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STUDY OF SCHRÖDINGER-CHOQUARD PROBLEM WITH $P(\cdot)$-LAPLACIAN OPERATOR

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Abstract In this paper, our focus is on a specific class of non-linear ψ-Hilfer fractional generalized Schrödinger-Choquard differential equations involving the $p(\cdot)$-Laplacian operator with Dirichlet boundary conditions. By employing the mountain pass theorem without the Palais–Smale condition, along with the Hardy-Littlewood-Sobolev inequality with variable exponents, we establish the existence of a weak solution to our problem. Our main results are novel and contribute to the literature on problems involving ψ-Hilfer derivatives with the $p(\cdot)$-Laplacian operator. This investigation enhances the scope of understanding in this specific class of problems.

Keywords Generalized ψ-Hilfer derivative, Schrödinger-Choquard differential equations, Hardy-Littlewood-Sobolev inequality, mountain pass theorem.

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1. Introduction

The equation known as the Choquard equation, given by

$$-\Delta u + u = \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad u \in H^1(\mathbb{R}^3), \tag{1.1}$$

was initially introduced by Choquard in 1976 and has since captured considerable attention in the realms of physics and Mathematical Analysis [27]. This equation serves as an approximation to the Hartree–Fock theory of a one-component plasma, providing insights into intricate interactions between particles. Lions in [28] studied the normalized solutions of the following problem

$$-\Delta u + \lambda u = \left(\int_{\mathbb{R}^3} u^2(y) V(|x-y|) dy \right) u(x), \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

where V is some given positive function. In the special case where $V = 1/|x|$, equation (1.2) return to equation (1.1). Furthermore, Penrose proposed it as a model for elucidating the

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self-gravitational collapse of a quantum mechanical wave function, underscoring its significance in comprehending essential quantum phenomena [31]. Recently Moroz and Van Schaftingen considered the special model

$$-\Delta u + \mu u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where I_α is the Riesz-potential. They proved in [29] that the equation above has solutions if and only if

$$\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}.$$

In the context of Choquard equations driven by a p-Laplacian operator, Le in [26], established the existence of weak solutions to the following semilinear Choquard equation, which appears as a model in quantum mechanics,

$$-\Delta_p u = \left(\frac{1}{|x|^{N-\alpha}} * |u|^q\right)|u|^{q-2}u, \quad u \in \mathbb{R}^N, \tag{1.3}$$

where $2 \leq p < q \leq N$ and $\max\{0, N-2p\} < \alpha < N$. In [3], the authors studied the existence of semiclassical ground state solutions to the following generalized Choquard equation

$$-\Delta_p u + |u|^{p-2} = \left(\int_{\mathbb{R}^3} \frac{F(u(y))}{|x-y|} dy\right) f(u(x)) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

In the context of fractional derivatives, Additionally, the authors in [24] have precisely solved the following fractional diffusion equation using Riemann-Liouville fractional derivatives,

$$D^\alpha_{0^+} f(r,t) = C_\alpha \Delta f(r,t), \tag{1.5}$$

where $f(r,t)$ denotes the unknown field and C_α denotes the fractional diffusion constant with dimensions $[\mathrm{cm}/s^\alpha]$ and $D^\alpha_{0^+}$ is the Riemann-Liouville derivative of order α.

Numerous researchers have proposed the utilization of fractional time derivatives to address issues related to linear or non-linear differential equations. A pivotal question arises regarding the connection between fractional derivatives and gradient terms. This question finds an answer in [41], where the authors extend gradient elasticity models to characterize materials exhibiting fractional non-locality and fractality. On a different note, pertaining to the Choquard problem, the associated Schrödinger-type evolution equation is expressed as follows:

$$i\partial_t \varphi = \Delta\varphi + (W * |\varphi|^2)\varphi. \tag{1.6}$$

This model represents a sizable system of non-relativistic bosonic atoms and molecules featuring an attractive interaction characterized by a weaker and longer-range nature compared to the nonlinear Schrödinger equation. In (1.6), the interaction potential W is formally expressed as Dirac's delta at the origin [21]. In the work presented in [32], the authors concentrate on the following Cauchy problem involving a Schrödinger-Choquard equation with a pure power non-linearity:

$$\begin{cases} i\dot{u} + \Delta u + (I_\alpha * |u|^p)|u|^{p-2} = 0, \\ u(0,\cdot) = u_0, \end{cases}$$

in three space dimensions, the previous problem has several physical origins such as quantum mechanics. For a class of Kirchhoff problem involving Choquard nonlinearity with real parameter we refer to [25].

In the context of fractional boundary value problems, the authors of [15] established the existence of a weak solution for a non-linear ψ-Hilfer fractional generalized double phase-Choquard differential equation by employing the mountain pass theorem. In 2023, Sousa et al. [38], discussed the existence and regularity of weak solutions for the ψ-Hilfer fractional boundary value problem by using an extension of the Lax-Milgram theorem to the following nonlinear boundary value problem

$$\begin{cases} {}^{\mathrm{H}}\mathrm{D}_{T}^{\alpha,\beta;\psi}\left(|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\alpha,\beta;\psi}\xi(t)|^{p-2}\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\alpha,\beta;\psi}\xi(t)\right)+\xi(t)=\lambda\Phi(t,\xi(t)), & t\in(0,T),\\ \mathbf{I}_{0^+}^{\beta(\beta-1);\psi}\xi(0)=\mathbf{I}_{T}^{\beta(\beta-1);\psi}\xi(T)=0, \end{cases}$$

where λ is a parameter and $\Phi:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function. For more existence results by using differents operator we refer to [1,4–9,11,13–15,22,23,33–37,39,42].

