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ASYMPTOTIC SOLUTION FOR A SYSTEM OF SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS*

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Abstract This paper is mainly aimed to investigate the asymptotic solution for a system of singularly perturbed delay differential equations. In order to establish the step–like asymptotic solution, the basic framework of contrast structure theory is employed. Some sufficient criteria for the existence of asymptotic solution will be proposed. After that, by means of the boundary layer functions method and sewing method, the expansion for the asymptotic solution will be constructed and the existence of a uniformly valid continuous solution is proved. Finally, the effectiveness of the established results is validated via a concrete example.

Keywords Asymptotic solution, singular perturbation, delay, boundary layer functions, contrast structure.

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1. Introduction

Singularly perturbed differential equations (SPDEs) have gradually come into the view of many scholars since Prandtl [31] studied the viscous incompressible flow in the beginning of the 20th century, and arise in many fields of science and technology, such as, life sciences [18], dissipative systems [2, 3], reaction–convection–diffusion problems [1, 25], optimal control problems [10, 40], etc. In the past few years, a large number of special purpose methods have been established to deal with SPDEs, for details, one may refer to the method of matched asymptotic expansion [4, 27], geometric singular perturbation theory [38, 39], averaging method [17, 42], multiple–scale analysis [20], renormalization group approach [32, 41], boundary layer functions method [36, 37], and so on. Additionally, in mathematics, delay differential equations (DDEs) are differential equations in which the derivative of the unknown function not only evaluated at present time but also evaluated in terms of the values of the function at previous times, the interest for DDEs keeps on growing in almost all scientific areas, and especially see, e.g. population dynamics [12, 19], epidemiology [16], human pupil–light reflex models [22, 23], bistable devices [34], etc.

With the rapid development of singular perturbation theory and delay differential equation theory, there has been a growing great interest in solving singularly perturbed delay differential

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equations (SPDDEs), in which the highest order derivative is multiplied by sufficiently small positive parameter ϵ, and contains at least one delay term. Such type of differential equation plays an important role in mathematical modeling of various practical phenomena, for example, H_∞ control system [11], first–exit time problem [21], evolutionary biology [26]. It is common knowledge that there exist boundary layer(s) and internal layer(s) for the solution of the SPDEs with or without delay terms. Owing to the exact results can not be obtained while the parameter ϵ small enough. In order to overcome this problem, ϵ–uniform numerical method, in which the error bound is independent of the small parameter ϵ, has been applied by many scholars to study this kinds of equations. In the existing literature, quite a good number of articles have been reported for handling SPDDEs by means of numerical method [6–8, 13–15, 24, 33], and a novel method has been applied in [5] for SPDDEs of reaction–diffusion type. With the exception of numerical method, the boundary layer functions method [36, 37] has been widely applied to construct uniformly valid asymptotic solution of SPDE(s), see, e.g. [9, 28–30, 35]. Among them, Ni et al. [29] considered the existence of spatial contrast structure for a class of semi–linear SPDDE, Feng and Ni [9] discussed the step–like type spatial contrast structure solution with internal layers for a kind of quasi–linear SPDDE, and Ni [28] is attracted by the asymptotic solution to a second–order fast–slow SPDDEs.

Back to the end of the last century, Vasil'eva [35] considered a system of first–order SPDEs with Neumann condition

$$\begin{cases} \epsilon u' = F(u, v, x), \quad \epsilon v' = G(u, v, x), \\ u'(a, \mu) = 0, \quad u'(b, \mu) = 0, \end{cases}$$

using the boundary layer functions method, under some certain conditions, the existence of step–like solution is proved. Based on Vasil'eva's work, a system of first–order SPDEs with discontinuous functions in the right–hand side has been investigated by Pang et al. in [30].

Motivated by the above work, in this paper, a system of SPDDEs will be investigated, under some sufficient assumptions, the existence of step–like type spatial contrast structures are proved and the uniformly valid asymptotic solution is constructed.

The remainder of this paper is organized as follows. In Section 2, the studied problem will be stated firstly, and then a necessary definition as well as some sufficient assumptions for the existence of step–like type spatial contrast structures will be presented. Under these assumptions, the formal asymptotic solution will be constructed in Section 3 as $\epsilon \to 0$, among which, Subsection 3.1 is aimed to construct the regular parts of asymptotic solutions, the transition and boundary layer parts will be constructed in Subsection 3.2, and the asymptotic solutions will be sewn continuously at point $t = \sigma$ in Subsection 3.3. The main result for the existence of solution and estimation of remainder are presented in Section 4. Whereafter in Section 5, a concrete example and numerical simulations for fixed small parameter ϵ will be carried out to demonstrate the effectiveness of the obtained algorithm.

2. System descriptions and conditions

Consider a system of SPDDEs as follows

$$\begin{cases} \epsilon y' = F\left(y(t), z(t), y(t-\sigma), z(t-\sigma), t\right), \\ \epsilon z' = G\left(y(t), z(t), y(t-\sigma), z(t-\sigma), t\right), \end{cases} \quad 0 \le t \le T,$$
$$\begin{cases} y(t) = \varphi(t), \quad z(t) = \psi(t), \quad -\sigma \le t \le 0, \\ y(T) = y^T, \quad z(T) = z^T, \end{cases} \tag{2.1}$$

in which, parameter $0 < \epsilon \ll 1$ is sufficiently small, $\sigma > 0$ is a constant delay, the initial functions $\varphi(t)$ and $\psi(t)$ are sufficiently smooth defined in the interval $[-\sigma, 0]$, T is a given positive constant. For simplicity, assume that $T \in (\sigma, 2\sigma)$ and denote $[u(t)] = u(t-\sigma)$.

The following definition of solution is natural for problem (2.1).

Definition 2.1. A solution to the system of equations (2.1) is defined as a vector function $x(t) = (y(t), z(t))^\top$ with the following properties,

(a) $x(t) \in C[-\sigma, T] \cap \left(C^1(0,\sigma) \cup C^1(\sigma, T)\right)$,

(b) $x(t) = (\varphi(t), \psi(t))^\top$, $-\sigma \le t \le 0$, and $x(T) = \left(y^T, z^T\right)^\top$,

(c) $x(t)$ satisfying the first two equations of (2.1) in the interval $(0,\sigma)$ and (σ, T).

Let $\epsilon = 0$, the so–called degenerate system for system (2.1) can be written as

$$\begin{cases} F\left(y(t), z(t), [y(t)], [z(t)], t\right) = 0, & \multirow{2}{*}{$0 \le t \le T,$} \\ G\left(y(t), z(t), [y(t)], [z(t)], t\right) = 0, & \\ y(t) = \varphi(t), \quad z(t) = \psi(t), \quad -\sigma \le t \le 0. \end{cases} \tag{2.2}$$

For ϵ sufficiently small, some sufficient criteria are established, under which problem (2.1) has a step–like spatial contrast structure, namely, a solution that tends in the interval $(0,\sigma)\cup(\sigma,T)$ to the solution of degenerate system (2.2) as $\epsilon \to 0$. In addition, the asymptotic expansion of this solution with respect to ϵ will be constructed in the entire interval $[0, T]$, including boundary layers at endpoint. These assumptions are listed as (**H1**), (**H2**), $\cdots$.

Generally speaking, the required order of smoothness for functions F and G are depended on the order of the asymptotic expansion to be constructed. For the sake of convenience, set

$$\mathcal{D} = \left\{(y_1, z_1, y_2, z_2, t) : |x_i| \le l,\ 0 \le t \le T,\ i = 1, 2\right\},$$

where l is a positive constant. Since the order of the constructed asymptotic expansion is up to arbitrary order, thus, the functions F and G are assumed to be infinitely differentiable on the given region $\mathcal{D}$. As is known to all, the method of Step–by–step is an effective method in solving DDEs. To this end, the given region $\mathcal{D}$ should be divided into two different regions $\mathcal{D}^{(-)}$ and $\mathcal{D}^{(+)}$ by the separatrix $t = \sigma$, and denote

$$\mathcal{D}^{(-)} = \left\{(y_1, z_1, y_2, z_2, t) :\ |x_i| \le l,\ 0 \le t \le \sigma,\ i = 1, 2\right\},$$
$$\mathcal{D}^{(+)} = \left\{(y_1, z_1, y_2, z_2, t) :\ |x_i| \le l,\ \sigma \le t \le T,\ i = 1, 2\right\}.$$

For system (2.2), it can be gained the so–called left degenerate equations on $\mathcal{D}^{(-)}$ as

$$\begin{cases} F\left(y^{(-)}(t), z^{(-)}(t), [\varphi], [\psi], t\right) = 0, \\ G\left(y^{(-)}(t), z^{(-)}(t), [\varphi], [\psi], t\right) = 0, \end{cases} \tag{2.3}$$

and the right degenerate equations on $\mathcal{D}^{(+)}$ as

$$\begin{cases} F\left(y^{(+)}(t), z^{(+)}(t), [y^{(-)}], [z^{(-)}], t\right) = 0, \\ G\left(y^{(+)}(t), z^{(+)}(t), [y^{(-)}], [z^{(-)}], t\right) = 0. \end{cases} \tag{2.4}$$

The following assumption is given for equations (2.3) and (2.4).

(**H1**) Assume that there exists a solution $\big(y^{(-)}(t), z^{(-)}(t)\big)^\top = \big(\alpha^{(-)}(t), \beta^{(-)}(t)\big)^\top$ for equations (2.3) on $\mathcal{D}^{(-)}$, and there exists a solution $\big(y^{(+)}(t), z^{(+)}(t)\big)^\top = \big(\alpha^{(+)}(t), \beta^{(+)}(t)\big)^\top$ for equations (2.4) on $\mathcal{D}^{(+)}$, where $\alpha^{(\mp)}(t)$, $\beta^{(\mp)}(t)$ are sufficiently smooth.

To be definite, we further assume that $\varphi(0) > \alpha^{(-)}(0)$, $\alpha^{(-)}(\sigma) > \alpha^{(+)}(\sigma)$ and $\alpha^{(+)}(T) < y^T$.

