approx 0.600546 > 0$, and we take $R = 100$. Thus hypothesis $(H_1), (H_2)$ are satisfied.

Also we have

$$\begin{aligned} |f_1(t, u_1, u_2, u_3, u_4)| \leqslant & \frac{2}{(9+t)^2} + \frac{te^{-t}|u_1|^{0.1}}{(1+t^{3.6})^{0.1}} + \frac{te^{-2t}|u_2|^{0.3}}{(1+t^{3.6})^{0.3}} + te^{-10t}|u_3|^{0.4} + \frac{|u_4|^{0.1}}{1+t^2}, \\ = & a_0(t) + a_1(t)|u_1|^{\iota_1} + a_2(t)|u_2|^{\iota_2} + a_3(t)|u_3|^{\iota_3} + a_4(t)|u_4|^{\iota_4}, \end{aligned}$$

$$
\begin{aligned}
|f_2(t,u_1,u_2,u_3,u_4)| \leqslant & \frac{1}{(1+t)^2} + \frac{te^{-3t}|u_1|^{0.2}}{(1+t^{3.6})^{0.2}} + \frac{te^{-4t}|u_2|^{0.4}}{(1+t^{3.6})^{0.4}} + \frac{t^3|u_3|^{0.2}}{(1+t^3)^2} + \frac{|u_4|^{0.3}}{5(1+t^2)}, \\
= & b_0(t) + b_1(t)|u_1|^{\tau_1} + b_2(t)|u_2|^{\tau_2} + b_3(t)|u_3|^{\tau_3} + b_4(t)|u_4|^{\tau_4},
\end{aligned}
$$

and

$$
\begin{aligned}
a_0^* &= \int_0^\infty a(t)a_0 dt \approx 0.0199, & b_0^* &= \int_0^\infty b(t)b_0(t)dt \approx 0.2325, \\
a_1^* &= \int_0^\infty a(t)a_1(t)(1+t^{3.6})^{0.1}dt = \frac{1}{4}, & b_1^* &= \int_0^\infty b(t)b_1(t)(1+t^{3.6})^{0.2}dt = \frac{1}{25}, \\
a_2^* &= \int_0^\infty a(t)a_2(t)(1+t^{3.6})^{0.3}dt = \frac{1}{9}, & b_2^* &= \int_0^\infty b(t)b_2(t)(1+t^{3.6})^{0.4}dt = \frac{1}{36}, \\
a_3^* &= \int_0^\infty a(t)a_3(t)dt = \frac{1}{121}, & b_3^* &= \int_0^\infty b(t)b_3(t)dt \approx 0.1343, \\
a_4^* &= \int_0^\infty a(t)a_4(t)dt \approx 0.4971, & b_4^* &= \int_0^\infty b(t)b_4(t)dt \approx 0.0994,
\end{aligned}
$$

so hypothesis (H_3) is satisfied.

It is easy to verify that f_1, f_2 are increasing with respect to the variables u, v, x, y and $f_1(t,0,0,0,0) \not\equiv 0, f_2(t,0,0,0,0) \not\equiv 0, \forall\ t \in \mathbb{R}_+$. Thus hypothesis (H_4) is satisfied. By Theorem 3.1, it follows that the fractional differential system (4.1)-(4.2) has positive maximal and minimal solutions, which can be established via two explicit monotone iterative sequences in 3.7 and 3.8.

Also the fractional differential system (4.1)-(4.2) simulate iterative process curve are provided using the iterative method and numerical simulation in Table 1, Table 2, Figure 1 and Figure 2.

Table 1. Iterative process x_n, $y_n (n = 1, 2, 3, 4, 5, 6)$ in Example 4.1.

	t					
	0.0	2.0	4.0	6.0	8.0	10.0
$x_1(t)$	0.000 000	0.225 935	0.616 439	1.108 379	1.692 811	2.353 756
$x_2(t)$	0.000 000	3.048 702	7.578 297	13.086 848	19.541 75	26.787 124
$x_3(t)$	0.000 000	4.041 659	10.074 574	17.421 678	26.035 894	35.707 984
$x_4(t)$	0.000 000	4.194 344	10.459 264	18.090 858	27.039 519	37.087 653
$x_5(t)$	0.000 000	4.215 256	10.511 972	18.182 569	27.177 086	37.276 785
$x_6(t)$	0.000 000	4.218 076	10.519 08	18.194 938	27.195 64	37.302 294
$y_1(t)$	0.000 000	0.312 061	0.571 329	0.837 741	1.114 685	1.397 957
$y_2(t)$	0.000 000	3.526 149	7.325 612	11.277 563	15.403 331	19.632 984
$y_3(t)$	0.000 000	4.695 581	9.722 778	14.949 905	20.406 363	25.999 91
$y_4(t)$	0.000 000	4.876 775	10.092 658	15.515 701	21.176 582	26.979 638
$y_5(t)$	0.000 000	4.901 619	10.143 339	15.593 207	21.282 077	27.113 819
$y_6(t)$	0.000 000	4.904 97	10.150 173	15.603 659	21.296 303	27.131 914