Inspired by these findings, we shift our focus to investigating the existence of a solution in an appropriate fractional ψ-Hilfer derivative space for the Schrödinger-Choquard problem with a $p(\cdot)$-Laplacian operator below

$$\begin{cases} {}^{\mathrm{H}}\mathrm{D}_{T}^{\gamma,\beta;\psi}\left(\left|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|^{p(x)-2}\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right)+G(x)|u|^{p(x)-2}=\left(\displaystyle\int_\Omega\frac{F(x,u)}{|x-y|^{\lambda(x,y)}}\mathrm{d}x\right)f(y,v) \text{ in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where ${}^{\mathrm{H}}\mathrm{D}_{T}^{\gamma,\beta;\psi}$ and ${}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}$ are ψ-Hilfer fractional derivatives of order $\frac{1}{p(x)}<\gamma<1$ and type $0\le\beta\le1$, $G:\Omega\longrightarrow\mathbb{R}$ and $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function satisfying:
(f_1) The growth condition i.e.,

$$|f(x,u)|\le c_1\left(|u|^{r_1(x)-1}+|u|^{r_2(x)-1}\right),\ \text{for all } (x,u)\in\Omega\times\mathbb{R},\ \text{and } c_1>0,$$

where

$$p(x)\ll r_i(x)q^-\le r_i(x)q^+\ll p^\star(x):=\frac{Np(x)}{N-\gamma p(x)},\quad \text{and } r_i^->\frac{p^+}{2}\quad \text{with } i=1,2 \tag{1.8}$$

and $\lambda:\Omega\times\Omega\longrightarrow\mathbb{R}$ be a function satisfying

$$\frac{1}{q(x)}+\frac{\lambda(x,y)}{N}+\frac{1}{q(y)}=2,\quad \text{for all } x,y\in\Omega. \tag{1.9}$$

(f_2) The Ambrosetti-Rabinowitz type condition:

$$0<\alpha F(x,u)\le 2f(x,u)u,\quad \text{where}\quad F(x,u):=\int_0^u f(x,v)dv,$$

where $\alpha>0$ is a fixed number with $\alpha>p^+$.
(G) $G\in L^{\alpha(x)}(\Omega)$ is a continuous non-negative weighted functions where $\alpha\in C(\bar\Omega)$ satisfies one of the following assumptions:

(*i*) $q\in C(\bar\Omega),\ p(x)<\frac{\alpha(x)}{\alpha(x)-1}q(x),\ 1<q(x)<\frac{\alpha(x)-1}{\alpha(x)}p_\alpha^*(x),$ for all $x\in\bar\Omega$,
or

(ii) $q \in C(\bar{\Omega})$, and $\dfrac{Np(x)}{Np(x) - q(x)(N - p(x))} < \alpha(x) < \dfrac{p(x)}{p(x) - q(x)}$.

In what follows we present the result obtained in this manuscript:

Theorem 1.1. *The problem* (1.7) *has a nontrivial solution under the conditions* (f_1)-(f_2) *and* *(G).*

The above result represent the first contribution available in the literature for the ψ-Hilfer fractional generalized Schrödinger-Choquard differential equations involving the $p(\cdot)$-Laplacian operator with Dirichlet boundary conditions within the framework of ψ-fractional derivative space $\mathcal{H}^{\gamma,\beta,\psi}_{p(x)}(\Omega)$. Our approach to establishing existence results for problem (1.7) hinge on utilizing the mountain pass theorem without the Palais–Smale condition [10]. One of the key challenges in this approach lies in utilizing the Hardy-Littlewood-Sobolev inequality for nonlinearities involving ψ-Hilfer fractional derivative.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of variable exponent (weighted) Lebesgue spaces and ψ-fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.7), along with their corresponding proofs.

2. Preliminary

In this section we collect preliminary concepts of the theory of variable exponent Lebesgue space, classical and fractional ψ-Hilfer derivative space (see [12, 16–18, 30]).

2.1. Variable exponent (weighted) Lebesgue space

In the following, we define

$$C^+(\bar{\Omega}) = \Big\{ g \in C(\Omega) : \ 1 < g^- \leq g^+ < +\infty \Big\},$$

where

$$g^- := \inf_{x \in \bar{\Omega}} g(x) \quad \text{and} \quad g^+ := \sup_{x \in \bar{\Omega}} g(x).$$

Denote by $\mathbf{U}(\Omega)$ the set of all measurable real-valued functions defined in Ω. For any $p \in C^+(\Omega)$, we denote the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \in \mathbf{U}(\Omega) : \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \ \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

then, the variable exponent Lebesgue space $\Big(L^{p(x)}(\Omega), \| \cdot \|_{p(x)} \Big)$ becomes a Banach space.

Let $A(x)$ be a measurable real valued function and $A(x) > 0$ for $x \in \Omega$. Then the weight variable exponent Lebesgue space $L^{p(x)}_{A(x)}(\Omega)$ is defined by

$$L^{p(x)}_{A(x)}(\Omega) = \left\{ u \in \mathbf{U}(\Omega) : \int_\Omega A(x)|u(x)|^{p(x)}\mathrm{d}x < \infty \right\},$$

which is equipped with the norm

$$\|u\|_{p(x),A(x)} = \inf\left\{ \lambda > 0 : \int_\Omega A(x)\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\mathrm{d}x \leq 1 \right\}.$$

We have the following generalized Hölder inequality

$$\Big|\int_\Omega u(x)v(x)\mathrm{d}x\Big| \leq 2\|u\|_{p(x)}\|v\|_{q(x)}, \tag{2.1}$$

for $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$ such that $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. At this point, let define the following map $\sigma^{p(x)} : L^{p(x)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\sigma^{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

Then, we can see the important relationship between the norm $\|\cdot\|_{p(x)}$ and the corresponding modular function $\sigma^{p(x)}(\cdot)$ given in the next proposition.

