Journal of Applied Analysis and Computation
Volume 16, Number 1, February 2026, 76–98
Website:http://www.jaac-online.com
DOI:10.11948/20250117

USING RELIABLE TECHNIQUES TO SOLVE NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS ALONG WITH THEORETICAL ANALYSIS

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Abstract Throughout the current work, we implement a novel method to provide exact and approximate solutions, called the fractional $\mathcal{J}-$transform Adomian decomposition method ($\mathcal{FJADM}$). The suggested technique is used to explore solutions to the nonlinear time-fractional diffusion, Harry-Dym, and Fisher equations, including their theoretical analysis. We present detailed proofs of the existence and uniqueness theorems applied to nonlinear fractional ODEs using the $\mathcal{FJADM}$. We combined in this work both the $\mathcal{J}$-transform method ($\mathcal{JTM}$) and the Adomian decomposition method (ADM). Clearly, from the results obtained, the new scheme proposed in this work is highly accurate and efficient. The results have shown how powerful and effective this method is and how straightforward it is for solving many types of fractional differential equations. The numerical calculations in the current work were carried out using Mathematica 13.

Keywords Caputo fractional derivative, Adomian decomposition method, J-transform method, Harry-Dym equation, Banach fixed point theorem.

MSC(2010) 26A33, 34A08, 34C60, 35R11, 26A33, 92D30.

1. Introduction

Due to the fact that fractional differential equations have so many great applications in a wide range of scientific fields, interest in them has recently increased [8, 15, 22, 30]. Fractional differential equations provide a clear description of a number of significant phenomena in the fields of electromagnetics, signal processing, biological population models, electrochemistry, and fluid mechanics. They are also used in the social sciences, which include economics, finance, climatology, and food supplements. To do that, we need a trustworthy and efficient method to determine the analytical solutions to differential equations of fractional order; see [6, 27].

In applied mathematics, finding exact, numerical solutions for these equations is crucial. Due to all this, it is still a major issue in applied mathematics and physics to find accurate solutions for nonlinear differential equations of fractional order. Several effective and potent techniques have been put forth to find approximate the analytical solutions of fractional differential equations, including the Elzaki transform method [11], the fractional complex transform [2, 14], the first integral method [10], the fractional Adomian decomposition method [5, 18], fractional matrix method [29], the fractional homotopy perturbation method [13, 24], the fractional Laplace

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decomposition method [12], and the fractional Sumudu transform method [1], and fractional natural transform method [3, 20, 21, 25, 26].

We explore the $\mathcal{FJADM}$ method as a new approach in order to derive approximations of solutions to nonlinear fractional PDEs. The present work studies the $\mathcal{FJADM}$, a new integral transform for differential equations of fractional order. The suggested mechanism is used to explore various aspects of fractional Caputo and Riemann-Liouville derivatives, including their properties, uses, and some basic theorems. Numerical examples are provided for a variety of fractional differential equations to demonstrate their validity, and the $\mathcal{FJADM}$ method is based on fractional $\mathcal{J}$-transform theorems and provides solutions in infinite series form. A series that, if an exact solution exists, may converge to a closed solution.

Numerous practical applications employ the Harry Dym equation [4], which is attributed to an unpublished article written by Harry Dym between 1973 and 1974. The Korteweg–de Vries equation and the Sturm–Liouville operator are connected to the Harry Dym equation. The Harry Dym equation illustrates a situation where nonlinearity and dispersion are intertwined. Harry Dym is a nonlinear evolution equation that is fully integrable. It is significant since it lacks the Painleve property and complies with an endless number of conservation necessities. In 1937, Fisher put forth the Fisher equation as a model for the temporal and spatial spread of a virulent gene in an infinite media. Chemical kinetics, autocatalytic chemical reactions, flame propagation, nuclear reactor theory, neurophysiology, and branching Brownian motion processes are among its areas of conflict. The Fisher equation solves problems like the nonlinear evolution of a population in a one-dimensional habitat by combining diffusion and logistic nonlinearity. Population growth and dispersion are modeled using the Fisher equation. Both the transmission of nerve impulses and power law delays between movements are modeled by the fractional derivative term. Fractional Harry-Dym and Fisher equations solutions were found through the application of the Mohand homotopy perturbation transform scheme ($\mathcal{MHPT}$) [19].