Denote the notation "$_{(\cdot)}$" as the derivative of the corresponding function with respect to the first variable or the second variable, for instance, $F_{(\cdot)}(t)$ denotes $F_y(t)$ or $F_z(t)$, and introduce the following two Jacobian matrices

$$\mathbf{J}^{(\mp)}(y, z, t) = \begin{pmatrix} F_y^{(\mp)}(t) \; F_z^{(\mp)}(t) \\ G_y^{(\mp)}(t) \; G_z^{(\mp)}(t) \end{pmatrix},$$

where abbreviations $F_{(\cdot)}^{(-)}(t)$, $G_{(\cdot)}^{(-)}(t)$ denote the functions $F_{(\cdot)}$, $G_{(\cdot)}$ take values at point $(y, z, [\varphi], [\psi], t)$, and abbreviations $F_{(\cdot)}^{(+)}(t)$, $G_{(\cdot)}^{(+)}(t)$ represent the functions $F_{(\cdot)}$, $G_{(\cdot)}$ take values at point $(y, z, [\alpha^{(-)}], [\beta^{(-)}], t)$.

Throughout this paper, the operator "det" is applied to express the determinant of corresponding matrix. For the Jacobian matrix $\mathbf{J}^{(\mp)}$, the following assumption is given.

(**H2**) Assume that the inequalities $\det \mathbf{J}^{(\mp)}\big(\alpha^{(\mp)}(t), \beta^{(\mp)}(t), t\big) < 0$ are fulfilled.

Consider the so–called associated systems (see, e.g. [36, pp.30]). In a small neighborhood of the left endpoint $t = 0$, it can be obtained

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}}{\mathrm{d}\zeta} = F\big(\tilde{y}, \tilde{z}, \varphi(-\sigma), \psi(-\sigma), 0\big), & \\ \dfrac{\mathrm{d}\tilde{z}}{\mathrm{d}\zeta} = G\big(\tilde{y}, \tilde{z}, \varphi(-\sigma), \psi(-\sigma), 0\big), & \zeta > 0, \\ \tilde{y}(0) = \varphi(0), \quad \tilde{y}(+\infty) = \alpha^{(-)}(0), & \\ \tilde{z}(0) = \psi(0), \quad \tilde{z}(+\infty) = \beta^{(-)}(0). & \end{cases} \tag{2.5}$$

In the transition layer region to the left of the point $t = \sigma$, it can be obtained

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}^{(-)}}{\mathrm{d}\xi} = F\big(\tilde{y}^{(-)}, \tilde{z}^{(-)}, \varphi(0), \psi(0), \sigma\big), & \\ \dfrac{\mathrm{d}\tilde{z}^{(-)}}{\mathrm{d}\xi} = G\big(\tilde{y}^{(-)}, \tilde{z}^{(-)}, \varphi(0), \psi(0), \sigma\big), & \xi < 0, \\ \tilde{y}^{(-)}(0) = p, \quad \tilde{y}^{(-)}(-\infty) = \alpha^{(-)}(\sigma), & \end{cases} \tag{2.6}$$

where $\alpha^{(+)}(\sigma) < p < \alpha^{(-)}(\sigma)$.

The following assumption is given.

(**H3**) Assume that there exist first integrals for the associated systems (2.5), (2.6) as follows, respectively,

$$\Phi(\tilde{y}, \tilde{z}) = C, \quad \Phi^{(-)}\big(\tilde{y}^{(-)}, \tilde{z}^{(-)}\big) = C,$$

herein C is a constant.

According to assumption (**H3**), it is shown that on $(\tilde{y}, \tilde{z})$–plane as well as $\left(\tilde{y}^{(-)}, \tilde{z}^{(-)}\right)$–plane, there exist separatrices with the following expressions

$$\Omega^l : \ \tilde{z} = \Phi^l(\tilde{y}), \tag{2.7}$$

$$\Omega^{(-)} : \ \tilde{z}^{(-)} = \Phi^{(-)}\left(\tilde{y}^{(-)}\right). \tag{2.8}$$

Some other conditions will be imposed during the construction of the asymptotic solution.

3. Algorithm of constructing the asymptotic expansions

Asymptotic expansion of the solution to system (2.1) will be constructed separately on the regions $\mathcal{D}^{(-)}$ and $\mathcal{D}^{(+)}$ and then sewn its left and right parts at the separatrix $t = \sigma$. To this end, consider the following two problems, the left problem on $\mathcal{D}^{(-)}$

$$\begin{cases} \epsilon y'^{(-)} = F\left(y^{(-)}(t), z^{(-)}(t), [\varphi], [\psi], t\right), \\ \epsilon z'^{(-)} = G\left(y^{(-)}(t), z^{(-)}(t), [\varphi], [\psi], t\right), \\ y^{(-)}(t) = \varphi(t), \quad z^{(-)}(t) = \psi(t), \quad y^{(-)}(\sigma, \epsilon) = p(\epsilon), \end{cases} \tag{3.1}$$

and the right problem on $\mathcal{D}^{(+)}$

$$\begin{cases} \epsilon y'^{(+)} = F\left(y^{(+)}(t), z^{(+)}(t), [y^{(-)}], [z^{(-)}], t\right), \\ \epsilon z'^{(+)} = G\left(y^{(+)}(t), z^{(+)}(t), [y^{(-)}], [z^{(-)}], t\right), \\ y^{(+)}(\sigma, \epsilon) = p(\epsilon), \quad y^{(+)}(T) = y^T, \quad z^{(+)}(T) = z^T, \end{cases} \tag{3.2}$$

in which $\alpha^{(+)}(\sigma) < p(\epsilon) < \alpha^{(-)}(\sigma)$ in systems (3.1), (3.2) is unknown.

Denote $x(t) = (y(t, z(t))^{\top}$. By means of the standard Vasil'eva boundary layer functions method [37, pp.15-29], the asymptotic solution to systems (3.1), (3.2) can be sought in the form of the sum of a regular part and boundary layer parts, in which the boundary layer parts describe the dynamical behavior of solution in a neighborhood of the boundary points, that is to say, the asymptotic expansion of the solution has the following form

$$\begin{aligned} x^{(-)}(t, \epsilon) &= \bar{x}^{(-)}(t, \epsilon) + Lx(\tau_0, \epsilon) + Q^{(-)}x(\tau, \epsilon), \\ x^{(+)}(t, \epsilon) &= \bar{x}^{(+)}(t, \epsilon) + Q^{(+)}x(\tau, \epsilon) + Rx(\tau_1, \epsilon), \end{aligned} \tag{3.3}$$

where $x^{(\mp)}(t, \epsilon)$ stand for the regular part; $Lx(\tau_0, \epsilon)$ denotes the left boundary layer function which is defined in a neighborhood of the left endpoint $t = 0$, and depended on the stretched variable $\tau_0 = t/\epsilon$; $Q^{(\mp)}x(\tau, \epsilon)$ represent the transition layer functions which are defined in the left (right) neighborhood of the separate point $t = \sigma$, and depended on the stretched variable $\tau = (t-\sigma)/\epsilon$; $Rx(\tau_1, \epsilon)$ denotes the right boundary layer function which is defined in a neighborhood of the point $t = T$ and depended on the stretched variable $\tau_1 = (t - T)/\epsilon$.

Each of the functions in expression (3.3) can be sought in the form of an expansion in powers of the small parameter

$$\bar{x}^{(\mp)}(t, \epsilon) = \bar{x}_0^{(\mp)}(t) + \epsilon \bar{x}_1^{(\mp)}(t) + \cdots + \epsilon^k \bar{x}_k^{(\mp)}(t) + \cdots, \tag{3.4}$$

$$Lx(\tau_0,\epsilon) = L_0x(\tau_0) + \epsilon L_1x(\tau_0) + \cdots + \epsilon^k L_kx(\tau_0) + \cdots , \tag{3.5}$$

$$Q^{(\mp)}x(\tau,\epsilon) = Q_0^{(\mp)}x(\tau) + \epsilon Q_1^{(\mp)}x(\tau) + \cdots + \epsilon^k Q_k^{(\mp)}x(\tau) + \cdots , \tag{3.6}$$

$$Rx(\tau_1,\epsilon) = R_0x(\tau_1) + \epsilon R_1x(\tau_1) + \cdots + \epsilon^k R_kx(\tau_1) + \cdots . \tag{3.7}$$

And the unknown quantity $p(\epsilon)$ can also be written as a series expansion in powers of ϵ as

$$p(\epsilon) = p_0 + \epsilon p_1 + \cdots + \epsilon^k p_k + \cdots . \tag{3.8}$$

The unknown coefficients p_i, $i = 0, 1, 2, \cdots$ in relation (3.8) will be determined from the sewing condition of asymptotic expansions $x^{(-)}(t,\epsilon)$ and $x^{(+)}(t,\epsilon)$.

By virtue of the boundary functions $L_kx(\tau_0)$, $Q_k^{(\mp)}x(\tau)$ and $R_kx(\tau_1)$ should approach zero as $\tau_0 \to +\infty$, $\tau \to \mp\infty$ and $\tau_1 \to -\infty$ respectively. Therefore, we introduce the conditions

$$L_kx(+\infty) = 0, \quad Q_k^{(\mp)}x(\mp\infty) = 0, \quad R_kx(-\infty) = 0, \quad k = 0, 1, \cdots .$$

3.1. Regular part

For the functions $\bar{x}^{(\mp)}(t,\epsilon)$, substituting the series (3.4) into the first two equations of systems (3.1), (3.2), and representing the right–hand sides of these equations as Taylor series with respect to small parameter ϵ, and then the equations for the functions in (3.4) can be obtained by equating the coefficients of like powers of ϵ in these relations.