Proposition 2.1. [20] *If u and $(u_n) \in L^{p(x)}(\Omega)$, we have*
(i) $\|u\|_{p(x)} < 1\ (= 1, > 1) \Longleftrightarrow \sigma^{p(x)}(u) < 1\ (= 1, > 1)$,
(ii) $\|u\|_{p(x)} > 1 \Longrightarrow \|u\|^{p^-}_{p(x)} \leq \sigma^{p(x)}(u) \leq \|u\|^{p^+}_{p(x)}$,
(iii) $\|u\|_{p(x)} < 1 \Longrightarrow \|u\|^{p^+}_{p(x)} \leq \sigma^{p(x)}(u) \leq \|u\|^{p^-}_{p(x)}$,
(iv) $\lim_{n\longrightarrow\infty} \|u_n - u\|_{p(x)} = 0 \Longleftrightarrow \lim_{n\longrightarrow\infty} \sigma^{p(x)}(u_n - u) = 0$.

Proposition 2.2. [20] *Let $p : \Omega \longrightarrow \mathbb{R}$ be a Lipschitz continuous function with $1 < p^- \leq p^+ < N$ and $r \in C^+(\Omega)$, then*
(i) If $p(x) \leq r(x) \leq p^\star(x)$, then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$.
(ii) If $p(x) \leq r(x) \ll p^\star(x)$, then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}_{loc}(\Omega)$.

Proposition 2.3. [20] *Assume that the boundary of Ω possesses the cone property and $p \in C(\bar{\Omega})$. If (G) is holds, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}_{a(x)}(\Omega)$ is compact.*

Proposition 2.4 (Hardy–Littlewood–Sobolev inequality)**.** [2] *Let $p, q \in C^+(\Omega)$, $f \in L^{p^+}(\Omega) \cap L^{p^-}(\Omega)$, $g \in L^{q^+}(\Omega) \cap L^{q^-}(\Omega)$ and $\lambda : \Omega \times \Omega \longrightarrow \mathbb{R}$ be a continuous function such that $0 < \lambda^- \leq \lambda^+ < N$ and $1/p(x) + \lambda(x,y)/N + 1/q(x) = 2$. Then there exists a sharp constant $C > 0$, independent of f, and g, such that*

$$\left|\int_{\Omega\times\Omega} \frac{f(u)g(y)}{|x-y|^{\lambda(x,y)}}dxdy\right| \leq C\Big(\|f\|_{p^+}\|g\|_{q^+} + \|f\|_{p^-}\|g\|_{q^-}\Big). \tag{2.2}$$

As a consequence of Proposition 2.4, we have the following results:

Corollary 2.1. [2] *Let q and λ be two function are given in (f_1). If $u \in W^{1,p(x)}(\Omega)$ and $r \in \mathbf{M}$ (the set of all continuous functions which satisfied* (1.8)*). Then*

$$|u(x)|^{r(x)} \in L^{q^+}(\Omega) \cap L^{q^-}(\Omega)$$

and

$$\left| \int_{\Omega\times\Omega} \frac{|u(x)|^{r(x)}|u(x)|^{r(y)}}{|x-y|^{\lambda(x,y)}} dxdy \right| \leq C\Big(\||u(x)|^{r(x)}\|^2_{q^+} \||u(y)|^{r(y)}\|^2_{q^-} \Big).$$

2.2. ψ-Hilfer fractional derivative space

Let $A := [c,d]$ $(-\infty \leq c < d \leq \infty)$, $N-1 < \gamma < N$, $N \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^N(A,\mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in A$.

• The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}^{\gamma;\psi}_{c^+}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \psi'(y)(\psi(x)-\psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y. \tag{2.3}$$

• The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}^{\gamma;\psi}_{d^-}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \psi'(y)(\psi(y)-\psi(x))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y. \tag{2.4}$$

• The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$${}^{\mathrm{H}}\mathrm{D}^{\gamma,\beta;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\beta(N-\gamma);\psi}_{c^+}\Big(\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^N \mathbf{I}^{(1-\beta)(N-\gamma);\psi}_{c^+}\mathbf{f}(x).$$

• The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$${}^{\mathrm{H}}\mathrm{D}^{\gamma,\beta;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\beta(N-\gamma);\psi}_{d^-}\Big(-\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^N \mathbf{I}^{(1-\beta)(N-\gamma);\psi}_{d^-}\mathbf{f}(x).$$

Choosing $\beta \longrightarrow 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathrm{D}^{\gamma;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{(N-\gamma);\psi}_{c^+}\Big(\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^N \mathbf{f}(x), \tag{2.5}$$

$$\mathrm{D}^{\gamma;\psi}_{d^-}\mathbf{f}(x) = \mathbf{I}^{(N-\gamma);\psi}_{d^-}\Big(-\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^N \mathbf{f}(x). \tag{2.6}$$

Remark 2.1. The ψ-Hilfer fractional derivatives defined as above can be written in the following form

$${}^{\mathrm{H}}\mathrm{D}^{\gamma,\beta;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\mu-\gamma;\psi}_{c^+}\mathrm{D}^{\gamma;\psi}_{c^+}\mathbf{f}(x),$$

and

$${}^{\mathrm{H}}\mathrm{D}^{\gamma,\beta;\psi}_{d^-}\mathbf{f}(x) = \mathbf{I}^{\mu-\gamma;\psi}_{d^-}\mathrm{D}^{\gamma;\psi}_{d^-}\mathbf{f}(x),$$

with $\mu = \gamma + \beta(N-\gamma)$ and $\mathbf{I}^{\mu-\gamma;\psi}_{c^+}$, $\mathbf{I}^{\mu-\gamma;\psi}_{d^-}$, $\mathrm{D}^{\gamma;\psi}_{c^+}$ and $\mathrm{D}^{\gamma;\psi}_{d^-}$ as defined in (2.3), (2.4), (2.5) and (2.6).

