Journal of Applied Analysis and Computation
Volume 16, Number 1, February 2026, 45–75
Website:http://www.jaac-online.com
DOI:10.11948/20250124

GENERALIZED NONISOSPECTRAL MULTI-COMPONENT SUPER INTEGRABLE HIERARCHY AND DARBOUX TRANSFORMATION

Haifeng Wang, Jinxiu Li and Huiqin Cao†

Abstract Based on Lie superalgebra spl(2N,1), a generalized nonisospectral multi-component super Ablowitz-Kaup-Newell-Segur (AKNS) integrable hierarchy is obtained. Then, we present a generalized nonisospectral three-component coupled super AKNS integrable hierarchy associated with Lie superalgebra spl(6,1) which is a a special case of the Lie superalgebra spl(2N,1) when $N = 3$. Using of supertrace identity, the super bi-Hamiltonian structures of the generalized multi-component and three-component coupled super AKNS integrable hierarchies are obtained. Additionally, we investigate the Darboux transformation of the generalized nonisospectral three-component coupled super AKNS integrable hierarchy.

Keywords Lie superalgebra, generalized multi-component super AKNS hierarchy, super bi-Hamiltonian structure.

MSC(2010) 37K05, 37K40, 35Q53.

1. Introduction

In recent years, super integrable systems have received much attention in the field of soliton and integrable systems. Super integrable systems have many features, including supersymmetry [17,24], conservation laws [19,25], and so on [5–7,9]. Various methods have been used to generate integrable hierarchies, such as the AKNS method [1,3,4,11,16], the Lax pair method [2], the Tu scheme [29], and so on. Among that, the Tu scheme is a powerful tool for generating the Hamiltonion structures by introducing trace identity. Many integrable hierarchies and their Hamiltonian structures have been obtained by means of Tu scheme [23,32,37]. Based on the Tu scheme, a method was proposed for deriving isospectral and nonisospectral integrable hierarchies by Zhang et al. [39,40]. Moreover, Wang and Zhang [33] investigated the generation of the infinite-dimensional isospectral and nonisospectral integrable hierarchies by constructing a multi-component non-semisimple Lie algebra.

Many mathematical physicists were interested in the derivation of the generalized hierarchies of soliton equations [8,10,30,36]. Shen [27] obtained a generalized integrable hierarchy by introducing a perturbation term in spectral problems for AKNS integrable coupling. Shen generalized the spatial spectral problem of the AKNS integrable system as follows:

$$\overline{Y} = \begin{pmatrix} Y_1 \; Y_2 \\ 0 \;\; Y_1 \end{pmatrix} = \begin{pmatrix} \lambda+h & p_1 & 0 & p_2 \\ q_1 & -\lambda-h & q_2 & 0 \\ 0 & 0 & \lambda+h & p_1 \\ 0 & 0 & q_1 & -\lambda-h \end{pmatrix}, \quad h = \eta(p_1q_2 + q_1p_2). \tag{1.1}$$

†The corresponding author.
School of Science, Jimei University, Xiamen, Fujian 361021, China
Email: tb19080005b4@cumt.edu.cn(H. Wang), 15070192968@163.com(J. Li), caohuiqin66@163.com(H. Cao)

Here, h is a nonlinear perturbation term, and η is an arbitrary even constant. In this paper, we extend the above two-dimensional generalized AKNS spectral problem to its N-dimensional counterpart. To our knowledge, this generalization remains largely unexplored in the literature. Moreover, Han and Yu [12] derived a generalized super integrable AKNS hierarchy associated with Lie superalgebra $spl(2,1)$. Then a nonlinear generalized super AKNS integrable coupling was derived by Hu et al. [14]. However, the investigation of multi-component integrable hierarchies is notably lacking in mathematical physics [15,26,38], and generalized multi-component systems have been virtually unexplored. Recently, Wang et al. [34] obtained a multi-component super integrable Dirac hierarchy by constructing a new type of multi-component Lie superalgebra $sl(2N,1)$. Then they also realized a multi-component extension of Lie superalgebra $spl(2,1)$ to Lie superalgebra $spl(2N,1)$ [31]. In this work, we construct a new class of higher-dimensional generalized nonisospectral AKNS hierarchy based on a newly developed higher-dimensional Lie superalgebra $spl(2N,1)$.

This paper is organized as follows. In Section 2, we review the form of the Lie superalgebra $spl(2N,1)$ and its commutation relations, along with some related concepts. Based on our previous work, we apply the Lie superalgebra $spl(2N,1)$ to a generalized spectral problem by introducing a perturbation term. In Section 3, we obtain a generalized nonisospectral multi-component super AKNS hierarchy associated with the Lie superalgebra $spl(2N,1)$. In Section 4, we obtain a generalized nonisospectral three-component coupled super AKNS integrable hierarchy associated with the Lie superalgebra $spl(6,1)$ and derive its Darboux transformation. Using the supertrace identity [21], the super bi-Hamiltonian structures of the generalized nonisospectral multi-component and three-component coupled super AKNS integrable hierarchies are obtained.

2. Preliminaries

Given a superalgebra $\mathcal{A}'$ with elements $x,y\in\mathcal{A}'$, where xy denotes its multiplicative operation, we define a new bracket operation $\langle x,y\rangle$ through the following graded (anti-)commutation relations [28]:
(1) When either $x\in\mathcal{A}'_0$ or $y\in\mathcal{A}'_0$:

$$\langle x,y\rangle=[x,y]=xy-yx.$$

(2) When both $x\in\mathcal{A}'_1$ and $y\in\mathcal{A}'_1$:

$$\langle x,y\rangle=[x,y]_+=xy+yx.$$

This bracket operation induces a Lie superalgebra $\mathcal{A}$. Here $\mathcal{A}'$=$\mathcal{A}'_0\oplus\mathcal{A}'_1$, $\mathcal{A}'_0$ is called the even subspace of $\mathcal{A}'$ and $\mathcal{A}'_1$ is called the odd.

Let us review a basis for the Lie superalgebra spl(2,1)

$$\begin{aligned}
&E_1=\begin{pmatrix} e_1 & 0\\ 0 & 0\end{pmatrix},\quad E_2=\begin{pmatrix} I_2 & 0\\ 0 & 2\end{pmatrix},\quad E_3=\begin{pmatrix} e_2 & 0\\ 0 & 0\end{pmatrix},\quad E_4=\begin{pmatrix} e_3 & 0\\ 0 & 0\end{pmatrix},\\
&E_5=\begin{pmatrix} 0 & e_{10}\\ 0 & 0\end{pmatrix},\quad E_6=\begin{pmatrix} 0 & e_{01}\\ 0 & 0\end{pmatrix},\quad E_7=\begin{pmatrix} 0 & 0\\ e_{10}^T & 0\end{pmatrix},\quad E_8=\begin{pmatrix} 0 & 0\\ e_{01}^T & 0\end{pmatrix},
\end{aligned}\tag{2.1}$$

where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_{10} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T,\ e_{01} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T. \quad (2.2)$$

The commutative relations of (2.1) are

$$\begin{aligned}
&[E_1, E_2] = 0, \quad [E_1, E_3] = 2E_3, \quad [E_1, E_4] = -2E_4,\\
&[E_1, E_5] = 0, \quad [E_1, E_6] = -E_6, \quad [E_1, E_7] = -E_7,\\
&[E_1, E_8] = E_8, \quad [E_2, E_3] = 0, \quad [E_2, E_4] = 0, \quad [E_2, E_5] = -E_5, \quad [E_2, E_6] = -E_6, \quad [E_2, E_7] = E_7,\\
&[E_2, E_8] = E_8, \quad [E_3, E_4] = E_1, \quad [E_3, E_5] = 0, \quad [E_3, E_6] = E_5, \quad [E_3, E_7] = -E_8, \quad [E_3, E_8] = 0,\\
&[E_4, E_5] = E_6, \quad [E_4, E_6] = 0, \quad [E_4, E_7] = 0, \quad [E_4, E_8] = -E_7, \quad [E_5, E_5]_+ = 0, \quad [E_5, E_6]_+ = 0,\\
&[E_5, E_7]_+ = \frac{1}{2}(E_1 + E_2), \quad [E_5, E_8]_+ = E_3, \quad [E_6, E_6]_+ = 0, \quad [E_6, E_7]_+ = E_4,\\
&[E_6, E_8]_+ = \frac{1}{2}(E_2 - E_1), \quad [E_7, E_7]_+ = 0, \quad [E_7, E_8]_+ = 0, \quad [E_8, E_8]_+ = 0.
\end{aligned}$$

Considering a N $\times$ N block matrix of the form:

$$W(A_1, A_2, \cdots, A_N, B, C, D) = \begin{pmatrix}
A_1 & A_N & A_{N-1} & \cdots & A_4 & A_3 & A_2 & B\\
A_2 & A_1 & A_N & \cdots & A_5 & A_4 & A_3 & B\\
A_3 & A_2 & A_1 & \cdots & A_6 & A_5 & A_4 & B\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
A_{N-2} & A_{N-3} & A_{N-4} & \cdots & A_1 & A_N & A_{N-1} & B\\
A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_2 & A_1 & A_N & B\\
A_N & A_{N-1} & A_{N-2} & \cdots & A_3 & A_2 & A_1 & B\\
C & C & C & \cdots & C & C & C & D
\end{pmatrix}, \quad (2.3)$$

where $A_k, 1 \le k \le N$, represent N arbitrary square matrices, B is a column vector, C is a row vector and D is a first-order matrix. The Lie superalgebra spl(2,1) can be enlarged to the Lie superalgebra spl(2N,1) by means of (2.3) in [31] and [35]:

$$E_{4m+1} = W\begin{pmatrix} 0 & \cdots & 0 & \overset{m+1}{e_1} & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{4m+2} = W\begin{pmatrix} 0 & \cdots & 0 & \overset{m+1}{I_2} & 0 & \cdots & 0 & 0 & 0 & 2 \end{pmatrix},$$

$$E_{4m+3} = W\begin{pmatrix} 0 & \cdots & 0 & \overset{m+1}{e_2} & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{4m+4} = W\begin{pmatrix} 0 & \cdots & 0 & \overset{m+1}{e_3} & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m = 0, 1, \ldots, N-1,$$

$$E_{4N+1} = W\left(0 \cdots 0\,0\,0 \cdots 0\, e_{10}\, 0\, 0\right), \quad E_{4N+2} = W\left(0 \cdots 0\,0\,0 \cdots 0\, e_{01}\, 0\, 0\right),$$

$$E_{4N+3} = W\left(0 \cdots 0\,0\,0 \cdots 0\,0\, e_{10}^T\, 0\right), \quad E_{4N+4} = W\left(0 \cdots 0\,0\,0 \cdots 0\,0\, e_{01}^T\, 0\right).$$

The above matrices satisfy the following (anti) commutative relations

$$\begin{aligned}
&[E_{4m+1}, E_{4n+1}] = [E_{4m+2}, E_{4n+2}] = [E_{4m+3}, E_{4n+3}] = [E_{4m+4}, E_{4n+4}] = 0,\\
&[E_{4m+1}, E_{4n+3}] = 2E_{4(m+n-\sigma N)+3},\\
&[E_{4m+1}, E_{4n+4}] = -2E_{4(m+n-\sigma N)+4},\\
&[E_{4m+3}, E_{4n+4}] = E_{4(m+n-\sigma N)+1},\\
&[E_{4m+3}, E_{4N+1}] = [E_{4m+3}, E_{4N+4}] = [E_{4m+4}, E_{4N+2}] = [E_{4m+4}, E_{4N+3}] = 0,\\
&[E_{4m+1}, E_{4N+1}] = -[E_{4m+2}, E_{4N+1}] = [E_{4m+3}, E_{4N+2}] = E_{4N+1},\\
&-[E_{4m+1}, E_{4N+2}] = -[E_{4m+2}, E_{4N+2}] = [E_{4m+4}, E_{4N+1}] = E_{4N+2},\\
&-[E_{4m+1}, E_{4N+3}] = [E_{4m+2}, E_{4N+3}] = -[E_{4m+4}, E_{4N+4}] = E_{4N+3},\\
&[E_{4m+1}, E_{4N+4}] = [E_{4m+2}, E_{4N+4}] = -[E_{4m+3}, E_{4N+3}] = E_{4N+4},\\
&[E_{4N+1}, E_{4N+1}]_+ = [E_{4N+1}, E_{4N+2}]_+\\
&\qquad = [E_{4N+2}, E_{4N+2}]_+\\
&\qquad = [E_{4N+3}, E_{4N+3}]_+\\
&\qquad = [E_{4N+3}, E_{4N+4}]_+\\
&\qquad = [E_{4N+4}, E_{4N+4}]_+\\
&\qquad = 0,\\
&[E_{4N+1}, \quad E_{4N+3}]_+ = \sum_{i=0}^{N-1} \frac{1}{2}(E_{4m+1} + E_{4m+2}),\\
&[E_{4N+1}, E_{4N+4}]_+ = \sum_{i=0}^{N-1} E_{4m+3},\\
&[E_{4N+2}, E_{4N+3}]_+ = \sum_{i=0}^{N-1} E_{4m+4},\\
&[E_{4N+2}, E_{4N+4}]_+ = \sum_{i=0}^{N-1} \frac{1}{2}(-E_{4m+1} + E_{4m+2}), \qquad m, n = 0, 1, ..., N-1,
\end{aligned}$$

where

$$\sigma = \begin{cases} 0, & 0 \le m+n \le N-1, \\ 1, & N \le m+n \le 2N-2. \end{cases} \tag{2.4}$$

3. A generalized nonisospectral multi-component super integrable AKNS hierarchy

In this section, we derive a new generalized nonisospectral multi-component super integrable AKNS hierarchy associated with the Lie superalgebra spl(2N,1). We consider the spectral problems with a perturbation term h_N

$$\begin{cases}\psi_x = \widetilde{Y}\psi, \quad \psi_t = \widetilde{Z}\psi, \quad \lambda_t = \sum_{i\geq 0} k_i(t)\lambda^{-i},\\ \widetilde{Y} = W(Y_1\ Y_2 \cdots\ Y_N\ Y_a\ Y_b\ 0)\\ \quad =(\lambda+h_N)E_1 + \sum_{i=1}^{N}(p_iE_{4i-1}+q_iE_{4i})\\ \qquad +\alpha E_{4N+1}+\beta E_{4N+2}+\gamma E_{4N+3}+\zeta E_{4N+4},\\ \widetilde{Z} = W(Z_1\ Z_2\cdots\ Z_N\ Z_a\ Z_b\ Z_c)\\ \quad =\sum_{i=1}^{N}(a_iE_{4i-3}+b_iE_{4i-2}+c_iE_{4i-1}+d_iE_{4i})\\ \qquad +\mu E_{4N+1}+\nu E_{4N+2}+\varrho E_{4N+3}+\delta E_{4N+4},\end{cases} \tag{3.1}$$

where

$$\begin{aligned} &Y_1 = \begin{pmatrix}(\lambda+h_N) & p_1\\ q_1 & -(\lambda+h_N)\end{pmatrix}, \quad Y_j = \begin{pmatrix}0 & p_j\\ q_j & 0\end{pmatrix},\\ &Y_a = \begin{pmatrix}\alpha\ \beta\end{pmatrix}^T, \quad Y_b = \begin{pmatrix}\gamma\ \zeta\end{pmatrix}, \quad j=2,...,N,\\ &Z_m = \begin{pmatrix}a_m+b_m & c_m\\ d_m & -a_m+b_m\end{pmatrix}, \quad Z_a = \begin{pmatrix}\mu\ \nu\end{pmatrix}^T,\\ &Z_b = \begin{pmatrix}\varrho\ \delta\end{pmatrix}, \quad Z_c = \sum_{i=1}^{N}(2b_i),\ m=1,2,...,N,\end{aligned}$$

here $h_N = \eta(\sum_{i\geq 1}(p_iq_{N+1-i}+\alpha\gamma+\beta\zeta))$, η is an arbitrary even constant. p_m, q_m, $m=1,\ldots,N$, are bosonic, while α, β, γ , ζ are fermionic. And a_m, b_m, c_m, d_m commuting, while μ, ν, ϱ, δ anticommuting.