For the leading term $\bar{x}_0^{(\mp)}(t)$ of the regular part of the asymptotics, it can be obtained

$$0 = F\left(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, [\varphi], [\psi], t\right), \quad 0 = G\left(\bar{y}_0^{(-)}, \bar{z}_0^{(-)}, [\varphi], [\psi], t\right),$$
$$0 = F\left(\bar{y}_0^{(+)}, \bar{z}_0^{(+)}, [\bar{y}_0^{(-)}], [\bar{z}_0^{(-)}], t\right), \quad 0 = G\left(\bar{y}_0^{(+)}, \bar{z}_0^{(+)}, [\bar{y}_0^{(-)}], [\bar{z}_0^{(-)}], t\right),$$

which readily coincide with the degenerate equations (2.3) and (2.4), in view of assumption (**H1**), it can be achieved

$$\bar{y}_0^{(\mp)}(t) = \alpha^{(\mp)}(t), \quad \bar{z}_0^{(\mp)}(t) = \beta^{(\mp)}(t).$$

For the higher order terms $\bar{x}_k^{(\mp)}(t)$, $k \geq 1$ of the regular part of the asymptotics, it can be obtained by the following systems of linear algebraic equations

$$\begin{cases} \bar{F}_y^{(\mp)}(t)\bar{y}_k^{(\mp)}(t) + \bar{F}_z^{(\mp)}(t)\bar{z}_k^{(\mp)}(t) = F_k^{(\mp)}(t), \\ \bar{G}_y^{(\mp)}(t)\bar{y}_k^{(\mp)}(t) + \bar{G}_z^{(\mp)}(t)\bar{z}_k^{(\mp)}(t) = G_k^{(\mp)}(t), \end{cases} \tag{3.9}$$

here, $\bar{F}_{(\cdot)}^{(-)}(t)$, $\bar{G}_{(\cdot)}^{(-)}(t)$ and $\bar{F}_{(\cdot)}^{(+)}(t)$, $\bar{G}_{(\cdot)}^{(+)}(t)$ stand for the functions $F_{(\cdot)}$, $G_{(\cdot)}$ take values at point $\left(\alpha^{(-)}(t), \beta^{(-)}(t), [\varphi], [\psi], t\right)$ and $\left(\alpha^{(+)}(t), \beta^{(+)}(t), [\alpha^{(-)}], [\beta^{(-)}], t\right)$, respectively. And $F_k^{(\mp)}(t)$, $G_k^{(\mp)}(t)$ are known functions which can be expressed recursively by $\bar{y}_j^{(\mp)}(t)$, $\bar{z}_j^{(\mp)}(t)$ with $j < k$. From assumption (**H2**), the coefficient determinant $\det \mathbf{J}^{(\mp)}$ of equations (3.9) are nonzero, so, the problems (3.9) are solvable and its solutions can be expressed as

$$\bar{y}_k^{(\mp)}(t) = \frac{\bar{G}_z^{(\mp)}(t)F_k^{(\mp)}(t) - \bar{F}_z^{(\mp)}(t)G_k^{(\mp)}(t)}{\det \mathbf{J}^{(\mp)}},$$
$$\bar{z}_k^{(\mp)}(t) = \frac{\bar{F}_y^{(\mp)}(t)G_k^{(\mp)}(t) - \bar{G}_y^{(\mp)}(t)F_k^{(\mp)}(t)}{\det \mathbf{J}^{(\mp)}}.$$

As so far, the regular partial series of asymptotic solution has been constructed completely.

3.2. Transition and boundary layer parts

The problem for the transition and boundary layer functions can also be obtained by the standard Vasil'eva's method [37, pp.15–29].

For the functions $Lx(\tau_0)$ in a neighborhood of the endpoint $t=0$, it can be obtained

$$\begin{cases} \dfrac{\mathrm{d}Ly}{\mathrm{d}\tau_0} = F\big(\bar{y}^{(-)}(\tau_0\epsilon,\epsilon)+Ly,\bar{z}^{(-)}(\tau_0\epsilon,\epsilon)+Lz,\varphi(\tau_0\epsilon-\sigma),\psi(\tau_0\epsilon-\sigma),\tau_0\epsilon\big) \\ \qquad - F\big(\bar{y}^{(-)}(\tau_0\epsilon,\epsilon),\bar{z}^{(-)}(\tau_0\epsilon,\epsilon),\varphi(\tau_0\epsilon-\sigma),\psi(\tau_0\epsilon-\sigma),\tau_0\epsilon\big), \\ \dfrac{\mathrm{d}Lz}{\mathrm{d}\tau_0} = G\big(\bar{y}^{(-)}(\tau_0\epsilon,\epsilon)+Ly,\bar{z}^{(-)}(\tau_0\epsilon,\epsilon)+Lz,\varphi(\tau_0\epsilon-\sigma),\psi(\tau_0\epsilon-\sigma),\tau_0\epsilon\big) \\ \qquad - G\big(\bar{y}^{(-)}(\tau_0\epsilon,\epsilon),\bar{z}^{(-)}(\tau_0\epsilon,\epsilon),\varphi(\tau_0\epsilon-\sigma),\psi(\tau_0\epsilon-\sigma),\tau_0\epsilon\big), \\ Lx(0,\epsilon)=\varphi(0)-\bar{x}^{(-)}(0,\epsilon), \quad Lx(+\infty,\epsilon)=0. \end{cases} \tag{3.10}$$

For brevity, denote $\varpi=\tau\epsilon+\sigma$. The problems for $Q^{(\mp)}x(\tau)$ can be obtained by the following system of equations

$$\begin{cases} \dfrac{\mathrm{d}Q^{(-)}y}{\mathrm{d}\tau} = F\big(\bar{y}^{(-)}(\varpi,\epsilon)+Q^{(-)}y,\bar{z}^{(-)}(\varpi,\epsilon)+Q^{(-)}z,\varphi(\tau\epsilon),\psi(\tau\epsilon),\varpi\big) \\ \qquad - F\big(\bar{y}^{(-)}(\varpi,\epsilon),\bar{z}^{(-)}(\varpi,\epsilon),\varphi(\tau\epsilon),\psi(\tau\epsilon),\varpi\big), \\ \dfrac{\mathrm{d}Q^{(-)}z}{\mathrm{d}\tau} = G\big(\bar{y}^{(-)}(\varpi,\epsilon)+Q^{(-)}y,\bar{z}^{(-)}(\varpi,\epsilon)+Q^{(-)}z,\varphi(\tau\epsilon),\psi(\tau\epsilon),\varpi\big) \\ \qquad - G\big(\bar{y}^{(-)}(\varpi,\epsilon),\bar{z}^{(-)}(\varpi,\epsilon),\varphi(\tau\epsilon),\psi(\tau\epsilon),\varpi\big), \\ Q^{(-)}(0,\epsilon)=p(\epsilon)-\bar{y}^{(-)}(\sigma,\epsilon), \quad Q^{(-)}(-\infty)=0, \end{cases} \tag{3.11}$$

and

$$\begin{cases} \dfrac{\mathrm{d}Q^{(+)}y}{\mathrm{d}\tau} = F\Big(\bar{y}^{(+)}(\varpi,\epsilon)+Q^{(+)}y,\bar{z}^{(+)}(\varpi,\epsilon)+Q^{(+)}z,\bar{y}^{(-)}(\tau\epsilon)+Ly(\tau), \\ \qquad \bar{z}^{(-)}(\tau\epsilon)+Lz(\tau),\varpi\Big) - F\left(\bar{y}^{(+)}(\varpi,\epsilon),\bar{z}^{(+)}(\varpi,\epsilon),\bar{y}^{(-)}(\tau\epsilon),\bar{z}^{(-)}(\tau\epsilon),\varpi\right), \\ \dfrac{\mathrm{d}Q^{(+)}z}{\mathrm{d}\tau} = G\Big(\bar{y}^{(+)}(\varpi,\epsilon)+Q^{(+)}y,\bar{z}^{(+)}(\varpi,\epsilon)+Q^{(+)}z,\bar{y}^{(-)}(\tau\epsilon)+Ly(\tau), \\ \qquad \bar{z}^{(-)}(\tau\epsilon)+Lz(\tau),\varpi\Big) - F\left(\bar{y}^{(+)}(\varpi,\epsilon),\bar{z}^{(+)}(\varpi,\epsilon),\bar{y}^{(-)}(\tau\epsilon),\bar{z}^{(-)}(\tau\epsilon),\varpi\right), \\ Q^{(+)}x(0,\epsilon)=p(\epsilon)-\bar{y}^{(+)}(\sigma,\epsilon), \quad Q^{(+)}x(+\infty)=0. \end{cases} \tag{3.12}$$

Similarly, denote $\rho=\tau_1\epsilon+T$. For $Rx(\tau_1)$, it can be obtained the problem

$$\begin{cases} \dfrac{\mathrm{d}Ry}{\mathrm{d}\tau_1} = F\Big(\bar{y}^{(+)}(\rho,\epsilon)+Ry,\bar{z}^{(+)}(\rho,\epsilon)+Rz,\bar{y}^{(-)}(\rho-\sigma),\bar{z}^{(-)}(\rho-\sigma),\rho\Big) \\ \qquad - F\Big(\bar{y}^{(+)}(\rho,\epsilon),\bar{z}^{(+)}(\rho,\epsilon),\bar{y}^{(-)}(\rho-\sigma),\bar{z}^{(-)}(\rho-\sigma),\rho\Big), \\ \dfrac{\mathrm{d}Rz}{\mathrm{d}\tau_1} = G\Big(\bar{y}^{(+)}(\rho,\epsilon)+Ry,\bar{z}^{(+)}(\rho,\epsilon)+Rz,\bar{y}^{(-)}(\rho-\sigma),\bar{z}^{(-)}(\rho-\sigma),\rho\Big) \\ \qquad - F\Big(\bar{y}^{(+)}(\rho,\epsilon),\bar{z}^{(+)}(\rho,\epsilon),\bar{y}^{(-)}(\rho-\sigma),\bar{z}^{(-)}(\rho-\sigma),\rho\Big), \\ Rx(0,\epsilon)=x^T-\bar{x}^{(+)}(T,\epsilon), \quad Rx(-\infty)=0. \end{cases} \tag{3.13}$$

Substituting (3.4)–(3.8) into (3.10)–(3.13) respectively, and expanding the right–hand sides of these equations as Taylor series with respect to small parameter ϵ, and then the equations can be obtained by equating the coefficients like powers of small parameter.