Moreover, the concept of $\mathcal{J}$-transform method was presented by Shehu Maitama and Wei dong Zhao [17]. In addition, Obeidat et al. presented proofs to some properties of the $\mathcal{J}$-transform in [23]. We presented proofs to the existence and uniqueness theorems along with error estimate using the new approach. In addition, we present exact and approximate solutions to harry-Dym and Fisher's equations.

The outline of this work is as follows. In Section 2, the basic definitions and some properties of the $\mathcal{J}$-transform and Adomian methods are discussed. Section 3 presents the background of fractional calculus and some theorems related to the fractional $\mathcal{J}$-transform, ending with a formula to calculate the Adomian polynomial. In Section 4, we present detailed proofs of the $\mathcal{FJADM}$ for some classes of nonlinear fractional partial differential equations. In Section 5, we present applications of the $\mathcal{FJADM}$ of some fractional nonlinear partial differential equations. Lastly, in Section 6, we provide our conclusion from this research work.

The original contribution of this work consists mainly of proving theorems and applications in sections four and five. Mathematica program was used to compute the Adomian polynomial terms.

2. Resources for fractional calculus background

In this section, we review some of the key terminologies pertaining to fractional calculus, see [15, 22, 30].

Definition 2.1. [22] A function $\phi(\tau) \in \mathbb{R}$, $\tau > 0$ is said to be in the space $\mathcal{C}_\alpha$, where $\alpha \in \mathbb{R}$, if $\exists q \in \mathbb{R}$ with $q > \alpha$, such that: $\phi(\tau) = \tau^q g(\tau)$, where $g(\tau) \in \mathcal{C}[0, \infty)$ and is said to be in the space $\mathcal{C}_\alpha^\iota$ if $\phi^{(\iota)}(\tau) \in \mathcal{C}_\alpha$, and $\iota \in \mathbb{N}$.

Definition 2.2. [15] The Riemann-Liouville operator of $f(s) \in \mathcal{C}_\alpha$ with order $\mu > 0$ is given as follows:

$$I_s^{\mu}[f(s)] = \frac{1}{\Gamma(\mu)} \int_0^s \frac{f(\tau)}{(s-\tau)^{1-\mu}} d\tau.$$

Definition 2.3. [15] The Caputo fractional derivative of $f(s) \in \mathcal{C}^{\iota}(0,b)$, $\mu \in (\iota - 1, \iota)$, where $\iota \in \mathbb{N}$, and $\mu > 0$ is given by:

$${}^{c}D_s^{\mu}[f(s)] = \frac{1}{\Gamma(\iota-\mu)} \left(\int_0^s \frac{f^{(\iota)}(\tau)}{(s-\tau)^{\mu-\iota+1}} d\tau \right).$$

Definition 2.4. [16] The Mittag-Leffler function with two parameters is given by:

$$E_{\alpha,\gamma}(\tau) = \sum_{\kappa=0}^{\infty} \frac{\tau^{\kappa}}{\Gamma(\alpha\kappa+\gamma)}, \qquad \alpha > 0,\ \gamma > 0,\ \tau \in \mathbb{C},$$

where the Gamma function Γ for $\iota \geq 0$ is given by [28]:

$$\Gamma(\iota) = \int_0^{\infty} e^{-\tau}\, \tau^{\iota-1} d\tau.$$

3. Adomian polynomials and $\mathcal{J}$-transform: An overview

In this section, we provide important properties and definitions related to the (Adomian and $\mathcal{J}$-transform history) in general, which will be used frequently throughout this research work.