In this paper we take $\Omega = A_1 \times \cdots \times A_N = [c_1, d_1] \times \cdot \cdot \times [c_N, d_N]$ where $-\infty < c_i < d_i < +\infty$ for all $i \in \mathbb{N}$, $0 < \gamma_1 < ... < \gamma_N < 1$. Consider also $\psi(\cdot)$ be an increasing and positive monotone function on $(c_1, d_1), ..., (c_N, d_N)$, having a continuous derivative $\psi'(\cdot)$ on $(c_1, d_1], ..., (c_N, d_N]$.

• The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1, ..., \mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_{A_1} \int_{A_2} \cdots \int_{A_N} \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y,$$

with $\psi'(y)(\psi(x) - \psi(y))^{\gamma-1} = \psi'(y_1)(\psi(x_1) - \psi(y_1))^{\gamma_1-1} \cdots \psi'(y_N)(\psi(x_N) - \psi(y_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $x = x_1x_2\cdots x_N$ and $\mathrm{d}y = \mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_N$.

• The ψ-Hilfer fractional partial derivative of N-variables of order γ and type β $(0 \le \beta \le 1)$ is defined by

$${}^{\mathrm{H}}\mathrm{D}_{c,x}^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_{c,x}^{\beta(N-\gamma);\psi}\Big(\frac{1}{\psi'(x)}\frac{\partial^N}{\partial x}\Big)\mathbf{I}_{c,x}^{(1-\beta)(N-\gamma);\psi}\mathbf{f}(x),$$

with $\partial x = \partial x_1, \partial x_2, ..., \partial x_N$ and $\psi'(x) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$. Analogously, it is defined ${}^{\mathrm{H}}\mathrm{D}_{d,x}^{\gamma,\beta;\psi}(\cdot)$.

• The left-sided ψ-Hilfer fractional derivative space is defined by

$$\mathcal{H}_{p(x)}^{\gamma,\beta,\psi}(\Omega) = \mathcal{H} := \Big\{u \in L^{p(x)}(\Omega) : \ \Big|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\Big| \in L^{p(x)}(\Omega)\Big\},$$

enduid with the norm

$$\|u\|_{\mathcal{H}_{p(x)}^{\gamma,\beta,\psi}} = \|u\|_{p(x),G} + \Big\|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi}u\Big\|_{p(x)},$$

where

$$\|u\|_{p(x),G} = \inf\left\{l > 0 : \ \int_{\Omega} G(x)\left|\frac{u(x)}{l}\right|^{p(x)} \le 1\right\}.$$

Note that $\mathcal{H}$ is the closure of $C_0^\infty(\Omega)$. Also, $\mathcal{H}$ is a separable and refexive Banach spaces. Moreover, due to Propositions 2.2 and (2.3), we deduce the following embedding:

$$\mathcal{H} \overset{\text{cpt}}{\hookrightarrow} L^{\tau(x)}(\Omega) \quad \text{for all } \tau \in C^+\left(\mathbb{R}^N\right) \text{ and } p \ll \tau \ll p^\star \text{ in } \mathbb{R}^N.$$

Using (1.8), we infer that

$$\mathcal{H} \overset{\text{cpt}}{\hookrightarrow} L^{r_i(x)q(x)}(\Omega).$$

In particular,

$$\mathcal{H} \overset{\text{cpt}}{\hookrightarrow} L^{r_i(x)q^{\pm}}(\Omega). \tag{2.7}$$

3. Main results

In the proof of Theorem 1.1 we will use variational methods. The relevant energy functional of our problem (1.7) is defined by

$$\mathfrak{E}(u) = \int_{\Omega} \frac{\left|\,{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi}u(x)\right|^{p(x)}}{p(x)}\mathrm{d}x + \int_{\Omega} G(x)\frac{|u(x)|^{p(x)}}{p(x)}\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u)F(y,u)}{|x-y|^{\lambda(x,y)}}dxdy.$$

Corollary 3.1. *The function $\mathfrak{E}$ belongs to $C^1(\mathcal{H},\mathbb{R})$, and we can express it as follows:*

$$\begin{aligned}\mathfrak{E}'(u)v &= \int_\Omega \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x)\right|^{p(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x)\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} v(x)dx \\ &+ \int_{\mathbb{R}^N} G(x)|u(x)|^{p(x)-2}u(x)v(x)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u)f(y,u)v(y)}{|x-y|^{\lambda(x,y)}}dxdy, \end{aligned} \tag{3.1}$$

for all u, $v\in\mathcal{H}$.

Proof. We can prove this corollary by using similar analysis presented in [3] Lemma 3.2. □

At this point, we are ready to prove our main results given in Theorem 1.1, the proof of which is divided into several steps.

Step 1: Mountain pass geometry

In this step we show that the energy functional $\mathfrak{E}$ associate to the problem (1.7) satisfies the mountain pass geometry, i.e. satisfies the following Lemma.

Lemma 3.1. *The functional $\mathfrak{E}$ exhibits the following characteristics:*
(i) For sufficiently small $\rho>0$, $\mathfrak{E}(u)\geq\eta$ holds for $u\in\mathcal{H}$ with $\|u\|_{\mathcal{H}}=\rho$, where $\eta>0$.
(ii) There exists an element $a\in\mathcal{H}$ such that $\|a\|_{\mathcal{H}}>\rho$ and $\mathfrak{E}(a)<0$.