3.2.1. Leading terms

For $L_0x(\tau_0)$, it can be obtained by equating the like powers of ϵ^0 in both sides of (3.10),

$$\begin{cases} \dfrac{\mathrm{d}L_0y}{\mathrm{d}\tau_0} = F\big(\alpha^{(-)}(0) + L_0y, \beta^{(-)}(0) + L_0z, \varphi(-\sigma), \psi(-\sigma), 0\big), \\ \dfrac{\mathrm{d}L_0z}{\mathrm{d}\tau_0} = G\big(\alpha^{(-)}(0) + L_0y, \beta^{(-)}(0) + L_0z, \varphi(-\sigma), \psi(-\sigma), 0\big), \\ L_0y(0) = \varphi(0) - \alpha^{(-)}(0), \quad L_0y(+\infty) = 0, \\ L_0z(0) = \psi(0) - \beta^{(-)}(0), \quad L_0z(+\infty) = 0. \end{cases} \tag{3.14}$$

Taking variable changes

$$\tilde{y}(\tau_0) = \alpha^{(-)}(0) + L_0y(\tau_0), \quad \tilde{z}(\tau_0) = \beta^{(-)}(0) + L_0z(\tau_0), \tag{3.15}$$

applying this notations, (3.14) can be written in the form (2.5), in which the variable ζ is replaced by τ_0, as follows

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau_0} = F\left(\tilde{y}, \tilde{z}, \varphi(-\sigma), \psi(-\sigma), 0\right), \\ \dfrac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau_0} = G\left(\tilde{y}, \tilde{z}, \varphi(-\sigma), \psi(-\sigma), 0\right), \\ \tilde{y}(0) = \varphi(0), \quad \tilde{y}(+\infty) = \alpha^{(-)}(0), \\ \tilde{z}(0) = \psi(0), \quad \tilde{z}(+\infty) = \beta^{(-)}(0), \end{cases} \tag{3.16}$$

by virtue of assumptions (**H1**)–(**H3**), equations (3.16) is solvable, and then there exists a $L_0x(\tau_0)$. By [36, pp.101–105], there hold the following inequalities

$$|\tilde{y} - \alpha^{(-)}(0)| \le c\exp(-\kappa|\tau_0|), \quad |\tilde{z} - \beta^{(-)}(0)| \le c\exp(-\kappa|\tau_0|), \quad c, \kappa > 0.$$

From variable changes (3.15), it can be achieved that the leading term $L_0x(\tau_0)$ satisfy an exponential estimate as follows

$$|L_0x(\tau_0)| \le c\exp(-\kappa\tau_0), \quad c, \kappa > 0, \quad \tau_0 \ge 0.$$

By equating the coefficients of ϵ^0 in equations (3.11), it can be obtained the following system of equations for the functions $Q_0^{(-)}x(\tau)$

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(-)}y}{\mathrm{d}\tau} = F\big(\alpha^{(-)}(\sigma) + Q_0^{(-)}y, \beta^{(-)}(\sigma) + Q_0^{(-)}z, \varphi(0), \psi(0), \sigma\big), \\ \dfrac{\mathrm{d}Q_0^{(-)}z}{\mathrm{d}\tau} = G\big(\alpha^{(-)}(\sigma) + Q_0^{(-)}y, \beta^{(-)}(\sigma) + Q_0^{(-)}z, \varphi(0), \psi(0), \sigma\big), \\ Q_0^{(-)}y(0) = p_0 - \alpha^{(-)}(\sigma), \quad Q_0^{(-)}y(-\infty) = 0. \end{cases} \tag{3.17}$$

By taking the following variable changes

$$\tilde{y}^{(-)}(\tau) = \alpha^{(-)}(\sigma) + Q_0^{(-)}y(\tau), \quad \tilde{z}^{(-)}(\tau) = \beta^{(-)}(\sigma) + Q_0^{(-)}z(\tau), \tag{3.18}$$

and then it can be obtained

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}^{(-)}}{\mathrm{d}\tau} = F\big(\tilde{y}^{(-)}, \tilde{z}^{(-)}, \varphi(0), \psi(0), \sigma\big), \\ \dfrac{\mathrm{d}\tilde{z}^{(-)}}{\mathrm{d}\tau} = G\big(\tilde{y}^{(-)}, \tilde{z}^{(-)}, \varphi(0), \psi(0), \sigma\big), \\ \tilde{y}^{(-)}(0) = p_0, \quad \tilde{y}^{(-)}(-\infty) = \alpha^{(-)}(\sigma). \end{cases} \tag{3.19}$$

Similar to (3.16), the functions $Q_0^{(-)}x(\tau)$ will be determined exclusively, and $Q_0^{(-)}x(\tau)$ satisfy an exponential estimate as follows

$$|Q_0^{(-)}x(\tau)| \leq c\exp(\kappa\tau), \quad c, \kappa > 0, \quad \tau \leq 0.$$

By equating the coefficients of ϵ^0 in equations (3.12), it can be obtained the system of equations for functions $Q_0^{(+)}x(\tau)$ as follows

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(+)}y}{\mathrm{d}\tau} = F\big(\alpha^{(+)}(\sigma) + Q_0^{(+)}y, \beta^{(+)}(\sigma) + Q_0^{(+)}z, \alpha^{(-)}(0) + L_0 y, \beta^{(-)}(0) + L_0 z, \sigma\big), \\ \dfrac{\mathrm{d}Q_0^{(+)}z}{\mathrm{d}\tau} = G\big(\alpha^{(+)}(\sigma) + Q_0^{(+)}y, \beta^{(+)}(\sigma) + Q_0^{(+)}z, \alpha^{(-)}(0) + L_0 y, \beta^{(-)}(0) + L_0 z, \sigma\big), \\ Q_0^{(+)}y(0) = p_0 - \alpha^{(+)}(\sigma), \quad Q_0^{(+)}y(+\infty) = 0. \end{cases} \tag{3.20}$$

By introducing variable changes

$$\tilde{y}^{(+)}(\tau) = \alpha^{(+)}(\sigma) + Q_0^{(+)}y(\tau), \quad \tilde{z}^{(+)}(\tau) = \beta^{(+)}(\sigma) + Q_0^{(+)}z(\tau), \tag{3.21}$$

and then, from equations (3.15) and (3.21), equations (3.20) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}^{(+)}}{\mathrm{d}\tau} = F\big(\tilde{y}^{(+)}, \tilde{z}^{(+)}, \tilde{y}, \tilde{z}, \sigma\big), \quad \dfrac{\mathrm{d}\tilde{z}^{(+)}}{\mathrm{d}\tau} = G\big(\tilde{y}^{(+)}, \tilde{z}^{(+)}, \tilde{y}, \tilde{z}, \sigma\big), \\ \tilde{y}^{(+)}(0) = p_0, \quad \tilde{y}^{(+)}(+\infty) = \alpha^{(+)}(\sigma), \end{cases} \tag{3.22}$$

in which $(\tilde{y}(\zeta), \tilde{z}(\zeta))^\top$ is a solution to equations (2.5), and it should be replaced ζ by τ in equations (3.22). It can be readily seen that equations (3.22) is a non-autonomous system, and hence, the existence of solutions to such system could not be discussed in the same way as for (3.14), (3.17). However, it can be considered together with equations (3.14). To this end, it can be obtained

$$\begin{cases} \dfrac{\mathrm{d}\tilde{y}^{(+)}}{\mathrm{d}\tau} = F\big(\tilde{y}^{(+)}, \tilde{z}^{(+)}, \tilde{y}, \tilde{z}, \sigma\big), \\ \dfrac{\mathrm{d}\tilde{z}^{(+)}}{\mathrm{d}\tau} = G\big(\tilde{y}^{(+)}, \tilde{z}^{(+)}, \tilde{y}, \tilde{z}, \sigma\big), \\ \dfrac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = F\big(\tilde{y}, \tilde{z}, \varphi(-\sigma), \psi(-\sigma), 0\big), \\ \dfrac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau} = G\big(\tilde{y}, \tilde{z}, \varphi(-\sigma), \psi(-\sigma), 0\big), \\ \tilde{y}^{(+)}(0) = p, \quad \tilde{y}^{(+)}(+\infty) = \alpha^{(+)}(\sigma), \\ \tilde{y}(0) = \varphi(0), \quad \tilde{y}(+\infty) = \alpha^{(-)}(0). \end{cases} \tag{3.23}$$

Here, the phase space $\Sigma_4 : \big(y^{(+)}, z^{(+)}, \tilde{y}, \tilde{z}\big)$ is the direct product of the spaces $\Sigma_2^{(+)} : \big(y^{(+)}, z^{(+)}\big)$ and $\widetilde{\Sigma}_2 : (\tilde{y}, \tilde{z})$, that is to say, $\Sigma_4 = \Sigma_2^{(+)} \oplus \widetilde{\Sigma}_2$. It can be easily found that the point $M\left(\alpha^{(+)}(\sigma), 0, \alpha^{(-)}(0), 0\right)$ is an equilibrium point to system (3.23). Introduce two matrices

$$A = \begin{pmatrix} F_{\tilde{y}^{(+)}} & F_{\tilde{z}^{(+)}} \\ G_{\tilde{y}^{(+)}} & G_{\tilde{z}^{(+)}} \end{pmatrix}, \quad B = \begin{pmatrix} F_{\tilde{y}} & F_{\tilde{z}} \\ G_{\tilde{y}} & G_{\tilde{z}} \end{pmatrix}.$$

Denote ∇_M as the corresponding matrix take values at the equilibrium point M, here, ∇ stand for the matrix A or B. Then, at the equilibrium point M, the characteristic equation to system (3.23) can be expressed as

$$\big[\lambda^2 - \mathrm{tr}(A_M)\lambda + \det A_M\big]\,\big[\lambda^2 - \mathrm{tr}(B_M)\lambda + \det B_M\big] = 0, \tag{3.24}$$

in which the operator "tr" represents the trace to the corresponding matrix.