Taking the following Laurent series expansions

$$\begin{aligned} &a_m = \sum_{i=0}^{\infty} a_{mi}\lambda^{-i}, \quad b_m = \sum_{i=0}^{\infty} b_{mi}\lambda^{-i}, \quad d_m = \sum_{i=0}^{\infty} d_{mi}\lambda^{-i},\\ &c_m = \sum_{i=0}^{\infty} c_{mi}\lambda^{-i}, \quad \mu = \sum_{i=0}^{\infty}\mu_i\lambda^{-i}, \quad \nu = \sum_{i=0}^{\infty}\nu_i\lambda^{-i},\\ &\varrho = \sum_{i=0}^{\infty}\varrho_i\lambda^{-i}, \quad \delta = \sum_{i=0}^{\infty}\delta_i\lambda^{-i}, \quad m=1,2,\ldots,N,\end{aligned} \tag{3.2}$$

in the stationary zero curvature equation

$$\widetilde{Z}_x = \frac{\partial\widetilde{Y}}{\partial\lambda}\lambda_t + [\widetilde{Y},\widetilde{Z}], \tag{3.3}$$

we have

$$\begin{cases} a_{1,x} = p_1 d_1 - q_1 c_1 + \sum\limits_{\substack{k+l=N+2\\2\le k,l\le N}} (p_k d_l - q_k c_l) + \frac{1}{2}(\alpha\varrho + \gamma\mu - \beta\delta - \zeta\nu) + \lambda_t, \\ a_{rx} = \sum\limits_{\substack{i+j=r+1\\1\le i,j\le r}} (p_i d_j - q_i c_j) + \sum\limits_{\substack{k+l=r+N+1\\r+1\le k,l\le N}} (p_k d_l - q_k c_l) + \frac{1}{2}(\alpha\varrho + \gamma\mu - \beta\delta - \zeta\nu), \\ b_{m,x} = \frac{1}{2}(\alpha\varrho + \gamma\mu + \beta\delta + \zeta\nu), \\ c_{m,x} = 2(\lambda + h_N)c_m - \sum\limits_{\substack{i+j=m+1\\1\le i,j\le m}} 2p_i a_j - \sum\limits_{\substack{k+l=m+N+1\\m+1\le k,l\le N}} 2p_k a_l + \alpha\delta - \mu\zeta, \\ d_{m,x} = -2(\lambda + h_N)d_m + \sum\limits_{\substack{i+j=m+1\\1\le i,j\le m}} 2q_i a_j + \sum\limits_{\substack{k+l=m+N+1\\m+1\le k,l\le N}} 2q_k a_l + \beta\varrho - \nu\gamma, \\ r = 2,\dots,N, \quad m = 1,2,\dots,N, \\ \mu_x = (\lambda + h_N)\mu + \sum\limits_{i=1}^{N}(p_i\nu + \alpha(-a_i + b_i) - \beta c_i), \\ \nu_x = -(\lambda + h_N)\nu + \sum\limits_{i=1}^{N}(q_i\mu + \beta(a_i + b_i) - \alpha d_i), \\ \varrho_x = -(\lambda + h_N)\varrho + \sum\limits_{i=1}^{N}(-q_i\delta + \gamma(a_i - b_i) + \zeta d_i), \\ \delta_x = (\lambda + h_N)\delta + \sum\limits_{i=1}^{N}(-p_i\varrho - \zeta(a_i + b_i) + \gamma c_i). \end{cases} \tag{3.4}$$

Taking $\widetilde{Z}^{(n)} = (\lambda^n \widehat{Z})_+ + \Delta_N$, $\lambda_t^{(n)} = \sum\limits_{i=0}^{n} k_i(t)\lambda^{n-i}$, where the modification term

$$\Delta_N = \begin{pmatrix} F_N & 0 & \cdots & 0 & 0 & 0 \\ 0 & -F_N & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & F_N & 0 & 0 \\ 0 & 0 & \cdots & 0 & -F_N & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Solving the zero curvature equation

$$\frac{\partial \widetilde{Y}}{\partial \widetilde{u}}\widetilde{u}_t + \frac{\partial \widetilde{Y}}{\partial \lambda}\lambda_t^{(n)} - \widetilde{Z}_x^{(n)} + [\widetilde{Y}, \widetilde{Z}^{(n)}] = 0,$$

we derive

$$\begin{cases} h_{N,t_n} = F_{Nx}, \\ p_{m,t_n} = 2c_{m,n+1} + 2p_m F_N, \\ q_{m,t_n} = -2d_{m,n+1} - 2q_m F_N, \\ \alpha_{t_n} = \mu_{n+1} + \alpha F_N, \\ \beta_{t_n} = -\nu_{n+1} - \beta F_N, \\ \gamma_{t_n} = -\varrho_{n+1} - \gamma F_N, \\ \zeta_{t_n} = \delta_{n+1} + \zeta F_N, \end{cases} \tag{3.6}$$

which give rise to the identity

$$\Big(\sum_{i\geq 1}(p_i q_{N+1-i} + \alpha\gamma + \beta\zeta)\Big)_{t_n} = (-2a_{N,n+1})_x. \tag{3.7}$$

Therefore, we obtain $F_N = -2\eta a_{N,n+1}$, here η is an arbitrary even constant. Then the generalized nonisospectral multi-component super AKNS hierarchy is derived:

$$\widetilde{u}_{t_n} = \begin{pmatrix} p_m \\ q_m \\ \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix}_{t_n} = \begin{pmatrix} 2c_{m,n+1} + 2p_m(-2\eta a_{N,n+1}) \\ -2d_{m,n+1} - 2q_m(-2\eta a_{N,n+1}) \\ \mu_{n+1} + \alpha(-2\eta a_{N,n+1}) \\ -\nu_{n+1} - \beta(-2\eta a_{N,n+1}) \\ -\varrho_{n+1} - \gamma(-2\eta a_{N,n+1}) \\ \delta_{n+1} + \zeta(-2\eta a_{N,n+1}) \end{pmatrix}, \quad m = 1, 2, \ldots, N. \tag{3.8}$$

In (3.2), the coefficients a_{mi}, b_{mi}, c_{mi}, etc. represent undetermined functions. These coefficients are uniquely determined through (3.4), which itself is specified by initial conditions. The AKNS hierarchy (3.8) is then obtained by applying these initial conditions to (3.4) and subsequently solving the resulting zero-curvature equation.

3.1. Super bi-Hamiltonian structure

Making use of the supertrace identity, we can obtain the super bi-Hamiltonian structure associated with the Lie superalgebra spl(2N,1). The following results are obtained by a direct calculation from (3.1):

$$\begin{aligned} &\langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \lambda}\rangle = 2Na_1, \quad \langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial p_1}\rangle = 2\eta N q_N a_1 + Nd_1, \\ &\langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial q_1}\rangle = 2\eta N p_N a_1 + Nc_1, \\ &\langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial p_r}\rangle = 2\eta N q_{N-r+1} a_1 + Nd_{N-r+2}, \\ &\langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial q_r}\rangle = 2\eta N p_{N-r+1} a_1 + Nc_{N-r+2}, \end{aligned}$$

$$\langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \alpha}\rangle = 2\eta N\gamma a_1 - N\varrho, \quad \langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \beta}\rangle = 2\eta N\zeta a_1 - N\delta,$$
$$\langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \gamma}\rangle = -2\eta N\alpha a_1 + N\mu, \quad \langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \zeta}\rangle = -2\eta N\beta a_1 + N\nu, \quad r = 2, \ldots, N. \tag{3.9}$$

Taking the Laurent series expansions (3.2) in the supertrace identity

$$\frac{\delta}{\delta \widetilde{u}} \int \langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \lambda}\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \widetilde{Z}, \frac{\partial \widetilde{Y}}{\partial \widetilde{u}}\rangle, \qquad \widetilde{u} = (p_m, q_m, \alpha, \beta, \gamma, \zeta)^T, \quad m = 1, 2, \ldots, N,$$

and comparing the powers of λ , we have

$$\begin{pmatrix} 2\eta q_N a_{1,n+1} + d_{1,n+1} \\ 2\eta p_N a_{1,n+1} + c_{1,n+1} \\ 2\eta q_{N-1} a_{1,n+1} + d_{N,n+1} \\ 2\eta p_{N-1} a_{1,n+1} + c_{N,n+1} \\ \vdots \\ 2\eta q_1 a_{1,n+1} + d_{2,n+1} \\ 2\eta p_1 a_{1,n+1} + c_{2,n+1} \\ 2\eta \gamma a_{1,n+1} - \varrho_{n+1} \\ 2\eta \zeta a_{1,n+1} - \delta_{n+1} \\ -2\eta \alpha a_{1,n+1} + \mu_{n+1} \\ -2\eta \beta a_{1,n+1} + \nu_{n+1} \end{pmatrix} = \frac{\delta \widetilde{H}_{n+1}}{\delta \widetilde{u}}, \ \widetilde{H}_{n+1} = \int \frac{1}{n+1}(-2a_{1,n+2})dx. \tag{3.10}$$

Therefore, the super bi-Hamiltonian structure of the generalized nonisospectral multi-component super integrable AKNS hierarchy is obtained:

$$\begin{aligned} \widetilde{u}_{t_n} &= \widetilde{Q}_N \widetilde{R}_N \frac{\delta \widetilde{H}_{n+1}}{\delta \widetilde{u}} - 2\eta \widetilde{Q}_N \widetilde{G}_N k_{n+1}(t)x \\ &= \widetilde{Q}_N \widetilde{L}_N \widetilde{R}_N \frac{\delta \widetilde{H}_n}{\delta \widetilde{u}} + (-2\eta \widetilde{Q}_N \widetilde{L}_N \widetilde{G}_N + \widetilde{Q}_N \widetilde{M}_N) k_n(t)x, \end{aligned} \tag{3.11}$$

where $\widetilde{G}_N = (q_N \ p_N \ q_{N-1} \ p_{N-1} \ \cdots \ q_1 \ p_1 \ \gamma \ \zeta \ -\alpha \ -\beta)^T$, and $\widetilde{M}_N = (q_1 \ p_1 \ q_N \ p_N \ q_{N-1} \ p_{N-1} \ \cdots \ q_2 \ p_2 \ -\gamma \ -\zeta \ \alpha \ \beta)^T$, $N \geq 2$, and the super Hamiltonian operators $\widetilde{Q}_N, \ \widetilde{R}_N, \ \widetilde{L}_N$ are presented in Appendix A.

4. A generalized nonisospectral three-component coupled super AKNS integrable hierarchy

In this section, when N=3, the Lie superalgebra spl(2N,1) could be reduced to the Lie superalgebra spl(6,1). In order to better understand our paper, we obtain the generalized nonisospectral three-component coupled super AKNS integrable hierarchy associated with the Lie superalgebra spl(6,1). Then, the super bi-Hamiltonian structure of the generalized nonisospectral three-component coupled super AKNS integrable hierarchy is derived.

Lie superalgebra spl(6,1) whose basis is

$$E_1=\begin{pmatrix} e_1 & 0 & 0 & 0\\ 0 & e_1 & 0 & 0\\ 0 & 0 & e_1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_2=\begin{pmatrix} I_2 & 0 & 0 & 0\\ 0 & I_2 & 0 & 0\\ 0 & 0 & I_2 & 0\\ 0 & 0 & 0 & 2\end{pmatrix},\quad E_3=\begin{pmatrix} e_2 & 0 & 0 & 0\\ 0 & e_2 & 0 & 0\\ 0 & 0 & e_2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$E_4=\begin{pmatrix} e_3 & 0 & 0 & 0\\ 0 & e_3 & 0 & 0\\ 0 & 0 & e_3 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_5=\begin{pmatrix} 0 & 0 & e_1 & 0\\ e_1 & 0 & 0 & 0\\ 0 & e_1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_6=\begin{pmatrix} 0 & 0 & I_2 & 0\\ I_2 & 0 & 0 & 0\\ 0 & I_2 & 0 & 0\\ 0 & 0 & 0 & 2\end{pmatrix},$$

$$E_7=\begin{pmatrix} 0 & 0 & e_2 & 0\\ e_2 & 0 & 0 & 0\\ 0 & e_2 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_8=\begin{pmatrix} 0 & 0 & e_3 & 0\\ e_3 & 0 & 0 & 0\\ 0 & e_3 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_9=\begin{pmatrix} 0 & e_1 & 0 & 0\\ 0 & 0 & e_1 & 0\\ e_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$E_{10}=\begin{pmatrix} 0 & I_2 & 0 & 0\\ 0 & 0 & I_2 & 0\\ I_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\end{pmatrix},\quad E_{11}=\begin{pmatrix} 0 & e_2 & 0 & 0\\ 0 & 0 & e_2 & 0\\ e_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_{12}=\begin{pmatrix} 0 & e_3 & 0 & 0\\ 0 & 0 & e_3 & 0\\ e_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$E_{13}=\begin{pmatrix} 0 & 0 & 0 & e_{10}\\ 0 & 0 & 0 & e_{10}\\ 0 & 0 & 0 & e_{10}\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_{14}=\begin{pmatrix} 0 & 0 & 0 & e_{01}\\ 0 & 0 & 0 & e_{01}\\ 0 & 0 & 0 & e_{01}\\ 0 & 0 & 0 & 0\end{pmatrix},\quad E_{15}=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ e_{10}^T & e_{10}^T & e_{10}^T & 0\end{pmatrix},$$

$$E_{16}=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ e_{01}^T & e_{01}^T & e_{01}^T & 0\end{pmatrix}, \tag{4.1}$$

which satisfy

$$[E_1,E_2]=0,\quad [E_1,E_3]=2E_3,\quad [E_1,E_4]=-2E_4,\quad [E_1,E_5]=0,\quad [E_1,E_6]=0,$$
$$[E_1,E_7]=2E_5,\quad [E_1,E_8]=-2E_8,\quad [E_1,E_9]=0,\quad [E_1,E_{10}]=0,\quad [E_1,E_{11}]=2E_{11},$$