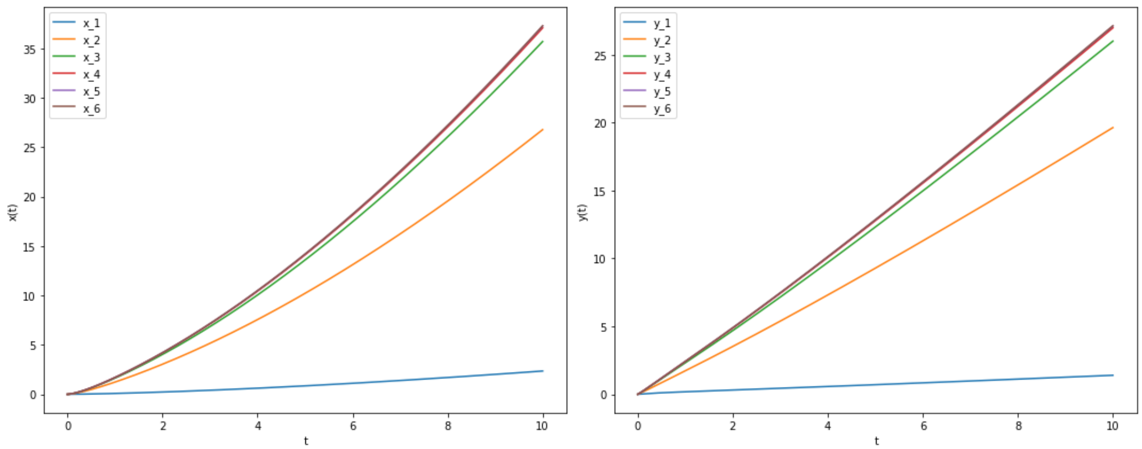

Figure 1. The iterative process of x_n, y_n.

Table 2. Iterative process u_n, $v_n (n = 1, 2, 3, 4, 5, 6)$ in Example 4.1.

	t					
	0.0	2.0	4.0	6.0	8.0	10.0
$u_1(t)$	0.000 000	7.301 772	18.345 892	31.859 098	47.731 960	65.571 558
$u_2(t)$	0.000 000	4.549 036	11.353 965	19.648 333	29.376 408	40.301 066
$u_3(t)$	0.000 000	4.261 714	10.629 085	18.386 363	27.482 800	37.697 106
$u_4(t)$	0.000 000	4.224 304	10.534 779	18.222 254	27.236 616	37.358 630
$u_5(t)$	0.000 000	4.219 293	10.522 148	18.200 275	27.203 647	37.313 302
$u_6(t)$	0.000 000	4.218 619	10.520 450	18.197 320	27.199 214	37.307 207
$v_1(t)$	0.000 000	8.613 898	17.645 020	27.025 490	36.814 290	46.847 301
$v_2(t)$	0.000 000	5.298 957	10.952 844	16.830 606	22.965 920	29.255 204
$v_3(t)$	0.000 000	4.956 835	10.255 948	15.765 405	21.516 449	27.411 912
$v_4(t)$	0.000 000	4.912 370	10.165 268	15.626 743	21.327 722	27.171 875
$v_5(t)$	0.000 000	4.906 416	10.153 123	15.608 170	21.302 443	27.139 723
$v_6(t)$	0.000 000	4.905 615	10.151 490	15.605 673	21.299 044	27.135 400