(**H4**) Assume that there holds the inequalities $\Delta \equiv [\mathrm{tr}(\nabla_M)]^2 - 4\det\nabla_M > 0$.

It is found that the characteristic equation (3.24) has real roots with opposite signs by virtue of assumption (**H4**), and

$$\lambda_{1,2,3,4} = \frac{\mathrm{tr}(\nabla_M) \pm \sqrt{\Delta}}{2}.$$

Therefore, the equilibrium point M of system (3.23) is a hyperbolic saddle, furthermore, there exists a stable 2–dimensional submanifold $W^s(M)$ and an unstable 2–dimensional submanifold $W^u(M)$ passing through the equilibrium point $M\left(\alpha^{(+)}(\sigma), 0, \alpha^{(-)}(0), 0\right)$.

Denote the unstable 2–dimensional submanifold as

$$W^s(M): \ \vec{z} = \Psi(\vec{y}),$$

where $\vec{z} = \big(\tilde{z}^{(+)}, \tilde{z}\big)^\top$, $\vec{y} = \big(\tilde{y}^{(+)}, \tilde{y}\big)^\top$, $\Psi = (\Psi_1, \Psi_2)^\top$ and $0 = \Psi_i\left(\alpha^{(+)}(\sigma), \alpha^{(-)}(0)\right)$, $(i = 1, 2)$. Clearly, it can be seen that the projections of $W^s(M)$ onto $\widetilde{\Sigma}_2$ is Ω^l, that is, $(W^s(M))^\perp_{\widetilde{\Sigma}_2} = \Omega^l$. Denote $(W^s(M))^\perp_{\Sigma_2^{(+)}} = \Omega^{(+)}$, and the following assumption is proposed.

(**H5**) Assume that there has $\big\{\tilde{y}^{(+)} = p_0\big\} \cap \Omega^{(+)} \neq \emptyset$ on the phase plane $\widetilde{\Sigma}_2$.

Therefore, the existence of solution to system (3.23) can be guaranteed, and there exists a solution to system (3.22). Furthermore, the separatrix orbit can be written as

$$\Omega^{(+)}: \quad \tilde{z}^{(+)} = \Phi^{(+)}\big(\tilde{y}^{(+)}\big). \tag{3.25}$$

Therefore, the function $Q_0^{(+)}x(\tau)$ can be uniquely determined, and $Q_0^{(+)}x(\tau)$ satisfy the decays exponentially as $\tau \to +\infty$

$$|Q_0^{(+)}x(\tau)| \le c\exp(-\kappa\tau), \quad c, \kappa > 0, \quad \tau \ge 0.$$

For the leading term $R_0x(\tau_1)$, by (3.13), one is led to

$$\begin{cases} \dfrac{\mathrm{d}R_0y}{\mathrm{d}\tau_1} = F\big(\alpha^{(+)}(T) + R_0y, \beta^{(+)}(T) + R_0z, \alpha^{(-)}(T-\sigma), \beta^{(-)}(T-\sigma), T\big), \\ \dfrac{\mathrm{d}R_0z}{\mathrm{d}\tau_1} = G\big(\alpha^{(+)}(T) + R_0y, \beta^{(+)}(T) + R_0z, \alpha^{(-)}(T-\sigma), \beta^{(-)}(T-\sigma), T\big). \end{cases}$$

Similar to (3.14), according to assumptions (**H1**)–(**H3**), there exist $R_0x(\tau_1)$ which satisfy the following exponential estimate

$$|R_0x(\tau_1)| \le c\exp(\kappa\tau_1), \quad c, \kappa > 0, \quad \tau_1 \le 0.$$

3.2.2. Higher order terms

By equating the coefficients of ϵ^k in equations (3.10), it can be obtained the system of equations for functions $L_k x(\tau)$, $k \geq 1$ as follows

$$\begin{cases} \dfrac{\mathrm{d}L_k y}{\mathrm{d}\tau_0} = \tilde{F}_y(\tau_0)L_k y + \tilde{F}_z(\tau_0)L_k z + \tilde{f}_k^{(-)}(\tau_0), \\ \dfrac{\mathrm{d}L_k z}{\mathrm{d}\tau_0} = \tilde{G}_y(\tau_0)L_k y + \tilde{G}_z(\tau_0)L_k z + \tilde{g}_k^{(-)}(\tau_0), \\ L_k y(0) = -\bar{y}_k^{(-)}(0), \quad L_k y(+\infty) = 0, \end{cases} \tag{3.26}$$

in which $\tilde{f}_k^{(-)}(\tau_0)$ and $\tilde{g}_k^{(-)}(\tau_0)$ are known functions which can be expressed recursively through $L_j x(\tau_0)$ $(j < k)$, and $\tilde{F}_{(\cdot)}$, $\tilde{G}_{(\cdot)}$ represent the functions $F_{(\cdot)}$, $G_{(\cdot)}$ take values at the point $(\tilde{y}_0, \tilde{z}_0, \varphi(-\sigma), \psi(-\sigma), 0)$. It is clear that the system of equations (3.26) is solvable. Indeed, by virtue of [36, pp.100–109], consider the variational equations

$$\begin{cases} \dfrac{\mathrm{d}L_k y}{\mathrm{d}\tau_0} = \tilde{F}_y(\tau_0)L_k y + \tilde{F}_z(\tau_0)L_k z + \tilde{f}_k^{(-)}(\tau_0), \\ \dfrac{\mathrm{d}L_k z}{\mathrm{d}\tau_0} = \tilde{G}_y(\tau_0)L_k y + \tilde{G}_z(\tau_0)L_k z + \tilde{g}_k^{(-)}(\tau_0), \\ L_k y(0) = 0, \quad L_k y(+\infty) = 0, \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}L_k y}{\mathrm{d}\tau_0} = \tilde{F}_y(\tau_0)L_k y + \tilde{F}_z(\tau_0)L_k z, \\ \dfrac{\mathrm{d}L_k z}{\mathrm{d}\tau_0} = \tilde{G}_y(\tau_0)L_k y + \tilde{G}_z(\tau_0)L_k z, \\ L_k y(0) = -\bar{y}_k^{(-)}(0), \quad L_k y(+\infty) = 0. \end{cases} \tag{3.27}$$

In order to obtain a solution of the system of equations (3.27), taking

$$L_k y = \delta_1, \quad L_k z = h(\tau_0)\delta_1 + \delta_2. \tag{3.28}$$

Substituting (3.28) into (3.27), it can be obtained

$$\frac{\mathrm{d}\delta_1}{\mathrm{d}\tau_0} = F_1\delta_1 + \tilde{F}_z\delta_2 + \tilde{f}_k^{(-)}, \quad \frac{\mathrm{d}\delta_2}{\mathrm{d}\tau_0} = F_2\delta_2 + \tilde{g}_k^{(-)} - h(\tau_0)\tilde{f}_k^{(-)},$$

with

$$F_1 = \tilde{F}_y(\tau_0) + \tilde{F}_z(\tau_0)h, \quad F_2 = \tilde{G}_z - h\tilde{F}_z(\tau_0), \quad h(\tau_0) = \frac{\mathrm{d}\Phi^l}{\mathrm{d}\tau_0},$$

in which Φ^l has been provided in (2.7). Furthermore, it can be found that

$$\begin{aligned} \delta_2(\tau_0) &= \int_{-\infty}^{\tau_0} \exp\left(\int_\eta^{\tau_0} F_2(s)\mathrm{d}s\right)\left(\tilde{g}_k^{(-)}(\tau_0) - h(\eta)\tilde{f}_k^{(-)}(\tau_0)\right)\mathrm{d}\eta, \\ \delta_1(\tau_0) &= \int_0^{\tau_0} \exp\left(\int_\eta^{\tau_0} F_1(s)\mathrm{d}s\right)\left(\tilde{F}_z(\eta)\delta_2(\eta) + \tilde{f}_k^{(-)}(\eta)\right)\mathrm{d}\eta. \end{aligned}$$

For the homogeneous system of equations (3.27), one has already obtained that $L_k z(\tau_0) = h(\tau_0)L_k y(\tau_0)$, hence, the solution to system of equations (3.27) can be obtained easily. According to the principle of superposition, the solution to equations (3.26) can be obtained

$$\begin{aligned} L_k y(\tau_0) &= -\bar{y}_k^{(-)}(0)\exp\left(\int_0^{\tau_0} F_1(s)\mathrm{d}s\right) + \delta_1(\tau_0), \\ L_k z(\tau_0) &= -\bar{y}_k^{(-)}(0)h(\tau_0)\exp\left(\int_0^{\tau_0} F_1(s)\mathrm{d}s\right) + h(\tau_0)\delta_1(\tau_0) + \delta_2(\tau_0). \end{aligned}$$

The higher order boundary layer functions $L_k x(\tau_0)$, $k \geq 1$ satisfy the exponential estimate

$$|L_k x(\tau_0)| \leq c\exp(-\kappa\tau_0), \quad c, \kappa > 0, \quad k \geq 1, \quad \tau_0 \geq 0.$$

As so far, the left boundary layer function (3.5) has been constructed completely.