To commence, it is necessary to establish the following useful property:

Proposition 3.1. *For each $v\in\mathcal{H}$, we have the following property: F and fv are belong on $L^{q^\pm}(\Omega)$.*

Proof. Due to (f_1), if $u\in\mathbb{R}$ and $u(x)\neq 0$, then

$$|f(x,v)|\leq c_1\Big(|v(x)|^{r_1(x)-1}+|v(x)|^{r_2(x)-1}\Big). \tag{3.2}$$

Also, from the last inequality, we deduce

$$\begin{aligned}|F(x,u)| &= \left|\int_0^u f(x,v)dv\right| \\ &\leq c_1\int_0^u \Big(|v(x)|^{r_1(x)-1}+|v(x)|^{r_2(x)-1}\Big)dx \\ &\leq c_1'\Big(|u(x)|^{r_1(x)}+|u(x)|^{r_2(x)}\Big).\end{aligned}$$

Therefore,

$$|F(x,v)|^{q^+}\leq c_2\Big(|v(x)|^{q^+r_1(x)}+|v(x)|^{q^+r_2(x)}\Big), \tag{3.3}$$

$$|F(x,v)|^{q^-}\leq c_2\Big(|v(x)|^{q^-r_1(x)}+|v(x)|^{q^-r_2(x)}\Big), \tag{3.4}$$

where $c_2={c_1'}^{q^+},{c_1'}^{q^-}$. Now, utilizing (2.7), we deduce that $F\in L^{q(x)}(\Omega)$, in particular, $F\in L^{q^+}(\Omega)$ and $F\in L^{q^-}(\Omega)$. By a similar argument as above, which applies to $f(u)v$, we deduce that $fv\in L^{q^\pm}(\Omega)$. □

Lemma 3.2. *For each $u \in \mathcal{H}$. We have the following properties:*
(i)

$$\int_{\Omega\times\Omega} \frac{|F(x,u)f(y,u)v(y)|}{|x-y|^{\lambda(x,y)}}dxdy < \infty.$$

(ii)

$$\int_{\Omega\times\Omega} \frac{|F(x,u)f(y,u)v(y)|}{|x-y|^{\lambda(x,y)}}dxdy \leq c_3\Big(\|F\|_{q^+}\|fv\|_{q^+} + \|F\|_{q^-}\|fv\|_{q^-}\Big).$$

Proof. Recall that, for every $a, b \geq 0$

$$(a+b)^p \leq 2^{p-1}(a^p + b^p). \tag{3.5}$$

This with Proposition 3.1, equations (2.7), (3.2), and the fact that, $1 \leq p < \infty$, as well as Proposition 2.4, we arrive at the proof of Lemma 3.2. □

Corollary 3.2. *For each $v \in \mathcal{H}$ such that $\|v\|_{\mathcal{H}} \leq 1$, then the sequence $\Big\{\|f(u_k)v\|_{q^\pm}\Big\}_{k\in\mathbb{N}}$ is bounded.*

Proof. Due to (f_1) and (1.8), we obtain

$$\begin{aligned}
&\int_\Omega |f(y,u_k)v(y)|^{q^+}dy \\
&\leq c_2\int_\Omega \Big(|u_k|^{q^+(r_1(x)-1)} + |u_k|^{q^+(r_2(x)-1)}\Big)|v(y)|^{q^+}dy \\
&\leq c_2\Bigg[\Big(\int_\Omega |u_k|^{q^+r_1(x)}dy\Big)^{\frac{r_1(x)-1}{r_1(x)}}\Big(\int_\Omega |v|^{q^+r_1(x)}dy\Big)^{\frac{1}{r_1(x)}} + \Big(\int_\Omega |u_k|^{q^+r_2(x)}dy\Big)^{\frac{r_2(x)-1}{r_2(x)}} \\
&\quad\times\Big(\int_\Omega |v|^{q^+r_2(x)}dy\Big)^{\frac{1}{r_2(x)}}\Bigg] \\
&= c_2\Big(\|u_k\|_{q^+r_1(x)}^{q^+(r_1(x)-1)}\|v\|_{q^+r_1(x)}^{q^+} + \|u_k\|_{q^+r_2(x)}^{q^+(r_2(x)-1)}\|v\|_{q^+r_2(x)}^{q^+}\Big) \\
&\leq c_2\Big(\|u_k\|_{q^+r_1(x)}^{q^+(r_1(x)-1)} + \|u_k\|_{q^+r_2(x)}^{q^+(r_2(x)-1)}\Big) \\
&< \infty.
\end{aligned} \tag{3.6}$$

Similarly, we can show that $\Big\{\|f(y,u_k)v\|_{q^-}\Big\}_{k\in\mathbb{N}}$ is bounded. □

Now, we are ready to prove the Lemma 3.1. Regarding part (i), note that from Proposition 3.1, Lemma 3.2 and Proposition 2.4, we have

$$\left|\int_{\Omega\times\Omega} \frac{F(x,u)F(y,u)}{|x-y|^{(\lambda(x,y))}}\mathrm{d}x\ \mathrm{d}y\right| \leq c_3\Big(\|F\|_{q^+}^2 + \|F\|_{q^-}^2\Big), \quad \text{for all } u \in \mathcal{H}.$$

Due to (3.2), (3.3), (3.4) and (2.7), we have

$$\|F\|_{q^+(\Omega)} \leq c_4\Big(\|u\|_{q^+r_1(x)}^{r_1(x)} + \|u\|_{q^+r_2(x)}^{r_2(x)}\Big)$$

$$\leq c_5\Bigg(\max\left(\|u\|_{\mathcal{H}}^{r_1^+},\|u\|_{\mathcal{H}}^{r_1^-}\right)+\max\left(\|u\|_{\mathcal{H}}^{r_2^+},\|u\|_{\mathcal{H}}^{r_2^-}\right)\Bigg) \tag{3.7}$$

and

$$\|F\|_{q^-(\Omega)} \leq c_6\Bigg(\max\left(\|u\|_{\mathcal{H}}^{r_1^+},\|u\|_{\mathcal{H}}^{r_1^-}\right)+\max\left(\|u\|_{\mathcal{H}}^{r_2^+},\|u\|_{\mathcal{H}}^{r_2^-}\right)\Bigg), \tag{3.8}$$

where c_4, c_5 and c_6 are constants that does not depend on $u \in \mathcal{H}$.