$$[E_1, E_{12}] = -2E_{12}, \quad [E_1, E_{13}] = E_{13}, \quad [E_1, E_{14}] = -E_{14},$$
$$[E_1, E_{15}] = -E_{15}, \quad [E_1, E_{16}] = E_{16},$$
$$[E_2, E_3] = 0, \quad [E_2, E_4] = 0, \quad [E_2, E_5] = 0, \quad [E_2, E_6] = 0, \quad [E_2, E_7] = 0,$$
$$[E_2, E_8] = 0, \quad [E_2, E_9] = 0,$$
$$[E_2, E_{10}] = 0, \quad [E_2, E_{11}] = 0, \quad [E_2, E_{12}] = 0, \quad [E_2, E_{13}] = -E_{13},$$
$$[E_2, E_{14}] = -E_{14}, \quad [E_2, E_{15}] = E_{15},$$
$$[E_2, E_{16}] = E_{16}, \quad [E_3, E_4] = E_1, \quad [E_3, E_5] = -2E_7, \quad [E_3, E_6] = 0,$$
$$[E_3, E_7] = 0, \quad [E_3, E_8] = E_7,$$
$$[E_3, E_9] = -2E_{11}, \quad [E_3, E_{10}] = 0, \quad [E_3, E_{11}] = 0, \quad [E_3, E_{12}] = E_9,$$
$$[E_3, E_{13}] = 0, \quad [E_3, E_{14}] = E_{13},$$
$$[E_3, E_{15}] = -E_{16}, \quad [E_3, E_{16}] = 0, \quad [E_4, E_5] = 2E_8, \quad [E_4, E_6] = 0,$$
$$[E_4, E_7] = -E_5, \quad [E_4, E_8] = 0,$$
$$[E_4, E_9] = 2E_{12}, \quad [E_4, E_{10}] = 0, \quad [E_4, E_{11}] = -E_9, \quad [E_4, E_{12}] = 0,$$
$$[E_4, E_{13}] = E_{14}, \quad [E_4, E_{14}] = 0,$$
$$[E_4, E_{15}] = 0, \quad [E_4, E_{16}] = -E_{15}, \quad [E_5, E_6] = 0, \quad [E_5, E_7] = 2E_{11},$$
$$[E_5, E_8] = -2E_{12}, \quad [E_5, E_9] = 0,$$
$$[E_5, E_{10}] = 0, \quad [E_5, E_{11}] = 2E_3, \quad [E_5, E_{12}] = -2E_4,$$
$$[E_5, E_{13}] = E_{13}, \quad [E_5, E_{14}] = -E_{14},$$
$$[E_5, E_{15}] = -E_{15}, \quad [E_5, E_{16}] = E_{16}, \quad [E_6, E_7] = 0,$$
$$[E_6, E_8] = 0, \quad [E_6, E_9] = 0, \quad [E_6, E_{10}] = 0,$$
$$[E_6, E_{11}] = 0, \quad [E_6, E_{12}] = 0, \quad [E_6, E_{13}] = -E_{13}, \quad [E_6, E_{14}] = -E_{14},$$
$$[E_6, E_{15}] = E_{15}, \quad [E_6, E_{16}] = E_{16},$$
$$[E_7, E_8] = E_9, \quad [E_7, E_9] = -2E_3, \quad [E_7, E_{10}] = 0, \quad [E_7, E_{11}] = 0,$$
$$[E_7, E_{12}] = E_1, \quad [E_7, E_{13}] = 0,$$
$$[E_7, E_{14}] = E_{13}, \quad [E_7, E_{15}] = -E_{16}, \quad [E_7, E_{16}] = 0, \quad [E_8, E_9] = 2E_4,$$
$$[E_8, E_{10}] = 0, \quad [E_8, E_{11}] = -E_1,$$
$$[E_8, E_{12}] = 0, \quad [E_8, E_{13}] = E_{14}, \quad [E_8, E_{14}] = 0, \quad [E_8, E_{15}] = 0,$$
$$[E_8, E_{16}] = -E_{15}, \quad [E_9, E_{10}] = 0,$$
$$[E_9, E_{11}] = 2E_7, \quad [E_9, E_{12}] = -2E_8, \quad [E_9, E_{13}] = E_{13},$$
$$[E_9, E_{14}] = -E_{14}, \quad [E_9, E_{15}] = -E_{15},$$
$$[E_9, E_{16}] = E_{16}, \quad [E_{10}, E_{11}] = 0, \quad [E_{10}, E_{12}] = 0, \quad [E_{10}, E_{13}] = -E_{13}, \quad [E_{10}, E_{14}] = -E_{14},$$
$$[E_{10}, E_{15}] = E_{15}, \quad [E_{10}, E_{16}] = E_{16}, \quad [E_{11}, E_{12}] = E_5, \quad [E_{11}, E_{13}] = 0, \quad [E_{11}, E_{14}] = E_{13},$$
$$[E_{11}, E_{15}] = -E_{16}, \quad [E_{11}, E_{16}] = 0, \quad [E_{12}, E_{13}] = E_{14}, \quad [E_{12}, E_{14}] = 0, \quad [E_{12}, E_{15}] = 0,$$
$$[E_{12}, E_{16}] = -E_{15}, \quad [E_{13}, E_{14}]_+ = 0, \quad [E_{13}, E_{15}]_+ = \frac{1}{2}(E_1 + E_2 + E_5 + E_6 + E_9 + E_{10}),$$
$$[E_{13}, E_{16}]_+ = E_3 + E_7 + E_{11}, \quad [E_{14}, E_{15}]_+ = E_4 + E_8 + E_{14},$$
$$[E_{14}, E_{16}]_+ = \frac{1}{2}(-E_1 + E_2 - E_5 + E_6 - E_9 + E_{10}),$$
$$[E_{15}, E_{16}]_+ = 0, \quad [E_{13}, E_{13}]_+ = 0, \quad [E_{14}, E_{14}]_+ = 0,$$

$[E_{15}, E_{15}]_+ = 0, \quad [E_{16}, E_{16}]_+ = 0,$

where e_1, e_2, e_3, e_{10}, e_{01} are given by (2.2), E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_{10}, E_{11}, E_{12} are even, E_{13}, E_{14}, E_{15}, E_{16} are odd, $\varepsilon \in \mathbb{R}$, $[.\ ,\ .]$ is commutator and $[.\ ,\ .]_+$ denotes anticommutator. Let G_1=span$\{\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4, \mathrm{E}_5, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8, \mathrm{E}_9, \mathrm{E}_{10}, \mathrm{E}_{11}, \mathrm{E}_{12}\}$, G_2=span$\{\mathrm{E}_{13}, \mathrm{E}_{14}, \mathrm{E}_{15}, \mathrm{E}_{16}\}$, then $\mathrm{spl}(6,1) = \mathrm{G}_1 \bigoplus \mathrm{G}_2$. Denoting

$$[G_i, G_j] = \{[C, D] | C \in G_i, D \in G_j\}, \quad [G_i, G_j]_+ = \{[C, D]_+ | C \in G_i, D \in G_j\},$$

the closure properties between G_1 and G_2 are given by

$$[G_1, G_1] \subseteq G_1, \quad [G_1, G_2] \subseteq G_2, \quad [G_2, G_2]_+ \subseteq G_1.$$

Considering the super AKNS spectral problems related to the Lie superalgebra spl(6,1):

$$\begin{cases} \psi_x = \overline{Y}\psi, \quad \psi_t = \overline{Z}\psi, \quad \lambda_t = \sum\limits_{i \geq 0} k_i(t)\lambda^{-i}, \\ \overline{Y} = \begin{pmatrix} Y_1 & Y_3 & Y_2 & Y_a \\ Y_2 & Y_1 & Y_3 & Y_a \\ Y_3 & Y_2 & Y_1 & Y_a \\ Y_b & Y_b & Y_b & 0 \end{pmatrix} \\ \quad = (\lambda + h_3)E_1 + p_1 E_3 + q_1 E_4 + p_2 E_7 + q_2 E_8 + p_3 E_{11} \\ \quad\quad + q_3 E_{12} + \alpha E_{13} + \beta E_{14} + \gamma E_{15} + \zeta E_{16}, \\ \overline{Z} = \begin{pmatrix} Z_1 & Z_2 & Z_3 & Z_a \\ Z_2 & Z_1 & Z_2 & Z_a \\ Z_3 & Z_2 & Z_1 & Z_a \\ Z_b & Z_b & Z_b & Z_c \end{pmatrix} \\ \quad = a_1 E_1 + b_1 E_2 + c_1 E_3 + d_1 E_4 + a_2 E_5 + b_2 E_6 + c_2 E_7 \\ \quad\quad + d_2 E_8 + a_3 E_9 + b_3 E_{10} + c_3 E_{11} + d_3 E_{12} \\ \quad\quad + \mu E_{13} + \nu E_{14} + \varrho E_{15} + \delta E_{16}, \end{cases} \tag{4.2}$$

where

$$Y_1 = \begin{pmatrix} (\lambda + h_3) & p_1 \\ q_1 & -(\lambda + h_3) \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 0 & p_2 \\ q_2 & 0 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & p_3 \\ q_3 & 0 \end{pmatrix},$$

$$Y_a = \begin{pmatrix} \alpha & \beta \end{pmatrix}^T, \quad Y_b = \begin{pmatrix} \gamma & \zeta \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} a_1 + b_1 & c_1 \\ d_1 & -a_1 + b_1 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} a_2 + b_2 & c_2 \\ d_2 & -a_2 + b_2 \end{pmatrix},$$
$$Z_3 = \begin{pmatrix} a_3 + b_3 & c_3 \\ d_3 & -a_3 + b_3 \end{pmatrix},$$
$$Z_a = \begin{pmatrix} \mu\,\nu \end{pmatrix}^T, \quad Z_b = \begin{pmatrix} \varrho\,\delta \end{pmatrix}, \quad Z_c = 2(b_1 + b_2 + b_3),$$

p_1, q_1, p_2, q_2, p_3, q_3 are even, while α, β, γ, ζ are odd, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3 are commuting fields, while μ, ν,ϱ, δ are anticommuting fields. Here $h_3 = \eta(p_1q_3 + p_3q_1 + \alpha\gamma + \beta\zeta)$, η is an arbitrary even constant.

Solving the stationary zero curvature equation

$$\overline{Z}_x = \frac{\partial \overline{Y}}{\partial \lambda}\lambda_t + [\overline{Y}, \overline{Z}], \tag{4.3}$$

we obtain the recursion relations

$$\begin{cases} a_{1x} = -q_1c_1 - q_3c_2 - q_2c_3 + p_1d_1 + p_3d_2 + p_2d_3 + \frac{1}{2}(\alpha\varrho + \gamma\mu - \beta\delta - \zeta\nu) + \lambda_t, \\ a_{2x} = -q_2c_1 - q_1c_2 - q_3c_3 + p_2d_1 + p_1d_2 + p_3d_3 + \frac{1}{2}(\alpha\varrho + \gamma\mu - \beta\delta - \zeta\nu), \\ a_{3x} = -q_3c_1 - q_2c_2 - q_1c_3 + p_3d_1 + p_2d_2 + p_1d_3 + \frac{1}{2}(\alpha\varrho + \gamma\mu - \beta\delta - \zeta\nu), \\ b_{1x} = \frac{1}{2}(\alpha\varrho + \gamma\mu + \beta\delta + \zeta\nu), \\ b_{2x} = \frac{1}{2}(\alpha\varrho + \gamma\mu + \beta\delta + \zeta\nu), \\ b_{3x} = \frac{1}{2}(\alpha\varrho + \gamma\mu + \beta\delta + \zeta\nu), \\ c_{1x} = 2(\lambda + h_3)c_1 - 2p_1a_1 - 2p_3a_2 - 2p_2a_3 + \alpha\delta + \zeta\mu, \\ c_{2x} = 2(\lambda + h_3)c_2 - 2p_2a_1 - 2p_1a_2 - 2p_3a_3 + \alpha\delta + \zeta\mu, \\ c_{3x} = 2(\lambda + h_3)c_3 - 2p_3a_1 - 2p_2a_2 - 2p_1a_3 + \alpha\delta + \zeta\mu, \\ d_{1x} = -2(\lambda + h_3)d_1 + 2q_1a_1 + 2q_3a_2 + 2q_2a_3 + \beta\varrho + \gamma\nu, \\ d_{2x} = -2(\lambda + h_3)d_2 + 2q_2a_1 + 2q_1a_2 + 2q_3a_3 + \beta\varrho + \gamma\nu, \\ d_{3x} = -2(\lambda + h_3)d_3 + 2q_3a_1 + 2q_2a_2 + 2q_1a_3 + \beta\varrho + \gamma\nu, \\ \mu_x = (\lambda + h_3)\mu + (p_1 + p_2 + p_3)\nu + (-a_1 - a_2 - a_3 + b_1 + b_2 + b_3)\alpha - (c_1 + c_2 + c_3)\beta, \\ \nu_x = -(\lambda + h_3)\nu + (q_1 + q_2 + q_3)\mu + (a_1 + a_2 + a_3 + b_1 + b_2 + b_3)\beta - (d_1 + d_2 + d_3)\alpha, \\ \varrho_x = -(\lambda + h_3)\varrho - (q_1 + q_2 + q_3)\delta + (a_1 + a_2 + a_3 - b_1 - b_2 - b_3)\gamma + (d_1 + d_2 + d_3)\zeta, \\ \delta_x = (\lambda + h_3)\delta - (p_1 + p_2 + p_3)\varrho + (-a_1 - a_2 - a_3 - b_1 - b_2 - b_3)\zeta + (c_1 + c_2 + c_3)\gamma. \end{cases} \tag{4.4}$$

By substituting the Laurent series expansions

$$a_m = \sum_{i=0}^{\infty} a_{mi}\lambda^{-i}, \quad b_m = \sum_{i=0}^{\infty} b_{mi}\lambda^{-i},$$
$$c_m = \sum_{i=0}^{\infty} c_{mi}\lambda^{-i}, \quad d_m = \sum_{i=0}^{\infty} d_{mi}\lambda^{-i},$$

$$m=1,2,\quad \varrho=\sum_{i=0}^{\infty}\varrho_i\lambda^{-i},\quad \delta=\sum_{i=0}^{\infty}\delta_i\lambda^{-i},$$
$$\mu=\sum_{i=0}^{\infty}\mu_i\lambda^{-i},\quad \nu=\sum_{i=0}^{\infty}\nu_i\lambda^{-i},$$

into (4.4), the recursion relations are derived as follows:

$$\begin{cases}
a_{1i,x}=-q_1c_{1i}-q_3c_{2i}-q_2c_{3i}+p_1d_{1i}+p_3d_{2i}+p_2d_{3i}+\frac{1}{2}(\alpha\varrho_i+\gamma\mu_i-\beta\delta_i-\zeta\nu_i)+k_i(t),\\
a_{2i,x}=-q_2c_{1i}-q_1c_{2i}-q_3c_{3i}+p_2d_{1i}+p_1d_{2i}+p_3d_{3i}+\frac{1}{2}(\alpha\varrho_i+\gamma\mu_i-\beta\delta_i-\zeta\nu_i),\\
a_{3i,x}=-q_3c_{1i}-q_2c_{2i}-q_1c_{3i}+p_3d_{1i}+p_2d_{2i}+p_1d_{3i}+\frac{1}{2}(\alpha\varrho_i+\gamma\mu_i-\beta\delta_i-\zeta\nu_i),\\
b_{1i,x}=\frac{1}{2}(\alpha\varrho_i+\gamma\mu_i+\beta\delta_i+\zeta\nu_i),\\
b_{2i,x}=\frac{1}{2}(\alpha\varrho_i+\gamma\mu_i+\beta\delta_i+\zeta\nu_i),\\
b_{3i,x}=\frac{1}{2}(\alpha\varrho_i+\gamma\mu_i+\beta\delta_i+\zeta\nu_i),\\
c_{1i,x}=2(\lambda+h_3)c_{1i}-2p_1a_{1i}-2p_3a_{2i}-2p_2a_{3i}+\alpha\delta_i+\zeta\mu_i,\\
c_{2i,x}=2(\lambda+h_3)c_{2i}-2p_2a_{1i}-2p_1a_{2i}-2p_3a_{3i}+\alpha\delta_i+\zeta\mu_i,\\
c_{3i,x}=2(\lambda+h_3)c_{3i}-2p_3a_{1i}-2p_2a_{2i}-2p_1a_{3i}+\alpha\delta_i+\zeta\mu_i,\\
d_{1i,x}=-2(\lambda+h_3)d_{1i}+2q_1a_{1i}+2q_3a_{2i}+2q_2a_{3i}+\beta\varrho_i+\gamma\nu_i,\\
d_{2i,x}=-2(\lambda+h_3)d_{2i}+2q_2a_{1i}+2q_1a_{2i}+2q_3a_{3i}+\beta\varrho_i+\gamma\nu_i,\\
d_{3i,x}=-2(\lambda+h_3)d_{3i}+2q_3a_{1i}+2q_2a_{2i}+2q_1a_{3i}+\beta\varrho_i+\gamma\nu_i,\\
\mu_{ix}=(\lambda+h_3)\mu_i+(p_1+p_2+p_3)\nu_i+(-a_{1i}-a_{2i}-a_{3i}+b_{1i}+b_{2i}+b_{3i})\alpha-(c_{1i}+c_{2i}+c_{3i})\beta,\\
\nu_{ix}=-(\lambda+h_3)\nu_i+(q_1+q_2+q_3)\mu_i+(a_{1i}+a_{2i}+a_{3i}+b_{1i}+b_{2i}+b_{3i})\beta-(d_{1i}+d_{2i}+d_{3i})\alpha,\\
\varrho_{ix}=-(\lambda+h_3)\varrho_i-(q_1+q_2+q_3)\delta_i+(a_{1i}+a_{2i}+a_{3i}-b_{1i}-b_{2i}-b_{3i})\gamma+(d_{1i}+d_{2i}+d_{3i})\zeta,\\
\delta_{ix}=(\lambda+h_3)\delta_i-(p_1+p_2+p_3)\varrho_i+(-a_{1i}-a_{2i}-a_{3i}-b_{1i}-b_{2i}-b_{3i})\zeta+(c_{1i}+c_{2i}+c_{3i})\gamma.
\end{cases} \tag{4.5}$$