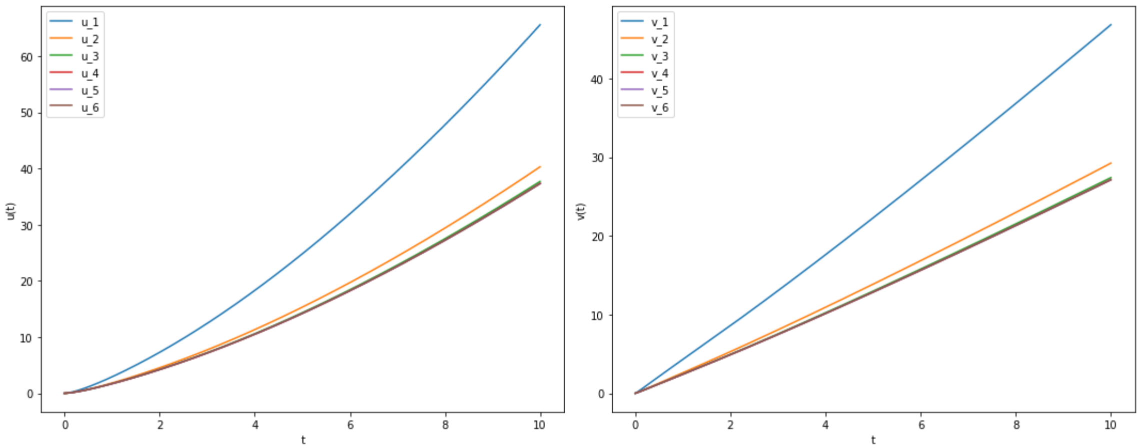

Figure 2. The iterative process of u_n, v_n.

Example 4.2. Consider the following coupled fractional differential system

$$\begin{cases} \mathcal{D}_{0^+}^{2.5}u(t) + \dfrac{1}{10t}\Bigg[\dfrac{5t}{(1+t)^2} + \dfrac{te^{-t}|u(t)|}{(1+t^{3.6})^{0.2}} + \dfrac{te^{-2t}|v(t)|}{(1+t^{3.6})^{0.4}} \\ \quad + te^{-10t}|\mathcal{D}_{0^+}^{1.5}u(t)| + \dfrac{|\mathcal{D}_{0^+}^{1.1}v(t)|}{1+e^{2t}}\Bigg] = 0, \\ \mathcal{D}_{0^+}^{2.1}v(t) + \dfrac{1}{10t^2}\Bigg[\dfrac{t^2}{(1+t)^2} + \dfrac{t^2e^{-3t}|u(t)|}{(1+t^{3.6})^{0.3}} + \dfrac{t^3e^{-4t}|v(t)|}{(1+t^{3.6})^{0.5}} \\ \quad + \dfrac{t^3|\mathcal{D}_{0^+}^{1.5}u(t)|}{(1+t^3)^2} + \dfrac{t^4|\mathcal{D}_{0^+}^{1.1}v(t)|}{5(1+e^{2t})}\Bigg] = 0, \end{cases} \tag{4.3}$$

with an improper integral and the infinite-point boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \ \lim\limits_{t\to\infty} \mathcal{D}_{0^+}^{1.5}u(t) = \displaystyle\int_0^\infty h_1(t)v'(t)dt + \sum_{i=1}^{\infty} \frac{1}{4^i}\mathcal{D}_{0^+}^{0.9}v(1-\frac{1}{i+2}), \\ v(0) = v'(0) = 0, \ \lim\limits_{t\to\infty} \mathcal{D}_{0^+}^{1.1}v(t) = \displaystyle\int_0^\infty h_2(t)u'(t)dt + \sum_{j=1}^{\infty} \frac{1}{2^j}\mathcal{D}_{0^+}^{1}u(1-\frac{1}{j+1}), \end{cases} \tag{4.4}$$

where $h_1(t) = \frac{1}{4}e^{-t}$, $h_2(t) = \frac{1}{4}e^{-2t}$, $a(t) = \frac{1}{10t}, b(t) = \frac{1}{10t^2}$ and

$$f_1(t,u_1,u_2,u_3,u_4) = \frac{5t}{(1+t)^2} + \frac{te^{-t}|u_1|}{(1+t^{3.6})^{0.2}} + \frac{te^{-2t}|u_2|}{(1+t^{3.6})^{0.4}} + te^{-10t}|u_3| + \frac{|u_4|}{1+e^{2t}},$$
$$f_2(t,u_1,u_2,u_3,u_4) = \frac{t^2}{(1+t)^2} + \frac{t^2e^{-3t}|u_1|}{(1+t^{3.6})^{0.3}} + \frac{t^3e^{-4t}|u_2|}{(1+t^{3.6})^{0.5}} + \frac{t^3|u_3|}{(1+t^3)^2} + \frac{t^4|u_4|}{5(1+e^{2t})},$$

and $\iota_1 = 0.2$, $\iota_2 = 0.4$, $\iota_3 = 0.4$, $\iota_4 = 0.1$, $\tau_1 = 0.3$, $\tau_2 = 0.5$, $\tau_3 = 0.2$, $\tau_4 = 0.3$.