Furthermore, due to (3.5), (3.7), (3.8), together with $\|u\|_{\mathcal{H}} < 1$, we infer that

$$\begin{aligned}
\mathfrak{E}(u) &\geq \int_\Omega \frac{1}{p^+}\left|\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi}u(x)\right|^{p(x)}\mathrm{d}x+\int_\Omega \frac{1}{p^+}G(x)|u(x)|^{p(x)}\mathrm{d}x \\
&\quad - c_6\Bigg(\max\left(\|u\|_{\mathcal{H}}^{2r_1^+},\|u\|_{\mathcal{H}}^{2r_1^-}\right)+\max\left(\|u\|_{\mathcal{H}}^{2r_2^+},\|u\|_{\mathcal{H}}^{2r_2^-}\right)\Bigg) \\
&\geq c_7\left(\left\|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right\|_{p(x)}^{p^+}+\|u\|_{p(x),G}^{p^+}\right)-c_6\left(\|u\|_{\mathcal{H}}^{2r_1^+}+\|u\|_{\mathcal{H}}^{2r_1^-}+\|u\|_{\mathcal{H}}^{2r_2^+}+\|u\|_{\mathcal{H}}^{2r_2^-}\right) \\
&\geq c_8\|u\|_{\mathcal{H}}^{p^+}-2c_6\left(\|u\|_{\mathcal{H}}^{2r_1^-}+\|u\|_{\mathcal{H}}^{2r_2^-}\right),
\end{aligned}$$

where c_7 and c_8 are positive constants that do not depend on u. The fact that $r_1^-, r_2^- > p^+/2$, then the result follows by fixing $\|u\|_{\mathcal{H}} = \rho$ with $\rho > 0$ small enough. For (ii), the assumption (f_2) owing to

$$F(x,u) \geq K u^{\frac{\alpha}{2}}, \qquad \text{for all } \ (x,u)\ \in \Omega\times\mathbb{R},$$

where K depends only on α. Now, considering a nonnegative function $\varphi \in C_c^\infty(\Omega)\backslash\{0\}$, the last inequality allows us to deduce that

$$\begin{aligned}
\mathfrak{E}(t\varphi) &\leq \frac{t^{p^+}}{p^-}\left(\left\|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi}u\right\|_{p(x)}^{p^+}+\|u(x)\|_{p(x),G}^{p^+}\right)-\frac{Kt^\alpha}{2}\int_{\Omega\times\Omega}\frac{(\varphi(x)\varphi(y))^{\alpha/2}}{|x-y|^{\lambda(x,y)}}dxdy \\
&\leq \frac{t^{p^+}}{p^-}\|u\|_{\mathcal{H}}^{p^+}-\frac{Kt^\alpha}{2}\int_{\Omega\times\Omega}\frac{(\varphi(x)\varphi(y))^{\alpha/2}}{|x-y|^{\lambda(x,y)}}dxdy.
\end{aligned}$$

Since, $\alpha > p^+$, then $\mathfrak{E}(t\varphi) < -\infty$ for t large enough. This finishes the proof of Lemma 3.1.

Step 2: Boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$

Recalling that the mountain pass theorem without the Palais-Smale condition (refer to [10], Theorem 5.4.1) states the existence of a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that:

$$\mathfrak{E}(u_k) \longrightarrow \theta \tag{3.9}$$

and

$$\mathfrak{E}'(u_k) \longrightarrow 0, \tag{3.10}$$

where $\theta > 0$ is the mountain pass level defined by

$$\theta := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\mathfrak{E}(\gamma(t)),$$

with

$$\Gamma := \Big\{\gamma \in C\Big([0,1],\mathcal{H}\Big);\gamma(0)=0,\gamma(1)=e\Big\}.$$

Note that

$$\mathfrak{E}(u_k) - \frac{\mathfrak{E}'(u_k)u_k}{\alpha} \le \theta + 1 + \|u_k\|_{\mathcal{H}}, \tag{3.11}$$

for k large enough. Moreover, from (f_2) we have for $\|u_k\|_{\mathcal{H}} \ge 1$ that

$$\begin{aligned}\mathfrak{E}(u_k) - \frac{\mathfrak{E}'(u_k)u_k}{\alpha} =& \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\alpha}\right)\left(\left|{}^{\mathrm{H}}\mathrm{D}_{0+}^{\gamma,\beta;\psi} u_k\right|^{p(x)} + G(x)|u_k(x)|^{p(x)}\right) dx \\ &+ \int_\Omega \int_\Omega \frac{F(u,u_k)}{|x-y|^{\lambda(x,y)}} \underbrace{\left(\frac{f(y,u_k)u_k(v)}{\alpha} - \frac{F(y,u_k)}{2}\right)}_{\ge 0} \mathrm{d}x\mathrm{d}y \\ \ge& c_{10} \int_\Omega \left(\left|\, {}^{\mathrm{H}}\mathrm{D}_{0+}^{\gamma,\beta;\psi} u_k\right|^{p(x)} + G(x)|u_k(x)|^{p(x)}\right) dx. \end{aligned} \tag{3.12}$$

Hence, (3.11) and (3.12) owing to the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$.