Taking initial values

$$a_{10}=a_{20}=a_{30}=1,$$
$$b_{10}=b_{20}=b_{30}=c_{10}=c_{20}=c_{30}=d_{10}=d_{20}=d_{30}=\varrho_0=\delta_0=\mu_0=\nu_0=k_0(t)=0,$$

then we obtain

$$a_{11}=k_1(t)x,\quad a_{21}=a_{31}=0,$$
$$b_{11}=b_{21}=b_{31}=0,$$
$$c_{11}=c_{21}=c_{31}=p_1+p_2+p_3,$$
$$d_{11}=d_{21}=d_{31}=q_1+q_2+q_3,\quad \mu_1=3\alpha,\quad \nu_1=3\beta,\quad \varrho_1=3\gamma,\quad \delta_1=3\zeta,$$
$$a_{12}=-\frac{1}{2}(p_1+p_2+p_3)(q_1+q_2+q_3)+\frac{3}{2}\gamma\alpha+\frac{3}{2}\zeta\beta+k_2(t)x,$$
$$a_{22}=-\frac{1}{2}(p_1+p_2+p_3)(q_1+q_2+q_3)+\frac{3}{2}\gamma\alpha+\frac{3}{2}\zeta\beta,$$

$$a_{32} = -\frac{1}{2}(p_1+p_2+p_3)(q_1+q_2+q_3) + \frac{3}{2}\gamma\alpha + \frac{3}{2}\zeta\beta,$$
$$b_{12} = b_{22} = b_{32} = \frac{3}{2}\gamma\alpha - \frac{3}{2}\zeta\beta,$$
$$c_{12} = \frac{1}{2}(p_{1x}+p_{2x}+p_{3x}) - h_3(p_1+p_2+p_3) + p_1k_1(t)x,$$
$$c_{22} = \frac{1}{2}(p_{1x}+p_{2x}+p_{3x}) - h_3(p_1+p_2+p_3) + p_2k_1(t)x,$$
$$c_{32} = \frac{1}{2}(p_{1x}+p_{2x}+p_{3x}) - h_3(p_1+p_2+p_3) + p_3k_1(t)x,$$
$$d_{12} = -\frac{1}{2}(q_{1x}+q_{2x}+q_{3x}) - h_3(q_1+q_2+q_3) + q_1k_1(t)x,$$
$$d_{22} = -\frac{1}{2}(q_{1x}+q_{2x}+q_{3x}) - h_3(q_1+q_2+q_3) + q_2k_1(t)x,$$
$$d_{32} = -\frac{1}{2}(q_{1x}+q_{2x}+q_{3x}) - h_3(q_1+q_2+q_3) + q_3k_1(t)x,$$
$$\mu_2 = 3\alpha_x - 3h_3\alpha + \alpha k_1(t)x, \quad \nu_2 = -3\beta_x - 3h_3\beta + \beta k_1(t)x,$$
$$\varrho_2 = -3\gamma_x - 3h_3\gamma + \gamma k_1(t)x, \quad \delta_2 = 3\zeta_x - 3h_3\zeta + \zeta k_1(t)x,$$
$$\begin{aligned} c_{13} = {} & \frac{1}{4}(p_{1xx}+p_{2xx}+p_{3xx}) - \frac{1}{2}h_{3x}(p_1+p_2+p_3) - h_3(p_{1x}+p_{2x}+p_{3x}) + h_3^2(p_1+p_2+p_3) \\ & - \frac{1}{2}(p_1+p_2+p_3)^2(q_1+q_2+q_3) + \frac{3}{2}(p_1+p_2+p_3)\gamma\alpha + \frac{3}{2}(p_1+p_2+p_3)\zeta\beta - \frac{3}{2}\alpha\zeta_x \\ & - \frac{3}{2}\zeta\alpha_x + (\frac{1}{2}p_{1x}x + \frac{1}{2}p_1 - h_3p_1x)k_1(t) + p_1k_2(t)x, \end{aligned}$$
$$\begin{aligned} c_{23} = {} & \frac{1}{4}(p_{1xx}+p_{2xx}+p_{3xx}) - \frac{1}{2}h_{3x}(p_1+p_2+p_3) - h_3(p_{1x}+p_{2x}+p_{3x}) + h_3^2(p_1+p_2+p_3) \\ & - \frac{1}{2}(p_1+p_2+p_3)^2(q_1+q_2+q_3) + \frac{3}{2}(p_1+p_2+p_3)\gamma\alpha + \frac{3}{2}(p_1+p_2+p_3)\zeta\beta - \frac{3}{2}\alpha\zeta_x \\ & - \frac{3}{2}\zeta\alpha_x + (\frac{1}{2}p_{2x}x + \frac{1}{2}p_2 - h_3p_2x)k_1(t) + p_2k_2(t)x, \end{aligned}$$
$$\begin{aligned} c_{33} = {} & \frac{1}{4}(p_{1xx}+p_{2xx}+p_{3xx}) - \frac{1}{2}h_{3x}(p_1+p_2+p_3) - h_3(p_{1x}+p_{2x}+p_{3x}) + h_3^2(p_1+p_2+p_3) \\ & - \frac{1}{2}(p_1+p_2+p_3)^2(q_1+q_2+q_3) + \frac{3}{2}(p_1+p_2+p_3)\gamma\alpha + \frac{3}{2}(p_1+p_2+p_3)\zeta\beta - \frac{3}{2}\alpha\zeta_x \\ & - \frac{3}{2}\zeta\alpha_x + (\frac{1}{2}p_{3x}x + \frac{1}{2}p_3 - h_3p_3x)k_1(t) + p_3k_2(t)x, \end{aligned}$$
$$\begin{aligned} d_{13} = {} & \frac{1}{4}(q_{1xx}+q_{2xx}+q_{3xx}) + \frac{1}{2}h_{3x}(q_1+q_2+q_3) + h_3(q_{1x}+q_{2x}+q_{3x}) - \frac{1}{2}(q_1+q_2+q_3)^2(p_1 \\ & + p_2+p_3) + \frac{3}{2}(q_1+q_2+q_3)\gamma\alpha + \frac{3}{2}(q_1+q_2+q_3)\zeta\beta + h_3^2(q_1+q_2+q_3) - \frac{3}{2}\beta\gamma_x - \frac{3}{2}\gamma\beta_x \\ & - (h_3q_1x + \frac{1}{2}q_{1x}x + \frac{1}{2}q_1)k_1(t) + q_1k_2(t)x, \end{aligned}$$
$$\begin{aligned} d_{23} = {} & \frac{1}{4}(q_{1xx}+q_{2xx}+q_{3xx}) + \frac{1}{2}h_{3x}(q_1+q_2+q_3) + h_3(q_{1x}+q_{2x}+q_{3x}) - \frac{1}{2}(q_1+q_2+q_3)^2(p_1 \\ & + p_2+p_3) + \frac{3}{2}(q_1+q_2+q_3)\gamma\alpha + \frac{3}{2}(q_1+q_2+q_3)\zeta\beta + h_3^2(q_1+q_2+q_3) - \frac{3}{2}\beta\gamma_x - \frac{3}{2}\gamma\beta_x \\ & - (h_3q_2x + \frac{1}{2}q_{2x}x + \frac{1}{2}q_2)k_1(t) + q_2k_2(t)x, \end{aligned}$$

$$
\begin{aligned}
d_{33} =& \frac{1}{4}(q_{1xx}+q_{2xx}+q_{3xx})+\frac{1}{2}h_{3x}(q_1+q_2+q_3)+h_3(q_{1x}+q_{2x}+q_{3x})-\frac{1}{2}(q_1+q_2+q_3)^2(p_1\\
&+p_2+p_3)+\frac{3}{2}(q_1+q_2+q_3)\gamma\alpha+\frac{3}{2}(q_1+q_2+q_3)\zeta\beta+h_3^2(q_1+q_2+q_3)-\frac{3}{2}\beta\gamma_x-\frac{3}{2}\gamma\beta_x\\
&-(h_3q_3x+\frac{1}{2}q_{3x}x+\frac{1}{2}q_3)k_1(t)+q_3k_2(t)x,\\
a_{33} =& -\frac{1}{4}\partial^{-1}(q_1+q_2+q_3)(p_{1xx}+p_{2xx}+p_{3xx})+\frac{1}{4}\partial^{-1}(p_1+p_2+p_3)(q_{1xx}+q_{2xx}+q_{3xx})\\
&+\frac{1}{2}\partial^{-1}(-q_3p_1-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)+\frac{3}{4}\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x})\\
&+3\alpha\gamma h_3+3\beta\zeta h_3-\frac{3}{2}\beta\zeta_x-\frac{3}{2}\zeta\beta_x+\frac{3}{2}\alpha\gamma_x+\frac{3}{2}\gamma\alpha_x+(p_1+p_2+p_3)h_3(q_1+q_2+q_3)\\
&+\frac{1}{2}(-p_1q_3-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)x,\\
\mu_3 =& 3\alpha_{xx}-3h_{3x}\alpha-6h_3\alpha_x+3h_3^2\alpha+3(p_1+p_2+p_3)\beta_x-\frac{3}{2}\alpha(p_1+p_2+p_3)(q_1+q_2+q_3)\\
&+\frac{3}{2}\beta(p_{1x}+p_{2x}+p_{3x})+9\alpha\zeta\beta+(\alpha_x x+\alpha-h_3\alpha x)k_1(t)+\alpha k_2(t)x,\\
\nu_3 =& 3\beta_{xx}+3h_{3x}\beta+6h_3\beta_x+3h_3^2\beta+3(q_1+q_2+q_3)\alpha_x-\frac{3}{2}\beta(p_1+p_2+p_3)(q_1+q_2+q_3)\\
&+\frac{3}{2}\alpha(q_{1x}+q_{2x}+q_{3x})+9\beta\gamma\alpha-(\beta_x x+\beta+h_3\beta x)k_1(t)+\beta k_2(t)x,\\
\varrho_3 =& 3\gamma_{xx}+3h_{3x}\gamma+6h_3\gamma_x+3h_3^2\gamma-3(q_1+q_2+q_3)\zeta_x-\frac{3}{2}\gamma(p_1+p_2+p_3)(q_1+q_2+q_3)\\
&-\frac{3}{2}\zeta(q_{1x}+q_{2x}+q_{3x})+9\gamma\zeta\beta-(\gamma_x x+\gamma+h_3\gamma x)k_1(t)+\gamma k_2(t)x,\\
\delta_3 =& 3\zeta_{xx}-3h_{3x}\zeta-6h_3\zeta_x+3h_3^2\zeta-3(p_1+p_2+p_3)\gamma_x-\frac{3}{2}\zeta(p_1+p_2+p_3)(q_1+q_2+q_3)\\
&-\frac{3}{2}\gamma(q_{1x}+q_{2x}+q_{3x})+9\gamma\alpha\zeta+(\zeta_x x+\zeta-h_3\zeta x)k_1(t)+\zeta k_2(t)x.
\end{aligned}
$$

We take $\overline{Z}^{(n)}=(\lambda^n\overline{Z})_++\Delta_n$, $\lambda_t^{(n)}=\sum\limits_{i=0}^{n}k_i(t)\lambda^{n-i}$, here Δ_3 is the modification term:

$$
\Delta_3=\begin{pmatrix}
F_3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -F_3 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & F_3 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -F_3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & F_3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -F_3 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{4.6}
$$

To obtain the generalized nonisospectral super AKNS hierarchy, we solve the zero curvature equation

$$
\frac{\partial\overline{Y}}{\partial\overline{u}}\overline{u}_t+\frac{\partial\overline{Y}}{\partial\lambda}\lambda_t^{(n)}-\overline{Z}_x^{(n)}+[\overline{Y},\overline{Z}^{(n)}]=0,
$$

and derive

$$\begin{cases} h_{3,t_n} = F_{3x}, \\ p_{1,t_n} = 2c_{1,n+1} + 2p_1F_3, \\ q_{1,t_n} = -2d_{1,n+1} - 2q_1F_3, \\ p_{2,t_n} = 2c_{2,n+1} + 2p_2F_3, \\ q_{2,t_n} = -2d_{2,n+1} - 2q_2F_3, \\ p_{3,t_n} = 2c_{3,n+1} + 2p_3F_3, \\ q_{3,t_n} = -2d_{3,n+1} - 2q_3F_3, \\ \alpha_{t_n} = \mu_{n+1} + \alpha F_3, \\ \beta_{t_n} = -\nu_{n+1} - \beta F_3, \\ \gamma_{t_n} = -\varrho_{n+1} - \gamma F_3, \\ \zeta_{t_n} = \delta_{n+1} + \zeta F_3, \end{cases} \tag{4.7}$$

which give rise to the identity

$$\begin{aligned}(p_1q_3 + p_2q_2 + p_3q_1 + \alpha\gamma + \beta\zeta)_{t_n} =& 2q_3c_{1,n+1} - 2p_1d_{3,n+1} - 2p_3d_{1,n+1} + 2q_1c_{3,n+1} + 2q_2c_{2,n+1} \\ & - 2p_2d_{2,n+1} + \mu_{n+1}\gamma - \alpha\varrho_{n+1} - \nu_{n+1}\zeta + \beta\sigma_{n+1} \\ =& (-2a_{3,n+1})_x. \end{aligned} \tag{4.8}$$

Hence, we find $F_3 = -2\eta a_{3,n+1}$, where η is an arbitrary even constant. Then the generalized nonisospectral super AKNS hierarchy is obtained:

$$\overline{u}_{t_n} = \begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \\ p_3 \\ q_3 \\ \alpha \\ \beta \\ \gamma \\ \zeta \end{pmatrix}_{t_n} = \begin{pmatrix} 2c_{1,n+1} - 4\eta p_1a_{3,n+1} \\ -2d_{1,n+1} + 4\eta q_1a_{3,n+1} \\ 2c_{2,n+1} - 4\eta p_2a_{3,n+1} \\ -2d_{2,n+1} + 4\eta q_2a_{3,n+1} \\ 2c_{3,n+1} - 4\eta p_3a_{3,n+1} \\ -2d_{3,n+1} + 4\eta q_3a_{3,n+1} \\ \mu_{n+1} - 2\eta\alpha a_{3,n+1} \\ -\nu_{n+1} + 2\eta\beta a_{3,n+1} \\ -\varrho_{n+1} + 2\eta\gamma a_{3,n+1} \\ \delta_{n+1} - 2\eta\zeta a_{3,n+1} \end{pmatrix}. \tag{4.9}$$

The terms a_{mi}, b_{mi}, c_{mi}, etc. in the Laurent expansion are undetermined coefficients. These coefficients are uniquely specified by (4.5), which is itself determined through initial conditions. By applying these initial conditions to (4.5) and solving the corresponding zero-curvature equation, we derive the complete 10-component system (4.9).