By calculating, we can obtain $\Omega_1 \approx 1.286806$, $\Omega_2 \approx 0.614382$, $\Delta = \Gamma(2.5)\Gamma(2.1) - \Omega_1\Omega_2 \approx 0.600546 > 0$. Thus hypothesis $(H_1), (H_2)$ are satisfied.

Also we have

$$|f_1(t,u_1,u_2,u_3,u_4) - f_1(t,\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)|$$

$$
\begin{aligned}
&\le \frac{te^{-t}}{(1+t^{3.6})^{0.2}}|u_1-\overline{u}_1| + \frac{te^{-2t}}{(1+t^{3.6})^{0.4}}|u_2-\overline{u}_2| + te^{-10t}|u_3-\overline{u}_3| + \frac{1}{1+e^{2t}}|u_4-\overline{u}_4| \\
&= c_{11}(t)|u_1-\overline{u}_1| + c_{12}(t)|u_2-\overline{u}_2| + c_{13}(t)|u_3-\overline{u}_3| + c_{14}(t)|u_4-\overline{u}_4|, \\
&\quad |f_2(t,u_1,u_2,u_3,u_4) - f_2(t,\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)| \\
&\le \frac{t^2e^{-3t}}{(1+t^{3.6})^{0.3}}|u_1-\overline{u}_1| + \frac{t^3e^{-4t}}{(1+t^{3.6})^{0.5}}|u_2-\overline{u}_2| + \frac{t^3}{(1+t^3)^2}|u_3-\overline{u}_3| + \frac{t^4}{5(1+e^{2t})}|u_4-\overline{u}_4| \\
&= c_{21}(t)|u_1-\overline{u}_1| + c_{22}(t)|u_2-\overline{u}_2| + c_{23}(t)|u_3-\overline{u}_3| + c_{24}(t)|u_4-\overline{u}_4|,
\end{aligned}
$$

and

$$
\begin{aligned}
c_{11}^* &= \int_0^\infty a(t)c_{11}(t)(1+t^{3.6})dt \approx 0.0455, & c_{12}^* &= \int_0^\infty a(t)c_{12}(t)(1+t^{3.6})dt \approx 0.0194, \\
c_{13}^* &= \int_0^\infty a(t)c_{13}(t)dt = \frac{1}{100}, & c_{14}^* &= \int_0^\infty a(t)c_{14}(t)dt \approx 0.0347, \\
c_{21}^* &= \int_0^\infty b(t)c_{21}(t)(1+t^{3.6})dt \approx 0.0067, & c_{22}^* &= \int_0^\infty b(t)c_{22}(t)(1+t^{3.6})dt \approx 0.0012, \\
c_{23}^* &= \int_0^\infty b(t)c_{23}(t)dt = \frac{1}{30}, & c_{24}^* &= \int_0^\infty b(t)c_{24}(t)dt \approx 0.0036,
\end{aligned}
$$

so hypothesis (H_5) is satisfied. By direct computation, we can obtain that $K_1 + K_2 \approx 2.7633$, $Q_1 + Q_2 \approx 3.1005$, $K_3 + K_4 \approx 4.0297$, $Q_3 + Q_4 \approx 4.1755$, $\Theta = 4.1755$, $\mathcal{B} = \Theta \max\big\{\sum_{k=1}^4 c_{1k}^*, \sum_{k=1}^4 c_{2k}^*\big\} \le \Theta \times \max\{0.1096, 0.0442\} \approx 0.4576 < 1$.

Hence all presupposed conditions of Theorem 3.2 are satisfied. Then the fractional differential system (4.3)-(4.4) has a unique positive solution.

5. Conclusion

In this article, we demonstrated the solvability of the coupled system of fractional boundary value problems involving an improper integral and the infinite-point on the half-line through the monotonic iterative method, obtaining the iterative relation of the system. Using Banach's contraction mapping principle, the existence and unique of the positive solution is demonstrated. Finally, a numerical simulation is given to illustrate the main results.

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