Comparing the like powers of ϵ^k on both sides of the system of equations (3.11), (3.12), and then the system of equation for the functions $Q_k^{(\mp)}x(\tau)$, $k \geq 1$ can be obtained as follows

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(-)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(-)}(\tau)Q_k^{(-)}y + \tilde{F}_z^{(-)}(\tau)Q_k^{(-)}z + \tilde{F}_k^{(-)}(\tau), \\ \dfrac{\mathrm{d}Q_k^{(-)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(-)}(\tau)Q_k^{(-)}y + \tilde{G}_z^{(-)}(\tau)Q_k^{(-)}z + \tilde{G}_k^{(-)}(\tau), \\ Q_k^{(-)}y(0) = p_k - \bar{y}_k^{(-)}(\sigma), \quad Q_k^{(-)}y(-\infty) = 0, \end{cases} \tag{3.29}$$

and

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(+)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(+)}(\tau)Q_k^{(+)}y + \tilde{F}_z^{(+)}(\tau)Q_k^{(+)}z \\ \qquad\qquad + \tilde{F}_{[y]}^{(+)}(\tau)L_k y(\tau) + \tilde{F}_{[z]}^{(+)}(\tau)L_k z(\tau) + \tilde{H}_k^{(+)}(\tau), \\ \dfrac{\mathrm{d}Q_k^{(+)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(+)}(\tau)Q_k^{(+)}y + \tilde{G}_z^{(+)}(\tau)Q_k^{(+)}z \\ \qquad\qquad + \tilde{G}_{[y]}^{(+)}(\tau)L_k y(\tau) + \tilde{G}_{[z]}^{(+)}(\tau)L_k z(\tau) + \tilde{W}_k^{(+)}(\tau), \\ Q_k^{(+)}y(0) = p_k - \bar{y}_k^{(+)}(\sigma), \quad Q_k^{(+)}y(+\infty) = 0, \end{cases} \tag{3.30}$$

here, $F_{[\cdot]}$, $G_{[\cdot]}$ denote the derivative of the function F, G with respect to the third variable or the fourth variable, and $\tilde{F}_k^{(-)}(\tau)$, $\tilde{G}_k^{(-)}(\tau)$ are known functions which can be expressed recursively through $Q_j^{(-)}x(\tau)$ $(j < k)$, $\tilde{H}_k^{(+)}(\tau)$, $\tilde{W}_k^{(+)}(\tau)$ are known functions which can be expressed recursively through $Q_j^{(+)}x(\tau)$ $(j < k)$, and $\tilde{F}_{(\cdot)}^{(-)}(\tau)$ as well as $\tilde{G}_{(\cdot)}^{(-)}(\tau)$ can be represented as

$$\begin{cases} \tilde{F}_{(\cdot)}^{(-)}(\tau) = F_{(\cdot)}\big(\tilde{y}_0^{(-)}, \tilde{z}_0^{(-)}, \varphi(0), \psi(0), \sigma\big), \\ \tilde{G}_{(\cdot)}^{(-)}(\tau) = G_{(\cdot)}\big(\tilde{y}_0^{(-)}, \tilde{z}_0^{(-)}, \varphi(0), \psi(0), \sigma\big), \end{cases}$$

by using (3.15), one found that $\tilde{F}_{(\cdot)}^{(+)}(\tau)$ and $\tilde{G}_{(\cdot)}^{(+)}(\tau)$ can be expressed as

$$\begin{cases} \tilde{F}_{(\cdot)}^{(+)}(\tau) = F_{(\cdot)}\big(\tilde{y}_0^{(+)}, \tilde{z}_0^{(+)}, \tilde{y}_0(\tau), \tilde{z}_0(\tau), \sigma\big), \\ \tilde{G}_{(\cdot)}^{(+)}(\tau) = G_{(\cdot)}\big(\tilde{y}_0^{(+)}, \tilde{z}_0^{(+)}, \tilde{y}_0(\tau), \tilde{z}_0(\tau), \sigma\big). \end{cases}$$

It is worth noting that the functions $L_k y$ and $L_k z$ appeared in (3.30) are taking values at τ rather than τ_0.

Introduce new notations as follows

$$\tilde{F}_k^{(+)}(\tau) \equiv \tilde{F}_{[y]}^{(+)}(\tau) L_k y(\tau) + \tilde{F}_{[z]}^{(+)}(\tau) L_k z(\tau) + \tilde{H}_k^{(+)}(\tau),$$
$$\tilde{G}_k^{(+)}(\tau) \equiv \tilde{G}_{[y]}^{(+)}(\tau) L_k y(\tau) + \tilde{G}_{[z]}^{(+)}(\tau) L_k z(\tau) + \tilde{W}_k^{(+)}(\tau),$$

it is easy to show that the functions $\tilde{F}_k^{(+)}(\tau)$ and $\tilde{G}_k^{(+)}(\tau)$ are known functions with respect to variable τ. Furthermore, (3.30) can be readily rewritten as

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(+)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(+)}(\tau) Q_k^{(+)} y + \tilde{F}_z^{(+)}(\tau) Q_k^{(+)} z + \tilde{F}_k^{(+)}(\tau), \\ \dfrac{\mathrm{d}Q_k^{(+)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(+)}(\tau) Q_k^{(+)} y + \tilde{G}_z^{(+)}(\tau) Q_k^{(+)} z + \tilde{G}_k^{(+)}(\tau), \\ Q_k^{(+)} y(0) = p_k - \bar{y}_k^{(+)}(\sigma), \quad Q_k^{(+)} y(+\infty) = 0. \end{cases} \tag{3.31}$$

Consider the homogeneous system of equations (3.29)

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(-)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(-)}(\tau) Q_k^{(-)} y + \tilde{F}_z^{(-)}(\tau) Q_k^{(-)} z, \\ \dfrac{\mathrm{d}Q_k^{(-)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(-)}(\tau) Q_k^{(-)} y + \tilde{G}_z^{(-)}(\tau) Q_k^{(-)} z. \end{cases} \tag{3.32}$$

It can be readily obtained that the pair of functions $\big(F(\tilde{y}_0^{(-)}, \tilde{z}_0^{(-)}, \varphi(0), \psi(0), \sigma),\ G(\tilde{y}_0^{(-)}, \tilde{z}_0^{(-)}, \varphi(0), \psi(0), \sigma)\big)$ plays as a solution to equations (3.32), which can be easily verified by employing the variable changes (3.18), it is omitted here for brevity. It can be seen from observation that the equations (3.31) and (3.29) have the similar forms, hence, the pair of functions $\big(F(\tilde{y}_0^{(+)}, \tilde{z}_0^{(+)}, \tilde{y}_0(\tau), \tilde{z}_0(\tau), \sigma),\ G(\tilde{y}_0^{(+)}, \tilde{z}_0^{(+)}, \tilde{y}_0(\tau), \tilde{z}_0(\tau), \sigma)\big)$ is a solution to the homogeneous system of equations that is corresponded to equations (3.31).

For the sake of simplicity, (3.29) and (3.31) can be unified to rewrite as

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(\mp)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(\mp)}(\tau) Q_k^{(\mp)} y + \tilde{F}_z^{(\mp)}(\tau) Q_k^{(\mp)} z + \tilde{F}_k^{(\mp)}(\tau), \\ \dfrac{\mathrm{d}Q_k^{(\mp)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(\mp)}(\tau) Q_k^{(\mp)} y + \tilde{G}_z^{(\mp)}(\tau) Q_k^{(\mp)} z + \tilde{G}_k^{(\mp)}(\tau), \\ Q_k^{(\mp)} y(0) = p_k - \bar{y}_k^{(\mp)}(\sigma), \quad Q_k^{(\mp)} y(\mp\infty) = 0. \end{cases} \tag{3.33}$$

Consider the following two systems

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(\mp)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(\mp)}(\tau) Q_k^{(\mp)} y + \tilde{F}_z^{(\mp)}(\tau) Q_k^{(\mp)} z + \tilde{F}_k^{(\mp)}(\tau), \\ \dfrac{\mathrm{d}Q_k^{(\mp)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(\mp)}(\tau) Q_k^{(\mp)} y + \tilde{G}_z^{(\mp)}(\tau) Q_k^{(\mp)} z + \tilde{G}_k^{(\mp)}(\tau), \\ Q_k^{(\mp)} y(0) = 0, \quad Q_k^{(\mp)} y(\mp\infty) = 0, \end{cases} \tag{3.34}$$

and

$$\begin{cases} \dfrac{\mathrm{d}Q_k^{(\mp)}y}{\mathrm{d}\tau} = \tilde{F}_y^{(\mp)}(\tau)Q_k^{(\mp)}y + \tilde{F}_z^{(\mp)}(\tau)Q_k^{(\mp)}z, \\ \dfrac{\mathrm{d}Q_k^{(\mp)}z}{\mathrm{d}\tau} = \tilde{G}_y^{(\mp)}(\tau)Q_k^{(\mp)}y + \tilde{G}_z^{(\mp)}(\tau)Q_k^{(\mp)}z, \\ Q_k^{(\mp)}y(0) = p_k - \bar{y}_k^{(\mp)}(\sigma), \quad Q_k^{(\mp)}y(\mp\infty) = 0. \end{cases} \tag{3.35}$$

According to [36, pp.100–109], let

$$Q_k^{(\mp)}y(\tau) = \delta_1^{(\mp)}, \quad Q_k^{(\mp)}z(\tau) = h^{(\mp)}(\tau)\delta_1^{(\mp)} + \delta_2^{(\mp)}, \tag{3.36}$$

with $h^{(\mp)}(\tau) = \frac{\mathrm{d}\Phi^{(\mp)}}{\mathrm{d}\tau}$, herein $\Phi^{(\mp)}$ have been exhibited in (2.8) and (3.25) respectively.