When n=1 in (4.7), we get

$$\begin{aligned} p_{1,t_1} =& p_{1x} + p_{2x} + p_{3x} - 2h_3^2(p_1 + p_2 + p_3) + 2\eta p_1(p_1 + p_2 + p_3)(q_1 + q_2 + q_3) - 6\eta p_1\gamma\alpha \\ & - 6\eta p_1\zeta\beta + 2p_1k_1(t)x, \end{aligned}$$

$$\begin{aligned}
q_{1,t_1} &= q_{1x}+q_{2x}+q_{3x}+2h_3(q_1+q_2+q_3)-2\eta q_1(p_1+p_2+p_3)(q_1+q_2+q_3)+6\eta q_1\gamma\alpha \\
&\quad +6\eta q_1\zeta\beta-2q_1k_1(t)x, \\
p_{2,t_1} &= p_{1x}+p_{2x}+p_{3x}-2h_3^2(p_1+p_2+p_3)+2\eta p_2(p_1+p_2+p_3)(q_1+q_2+q_3)-6\eta p_2\gamma\alpha \\
&\quad -6\eta p_2\zeta\beta+2p_2k_1(t)x, \\
q_{2,t_1} &= q_{1x}+q_{2x}+q_{3x}+2h_3(q_1+q_2+q_3)-2\eta q_2(p_1+p_2+p_3)(q_1+q_2+q_3)+6\eta q_2\gamma\alpha \\
&\quad +6\eta q_2\zeta\beta-2q_2k_1(t)x, \\
p_{3,t_1} &= p_{1x}+p_{2x}+p_{3x}-2h_3^2(p_1+p_2+p_3)+2\eta p_3(p_1+p_2+p_3)(q_1+q_2+q_3)-6\eta p_3\gamma\alpha \\
&\quad -6\eta p_3\zeta\beta+2p_3k_1(t)x, \\
q_{3,t_1} &= q_{1x}+q_{2x}+q_{3x}+2h_3(q_1+q_2+q_3)-2\eta q_3(p_1+p_2+p_3)(q_1+q_2+q_3)+6\eta q_3\gamma\alpha \\
&\quad +6\eta q_3\zeta\beta-2q_3k_1(t)x, \\
\alpha_{t_1} &= 3\alpha_x-3h_3\alpha+\alpha\eta(p_1+p_2+p_3)(q_1+q_2+q_3)-3\eta\alpha\zeta\beta+\alpha k_1(t)x, \\
\beta_{t_1} &= 3\beta_x+3h_3\beta-\eta\beta(p_1+p_2+p_3)(q_1+q_2+q_3)+3\eta\beta\gamma\alpha-\beta k_1(t)x, \\
\gamma_{t_1} &= 3\gamma_x+3h_3\gamma-\eta\gamma(p_1+p_2+p_3)(q_1+q_2+q_3)+3\eta\gamma\zeta\beta-\gamma k_1(t)x, \\
\zeta_{t_1} &= 3\zeta_x-3h_3\zeta+\eta\zeta(p_1+p_2+p_3)(q_1+q_2+q_3)-3\eta\zeta\gamma\alpha+\zeta k_1(t)x.
\end{aligned}$$

When n=2 in (4.7), we have

$$\begin{aligned}
p_{1,t_2} &= \frac{1}{2}(p_{1xx}+p_{2xx}+p_{3xx})-h_{3x}(p_1+p_2+p_3)-2h_3(p_{1x}+p_{2x}+p_{3x})+2h_3^2(p_1+p_2+p_3) \\
&\quad -(p_1+p_2+p_3)^2(q_1+q_2+q_3)+3(p_1+p_2+p_3)\gamma\alpha+3(p_1+p_2+p_3)\zeta\beta-3\alpha\zeta_x-3\zeta\alpha_x \\
&\quad +\eta p_1(q_1+q_2+q_3)(p_{1x}+p_{2x}+p_{3x})-\eta p_1(p_1+p_2+p_3)(q_{1x}+q_{2x}+q_{3x}) \\
&\quad -3\eta p_1\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x})-2\eta p_1(6\alpha\gamma h_3+6\beta\zeta h_3-3\beta\zeta_x-3\zeta\beta_x \\
&\quad +3\alpha\gamma_x+3\gamma\alpha_x+2(p_1+p_2+p_3)h_3(q_1+q_2+q_3))+(2\eta p_1^2q_3x+2\eta p_1p_2q_2x+2\eta p_1p_3q_1x \\
&\quad -2\eta p_1\gamma\alpha x-2\eta p_1\zeta\beta x+p_{1x}x+p_1-2h_3p_1x)k_1(t)-2\eta p_1\partial^{-1}(-q_3p_1-p_2q_2-p_3q_1+\gamma\alpha \\
&\quad +\zeta\beta)k_1(t)+2p_1k_2(t)x, \\
q_{1,t_2} &= -\frac{1}{2}(q_{1xx}+q_{2xx}+q_{3xx})-h_{3x}(q_1+q_2+q_3)-2h_3(q_{1x}+q_{2x}+q_{3x})-2h_3^2(q_1+q_2+q_3) \\
&\quad +(q_1+q_2+q_3)^2(p_1+p_2+p_3)-3(q_1+q_2+q_3)\gamma\alpha-3(q_1+q_2+q_3)\zeta\beta+3\beta\gamma_x+3\gamma\beta_x \\
&\quad -\eta q_1(q_1+q_2+q_3)(p_{1x}+p_{2x}+p_{3x})+\eta q_1(p_1+p_2+p_3)(q_{1x}+q_{2x}+q_{3x}) \\
&\quad +3\eta q_1\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x}) \\
&\quad +2\eta q_1(6\alpha\gamma h_3+6\beta\zeta h_3-3\beta\zeta_x-3\zeta\beta_x+3\alpha\gamma_x+3\gamma\alpha_x \\
&\quad +2(p_1+p_2+p_3)h_3(q_1+q_2+q_3))+(-2\eta q_1p_1q_3x-2\eta q_1p_2q_2x-2\eta q_1^2p_3x+2\eta q_1\gamma\alpha x \\
&\quad +2\eta q_1\zeta\beta x+2h_3q_1x+q_{1x}x+q_1)k_1(t)+2\eta q_1\partial^{-1}(-q_3p_1-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t) \\
&\quad -2q_1k_2(t)x, \\
p_{2,t_2} &= \frac{1}{2}(p_{1xx}+p_{2xx}+p_{3xx})-h_{3x}(p_1+p_2+p_3)-2h_3(p_{1x}+p_{2x}+p_{3x})+2h_3^2(p_1+p_2+p_3) \\
&\quad -(p_1+p_2+p_3)^2(q_1+q_2+q_3)+3(p_1+p_2+p_3)\gamma\alpha+3(p_1+p_2+p_3)\zeta\beta-3\alpha\zeta_x-3\zeta\alpha_x \\
&\quad +\eta p_2(q_1+q_2+q_3)(p_{1x}+p_{2x}+p_{3x})-\eta p_2(p_1+p_2+p_3)(q_{1x}+q_{2x}+q_{3x}) \\
&\quad -3\eta p_2\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x})-2\eta p_2(6\alpha\gamma h_3+6\beta\zeta h_3-3\beta\zeta_x-3\zeta\beta_x \\
&\quad +3\alpha\gamma_x+3\gamma\alpha_x+2(p_1+p_2+p_3)h_3(q_1+q_2+q_3))+(2\eta p_2p_1q_3x+2\eta p_2^2q_2x+2\eta p_2p_3q_1x
\end{aligned}$$

$$- 2\eta p_2\gamma\alpha x - 2\eta p_2\zeta\beta x + p_{2x}x + p_2 - 2h_3p_2x)k_1(t) - 2\eta p_2\partial^{-1}(-q_3p_1 - p_2q_2 - p_3q_1 + \gamma\alpha + \zeta\beta)k_1(t) + 2p_2k_2(t)x,$$

$$\begin{aligned}
q_{2,t_2} = & -\frac{1}{2}(q_{1xx} + q_{2xx} + q_{3xx}) - h_{3x}(q_1 + q_2 + q_3) - 2h_3(q_{1x} + q_{2x} + q_{3x}) - 2h_3^2(q_1 + q_2 + q_3) \\
& + (q_1 + q_2 + q_3)^2(p_1 + p_2 + p_3) - 3(q_1 + q_2 + q_3)\gamma\alpha - 3(q_1 + q_2 + q_3)\zeta\beta + 3\beta\gamma_x + 3\gamma\beta_x \\
& - \eta q_2(q_1 + q_2 + q_3)(p_{1x} + p_{2x} + p_{3x}) + \eta q_2\partial^{-1}(p_1 + p_2 + p_3)(q_{1x} + q_{2x} + q_{3x}) \\
& + 3\eta q_2\partial^{-1}\gamma\beta(p_{1x} + p_{2x} + p_{3x} - q_{1x} - q_{2x} - q_{3x}) \\
& + 2\eta q_2(6\alpha\gamma h_3 + 6\beta\zeta h_3 - 3\beta\zeta_x - 3\zeta\beta_x + 3\alpha\gamma_x + 3\gamma\alpha_x \\
& + 2(p_1 + p_2 + p_3)h_3(q_1 + q_2 + q_3)) + (-2\eta q_2p_1q_3x - 2\eta q_2p_2q_2x - 2\eta q_2p_3q_1x + 2\eta q_2\gamma\alpha x \\
& + 2\eta q_2\zeta\beta x + 2h_3q_2x + q_{2x}x + q_2)k_1(t) + 2\eta q_2\partial^{-1}(-q_3p_1 - p_2q_2 - p_3q_1 + \gamma\alpha + \zeta\beta)k_1(t) \\
& - 2q_2k_2(t)x,
\end{aligned}$$

$$\begin{aligned}
p_{3,t_2} = & \frac{1}{2}(p_{1xx} + p_{2xx} + p_{3xx}) - h_{3x}(p_1 + p_2 + p_3) - 2h_3(p_{1x} + p_{2x} + p_{3x}) + 2h_3^2(p_1 + p_2 + p_3) \\
& - (p_1 + p_2 + p_3)^2(q_1 + q_2 + q_3) + 3(p_1 + p_2 + p_3)\gamma\alpha + 3(p_1 + p_2 + p_3)\zeta\beta - 3\alpha\zeta_x - 3\zeta\alpha_x \\
& + \eta p_3(q_1 + q_2 + q_3)(p_{1x} + p_{2x} + p_{3x}) - \eta p_3(p_1 + p_2 + p_3)(q_{1x} + q_{2x} + q_{3x}) \\
& - 3\eta p_3\partial^{-1}\gamma\beta(p_{1x} + p_{2x} + p_{3x} - q_{1x} - q_{2x} - q_{3x}) - 2\eta p_3(6\alpha\gamma h_3 + 6\beta\zeta h_3 - 3\beta\zeta_x - 3\zeta\beta_x \\
& + 3\alpha\gamma_x + 3\gamma\alpha_x + 2(p_1 + p_2 + p_3)h_3(q_1 + q_2 + q_3)) + (2\eta p_3p_1q_3x + 2\eta p_3p_2q_2x + 2\eta p_3^2q_1x \\
& - 2\eta p_3\gamma\alpha x - 2\eta p_3\zeta\beta x + p_{3x}x + p_3 - 2h_3p_3x)k_1(t) - 2\eta p_3\partial^{-1}(-q_3p_1 - p_2q_2 - p_3q_1 + \gamma\alpha \\
& + \zeta\beta)k_1(t) + 2p_3k_2(t)x,
\end{aligned}$$

$$\begin{aligned}
q_{3,t_2} = & -\frac{1}{2}(q_{1xx} + q_{2xx} + q_{3xx}) - h_{3x}(q_1 + q_2 + q_3) - 2h_3(q_{1x} + q_{2x} + q_{3x}) - 2h_3^2(q_1 + q_2 + q_3) \\
& + (q_1 + q_2 + q_3)^2(p_1 + p_2 + p_3) - 3(q_1 + q_2 + q_3)\gamma\alpha - 3(q_1 + q_2 + q_3)\zeta\beta + 3\beta\gamma_x + 3\gamma\beta_x \\
& - \eta q_3(q_1 + q_2 + q_3)(p_{1x} + p_{2x} + p_{3x}) + \eta q_3\partial^{-1}(p_1 + p_2 + p_3)(q_{1x} + q_{2x} + q_{3x}) \\
& + 3\eta q_3\partial^{-1}\gamma\beta(p_{1x} + p_{2x} + p_{3x} - q_{1x} - q_{2x} - q_{3x}) \\
& + 2\eta q_3(6\alpha\gamma h_3 + 6\beta\zeta h_3 - 3\beta\zeta_x - 3\zeta\beta_x + 3\alpha\gamma_x + 3\gamma\alpha_x \\
& + 2(p_1 + p_2 + p_3)h_3(q_1 + q_2 + q_3)) + (-2\eta p_1q_3^2x - 2\eta q_2p_2q_3x - 2\eta q_3p_3q_1x + 2\eta q_3\gamma\alpha x \\
& + 2\eta q_3\zeta\beta x + 2h_3q_3x + q_{3x}x + q_3)k_1(t) + 2\eta q_3\partial^{-1}(-q_3p_1 - p_2q_2 - p_3q_1 + \gamma\alpha + \zeta\beta)k_1(t) \\
& - 2q_3k_2(t)x,
\end{aligned}$$

$$\begin{aligned}
\alpha_{t_2} = & 3\alpha_{xx} - 3h_{3x}\alpha - 6h_3\alpha_x + 3h_3^2\alpha + 3(p_1 + p_2 + p_3)\beta_x - \frac{3}{2}\alpha(p_1 + p_2 + p_3)(q_1 + q_2 + q_3) \\
& + \frac{3}{2}\beta(p_{1x} + p_{2x} + p_{3x}) + 9\alpha\zeta\beta + \frac{1}{2}\eta\alpha(q_1 + q_2 + q_3)(p_{1x} + p_{2x} + p_{3x}) \\
& - \frac{1}{2}\eta\alpha(p_1 + p_2 + p_3)(q_{1x} + q_{2x} + q_{3x}) - \frac{3}{2}\eta\alpha\partial^{-1}\gamma\beta(p_{1x} + p_{2x} + p_{3x} - q_{1x} - q_{2x} - q_{3x}) \\
& - \eta\alpha(6\alpha\gamma h_3 + 6\beta\zeta h_3 - 3\beta\zeta_x - 3\zeta\beta_x + 3\alpha\gamma_x \\
& + 3\gamma\alpha_x + 2(p_1 + p_2 + p_3)h_3(q_1 + q_2 + q_3)) - \eta\alpha(-p_1q_3 - p_2q_2 - p_3q_1 + \gamma\alpha + \zeta\beta)k_1(t)x \\
& - \eta\alpha\partial^{-1}(-q_3p_1 - p_2q_2 - p_3q_1 + \gamma\alpha + \zeta\beta)k_1(t) + (\alpha_x x + \alpha - h_3\alpha x)k_1(t) + \alpha k_2(t)x,
\end{aligned}$$

$$\begin{aligned}
\beta_{t_2} = & 3\beta_{xx} - 3h_{3x}\beta - 6h_3\beta_x + 3h_3^2\beta - 3(q_1 + q_2 + q_3)\alpha_x + \frac{3}{2}\beta(p_1 + p_2 + p_3)(q_1 + q_2 + q_3) \\
& - \frac{3}{2}\alpha(q_{1x} + q_{2x} + q_{3x}) - 9\beta\gamma\alpha - \frac{1}{2}\eta\beta(q_1 + q_2 + q_3)(p_{1x} + p_{2x} + p_{3x})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\eta\beta(p_1+p_2+p_3)(q_{1x}+q_{2x}+q_{3x})+\frac{3}{2}\eta\beta\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x})\\
&+\eta\beta(6\alpha\gamma h_3+6\beta\zeta h_3-3\beta\zeta_x-3\zeta\beta_x+3\alpha\gamma_x\\
&+3\gamma\alpha_x+2(p_1+p_2+p_3)h_3(q_1+q_2+q_3))+\eta\beta(-p_1q_3-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)x\\
&+\eta\beta\partial^{-1}(-q_3p_1-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)+(\beta_x x+\beta+h_3\beta x)k_1(t)-\beta k_2(t)x,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{t_2}=&-3\gamma_{xx}-3h_{3x}\gamma-6h_3\gamma_x-3h_3^2\gamma+3(q_1+q_2+q_3)\zeta_x+\frac{3}{2}\gamma(p_1+p_2+p_3)(q_1+q_2+q_3)\\
&+\frac{3}{2}\zeta(q_{1x}+q_{2x}+q_{3x})-9\gamma\zeta\beta-\frac{1}{2}\eta\gamma(q_1+q_2+q_3)(p_{1x}+p_{2x}+p_{3x})\\
&+\frac{1}{2}\eta\gamma(p_1+p_2+p_3)(q_{1x}+q_{2x}+q_{3x})+\frac{3}{2}\eta\gamma\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x})\\
&+\eta\gamma(6\alpha\gamma h_3+6\beta\zeta h_3-3\beta\zeta_x-3\zeta\beta_x+3\alpha\gamma_x\\
&+3\gamma\alpha_x+2(p_1+p_2+p_3)h_3(q_1+q_2+q_3))+\eta\gamma(-p_1q_3-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)x\\
&+\eta\gamma\partial^{-1}(-q_3p_1-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)+(\gamma_x x+\gamma+h_3\gamma x)k_1(t)-\gamma k_2(t)x,
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{t_2}=&\,3\zeta_{xx}-3h_{3x}\zeta-6h_3\zeta_x+3h_3^2\zeta-3(p_1+p_2+p_3)\gamma_x-\frac{3}{2}\zeta(p_1+p_2+p_3)(q_1+q_2+q_3)\\
&-\frac{3}{2}\gamma(q_{1x}+q_{2x}+q_{3x})+9\gamma\alpha\zeta+\frac{1}{2}\eta\zeta(q_1+q_2+q_3)(p_{1x}+p_{2x}+p_{3x})\\
&-\frac{1}{2}\eta\zeta(p_1+p_2+p_3)(q_{1x}+q_{2x}+q_{3x})-\frac{3}{2}\eta\zeta\partial^{-1}\gamma\beta(p_{1x}+p_{2x}+p_{3x}-q_{1x}-q_{2x}-q_{3x})\\
&-\eta\zeta(6\alpha\gamma h_3+6\beta\zeta h_3-3\beta\zeta_x-3\zeta\beta_x+3\alpha\gamma_x\\
&+3\gamma\alpha_x+2(p_1+p_2+p_3)h_3(q_1+q_2+q_3))-\eta\zeta(-p_1q_3-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)x\\
&-\eta\zeta\partial^{-1}(-q_3p_1-p_2q_2-p_3q_1+\gamma\alpha+\zeta\beta)k_1(t)+(\zeta_x x+\zeta+h_3\zeta x)k_1(t)+\zeta k_2(t)x.
\end{aligned}
$$