Definition 3.1. [23] Assume that $\mathcal{M}, c > 0$ and $\Phi(\tau)$ is a piece-wise continuous function over $\mathbb{R}$. Suppose that $\mathcal{B} = \left\{ \Phi(\tau) : |\Phi(\tau)| < \mathcal{M} e^{c\,\tau} \mathcal{X}_{(0,\infty)}(\tau) \right\}$, where $\mathcal{X}_{(0,\infty)}(\tau)$ is the characteristic map. So, $|\Phi(\tau)| \leq \mathcal{M} e^{c\,\tau}$ for $\tau \longrightarrow \infty$ i.e. given any $\Phi(\tau) \in \mathcal{B}$, where $s, \mathtt{u} > 0$, we've got:

$$\begin{aligned} \left| \int_0^{\infty} e^{-s\,\tau} \Phi(\tau \mathtt{u}) dt \right| &\leq \mathcal{M} \int_0^{\infty} e^{-s\,\tau} e^{c|\tau\,\mathtt{u}|} d\tau \\ &= \mathcal{M} \int_0^{\infty} e^{(c\mathtt{u}-s)\tau} d\tau. \end{aligned}$$

If $c\,\mathtt{u} - s < 0$, the above is convergent. Therefore, $\Phi(\tau)$ is of exponentially order.

The $\mathcal{J}$-transform is then provided as follows:

$$\mathcal{J}\left(\Phi(\tau)\right) = \mathcal{H}(s,\mathtt{u}) = \mathtt{u} \int_0^{\infty} e^{\frac{-s\,\tau}{\mathtt{u}}} \Phi(\tau) d\tau, \quad s, \mathtt{u} > 0, \tag{3.1}$$

where $\mathcal{J}$ stands for the $\mathcal{J}-$transformation of $\Phi(\tau)$ and $\mathtt{u}$, s are the variables of $\mathcal{J}-$transformation.

So, Eq. (3.1), can be expressed as,

$$\mathcal{J}\left(\Phi(\tau)\right) = \mathcal{H}(s,\mathtt{u}) = \mathtt{u}^2 \int_0^{\infty} e^{-s\,\tau}\, \Phi(\mathtt{u}\tau)\, d\tau, \quad s, \mathtt{u} \in (0,\infty). \tag{3.2}$$

Definition 3.2. Assume that $\mathcal{H}(s,u)$ be the $(\mathcal{JT})$ of the function $\Phi(\tau)$, then $\mathcal{J}^{-1}$ is called the $(\mathcal{IJT})$ of $\mathcal{H}(s,u)$, that is

$$\mathcal{J}^{-1}[\mathcal{H}(s,u)] = \Phi(\tau), \text{ for } \tau \geq 0.$$

Some of the $\mathcal{J}$-Transform's Properties

We first start by introducing few properties of the $\mathcal{J}$-transform ($\mathcal{JT}$) and inverse $\mathcal{J}$-transform ($\mathcal{IJT}$), which we will use throughout this work [17].

1. Suppose that $e^{(\tau\rho)}\ \Phi(\tau) \in \mathcal{B}$, where ρ is constant. Then

$$\mathcal{J}[e^{\rho\tau}\Phi(\tau)] = \frac{s-\rho u}{s}\ \mathcal{H}\left(s, \frac{su}{s-\rho u}\right).$$

2. Suppose $\mathcal{H}(s,u)$ be the ($\mathcal{JT}$) of $\Phi(\tau)$, and $\rho > 0$. Then,

$$\mathcal{J}[\Phi(\rho\tau)] = \frac{1}{\rho}\mathcal{H}\left(\frac{s}{\rho}, u\right).$$

3. Let $\mathcal{H}(s,u) = \dfrac{u^{n+2}}{s^{n+1}}$, $u, s > 0$, n=0,1,2,3,..., then ($\mathcal{IJT}$) is given by:

$$\mathcal{J}^{-1}\left[\frac{u^{n+2}}{s^{n+1}}\right] = \frac{\tau^n}{n!} = \frac{\tau^n}{\Gamma(n+1)}, \quad n = 0, 1, 2, 3, ...$$

4. Consider $\iota - 1 \le \alpha < \iota$, where $\iota \in \mathbb{Z}^+$ and $\mathcal{H}(s,u)$ is the ($\mathcal{JT}$) of $\Phi(\tau)$, then ($\mathcal{JT}$) of fractional Caputo derivative is given by:

$$\mathcal{J}\left[{}^cD_\tau^\alpha \nu(\tau)\right] = \left[\frac{s}{u}\right]^\alpha \mathcal{H}(s,u) - \sum_{\kappa=0}^{\iota-1} u\left[\frac{s}{u}\right]^{\alpha-\kappa-1} \left(D^\kappa \Phi(\tau)\right)_{\tau=0}.$$