Substituting (3.36) into (3.34), it is obtained that

$$\begin{aligned} \frac{\mathrm{d}\delta_1^{(\mp)}}{\mathrm{d}\tau} &= \left[\tilde{F}_y^{(\mp)} + \tilde{F}_z^{(\mp)}h^{(\mp)}\right]\delta_1^{(\mp)} + \tilde{F}_z^{(\mp)}\delta_2^{(\mp)} + \tilde{F}_k^{(\mp)}(\tau), \\ \frac{\mathrm{d}\delta_2^{(\mp)}}{\mathrm{d}\tau} &= \left[\tilde{G}_z^{(\mp)} - \tilde{F}_z^{(\mp)}h^{(\mp)}\right]\delta_2^{(\mp)} - h^{(\mp)}\tilde{F}_k^{(\mp)}(\tau) + \tilde{G}_k^{(\mp)}(\tau). \end{aligned}$$

Therefore, it can be solved as

$$\begin{aligned} \delta_2^{(\mp)} &= \int_{\mp\infty}^{\tau} \exp\left[\int_s^{\tau}\left(\tilde{G}_z^{(\mp)}(\eta) - \tilde{F}_z^{(\mp)}(\eta)h^{(\mp)}(\eta)\right)\mathrm{d}\eta\right] \\ &\quad \times \left(\tilde{G}_k^{(\mp)}(s) - h^{(\mp)}(s)\tilde{F}_k^{(\mp)}(s)\right)\mathrm{d}s, \\ \delta_1^{(\mp)} &= \int_0^{\tau} \exp\left[\int_s^{\tau}\left(\tilde{F}_y^{(\mp)}(\eta) + \tilde{F}_z^{(\mp)}(\eta)h^{(\mp)}(\eta)\right)\mathrm{d}\eta\right] \\ &\quad \times \left(\tilde{F}_z^{(\mp)}(s)\delta_2^{(\mp)}(s) + \tilde{F}_k^{(\mp)}(s)\right)\mathrm{d}s. \end{aligned}$$

The homogeneous equations (3.35) can be readily solved since $Q_k^{(\mp)}z(\tau) = h^{(\mp)}(\tau)\cdot Q_k^{(\mp)}y(\tau)$. According to the principle of superposition, solution to system (3.33) can be expressed as

$$Q_k^{(\mp)}y(\tau) = \left[p_k - \bar{y}_k^{(\mp)}(\sigma)\right]\exp\left[\int_0^{\tau}\left(\tilde{F}_y^{(\mp)}(s) + \tilde{F}_z^{(\mp)}(s)h^{(\mp)}(s)\right)\mathrm{d}s\right] + \delta_1^{(\mp)}(\tau), \tag{3.37}$$

$$\begin{aligned} Q_k^{(\mp)}z(\tau) &= \left[p_k - \bar{y}_k^{(\mp)}(\sigma)\right]h^{(\mp)}(\tau)\exp\left[\int_0^{\tau}\left(\tilde{F}_y^{(\mp)}(s) + \tilde{F}_z^{(\mp)}(s)h^{(\mp)}(s)\right)\mathrm{d}s\right] \\ &\quad + h^{(\mp)}(\tau)\delta_1^{(\mp)}(\tau) + \delta_2^{(\mp)}(\tau). \end{aligned} \tag{3.38}$$

And the functions $Q_kx^{(\mp)}(\tau)$, $k \geq 1$ satisfy the following exponential estimate

$$\left|Q_kx^{(\mp)}(\tau)\right| \leq c\exp(-\kappa|\tau|), \quad c,\kappa > 0, \quad k \geq 1.$$

This means that the transition layer functions have been constructed completely, however, the values p_i, $i = 1, 2, \cdots, k, \cdots$ appeared in transition layer functions are still unknown.

System of equations for determining the higher order terms of the right boundary layer functions $R_kx(\tau_1)$, $k \geq 1$ are similar to (3.26), and satisfy the exponential estimate

$$\left|R_kx(\tau_1)\right| \leq c\exp(\kappa\tau_1), \quad c,\kappa > 0, \quad k \geq 1, \quad \tau_1 \leq 0.$$

3.3. Sewing of asymptotic expansions

From the boundary conditions for $t=\sigma$ in system of equations (3.19), (3.22) and (3.33), it can be readily seen that the y–components of the constructed asymptotics on the left and on the right of point $t=\sigma$ are matched continuously. The sewing condition for z–components of the asymptotics can be written as $z^{(-)}(t,\epsilon)=z^{(+)}(t,\epsilon)$, which yields

$$\begin{aligned}&z_0^{(-)}(\sigma)+Q_0^{(-)}z(0)+\epsilon\big(z_1^{(-)}(\sigma)+Q_1^{(-)}z(0)\big)+\cdots\\=&z_0^{(+)}(\sigma)+Q_0^{(+)}z(0)+\epsilon\big(z_1^{(+)}(\sigma)+Q_1^{(+)}z(0)\big)+\cdots.\end{aligned}\tag{3.39}$$

In the zeroth order, it can be obtained from (3.39) that

$$z_0^{(-)}(\sigma)+Q_0^{(-)}z(0)=z_0^{(+)}(\sigma)+Q_0^{(+)}z(0).\tag{3.40}$$

Let

$$H(p)=\beta^{(-)}(\sigma)+\Phi^{(-)}\big(p-\alpha^{(-)}(\sigma)\big)-\beta^{(+)}(\sigma)-\Phi^{(+)}\big(p-\alpha^{(+)}(\sigma)\big).$$

(**H6**) Assume that $H(p)=0$ has a solution $p=p_0$, $p_0\in\big(\alpha^{(+)}(\sigma),\alpha^{(-)}(\sigma)\big)$, and $\frac{\mathrm{d}H(p_0)}{\mathrm{d}p}\neq 0$.

It can be found that (3.40) is equivalent to $H(p)=0$, from assumption (**H6**), (3.40) has a solution $p=p_0$.

In the following, we are led to seek the unknown values p_k, $k\geq 1$ appeared in transition layer functions $Q_k^{(\mp)}x(\tau)$. As a matter of fact, from the sewing condition (3.39) in the orders $k\geq 1$, it can be achieved

$$z_k^{(-)}(\sigma)+Q_k^{(-)}z(0)=z_k^{(+)}(\sigma)+Q_k^{(+)}z(0).\tag{3.41}$$

Combining (3.41) with (3.38), one is led to

$$Q_k^{(\mp)}z(0)=\big(p_k-\bar{y}^{(\mp)}(\sigma)\big)h^{(\mp)}(0)+\delta_2^{(\mp)}(0).$$

Substituting it into (3.41), the following equations for the coefficients p_k, $i=1,2,\cdots$, can be obtained

$$\begin{aligned}&z_k^{(-)}(\sigma)+\big(p_k-\bar{y}^{(-)}(\sigma)\big)h^{(-)}(0)+\delta_2^{(-)}(0)\\=&z_k^{(+)}(\sigma)+\big(p_k-\bar{y}^{(+)}(\sigma)\big)h^{(+)}(0)+\delta_2^{(+)}(0),\end{aligned}$$

therefore,

$$p_k=\frac{z_k^{(-)}(\sigma)+\delta_2^{(-)}(0)-\bar{y}^{(-)}(\sigma)h^{(-)}(0)-z_k^{(+)}(\sigma)-\delta_2^{(+)}(0)+\bar{y}^{(+)}(\sigma)h^{(+)}(0)}{\Theta(p_0)},$$

with $\Theta(p_0)=h^{(+)}(0)-h^{(-)}(0)$. According to (3.36) and assumption (**H6**), it can be obtained

$$\Theta(p_0)=\frac{\mathrm{d}H(p_0)}{\mathrm{d}p}\neq 0.$$

Hence, the coefficient functions $Q_k^{(\mp)}x(\tau)$ and p_k have already determined for any number k.

4. Main result

Theorem 4.1. *Suppose that the assumptions* (**H1**)–(**H6**) *hold, for* $0 < \epsilon \ll 1$*, there exists a continuous solution* $x(t,\epsilon) = (y(t,\epsilon), z(t,\epsilon))^\top$ *to the equations* (2.1) *in the interval* $[0,T]$*, for which the following asymptotic expansion holds*

$$x(t,\epsilon) = \begin{cases} \sum_{k=0}^{n} \epsilon^k \left[\bar{x}_k^{(-)}(t) + L_k x(\tau_0) + Q_k^{(-)} x(\tau)\right] + O(\epsilon^{n+1}), & 0 \le t \le \sigma, \\ \sum_{k=0}^{n} \epsilon^k \left[\bar{x}_k^{(+)}(t) + Q_k^{(+)} x(\tau) + R_k x(\tau_1)\right] + O(\epsilon^{n+1}), & \sigma \le t \le T, \end{cases}$$

with $\tau_0 = t/\epsilon$*,* $\tau = (t-\sigma)/\epsilon$*,* $\tau_1 = (t-T)/\epsilon$*, are stretched variables.*

Proof. From Definition 2.1, consider the following two systems

$$\begin{cases} \epsilon y'^{(-)} = F\left(y^{(-)}(t), z^{(-)}(t), [\varphi], [\psi], t\right), \\ \epsilon z'^{(-)} = G\left(y^{(-)}(t), z^{(-)}(t), [\varphi], [\psi], t\right), \\ y^{(-)}(t) = \varphi(t), \quad z^{(-)}(t) = \psi(t), \quad y^{(-)}(\sigma,\epsilon) = \bar{p}(\epsilon), \end{cases}$$

and

$$\begin{cases} \epsilon y'^{(+)} = F\left(y^{(+)}(t), z^{(+)}(t), [y^{(-)}], [z^{(-)}], t\right), \\ \epsilon z'^{(+)} = G\left(y^{(+)}(t), z^{(+)}(t), [y^{(-)}], [z^{(-)}], t\right), \\ y^{(+)}(\sigma,\epsilon) = \bar{p}(\epsilon), \quad y^{(+)}(T) = y^T, \quad z^{(+)}(T) = z^T. \end{cases}$$

Expanding $\bar{p}(\epsilon)$ into the series in powers of small parameter ϵ as follows

$$\bar{p}(\epsilon) = p_0 + \epsilon p_1 + \cdots + \epsilon^{n+1}(p_{n+1} + \theta), \tag{4.1}$$

with θ denoting a parameter.

By Section 3, solutions to the above two systems can be expressed as

$$x^{(-)}(t,\epsilon) = \sum_{k=0}^{n} \epsilon^k \left[\bar{x}_k^{(-)}(t) + L_k x(\tau_0) + Q_k^{(-)} x(\tau)\right] + O(\epsilon^{n+1}),$$

$$x^{(+)}(t,\epsilon) = \sum_{k=0}^{n} \epsilon^k \left[\bar{x}_k^{(+)}(t) + Q_k^{(+)} x(\tau) + R_k x(\tau_1)\right] + O(\epsilon^{n+1}),$$

with $\tau_0 = t/\epsilon$, $\tau = (t-\sigma)/\epsilon$, $\tau_1 = (t-T)/\epsilon$ are stretched variables.