4.1. Super bi-Hamiltonian structure

Making use of (4.2), we get

$$
\begin{aligned}
&\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\lambda}\rangle=6a_1,\quad\langle\overline{Z},\frac{\partial\overline{Y}}{\partial p_1}\rangle=6\eta q_3a_1+3d_1,\\
&\langle\overline{Z},\frac{\partial\overline{Y}}{\partial q_1}\rangle=6\eta p_3a_1+3c_1,\quad\langle\overline{Z},\frac{\partial\overline{U}}{\partial p_2}\rangle=6\eta q_2a_1+3d_3,\\
&\langle\overline{Z},\frac{\partial\overline{Y}}{\partial q_2}\rangle=6\eta p_2a_1+3c_3,\quad\langle\overline{Z},\frac{\partial\overline{Y}}{\partial p_3}\rangle=6\eta q_1a_1+3d_2,\\
&\langle\overline{Z},\frac{\partial\overline{Y}}{\partial q_3}\rangle=6\eta p_1a_1+3c_2,\quad\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\alpha}\rangle=6\eta\gamma a_1-3\varrho,\\
&\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\beta}\rangle=6\eta\zeta a_1-3\delta,\quad\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\gamma}\rangle=-6\eta\alpha a_1+3\mu,\\
&\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\zeta}\rangle=-6\eta\beta a_1+3\nu.
\end{aligned}
\tag{4.10}
$$

Substituting (4.10) into the supertrace identity

$$
\frac{\delta}{\delta\overline{u}}\int\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\lambda}\rangle dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\langle\overline{Z},\frac{\partial\overline{Y}}{\partial\overline{u}}\rangle,\qquad\overline{u}=(p_1,q_1,p_2,q_2,p_3,q_3,\alpha,\beta,\gamma,\zeta)^T,
$$

then comparing the powers of λ , we have

$$\frac{\delta}{\delta\overline{u}}\int 2a_{1,n+2}dx=(\gamma-n-1)\begin{pmatrix}2\eta q_3a_{1,n+1}+d_{1,n+1}\\ 2\eta p_3a_{1,n+1}+c_{1,n+1}\\ 2\eta q_2a_{1,n+1}+d_{3,n+1}\\ 2\eta p_2a_{1,n+1}+c_{3,n+1}\\ 2\eta q_1a_{1,n+1}+d_{2,n+1}\\ 2\eta p_1a_{1,n+1}+c_{2,n+1}\\ 2\eta\gamma a_{n+1}-\varrho_{n+1}\\ 2\eta\zeta a_{n+1}-\delta_{n+1}\\ -2\eta\alpha a_{n+1}+\mu_{n+1}\\ -2\eta\beta a_{n+1}+\nu_{n+1}\end{pmatrix}. \tag{4.11}$$

Substituting $n=0$ into (4.11), we find $\gamma=0$. Hence, we derive

$$\begin{pmatrix}2\eta q_3a_{1,n+1}+d_{1,n+1}\\ 2\eta p_3a_{1,n+1}+c_{1,n+1}\\ 2\eta q_2a_{1,n+1}+d_{3,n+1}\\ 2\eta p_2a_{1,n+1}+c_{3,n+1}\\ 2\eta q_1a_{1,n+1}+d_{2,n+1}\\ 2\eta p_1a_{1,n+1}+c_{2,n+1}\\ 2\eta\gamma a_{n+1}-\varrho_{n+1}\\ 2\eta\zeta a_{n+1}-\delta_{n+1}\\ -2\eta\alpha a_{n+1}+\mu_{n+1}\\ -2\eta\beta a_{n+1}+\nu_{n+1}\end{pmatrix}=\frac{\delta\overline{H}_{n+1}}{\delta\overline{u}},\quad \overline{H}_{n+1}=\int-\frac{2a_{1,n+2}}{n+1}dx. \tag{4.12}$$

By using the recursion relations (4.5), we obtain

$$\begin{pmatrix}d_{1,n+1}\\ c_{1,n+1}\\ d_{3,n+1}\\ c_{3,n+1}\\ d_{2,n+1}\\ c_{2,n+1}\\ -\varrho_{n+1}\\ -\delta_{n+1}\\ \mu_{n+1}\\ \nu_{n+1}\end{pmatrix}=\widetilde{L}_3\begin{pmatrix}d_{1,n}\\ c_{1,n}\\ d_{3,n}\\ c_{3,n}\\ d_{2,n}\\ c_{2,n}\\ -\varrho_{n}\\ -\delta_{n}\\ \mu_{n}\\ \nu_{n}\end{pmatrix}+\widetilde{M}_3k_n(t)x,$$

where

$$\widetilde{L}_3 = \begin{pmatrix} L_1 & L_2 & L_3 & G & H \\ L_3 & L_1 & L_2 & G & H \\ L_2 & L_3 & L_1 & G & H \\ P & P & P & S & T \\ O & O & O & U & V \end{pmatrix},$$

with

$$L_1 = \begin{pmatrix} q_1\partial^{-1}p_1 + q_3\partial^{-1}p_2 + q_2\partial^{-1}p_3 - \frac{1}{2}\partial - h_3 & -q_1\partial^{-1}q_1 - q_3\partial^{-1}q_2 - q_2\partial^{-1}q_3 \\ p_1\partial^{-1}p_1 + p_3\partial^{-1}p_2 + p_2\partial^{-1}p_3 & -p_1\partial^{-1}q_1 - p_3\partial^{-1}q_2 - p_2\partial^{-1}q_3 + \frac{1}{2}\partial - h_3 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} q_1\partial^{-1}p_2 + q_3\partial^{-1}p_3 + q_2\partial^{-1}p_1 & -q_1\partial^{-1}q_2 - q_3\partial^{-1}q_3 - q_2\partial^{-1}q_1 \\ p_1\partial^{-1}p_2 + p_3\partial^{-1}p_3 + p_2\partial^{-1}p_1 & -p_1\partial^{-1}q_2 - p_3\partial^{-1}q_3 - p_2\partial^{-1}q_1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} q_1\partial^{-1}p_3 + q_3\partial^{-1}p_1 + q_2\partial^{-1}p_2 & -q_1\partial^{-1}q_3 - q_3\partial^{-1}q_1 - q_2\partial^{-1}q_2 \\ p_1\partial^{-1}p_3 + p_3\partial^{-1}p_1 + p_2\partial^{-1}p_2 & -p_1\partial^{-1}q_3 - p_3\partial^{-1}q_1 - p_2\partial^{-1}q_2 \end{pmatrix},$$

$$G = \begin{pmatrix} -\frac{1}{2}(q_1+q_2+q_3)\partial^{-1}\alpha - \frac{1}{2}\beta & \frac{1}{2}(q_1+q_2+q_3)\partial^{-1}\beta \\ -\frac{1}{2}(p_1+p_2+p_3)\partial^{-1}\alpha & \frac{1}{2}(p_1+p_2+p_3)\partial^{-1}\beta + \frac{1}{2}\alpha \end{pmatrix},$$

$$H = \begin{pmatrix} \frac{1}{2}(q_1+q_2+q_3)\partial^{-1}\gamma & -\frac{1}{2}(q_1+q_2+q_3)\partial^{-1}\zeta + \frac{1}{2}\gamma \\ \frac{1}{2}(p_1+p_2+p_3)\partial^{-1}\gamma - \frac{1}{2}\zeta & -\frac{1}{2}(p_1+p_2+p_3)\partial^{-1}\zeta \end{pmatrix},$$

$$P = \begin{pmatrix} -\gamma\partial^{-1}(p_1+p_2+p_3) - \zeta & \gamma\partial^{-1}(q_1+q_2+q_3) \\ -\zeta\partial^{-1}(p_1+p_2+p_3) & \zeta\partial^{-1}(q_1+q_2+q_3) + \gamma \end{pmatrix},$$

$$O = \begin{pmatrix} \alpha\partial^{-1}(p_1+p_2+p_3) & -\alpha\partial^{-1}(q_1+q_2+q_3) + \beta \\ \beta\partial^{-1}(p_1+p_2+p_3) - \alpha & -\beta\partial^{-1}(q_1+q_2+q_3) \end{pmatrix},$$

$$S = \begin{pmatrix} -\partial - h_3 & -3\gamma\partial^{-1}\beta - (q_1+q_2+q_3) \\ 3\zeta\partial^{-1}\alpha + p_1+p_2+p_3 & \partial - h_3 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & 3\gamma\partial^{-1}\zeta \\ -3\zeta\partial^{-1}\gamma & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 3\alpha\partial^{-1}\beta \\ -3\beta\partial^{-1}\alpha & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \partial - h_3 & -3\alpha\partial^{-1}\zeta - (p_1+p_2+p_3) \\ 3\beta\partial^{-1}\gamma + q_1+q_2+q_3 & -v\partial - h_3 \end{pmatrix}.$$

So, the generalized nonisospectral super AKNS hierarchy (4.9) has the following super bi-Hamiltonian structure:

$$\begin{aligned}\overline{u}_{t_n} &= \widetilde{Q}_3\widetilde{R}_3\frac{\delta\overline{H}_{n+1}}{\delta\overline{u}} - 2\eta\widetilde{Q}_3\widetilde{G}_3k_{n+1}(t)x \\ &= \widetilde{Q}_3\widetilde{L}_3\widetilde{R}_3\frac{\delta\overline{H}_n}{\delta\overline{u}} + (-2\eta\widetilde{Q}_3\widetilde{L}_3\widetilde{G}_3 + \widetilde{Q}_3\widetilde{M}_3)k_n(t)x,\end{aligned} \tag{4.13}$$

where $\widetilde{G}_3 = (q_3\ p_3\ q_2\ p_2\ \ q_1\ p_1\ \gamma\ \zeta\ \ -\alpha\ \ -\beta)^T$, $\widetilde{M}_3 = (q_1\ p_1\ q_3\ p_3\ q_2\ p_2\ \ -\gamma\ \ -\zeta\ \alpha\ \beta)^T$. The super Hamiltonian operators $\widetilde{Q}_3, \widetilde{R}_3$ are given by:

$$\widetilde{Q}_3 = \begin{pmatrix} Q_{13} & Q_{11} & Q_{12} & G_1 & H_1 \\ Q_{23} & Q_{21} & Q_{22} & G_2 & H_2 \\ Q_{33} & Q_{31} & Q_{32} & G_3 & H_3 \\ P_3 & P_1 & P_2 & S & T \\ O_3 & O_1 & O_2 & U & V \end{pmatrix} + \overline{J},$$

where

$$\begin{aligned}
Q_{ij} &= \begin{pmatrix} -4\eta p_i\partial^{-1}p_j & 4\eta p_i\partial^{-1}q_j \\ 4\eta q_i\partial^{-1}p_j & -4\eta q_i\partial^{-1}q_j \end{pmatrix}, \quad P_j = \begin{pmatrix} -2\eta\alpha\partial^{-1}p_j & 2\eta\alpha\partial^{-1}q_j \\ 2\eta\beta\partial^{-1}p_j & -2\eta\beta\partial^{-1}q_j \end{pmatrix}, \\
O_j &= \begin{pmatrix} 2\eta\gamma\partial^{-1}p_j & -2\eta\gamma\partial^{-1}q_j \\ -2\eta\zeta\partial^{-1}p_j & 2\eta\zeta\partial^{-1}q_j \end{pmatrix}, \quad G_i = \begin{pmatrix} 2\eta\ p_i\partial^{-1}\alpha & -2\eta p_i\partial^{-1}\beta \\ -2\eta q_i\partial^{-1}\alpha & 2\eta q_i\partial^{-1}\beta \end{pmatrix}, \\
H_i &= \begin{pmatrix} -2\eta p_i\partial^{-1}\gamma & 2\eta p_i\partial^{-1}\zeta \\ 2\eta q_i\partial^{-1}\gamma & -2\eta q_i\partial^{-1}\zeta \end{pmatrix}, \quad S = \begin{pmatrix} \eta\alpha\partial^{-1}\alpha & -\eta\alpha\partial^{-1}\beta \\ -\eta\beta\partial^{-1}\alpha & \eta\beta\partial^{-1}\beta \end{pmatrix}, \quad 0 \leq i, j \leq 3, \\
T &= \begin{pmatrix} -\eta\alpha\partial^{-1}\gamma & \eta\alpha\partial^{-1}\zeta \\ \eta\beta\partial^{-1}\gamma & -\eta\beta\partial^{-1}\zeta \end{pmatrix}, \quad U = \begin{pmatrix} -\eta\gamma\partial^{-1}\alpha & \eta\gamma\partial^{-1}\beta \\ \eta\zeta\partial^{-1}\alpha & -\eta\zeta\partial^{-1}\beta \end{pmatrix}, \quad V = \begin{pmatrix} \eta\gamma\partial^{-1}\gamma & -\eta\gamma\partial^{-1}\zeta \\ -\eta\zeta\partial^{-1}\gamma & \eta\zeta\partial^{-1}\zeta \end{pmatrix},
\end{aligned}$$

and

$$\overline{J} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

4.2. Darboux transformation of the spectral problem

In the following, we would discuss Darboux transformations [13, 20, 22] of the spectral problem $\overline{Y}$ in (4.2). The transformation can be set:

$$\phi^{'} = T\phi,$$

where $\phi^{'}$ and ϕ satisfying the following equations

$$\begin{cases} \phi_x = Y\phi, \\ \phi^{'}_x = Y^{'}\phi^{'}, \end{cases} \tag{4.14}$$

$Y^{'}$ has the same form as Y but with p_1, p_2, p_3, q_1, q_2, q_3, α, β, γ, ζ, and h_3, replaced by $p^{'}_1$, $p^{'}_2$, $p^{'}_3$, $q^{'}_1$, $q^{'}_2$, $q^{'}_3$, $\alpha^{'}$, $\beta^{'}$, $\gamma^{'}$, $\zeta^{'}$, and $h^{'}_3$.