Remark 3.1. From the fact that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}$, $\{u_n\}_{n\in\mathbb{N}}$ is also bounded in $W^{1,p(x)}(\Omega)$. Then, Proposition 2.1 implies that there exists $u \in \mathcal{H}$ and a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$u_n(x) \longrightarrow u(x) \text{ a.e in } L^{p(x)}(\Omega), \tag{3.13}$$

$$u_n(x) \longrightarrow u(x) \text{ a.e in } \Omega, \tag{3.14}$$

$${}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_n \rightharpoonup {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_n \quad \text{in } \left(L^{r(x)}(\Omega)\right)^N. \tag{3.15}$$

Step 3: Existence of solution

Now, we show the existence of a critical point of $\mathfrak{E}$, which is a weak solution of the problem (1.7). First of all, we show the following property.

Lemma 3.3. *The following limits hold for a subsequence:*
(i)

$$\int_{\Omega\times\Omega} \frac{F(x,u_k)f(y,u)v(y)}{|x-y|^{\lambda(x,y)}} dxdy \longrightarrow \int_{\Omega\times\Omega} \frac{f(x,u)f(y,u)v(y)}{|x-y|^{\lambda(x,y)}} dxdy,$$

for all $v \in C_c^\infty(\Omega)$,
(ii)

$$\int_\Omega \int_\Omega \frac{f(x,u_k)\big(f(y,u_k)v(y) - f(y,u)v(y)\big)}{|x-y|^{\lambda(x,y)}} dxdy \longrightarrow 0,$$

for all $v \in C_c^\infty(\Omega)$,
(iii)

$$\int_\Omega \int_\Omega \frac{F(x,u_k)f(y,u_k)v(y)}{|x-y|^{\lambda(x,y)}} dxdy \longrightarrow \int_\Omega \int_\Omega \frac{F(x,u)f(y,u)v(y)}{|x-y|^{\lambda(x,y)}} dxdy,$$

for all $v \in C_c^\infty(\Omega)$.

Proof. (i) From Step 2 and (2.7) we conclude that $F(\cdot, v_n)$ is bounded in $L^{q^\pm}(\Omega)$. In addition, due to (3.14) and the continuity of F imply that $F(x,v_n) \longrightarrow F(x,v)$ pointwise a.e. in Ω we deduce that $F(u_k) \rightharpoonup F(u)$ in $L^{q^\pm}(\Omega)$. By virtue of Proposition 2.4, it follows that the function

$$H(u) := \int_\Omega \int_\Omega \frac{h(x)f(u(y))v(y)}{|x-y|^\lambda},\ h \in L^{q^+}(\Omega) \cap L^{q^-}(\Omega),$$

defines a continuous linear functional. Since $F(u_k) \rightharpoonup F(u)$ in $L^{q^{\pm}}(\Omega)$, it follows that

$$\int_{\mathbb{R}^{2N}} \frac{F(u_k(x))f(u_k(y))v(y)}{|x-y|^{\lambda}} \mathrm{d}x\ \mathrm{d}y \longrightarrow \int_{\mathbb{R}^{2N}} \frac{F(u(x))f(u(y))v(y)}{|x-y|^{\lambda}} \mathrm{d}x\ \mathrm{d}y, \text{ for all } v \in C_c^{\infty}(\Omega).$$

For (ii). First, let define the convolution operator $K : L^{q^{\pm}}(\Omega) \longrightarrow L^{\frac{2N}{\lambda}}(\Omega)$ by

$$K(w)(x) := \frac{1}{|x|^{\lambda}} * w(x).$$

Due to Proposition 2.4, we obtain that K is a linear and bounded operator. Hence, up to a subsequence, $\{K(F(\cdot, v_n))\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{2N}{\lambda}}(\Omega)$, by Hölder's inequality, it is enough to show that

$$\|(f(\cdot, v_n) - f(\cdot, v))\,\phi\|_{q^{\pm}} \longrightarrow 0. \tag{3.16}$$

Due to (3.6), Lebesgue's dominated convergence theorem, we obtain the required assertion in (3.16) and so (ii). (iii) is a direct consequence of (i) and (ii). □

Proposition 3.2. *Assume that (f_1), (f_2) and (G) are hold. For a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, we have*

$${}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_k(x) \longrightarrow {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x), \quad \textit{pointwise a.e. in}\ \Omega.$$

Consequently, it holds

$$\left|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_k\right|^{p(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_k \rightharpoonup \left|{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u\right|^{p(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u, \quad \textit{in}\ \left[L^{\frac{p(x)}{p(x)-1}}(\Omega)\right]^N. \tag{3.17}$$

Proof. We refer to the proof of Lemma 13 in [40]. □

Lemma 3.4. *The function u is a critical point of $\mathfrak{E}$.*

Proof. First, we assert that

$$\mathfrak{E}'(u_k)u \longrightarrow \mathfrak{E}'(u)v, \quad \text{for all} \quad v \in C_c^{\infty}(\Omega).$$

To determine this limit, observe that

$$\begin{aligned}\mathfrak{E}'(u)v = &\int_{\Omega} \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x)\right|^{p(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x)\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} v(x)dx \\ &+ \int_{\mathbb{R}^N} G(x)|u(x)|^{p(x)-2}u(x)v(x)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u)f(y,u)v(y)}{|x-y|^{\lambda(x,y)}} dxdy.\end{aligned}$$

Lemma 3.3 and Proposition 3.2 owing to

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u_k)f(y,u_k)v(y)}{|x-y|^{\lambda(x,y)}} dxdy \longrightarrow \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(x,u)f(y,u)v(y)}{|x-y|^{\lambda(x,y)}} dxdy \tag{3.18}$$

and

$$\begin{aligned}&\int_{\Omega} \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_k(x)\right|^{p(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_k(x)\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} v(x)dx \\ \longrightarrow &\int_{\Omega} \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x)\right|^{p(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u(x)\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} v(x)dx.\end{aligned} \tag{3.19}$$