It can be found that the function $y(t,\epsilon)$ is continuously connected at the point $t = \sigma$ by the boundary value conditions in equations (3.1) and (3.2). To finish the proof of Theorem 4.1, one need to choose θ in such a way that

$$z^{(-)}(\sigma,\epsilon) = z^{(+)}(\sigma,\epsilon). \tag{4.2}$$

Let $I(\theta,\epsilon) = z^{(-)}(\sigma,\epsilon) - z^{(+)}(\sigma,\epsilon)$. By virtue of the algorithm for constructing the asymptotics and sewing conditions, one is led to

$$I(\theta,\epsilon) = \epsilon^n \left[z_k^{(-)}(\sigma) + Q_k^{(-)} z(0) - z_k^{(+)}(\sigma) - Q_k^{(+)} z(0)\right] + O(\epsilon^{n+1})$$

$$= \epsilon^n \Theta(p_0)\theta + O(\epsilon^{n+1})$$
$$= \epsilon^n \left[\Theta(p_0)\theta + O(\epsilon)\right]. \tag{4.3}$$

Together with assumption (**H6**), it can be obtained that (4.3) shows there exists a parameter θ^* satisfies $I(\theta^*, \epsilon) = 0$ for ϵ sufficiently small. Consequently, by taking $\theta = \theta^*$ in (4.1), the relation (4.2) can be guaranteed and the function $z(t, \epsilon)$ is continuous. As a result, a continuous solution is obtained to the problem (2.1) and the proof is completed. □

5. Application

Example 5.1. Consider a system of SPDDEs as follows

$$\begin{cases} \epsilon y' = -y + 2z - 4y(t-1), & \\ \epsilon z' = 2y - z + 2z(t-1), & 0 < t < 2, \\ y(t) = 1, \quad z(t) = 1, & -1 \le t \le 0, \\ y(2) = -1, \quad z(2) = \dfrac{1}{3}. & \end{cases} \tag{5.1}$$

It can be easily found that the functions in the right–hand sides are continuous in the interval (0,2). The solutions to the system of degenerate equations for SPDDEs (5.1) can be obtained

$$\alpha^{(-)}(t) = 0, \quad \beta^{(-)}(t) = 2, \quad \alpha^{(+)}(t) = -\frac{8}{3}, \quad \beta^{(+)}(t) = -\frac{4}{3},$$

obviously, there have $\varphi(0) > \alpha^{(-)}(0)$, $\alpha^{(-)}(1) > \alpha^{(+)}(1)$, $\alpha^{(+)}(2) < y(2)$. With a simple computation, it is obtained that $\det \mathbf{J}^{(\mp)} = -3 < 0$, hence assumptions (**H1**), (**H2**) are satisfied.

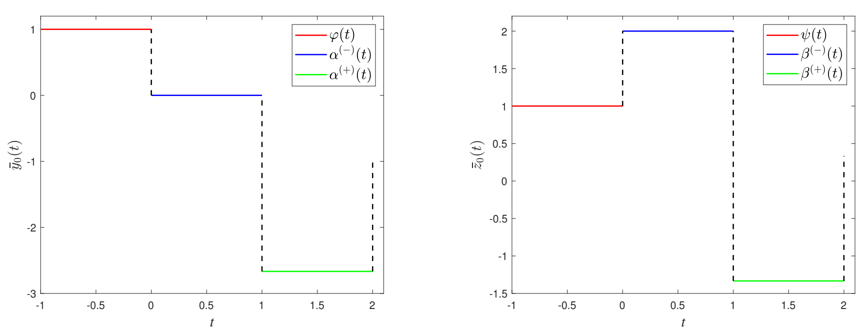

Figure 1. Graph of the degenerate solutions $\bar{y}_0(t)$, $\bar{z}_0(t)$ to SPDDEs (5.1) with respect to t.

In the zeroth order, the left boundary layer functions can be obtained by the following system of equations

$$\frac{\mathrm{d}L_0 y}{\mathrm{d}\tau_0} = -L_0 y + 2L_0 z, \quad \frac{\mathrm{d}L_0 z}{\mathrm{d}\tau_0} = 2L_0 y - L_0 z,$$
$$L_0 y(0) = 1, \quad L_0 y(+\infty) = 0, \quad L_0 z(0) = -1, \quad L_0 z(+\infty) = 0,$$

which is led to

$$L_0 y(\tau_0) = \exp(-3\tau_0), \quad L_0 z(\tau_0) = -\exp(-3\tau_0).$$

For transition layer functions, it can be reached

$$\frac{\mathrm{d}Q_0^{(-)}y}{\mathrm{d}\tau} = -Q_0^{(-)}y + 2Q_0^{(-)}z, \quad \frac{\mathrm{d}Q_0^{(-)}z}{\mathrm{d}\tau} = 2Q_0^{(-)}y - Q_0^{(-)}z,$$
$$Q_0^{(-)}y(0) = p_0, \quad Q_0^{(-)}y(-\infty) = 0,$$

and

$$\frac{\mathrm{d}Q_0^{(+)}y}{\mathrm{d}\tau} = -Q_0^{(+)}y + 2Q_0^{(+)}z - 4\exp(-3\tau),$$
$$\frac{\mathrm{d}Q_0^{(+)}z}{\mathrm{d}\tau} = 2Q_0^{(+)}y - Q_0^{(+)}z - 2\exp(-3\tau),$$
$$Q_0^{(+)}y(0) = p_0 + 8/3, \quad Q_0^{(+)}y(+\infty) = 0,$$

hence, it can be obtained

$$Q_0^{(-)}y(\tau) = p_0\exp(\tau), \quad Q_0^{(-)}z(\tau) = p_0\exp(\tau),$$

and

$$Q_0^{(+)}y(\tau) = \frac{3p_0 - 3\tau + 8}{3\exp(3\tau)}, \quad Q_0^{(+)}z(\tau) = \frac{-6p_0 + 6\tau - 7}{6\exp(3\tau)}.$$

This means that the assumptions (**H3**)–(**H5**) are fulfilled well.

By (3.40), the equation for determining p_0 can be obtained by

$$H(p) = 2p + \frac{9}{2},$$

obviously, the equation $H(p) = 0$ has a solution $p_0 = -\dfrac{9}{4} \in \left(-\dfrac{8}{3}, 0\right)$ and $\dfrac{\mathrm{d}H(p_0)}{\mathrm{d}p} = 2 \neq 0$, which implies that the assumption (**H6**) is satisfied. As a result,

$$Q_0^{(-)}y(\tau) = -\frac{9}{4}\exp(\tau), \quad Q_0^{(-)}z(\tau) = -\frac{9}{4}\exp(\tau),$$
$$Q_0^{(+)}y(\tau) = \frac{5 - 12\tau}{12\exp(3\tau)}, \quad Q_0^{(+)}z(\tau) = \frac{13 + 12\tau}{12\exp(3\tau)}.$$

For the leading term of $Rx(\tau_1)$, it is achieved that

$$\frac{\mathrm{d}R_0y}{\mathrm{d}\tau_1} = -R_0y + 2R_0z, \quad \frac{\mathrm{d}R_0z}{\mathrm{d}\tau_1} = 2R_0y - R_0z,$$
$$R_0y(0) = \frac{5}{3}, \quad R_0y(-\infty) = 0, \quad R_0z(0) = \frac{5}{3}, \quad R_0z(-\infty) = 0,$$

and then

$$R_0y(\tau_1) = \frac{5}{3}\exp(\tau_1), \quad R_0z(\tau_1) = \frac{5}{3}\exp(\tau_1).$$

Hence, the form asymptotic solution to SPDDEs (5.1) can be expressed as

$$y(t,\epsilon) = \begin{cases} \exp(-3\tau_0) - \dfrac{9}{4}\exp(\tau) + O(\epsilon), & 0 \le t \le 1, \\ -\dfrac{8}{3} + \dfrac{5 - 12\tau}{12\exp(3\tau)} + \dfrac{5}{3}\exp(\tau_1) + O(\epsilon), & 1 < t \le 2, \end{cases}$$

$$z(t,\epsilon) = \begin{cases} 2 - \exp(-3\tau_0) - \dfrac{9}{4}\exp(\tau) + O(\epsilon), & 0 \le t \le 1, \\ -\dfrac{4}{3} + \dfrac{13+12\tau}{12\exp(3\tau)} + \dfrac{5}{3}\exp(\tau_1) + O(\epsilon), & 1 < t \le 2, \end{cases}$$

in which $\tau_0 = \frac{t}{\epsilon}$, $\tau = \frac{t-1}{\epsilon}$, $\tau_1 = \frac{t-2}{\epsilon}$.

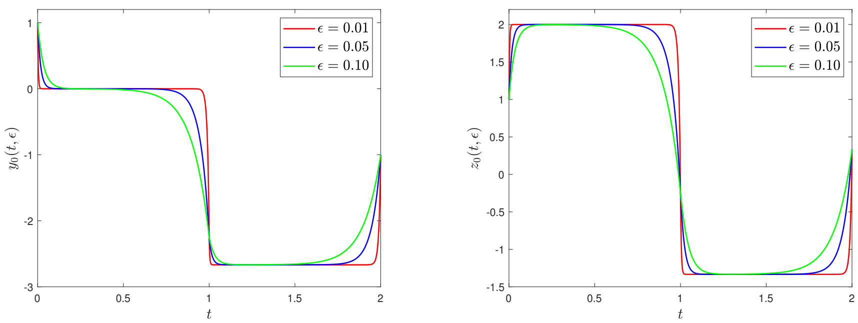

Figure 2. Graph of the first–order continuous asymptotic solution $(y_0(t,\epsilon), z_0(t,\epsilon))^\top$ for some given ϵ.

By means of numerical simulation, from Figures 1–2, it can be found that the constructed continuous asymptotic solution is a well depiction to the considered problem (5.1).

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