We can express the spectral matrices Y and $Y^{'}$ in the following forms:

$$\begin{aligned}
\overline{Y} = &\begin{pmatrix} \lambda+h_3 & p_1 & 0 & p_3 & 0 & p_2 & \alpha \\ q_1 & -(\lambda+h_3) & q_3 & 0 & q_2 & 0 & \beta \\ 0 & p_2 & \lambda+h_3 & p_1 & 0 & p_3 & \alpha \\ q_2 & 0 & q_1 & -(\lambda+h_3) & q_3 & 0 & \beta \\ 0 & p_3 & 0 & p_2 & \lambda+h_3 & p_1 & \alpha \\ q_3 & 0 & q_2 & 0 & q_1 & -(\lambda+h_3) & \beta \\ \gamma & \zeta & \gamma & \zeta & \gamma & \zeta & 0 \end{pmatrix} \\
=&\lambda Y_0 + Y_1 \\
=&\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} h_3 & p_1 & 0 & p_3 & 0 & p_2 & \alpha \\ q_1 & -h_3 & q_3 & 0 & q_2 & 0 & \beta \\ 0 & p_2 & h_3 & p_1 & 0 & p_3 & \alpha \\ q_2 & 0 & q_1 & -h_3 & q_3 & 0 & \beta \\ 0 & p_3 & 0 & p_2 & h_3 & p_1 & \alpha \\ q_3 & 0 & q_2 & 0 & q_1 & -h_3 & \beta \\ \gamma & \zeta & \gamma & \zeta & \gamma & \zeta & 0 \end{pmatrix},
\end{aligned}$$

$$\overline{Y}^{'} = \begin{pmatrix} \lambda+h^{'}_3 & p^{'}_1 & 0 & p^{'}_3 & 0 & p^{'}_2 & \alpha^{'} \\ q^{'}_1 & -(\lambda+h^{'}_3) & q^{'}_3 & 0 & q^{'}_2 & 0 & \beta^{'} \\ 0 & p^{'}_2 & \lambda+h^{'}_3 & p^{'}_1 & 0 & p^{'}_3 & \alpha^{'} \\ q^{'}_2 & 0 & q^{'}_1 & -(\lambda+h^{'}_3) & q^{'}_3 & 0 & \beta^{'} \\ 0 & p^{'}_3 & 0 & p^{'}_2 & \lambda+h^{'}_3 & p^{'}_1 & \alpha^{'} \\ q^{'}_3 & 0 & q^{'}_2 & 0 & q^{'}_1 & -(\lambda+h^{'}_3) & \beta^{'} \\ \gamma^{'} & \zeta^{'} & \gamma^{'} & \zeta^{'} & \gamma^{'} & \zeta^{'} & 0 \end{pmatrix}$$

$$
\begin{aligned}
&=\lambda Y_0 + Y_1^{'}\\
&=\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} h_3^{'} & p_1^{'} & 0 & p_3^{'} & 0 & p_2^{'} & \alpha^{'}\\ q_1^{'} & -h_3^{'} & q_3^{'} & 0 & q_2^{'} & 0 & \beta^{'}\\ 0 & p_2^{'} & h_3^{'} & p_1^{'} & 0 & p_3^{'} & \alpha^{'}\\ q_2^{'} & 0 & q_1^{'} & -h_3^{'} & q_3^{'} & 0 & \beta^{'}\\ 0 & p_3^{'} & 0 & p_2^{'} & h_3^{'} & p_1^{'} & \alpha^{'}\\ q_3^{'} & 0 & q_2^{'} & 0 & q_1^{'} & -h_3^{'} & \beta^{'}\\ \gamma^{'} & \zeta^{'} & \gamma^{'} & \zeta^{'} & \gamma^{'} & \zeta^{'} & 0 \end{pmatrix}.
\end{aligned}
$$

For (4.14), T satisfies the following equation:

$$T_x + T\overline{Y} = \overline{Y}^{'} T. \tag{4.15}$$

Assume that

$$
\begin{aligned}
T =&\lambda T_1 + T_0\\
=&\lambda \begin{pmatrix} A_{11} & A_{12} & 0 & A_{14} & 0 & A_{16} & A_{17}\\ A_{21} & A_{22} & A_{23} & 0 & A_{25} & 0 & A_{27}\\ 0 & A_{16} & A_{11} & A_{12} & 0 & A_{14} & A_{17}\\ A_{25} & 0 & A_{21} & A_{22} & A_{23} & 0 & A_{27}\\ 0 & A_{14} & 0 & A_{16} & A_{11} & A_{12} & A_{17}\\ A_{23} & 0 & A_{25} & 0 & A_{21} & A_{22} & A_{27}\\ A_{71} & A_{72} & A_{71} & A_{72} & A_{71} & A_{72} & 0 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & 0 & B_{14} & 0 & B_{16} & B_{17}\\ B_{21} & B_{22} & B_{23} & 0 & B_{25} & 0 & B_{27}\\ 0 & B_{16} & B_{11} & B_{12} & 0 & B_{14} & B_{17}\\ B_{25} & 0 & B_{21} & B_{22} & B_{23} & 0 & B_{27}\\ 0 & B_{14} & 0 & B_{16} & B_{11} & B_{12} & B_{17}\\ B_{23} & 0 & B_{25} & 0 & B_{21} & B_{22} & B_{27}\\ B_{71} & B_{72} & B_{71} & B_{72} & B_{71} & B_{72} & 0 \end{pmatrix}.
\end{aligned} \tag{4.16}
$$

Form (4.15), we have

$$T_x = \lambda T_{1,x} + T_{0,x} = \overline{Y}^{'} T - T\overline{Y}. \tag{4.17}$$

Comparing the powers of λ_j $(j = 0, 1, 2)$ in the above equations, the case of $j = 2$ gives rise to

$$Y_0 T_1 - T_1 Y_0 = 0, \tag{4.18}$$

and

$$
\begin{aligned}
&A_{12} = A_{14} = A_{16} = A_{17} = A_{21} = A_{23} = A_{25} = A_{27} = A_{32} = A_{34} = A_{36} = 0,\\
&A_{41} = A_{43} = A_{45} = A_{52} = A_{54} = A_{56} = A_{61} = A_{63} = A_{65} = A_{71} = A_{72} = 0.
\end{aligned} \tag{4.19}
$$

When $j = 1$, the following equations are derived:

$$T_{1,x} = Y_0 T_0 + Y_1^{'} T_1 - T_1 Y_1 - T_0 Y_0, \tag{4.20}$$

and

$$\begin{cases} A_{11,x} = (h_3' - h_3)A_{11} \Longrightarrow A_{11} = e^{h_3' - h_3}, \\ A_{22,x} = (h_3 - h_3')A_{22} \Longrightarrow A_{22} = e^{h_3 - h_3'}, \\ B_{12} = \frac{1}{2}(p_1' A_{22} + p_1 A_{11}), \\ B_{14} = \frac{1}{2}(p_3' A_{22} + p_3 A_{11}), \\ B_{16} = \frac{1}{2}(p_2' A_{22} + p_2 A_{11}), \\ B_{21} = \frac{1}{2}(q_1' A_{11} + q_1 A_{22}), \\ B_{23} = \frac{1}{2}(q_3' A_{11} - q_3 A_{22}), \\ B_{25} = \frac{1}{2}(q_2' A_{11} - q_2 A_{22}), \\ B_{17} = A_{11}\alpha, \\ B_{27} = A_{22}\beta. \end{cases} \tag{4.21}$$

For $j = 0$, the results are given by

$$T_{0,x} = Y_1' T_0 - T_0 Y_1, \tag{4.22}$$

and

$$\begin{cases} B_{11,x} = (h_3' - h_3)B_{11} + p_1' B_{21} + p_3' B_{25} + p_2' B_{23} + \alpha' B_{71} - (q_1 B_{12} + q_2 B_{14} + q_3 B_{16} + \gamma B_{17}), \\ B_{12,x} = (h_3' + h_3)B_{12} + p_1' B_{22} + \alpha' B_{72} - p_1 B_{11} - \zeta B_{17}, \\ B_{14,x} = (h_3' + h_3)B_{14} + p_3' B_{22} + \alpha' B_{72} - p_3 B_{11} - \zeta B_{17}, \\ B_{16,x} = (h_3' + h_3)B_{16} + p_2' B_{22} + \alpha' B_{72} - B_{11} p_2 - \zeta B_{17}, \\ B_{17,x} = h_3' B_{17} + (p_1' + p_3' + p_2')B_{27} - \alpha B_{11} - \beta(B_{12} + B_{14} + B_{16}), \\ B_{21,x} = q_1' B_{11} - (h_3' + h_3)B_{21} + \beta' B_{71} - q_1 B_{22} - \gamma B_{27}, \\ B_{22,x} = q_1' B_{12} + (h_3 - h_3')B_{22} + q_3' B_{16} + q_2' B_{14} + \beta' B_{72} - (p_1 B_{21} + p_2 B_{23} + p_3 B_{25} + \zeta B_{27}), \\ B_{23,x} = -(h_3' + h_3)B_{23} + q_3' B_{11} + \beta' B_{71} - q_3 B_{22} - \gamma B_{27}, \\ B_{25,x} = -(h_3' + h_3)B_{25} + q_2' B_{11} + \beta' B_{71} - (q_2 B_{22} + \gamma B_{27}), \\ B_{27,x} = (q_1' + q_2' + q_3')B_{17} - h_3' B_{27} - (\alpha B_{21} + \alpha B_{23} + \alpha B_{25} + \beta B_{22}), \\ B_{71,x} = \gamma' B_{11} + \zeta' B_{21} + \zeta' B_{23} + \zeta' B_{25} - h_3 B_{71} - (q_1 + q_2 + q_3)B_{72}, \\ B_{72,x} = \zeta' B_{22} + \gamma' B_{12} + \gamma' B_{14} + \gamma' B_{16} - (p_1 + p_2 + p_3)B_{71} + h_3 B_{72}. \end{cases} \tag{4.23}$$

From (4.21), we get

$$\begin{aligned} B_{12} =& \frac{1}{2}(p_1' e^{h_3 - h_3'} + e^{h_3' - h_3} p_1),\ B_{14} = \frac{1}{2}(p_3' e^{h_3 - h_3'} + e^{h_3' - h_3} p_3),\ B_{16} = \frac{1}{2}(p_2' e^{h_3 - h_3'} + e^{h_3' - h_3} p_2), \\ B_{21} =& \frac{1}{2}(q_1' e^{h_3' - h_3} + e^{h_3 - h_3'} q_1),\ B_{23} = \frac{1}{2}(q_3' e^{h_3' - h_3} - e^{h_3 - h_3'} q_3),\ B_{25} = \frac{1}{2}(q_2' e^{h_3' - h_3} - e^{h_3 - h_3'} q_2), \\ B_{11} =& \frac{1}{\gamma'}\partial_x(\gamma' e^{h_3' - h_3}) - \frac{1}{2}\frac{\zeta'}{\gamma'}(q_1' e^{h_3' - h_3} + e^{h_3 - h_3'} q_1 + q_2' e^{h_3' - h_3} - e^{h_3 - h_3'} q_2 + q_3' e^{h_3' - h_3} - e^{h_3 - h_3'} q_3) \\ & + e^{h_3' - h_3} h_3 + \frac{\zeta'}{\gamma'} e^{h_3 - h_3'}(q_1 + q_2 + q_3), \end{aligned}$$

$$
\begin{aligned}
B_{22} =& \frac{1}{\zeta'}\partial_x(\zeta' e^{h_3-h_3'}) - \frac{1}{2}\frac{\gamma'}{\zeta'}(p_1' e^{h_3-h_3'} + e^{h_3'-h_3}p_1 + p_2' e^{h_3-h_3'} + e^{h_3'-h_3}p_2 + p_3' e^{h_3-h_3'} + e^{h_3'-h_3}p_3) \\
&+ \frac{\gamma'}{\zeta'}e^{h_3'-h_3}(p_1+p_2+p_3) - e^{h_3-h_3'}h_3, \\
B_{17} =& \alpha e^{h_3'-h_3},\ B_{27} = \beta e^{h_3-h_3'},\ B_{71} = \gamma' e^{h_3'-h_3},\ B_{72} = \zeta' e^{h_3-h_3'}.
\end{aligned} \tag{4.24}
$$

Therefore, the matrix T becomes the following form:

$$
\begin{aligned}
T =& \lambda \begin{pmatrix}
e^{h_3'-h_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -e^{h_3-h_3'} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{h_3'-h_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -e^{h_3-h_3'} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{h_3'-h_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -e^{h_3-h_3'} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&+ \begin{pmatrix}
B_{11} & B_{12} & 0 & B_{14} & 0 & B_{16} & B_{17} \\
B_{21} & B_{22} & B_{23} & 0 & B_{25} & 0 & B_{27} \\
0 & B_{16} & B_{11} & B_{12} & 0 & B_{14} & B_{17} \\
B_{25} & 0 & B_{21} & B_{22} & B_{23} & 0 & B_{27} \\
0 & B_{14} & 0 & B_{16} & B_{11} & B_{12} & B_{17} \\
B_{23} & 0 & B_{25} & 0 & B_{21} & B_{22} & B_{27} \\
B_{71} & B_{72} & B_{71} & B_{72} & B_{71} & B_{72} & 0
\end{pmatrix}.
\end{aligned} \tag{4.25}
$$

5. Conclusions and discussions

In this paper, we considered the N×N matrix spectral problems associated with the Lie superalgebra $spl(2N,1)$. Then we obtained the generalized nonisospectral multi-component super AKNS hierarchy associated with the Lie superalgebra $spl(2N,1)$. The generalized nonisospectral three-component coupled super AKNS hierarchy associated with the Lie superalgebra $spl(6,1)$ was also obtained. By using the supertrace identity, we derived the bi-Hamiltonian structures of the generalized nonisospectral multi-component and the three-component coupled super AKNS integrable hierarchies. The Darboux transformation of the generalized nonisospectral three-component coupled super AKNS integrable hierarchy is obtained. When N=1, the Lie superalgebra $spl(2N,1)$ is reduced to Lie superalgebra $spl(2,1)$. When N=1 and $\eta=0$ in the generalized multi-component hierarchy (3.8), we found that (3.8) in our paper can be reduced to (11) in [41]. Comparing (17) with $\varepsilon=0$ in [12], we found that it is equivalent to generalized hierarchy (3.8) with $\eta=0$ in our paper. Moreover, we have derived a new generalized multi-component hierarchy associated with the Lie superalgebra $sl(2N,1)$ [18]. The extension of the Lie superalgebra

$spl(2N,1)$ may have contributed to the development of multi-component classification.

Data availability statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest. The author declare that they have no conflicts of interest.

Appendix A. The super Hamiltonian operators $\widetilde{Q}_N,\ \widetilde{R}_N,\ \widetilde{L}_N$ are given by:

$$\widetilde{Q}_N = N\overset{N}{\begin{pmatrix} Q_{1,N} & Q_{1,1} & Q_{1,2} & \cdots & Q_{1,N-1} & G_1 & H_1 \\ Q_{2,N} & Q_{2,1} & Q_{2,2} & \cdots & Q_{2,N-1} & G_2 & H_2 \\ Q_{3,N} & Q_{3,1} & Q_{3,2} & \cdots & Q_{3,N-1} & G_3 & H_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ Q_{N,N} & Q_{N,1} & Q_{N,2} & \cdots & Q_{N,N-1} & G_N & H_N \\ P_N & P_1 & P_2 & \cdots & P_{N-1} & S & T \\ O_N & O_1 & O_2 & \cdots & O_{N-1} & U & V \end{pmatrix}} + \widetilde{J},$$

where

$$Q_{i,j} = \begin{pmatrix} -4\eta p_i\partial^{-1}p_j & 4\eta p_i\partial^{-1}q_j \\ 4\eta q_i\partial^{-1}p_j & -4\eta q_i\partial^{-1}q_j \end{pmatrix}, \quad P_j = \begin{pmatrix} -2\eta\alpha\partial^{-1}p_j & 2\eta\alpha\partial^{-1}q_j \\ 2\eta\beta\partial^{-1}p_j & -2\eta\beta\partial^{-1}q_j \end{pmatrix},$$

$$O_j = \begin{pmatrix} 2\eta\gamma\partial^{-1}p_j & -2\eta\gamma\partial^{-1}q_j \\ -2\eta\zeta\partial^{-1}p_j & 2\eta\zeta\partial^{-1}q_j \end{pmatrix}, \quad G_i = \begin{pmatrix} 2\eta\ p_i\partial^{-1}\alpha & -2\eta p_i\partial^{-1}\beta \\ -2\eta q_i\partial^{-1}\alpha & 2\eta q_i\partial^{-1}\beta \end{pmatrix}, \quad 1 \le i,j \le N,$$

$$H_i = \begin{pmatrix} -2\eta p_i\partial^{-1}\gamma & 2\eta p_i\partial^{-1}\zeta \\ 2\eta q_i\partial^{-1}\gamma & -2\eta q_i\partial^{-1}\zeta \end{pmatrix}, \quad S = \begin{pmatrix} \eta\alpha\partial^{-1}\alpha & -\eta\alpha\partial^{-1}\beta \\ -\eta\beta\partial^{-1}\alpha & \eta\beta\partial^{-1}\beta \end{pmatrix},$$

$$T = \begin{pmatrix} -\eta\alpha\partial^{-1}\gamma & \eta\alpha\partial^{-1}\zeta \\ \eta\beta\partial^{-1}\gamma & -\eta\beta\partial^{-1}\zeta \end{pmatrix}, \quad U = \begin{pmatrix} -\eta\gamma\partial^{-1}\alpha & \eta\gamma\partial^{-1}\beta \\ \eta\zeta\partial^{-1}\alpha & -\eta\zeta\partial^{-1}\beta \end{pmatrix}, \quad V = \begin{pmatrix} \eta\gamma\partial^{-1}\gamma & -\eta\gamma\partial^{-1}\zeta \\ -\eta\zeta\partial^{-1}\gamma & \eta\zeta\partial^{-1}\zeta \end{pmatrix},$$

$$\widetilde{J} = N\overset{N}{\begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & J_1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & J_1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & J_1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & J_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & J_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & J_2 & 0 \end{pmatrix}}, J_1 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The super Hamiltonian operator $\widetilde{R}_N$:

$$\widetilde{R}_N = \begin{array}{c} \\ \\ \\ \\ N \\ \\ \\ \end{array} \left(\begin{array}{cccccccc} \multicolumn{5}{c}{\overbrace{\hphantom{R_{N,1} \quad R_{N,2} \quad R_{N,3} \quad \cdots \quad R_{N,N}}}^{N}} & & \\ R_{N,1} & R_{N,2} & R_{N,3} & \cdots & R_{N,N} & G_N & H_N \\ R_{N-1,1} & R_{N-1,2} & R_{N-1,3} & \cdots & R_{N-1,N} & G_{N-1} & H_{N-1} \\ R_{N-2,1} & R_{N-2,2} & R_{N-2,3} & \cdots & R_{N-2,N} & G_{N-2} & H_{N-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ R_{1,1} & R_{1,2} & R_{1,3} & \cdots & R_{1,N} & G_1 & H_1 \\ P_1 & P_2 & P_3 & \cdots & P_N & S & T \\ O_1 & O_2 & O_3 & \cdots & O_N & U & V \end{array} \right) + I,$$