Computational Adomian polynomials. The Adomian polynomials are an invaluable tool for effectively decomposing a complex nonlinear component into smaller, easier-to-manage components that can be integrable as a Taylor series. Following [7, 9], the representation of the unknown function Θ can be expressed as follows:

$$\Theta = \sum_{m=0}^{\infty} \Theta_m. \tag{3.3}$$

In order to find Θ_m , $m \ge 0$, a recursive relation must be established. When handling nonlinear components, $G(\Theta)$ shall be defined as an infinite series, or Adomian polynomials B_m, using the formula below:

$$G(\Theta) = \sum_{m=0}^{\infty} B_m(\Theta_0, \Theta_1, ..., \Theta_m). \tag{3.4}$$

Additionally, the nonlinear term B_m of $G(\Theta)$ can be obtained using the formula in [9]:

$$B_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[G\left(\sum_{i=0}^{m}\lambda^i\Theta_i\right)\right]_{\lambda=0}, \ m = 0, 1, 2, \cdots. \tag{3.5}$$

The following is an expression for the general formula for Eq. (3.5): Let $G(\Theta)$ be the nonlinear function, for instance. The following outcomes can be achieved by applying Eq. (3.4) and the definition of an Adomian polynomial:

$$\begin{aligned} B_0 &= G(\Theta_0), \\ B_1 &= \Theta_1 G'(\Theta_0), \\ B_2 &= \Theta_2 G'(\Theta_0) + \frac{1}{2!}\Theta_1^2 G''(\Theta_0). \end{aligned} \tag{3.6}$$

Finally, the other terms can be constructed using a similar procedure. The polynomials previously presented in Eq. (3.6) provide two significant observations. While B_2 depends solely on Θ_0, Θ_1, and Θ_2, etc., B_0 and Θ_0 are the only variables on which B_1, B_0, and Θ_1 rely. Substituting Eq. (3.6) into Eq. (3.4), one can observe:

$$\begin{aligned} G(\Theta) &= B_0 + B_1 + B_2 + ... \\ &= G(\Theta_0) + (\Theta_1 + \Theta_2 + \Theta_3 + ...)G'(\Theta_0) \\ &\quad + \frac{1}{2!}(\Theta_1^2 + 2\Theta_1\Theta_2 + 2\Theta_1\Theta_3 + \Theta_2^2 + ...)G''(\Theta_0) \\ &\quad + \frac{1}{3!}(\Theta_1^3 + 3\Theta_1^2\Theta_2 + 3\Theta_1^2\Theta_3 + 6\Theta_1\Theta_2\Theta_3 + ...)G'''(\Theta_0) + ... \\ &= G(\Theta_0) + (\Theta - \Theta_0)G'(\Theta_0) + \frac{1}{2!}(\Theta - \Theta_0)^2 G''(\Theta_0) + ... \end{aligned}$$

4. Convergence analysis using $\mathcal{FJADM}$

We shall demonstrate the proofs of the convergence and uniqueness theorems and then provide an estimate error using the $\mathcal{FJADM}$. Consider the nonlinear fractional order of the ODEs:

$$^{c}D_{\eta}^{\sigma}\varphi(\eta) + N(\varphi(\eta)) + L(\varphi(\eta)) = \psi(\eta), \quad 0 < \sigma \leq 1. \tag{4.1}$$

Accompanied by its I.C.:

$$\varphi(0) = \varphi_0. \tag{4.2}$$

Note that the non-linear part is $N(\varphi(\eta))$, the linear term is $L(\varphi(\eta))$, and $\psi(\eta)$ is the source term. Apply the $\mathcal{J}$-transformation and property 4 on Eq. (4.1):

$$\varphi(r, u) = \frac{\varphi_0 u^2}{r} - \left(\frac{u}{r}\right)^{\sigma} \mathcal{J}\left[L(\varphi(\eta)) + N(\varphi(\eta)) - \psi(\eta)\right]. \tag{4.3}$$