Furthermore, applying Lebesgue's Dominated Convergence Theorem, we also obtain

$$\int_{\mathbb{R}^N} G(x)|u_k(x)|^{p(x)-2}u_k(x)v(x)dx \longrightarrow \int_{\mathbb{R}^N} G(x)|u(x)|^{p(x)-2}u(x)v(x)dx.$$

From (3.18) and (3.19), the claim follows. As $\mathfrak{E}'(u_k)v \longrightarrow 0$, this claim implies that $\mathfrak{E}'(u)v = 0$ for all $v \in C_c^\infty(\mathbb{R}^N)$. With the knowledge that $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{H}$, the lemma follows. □

Proof of Theorem 1.1

If $u \neq 0$, then u serves as a nontrivial solution, concluding the theorem. However, if $u = 0$, the task is to locate another solution $v \in \mathcal{H}\backslash\{0\}$ for equation (1.7). In pursuit of this objective, the assertion presented below plays a pivotal role in our reasoning.

Claim. *There exist $s > 0$, $\vartheta > 0$ and a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \Omega$ such that*

$$\liminf_{n\longrightarrow+\infty} \int_{B_s(y_n)} |u_k(x)|^{q(x)}dx \geq \vartheta > 0. \tag{3.20}$$

Proof. In fact, if the above claim does not hold. by Lions's lemma [19], one has

$$u_k \longrightarrow 0, \quad \text{in} \quad L^{q^{\pm}}(\Omega). \tag{3.21}$$

Moreover, Proposition 2.4 owing to,

$$\left| \int_{\Omega\times\Omega} \frac{F(x,u_k)(f(y,u_k)u_k(y)}{|x-y|^{\lambda(x,y)}} \mathrm{d}x\mathrm{d}y \right| \leq c_{11}\|F\|_{q^+}\|fu_k\|_{q^+} + c_{13}\|F\|_{q^-}\|fu_k\|_{q^-}.$$

By (3.7), (3.8), 3.21 and Corollary 3.2, we obtain that

$$\int_\Omega \left|F(x,u_k)\right|^{q^+} \mathrm{d}x \longrightarrow 0 \quad \text{and} \quad \int_\Omega |F(x,u_k)|^{q^-} \mathrm{d}x \longrightarrow 0,$$
$$\int_\Omega \left|f(y,u_k)u_k(y)\right|^{q^+} \mathrm{d}y \longrightarrow 0 \quad \text{and} \quad \int_\Omega \left|f(y,u_k)u_k(y)\right|^{q^-} \mathrm{d}y \longrightarrow 0.$$

Therefore,

$$\int_\Omega \frac{F(x,u_k)f(y,u_k)u_k(y)}{|x-y|^{\lambda(x,y)}} \mathrm{d}x \ \mathrm{d}y \longrightarrow 0. \tag{3.22}$$

Using (3.10) together with (3.22) give

$$\int_\Omega \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\beta,\psi} u_k(x)\right|^{p(x)} dx + \int_{\mathbb{R}^N} G(x)|u_k(x)|^{p(x)}dx \longrightarrow 0.$$

This limit leads to $\mathfrak{E}(u_k) \longrightarrow 0$, which contradicts with (3.9). □

Due to the next Lemma, we finish the prove of Theorem 1.1.

Lemma 3.5. *Let $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ be such that $\mathfrak{E} \longrightarrow \theta$. Then there exists $\{\tilde{y}_k\}_{k\in\mathbb{N}} \subset \Omega$ such that the translated sequence*

$$\tilde{v} := u_k(x + \tilde{y}_k)$$

has a subsequence which converges in $\mathcal{H}$.

Proof. By utilizing the fact that $\mathfrak{E}'(u_k)u_k \longrightarrow 0$ and $\mathfrak{E}(u_k) \longrightarrow \theta$, we can employ the same reasoning as in the proof of Lemma 3 to demonstrate that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{H}$. Then, considering $\tilde{u}_k(u) = u_k(x+\tilde{y}_k)$ as a subsequence, we can find $\tilde{u} \in \mathcal{H}$ such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in $\mathcal{H}$ and $\tilde{u} \neq 0$ according to (3.20). Furthermore, for $(t_k)_{k\in\mathbb{N}} > 0$, we can construct $\tilde{v}_k = t_k\tilde{u}_k \in \mathcal{H}$. Then

$$\mathfrak{E}(\tilde{v}_k) \leq \max_{t\geq 0} \mathfrak{E}(tu_k) = \mathfrak{E}(u_k),$$

and so

$$\mathfrak{E}(\tilde{v}_k) \longrightarrow \theta. \tag{3.23}$$

Since (3.23) holds, we have that $\{\tilde{v}_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{H}$, which implies that we can assume $\tilde{v}_k \rightharpoonup \tilde{v}$ in $\mathcal{H}$. Moreover, $\{t_k\}_{k\in\mathbb{N}}$ is bounded and converges to $t_0 > 0$. Suppose for contradiction that $t_0 = 0$. Then, by the boundedness of $\{\tilde{u}_k\}_{k\in\mathbb{N}}$, we have $\|\tilde{v}_k\|_{0,\mathcal{H}} = t_k\|\tilde{u}_k\|_{\mathcal{H}} \longrightarrow 0$, which contradicts $\mathfrak{E}(\tilde{v}_k) \longrightarrow \theta > 0$. Hence, $t_0 > 0$. Since the weak limit is unique, we have $\tilde{v} = t_0\tilde{u}$ and $\tilde{u} \neq 0$. Thus, $\tilde{v}_k \longrightarrow \tilde{v}$ in $\mathcal{H}$, and consequently $\tilde{u}_k \longrightarrow \tilde{u}$ in $\mathcal{H}$. □

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