where

$$R_{i,j} = \begin{pmatrix} -2\eta q_i\partial^{-1}p_j & 2\eta q_i\partial^{-1}q_j \\ -2\eta p_i\partial^{-1}p_j & 2\eta p_i\partial^{-1}q_j \end{pmatrix}, \quad O_j = \begin{pmatrix} 2\eta\alpha\partial^{-1}p_j & -2\eta\alpha\partial^{-1}q_j \\ 2\eta\beta\partial^{-1}p_j & -2\eta\beta\partial^{-1}q_j \end{pmatrix},$$

$$P_j = \begin{pmatrix} -2\eta\gamma\partial^{-1}p_j & 2\eta\gamma\partial^{-1}q_j \\ -2\eta\zeta\partial^{-1}p_j & 2\eta\zeta\partial^{-1}q_j \end{pmatrix}, \quad G_i = \begin{pmatrix} \eta q_i\partial^{-1}\alpha & -\eta q_i\partial^{-1}\beta \\ \eta p_i\partial^{-1}\alpha & -\eta p_i\partial^{-1}\beta \end{pmatrix},$$

$$H_i = \begin{pmatrix} -\eta q_i\partial^{-1}\gamma & \eta q_i\partial^{-1}\zeta \\ -\eta p_i\partial^{-1}\gamma & \eta p_i\partial^{-1}\zeta \end{pmatrix}, \quad S = \begin{pmatrix} \eta\gamma\partial^{-1}\alpha & -\eta\gamma\partial^{-1}\beta \\ \eta\zeta\partial^{-1}\alpha & -\eta\zeta\partial^{-1}\beta \end{pmatrix}, \quad 1 \le i,j \le N,$$

$$T = \begin{pmatrix} -\eta\gamma\partial^{-1}\gamma & \eta\gamma\partial^{-1}\zeta \\ -\eta\zeta\partial^{-1}\gamma & \eta\zeta\partial^{-1}\zeta \end{pmatrix}, \quad U = \begin{pmatrix} -\eta\alpha\partial^{-1}\alpha & \eta\alpha\partial^{-1}\beta \\ -\eta\beta\partial^{-1}\alpha & \eta\beta\partial^{-1}\beta \end{pmatrix}, \quad V = \begin{pmatrix} \eta\alpha\partial^{-1}\gamma & -\eta\alpha\partial^{-1}\zeta \\ \eta\beta\partial^{-1}\gamma & -\eta\beta\partial^{-1}\zeta \end{pmatrix},$$

and I is an identity matrix with $N+2$ degree.

The super Hamiltonian operator

$$\widetilde{L}_N = \begin{array}{c} \\ \\ \\ \\ \\ N \\ \\ \\ \end{array} \left(\begin{array}{cccccccc} \multicolumn{6}{c}{\overbrace{\hphantom{L_{N-1} \quad L_N \quad L_1 \quad \cdots \quad L_{N-3} \quad L_{N-2}}}^{N}} & & \\ L_1 & L_2 & L_3 & \cdots & L_{N-1} & L_N & G & H \\ L_N & L_1 & L_2 & \cdots & L_{N-2} & L_{N-1} & G & H \\ L_{N-1} & L_N & L_1 & \cdots & L_{N-3} & L_{N-2} & G & H \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ L_3 & L_4 & L_5 & \cdots & L_1 & L_2 & G & H \\ L_2 & L_3 & L_4 & \cdots & L_N & L_1 & G & H \\ P & P & P & \cdots & P & P & S & T \\ O & O & O & \cdots & O & O & U & V \end{array} \right),$$

with

$$L_1 = \begin{pmatrix} q_1\partial^{-1}p_1 + q_N\partial^{-1}p_2 + \cdots + q_2\partial^{-1}p_N - \frac{1}{2}\partial - h_3 & -q_1\partial^{-1}q_1 - q_N\partial^{-1}q_2 - \cdots - q_2\partial^{-1}q_N \\ p_1\partial^{-1}p_1 + p_N\partial^{-1}p_2 + \cdots + p_2\partial^{-1}p_N & -p_1\partial^{-1}q_1 - p_N\partial^{-1}q_2 - \cdots - p_2\partial^{-1}q_N + \frac{1}{2}\partial - h_3 \end{pmatrix},$$

$$L_i = \begin{pmatrix} q_i\partial^{-1}p_1 + q_{i-1}\partial^{-1}p_2 + \cdots + q_1\partial^{-1}p_N & -q_i\partial^{-1}q_1 - q_{i-1}\partial^{-1}q_2 - \cdots - q_1\partial^{-1}q_N \\ p_i\partial^{-1}p_1 + p_{i-1}\partial^{-1}p_2 + \cdots + p_1\partial^{-1}p_N & -p_i\partial^{-1}q_1 - p_{i-1}\partial^{-1}q_2 - \cdots - p_1\partial^{-1}q_N \end{pmatrix}, \ 2 \le i \le N,$$

$$G = \begin{pmatrix} -\frac{1}{2}(q_1 + q_2 + \cdots + q_N)\partial^{-1}\alpha - \frac{1}{2}\beta & \frac{1}{2}(q_1 + q_2 + \cdots + q_N)\partial^{-1}\beta \\ -\frac{1}{2}(p_1 + p_2 + \cdots + p_N)\partial^{-1}\alpha & \frac{1}{2}(p_1 + p_2 + \cdots + p_N)\partial^{-1}\beta + \frac{1}{2}\alpha \end{pmatrix},$$

$$H = \begin{pmatrix} \frac{1}{2}(q_1 + q_2 + \cdots + q_N)\partial^{-1}\gamma & -\frac{1}{2}(q_1 + q_2 + \cdots + q_N)\partial^{-1}\zeta + \frac{1}{2}\gamma \\ \frac{1}{2}(p_1 + p_2 + \cdots + p_N)\partial^{-1}\gamma - \frac{1}{2}\zeta & -\frac{1}{2}(p_1 + p_2 + \cdots + p_N)\partial^{-1}\zeta \end{pmatrix},$$

$$P = \begin{pmatrix} -\gamma\partial^{-1}(p_1 + p_2 + \cdots + p_N)) - \zeta & \gamma\partial^{-1}(q_1 + q_2 + \cdots + q_N) \\ -\zeta\partial^{-1}(p_1 + p_2 + \cdots + p_N) & \zeta\partial^{-1}(q_1 + q_2 + \cdots + q_N) + \gamma \end{pmatrix},$$

$$O = \begin{pmatrix} \alpha\partial^{-1}(p_1 + p_2 + \cdots + p_N) & -\alpha\partial^{-1}(q_1 + q_2 + \cdots + q_N) + \beta \\ \beta\partial^{-1}(p_1 + p_2 + \cdots + p_N) - \alpha & -\beta\partial^{-1}(q_1 + q_2 + \cdots + q_N) \end{pmatrix},$$

$$S = \begin{pmatrix} -\partial - h_N & -N\gamma\partial^{-1}\beta - (q_1 + q_2 + \cdots + q_N) \\ N\zeta\partial^{-1}\alpha + p_1 + p_2 + \cdots + p_N & \partial - h_N \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & N\gamma\partial^{-1}\zeta \\ -N\zeta\partial^{-1}\gamma & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & N\alpha\partial^{-1}\beta \\ -N\beta\partial^{-1}\alpha & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \partial - h_N & -N\alpha\partial^{-1}\zeta - (p_1 + p_2 + \cdots + p_N) \\ N\beta\partial^{-1}\gamma + q_1 + q_2 + \cdots + q_N & -\partial - h_N \end{pmatrix}.$$

References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newelland and H. Segur, *The inverse scattering Transform-Fourier analysis for nonlinear problems*, Stud. Appl. Math., 1974, 53, 249–315.

[2] S. Carillo and O. Ragnisco, *Nonlinear Evolution Equations and Dynamical Systems*, Springer Science and Business Media, Berlin, 2012.

[3] L. Debnath, *Solitons and the Inverse Scattering Transform*, Society for Industrial and Applied Mathematics, Philadelphia, 1981.

[4] F. Demontis and C. van der Mee, *From the AKNS system to the matrix Schrödinger equation with vanishing potentials: Direct and inverse problems*, Stud. Appl. Math., 2023, 150, 481–519.

[5] C. Devchand and J. Schiff, *The supersymmetric Camassa-Holm equation and geodesic flow on the superconformal group*, J. Math. Phys., 2001, 42(1), 260–273.

[6] B. Dubrovin, S. Q. Liu and Y. J. Zhang, *On Hamiltonian perturbations of hyperbolic systems of conservation laws: I. Quasi-triviality of bi-Hamiltonian perturbations*, Commun. Pure Appl. Math., 2005, 59(4), 559–615.

[7] G. Falqui, F. Magri and M. Pedroni, *Bi-Hamiltonian geometry, Darboux coverings, and linearization of the KP hierarchy*, Commun. Math. Phys., 1998, 197(2), 303–324.

[8] X. G. Geng, *A hierarchy of nonlinear evolution equations, its Hamiltonian structure and classical integrable system*, Physica A, 1992, 180(1–2), 241–251.

[9] X. G. Geng, B. Xue and L. H. Wu, *A Super Camassa-Holm Equation with N-Peakon Solutions*, Stud. Appl. Math., 2013, 130, 1–16.

[10] V. S. Gerdjikov and M. I. Ivanov, *The quadratic bundle of general form and the nonlinear evolution equations. I. Expansions over the "squared" solutions are generalized Fourier transforms*, Bulg. J. Phys., 1983, 10, 13–26.

[11] M. Gürses and O. Oguz, *A super AKNS scheme*, Phys. Lett. A, 1985, 108, 437–440.

[12] J. W. Han and J. Yu, *A generalized super AKNS hierarchy associated with Lie superalgebra sl(2|1) and its super bi-Hamiltonian structure*, Commun. Nonlinear Sci. Numer. Simul., 2017, 44, 258–265.

[13] B. Y. He and L. Y. Chen, *Super integrable hierarchies associated with spl(2,1) and Darboux transformations*, J. Math. Phys., 2017, 58(12), 123503. DOI: 10.1063/1.5018031.

[14] B. B. Hu, W. X. Ma, T. C. Xia and L. Zhang, *Nonlinear integrable couplings of a generalized super Ablowitz-Kaup-Newell-Segur hierarchy and its super bi-Hamiltonian structures*, Math. Method Appl. Sci., 2018, 41, 1565–1577.

[15] A. Inam and M. ul Hassan, *Exact solitons of a N-component discrete coupled integrable system*, Theor. Math. Phys., 2023, 214, 36–71.

[16] K. H. Karlsen and Y. Rybalko, *On the well-posedness of a nonlocal (two-place) FORQ equation via a two-component peakon system*, J. Math. Anal. Appl., 2024, 529, 127601. DOI: 10.1016/j.jmaa.2023.127601.

[17] B. Kupershmidt, *Superintegrable systems*, Proc. Nat. Acad. Sci. USA, 1984, 81(20), 6562–6563.

[18] J. X. Li and H. F. Wang, *A multicomponent generalized nonisospectral super AKNS integrable hierarchy*, Theor. Math. Phys., 2024, 221(3), 2083–2108.

[19] Q. P. Liu, *Supersymmetric Harry Dym type equations*, J. Phys. A: Math. Gen., 1995, 28(8), 245–248.

[20] W. X. Ma, *Darboux transformations for a Lax integrable system in 2n-dimensions*, Lett. Math. Phys., 1997, 39(1), 33–49.

[21] W. X. Ma, J. S. He and Z. Y. Qin, *A supertrace identity and its applications to superingrable systems*, J. Math. Phys., 2008. DOI: 10.1063/1.2897036.

[22] W. X. Ma, J. H. Meng and H. Q. Zhang, *Integrable couplings, variational identities and Hamiltonian formulations*, Global J. Math. Sci., 2012, 1(1), 1–17.

[23] W. X. Ma, C. G. Shi, E. A. Appiah, C. X. Li and S. F. Sheng, *An integrable generalization of the Kaup-Newell soliton hierarchy*, Phys. Scripta, 2014, 89, 085203. DOI: 10.1088/0031-8949/89/8/085203.

[24] Z. Popowicz, *The fully supersymmetric AKNS equations*, J. Phys. A: Math. Gen., 1990, 23(7), 1127–1136.

[25] G. H. M. Roelofs and P. H. M. Kersten, *Supersymmetric extensions of the nonlinear Schrödinger equation: Symmetries and coverings*, J. Math. Phys., 1992, 33(6), 2185–2206.

[26] A. Savchenko and A. Zabrodin, *Multicomponent DKP hierarchy and its dispersionless limit*, Lett. Math. Phys., 2025, 115, 1–26.

[27] S. F. Shen, C. X. Li, Y. Y. Jin and W. X. Ma, *Completion of the Ablowitz-Kaup-Newell-Segur integrable coupling*, J. Math. Phys., 2017, 59, 103503. DOI: 10.1063/1.4990534.

[28] H. Z. Sun, *Lie Algebras, Lie Superalgebras and their Applications in Physics*, Peking University, Beijing, 1999.

[29] G. Z. Tu, *The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems*, J. Math. Phys., 1989, 30(2), 330–338.

[30] H. F. Wang and B. Y. He, *A class of extended Lie superalgebras and their applications*, Chaos Solitons Fract., 2023, 168, 113145. DOI: 10.1016/j.chaos.2023.113145.

[31] H. F. Wang, J. X. Li, Z. Z. Fang and H. J. Zhou, *A novel kind of multi-component non-isospectral super soliton hierarchy with self-consistent sources*, to appear in Phys. Fluids.

[32] H. F. Wang and Y. F. Zhang, *Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations*, J. Comput. Appl. Math., 2023, 420, 114812. DOI: 10.1016/j.cam.2022.114812.

[33] H. F. Wang and Y. F. Zhang, *A new multi-component integrable coupling and its application to isospectral and nonisospectral problems*, Commun. Nonlinear Sci. Numer. Simul., 2022, 105, 106075. DOI: 10.1016/j.cnsns.2021.106075.

[34] H. F. Wang, Y. F. Zhang and C. Z. Li, *A multi-component super integrable Dirac hierarchy*, Phys. Lett. B, 2023, 847, 138323. DOI: 10.1016/j.physletb.2023.138323.

[35] H. F. Wang, Y. F. Zhang and C. Z. Li, *Multi-component super integrable Hamiltonian hierarchies*, Physica D, 2023, 456, 133918. DOI: 10.1016/j.physd.2023.133918.

[36] Z. Y. Yan and H. Q. Zhang, *A hierarchy of generalized AKNS equations, N-Hamiltonian structures and finite-dimensional involutive systems and integrable systems*, J. Math. Phys., 2001, 42(1), 330–339.

[37] Y. J. Ye, Z. H. Li, C. X. Li, S. F. Shen and W. X. Ma, *A generalized Dirac soliton hierarchy and its bi-Hamiltonian structure*, Appl. Math. Lett., 2016, 60, 67–72.

[38] A. Zabrodin, *Tau-Function of the Multi-Component CKP Hierarchy*, Math. Phys. Anal. Geom., 2024. DOI: 10.1007/s11040-023-09473-6.

[39] Y. F. Zhang, J. Q. Mei and H. Y. Guan, *A method for generating isospectral and nonisospectral hierarchies of equations as well as symmetries*, J. Geom. Phys., 2020, 147, 103538. DOI: 10.1016/j.geomphys.2019.103538.

[40] Y. F. Zhang and X. Z. Zhang, *A scheme for generating nonisospectral integrable hierarchies and its related applications*, Acta. Math. Sin.-English Ser., 2021, 37(5), 707–730.

[41] R. G. Zhou, *A hierarchy of super AKNS hierarchy related to Lie superalgebra sl(2|1) and a finite dimensional super Hamiltonian system*, Mod. Phys. Lett. B, 2015, 29(22), 1550126. DOI: 10.1142/S0217984915501262.

Received April 2025; Accepted July 2025; Available online July 2025.