Apply the inverse $\mathcal{J}$-transform on Eq. (4.3) to obtain:

$$\varphi(\eta) = \Psi(\eta) + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\sigma} \mathcal{J}\left[L(\varphi(\eta)) + N(\varphi(\eta))\right]\right]. \tag{4.4}$$

$\Psi(\eta)$ represents the nonhomogeneous part as well as the I.C. Assume we have an infinite series solution to the function, $\varphi(\eta)$, as follows:

$$\varphi(\eta) = \sum_{k=0}^{\infty} \varphi_k(\eta). \tag{4.5}$$

The nonlinear term $N(\varphi(\eta)) = \sum_{j=0}^{\infty} A_j$ represents the Adomian polynomials A_j. Substitute Eq. (4.5) into Eq. (4.4) to obtain:

$$\sum_{j=0}^{\infty} \varphi_j(\eta) = \Psi(\eta) + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\sigma} \mathcal{J}\left[\sum_{j=0}^{\infty} A_j + \sum_{j=0}^{\infty} \varphi_j\right]\right]. \tag{4.6}$$

Comparing both sides of Eq. (4.6) to obtain the following general relation:

$$\varphi_{j+1}(\eta) = \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\sigma} \mathcal{J}\left[A_j + \varphi_j\right]\right], \ j \geq 0. \tag{4.7}$$

Finally, the exact solution is given by:

$$\varphi(\eta) = \sum_{j=0}^{\infty} \varphi_j(\eta). \tag{4.8}$$

Theorem 4.1. (Uniqueness Theorem). *If $0 < \lambda < 1$, then there exists a unique solution to Eq. (4.1), with $\lambda = \frac{(C_1+C_2)\eta^{\xi}}{\Gamma(\xi+1)}$, $\forall \xi \in [0, \beta]$.*

Proof. Considering the Banach space of every continuous function on $\Delta = [0, \beta]$ is $\mathbb{K} = (C[\Delta], \|.\|)$ and consider the norm $\|.\|$, then we define $\zeta : \mathbb{K} \to \mathbb{K}$ by

$$\varphi_{k+1}(\eta) = \Psi(\eta) + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi} \mathcal{J}\left[N(\varphi_k(\eta)) + L(\varphi_k(\eta))\right]\right].$$

Suppose that $L[\varphi(\eta)] = \varphi(\eta)$ and $N[\varphi(\eta)] = N(\varphi(\eta))$. Further, let $|N(\varphi) - N(\tilde{\varphi})| < C_1 |\varphi - \tilde{\varphi}|$ and $|L(\varphi) - L(\tilde{\varphi})| < C_2 |\varphi - \tilde{\varphi}|$, where C_1, C_2 are the Lipschitz constants with $0 \le C_1, C_2 < 1$ and $\varphi, \tilde{\varphi}$ are distinct solutions of Eq. (4.1). Then,

$$\begin{aligned}
\|\zeta(\varphi) - \zeta(\tilde{\varphi})\| &= \max_{\eta\in\Delta}\left|\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[L(\varphi)+N(\varphi)\right]\right] - \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[L(\tilde{\varphi})+N(\tilde{\varphi})\right]\right]\right| \\
&= \max_{\eta\in\Delta}\left|\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[L(\varphi)-L(\tilde{\varphi})\right]\right] + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[N(\varphi)-N(\tilde{\varphi})\right]\right]\right| \\
&\le \max_{\eta\in\Delta}\left[C_1\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[|\varphi-\tilde{\varphi}|\right]\right] + C_2\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[|\varphi-\tilde{\varphi}|\right]\right]\right] \\
&\le \max_{\eta\in\Delta}(C_1+C_2)\left[\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[|\varphi-\tilde{\varphi}|\right]\right]\right] \\
&\le (C_1+C_2)\left[\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[\|\varphi(\eta)-\tilde{\varphi}(\eta)\|\right]\right]\right] \\
&= \|\varphi-\tilde{\varphi}\|\frac{(C_1+C_2)}{\Gamma(\xi+1)}\eta^{\xi}.
\end{aligned}$$

Consequently, since $0 < \lambda < 1$, $\exists!$ solution for Eq. (4.1). By the Banach fixed-point theorem for contraction we conclude that ζ is a contraction mapping. This leads to the proof of Theorem 4.1. □

Theorem 4.2. (Convergence Theorem). *Equation (4.8) of Equation (4.1) has a convergent series solution for every $|\varphi_1| < \infty$ and $0 < \lambda < 1$.*

Proof. Given $\omega_i = \sum_{k=0}^{i} \varphi_k(\eta)$. We shall show that $\{\omega_i\}$ is a Cauchy sequence in the Banach space $\mathcal{B}$. Consider the Adomian polynomial in its most recent version (see [10]). Let $N(\omega_i) = \tilde{A}_i + \sum_{k=0}^{n-1} \tilde{A}_k$, $i \ge n$ and choose two partial sums ω_n and ω_i. Then,

$$\begin{aligned}
\|\omega_i - \omega_n\| &= \max_{\eta\in\Delta}|\omega_i - \omega_n| \\
&= \max_{\eta\in\Delta}\left|\sum_{k=n+1}^{i}\tilde{\varphi}_k(\eta)\right|, \ i = 1, 2, ... \\
&\le \max_{\eta\in\Delta}\left|\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[C\left(\sum_{k=n+1}^{i}\varphi_{k-1}(\eta)\right)\right]\right] + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[\sum_{k=n+1}^{i}A_{i-1}(\eta)\right]\right]\right| \\
&= \max_{\eta\in\Delta}\left|\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[C\left(\sum_{k=n}^{i-1}\varphi_k(\eta)\right)\right]\right] + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[\sum_{k=n}^{i-1}A_i(\eta)\right]\right]\right| \\
&\le \max_{\eta\in\Delta}\left|\mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[C(\omega_{i-1}) - C(\omega_{n-1})\right]\right] + \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[N(\omega_{i-1}) - N(\omega_{n-1})\right]\right]\right|
\end{aligned}$$

$$\leq C_1 \max_{\eta\in\Delta} \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[\|\omega_{i-1}-\omega_{n-1}\|\right]\right] + C_2 \max_{\eta\in\Delta} \mathcal{J}^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\mathcal{J}\left[\|\omega_{i-1}-\omega_{n-1}\|\right]\right]$$
$$= \frac{(C_1+C_2)\,\eta^{\xi}}{\Gamma(\xi+1)}\|\omega_{i-1}-\omega_{n-1}\|.$$

Now, $\|\omega_i-\omega_n\| \leq \lambda\|\omega_{i-1}-\omega_{n-1}\|$. Choose $i=n+1$, then

$$\|\omega_{n+1}-\omega_n\| \leq \lambda\|\omega_n-\omega_{n-1}\| \leq \lambda^2\|\omega_{n-1}-\omega_{n-2}\| \leq ... \leq \lambda^n\|\omega_1-\omega_0\|.$$

Additionally, using the triangle inequality one can obtain:

$$\begin{aligned}\|\omega_i-\omega_n\| &\leq \|\omega_{n+1}-\omega_n\| + \|\omega_{n+2}-\omega_{n+1}\| + ... + \|\omega_i-\omega_{i-1}\|\\ &\leq \left[\lambda^n+\lambda^{n+1}+...+\lambda^{i-1}\right]\|\omega_1-\omega_0\|\\ &\leq \lambda^n\left[\frac{1-\lambda^{i-n}}{1-\lambda}\right]\|\varphi_1\|.\end{aligned}$$

But, $0<\lambda<1$, then $1-\lambda^{i-n}<1$. So,

$$\|\omega_i-\omega_n\| \leq \frac{\lambda^n}{1-\lambda}\max_{\eta\in\Delta}|\varphi_1|. \tag{4.9}$$

since $\varphi(\eta)$ is bounded, then $|\varphi_1|<\infty$. Thus, $\|\omega_i-\omega_n\|\to 0$ as $n\to\infty$. Consequently, the sequence $\{\omega_i\}$ is a Cauchy sequence in $\mathbb{K}$. Hence, $\varphi(\eta)=\sum_{k=0}^{\infty}\varphi_k(\eta)$ converges. We've established Theorem 4.2. □

Theorem 4.3. (Error Estimate). *The series solution in Eq. (4.8) of Eq. (4.1) has a maximum absolute error:*

$$\max_{\eta\in\Delta}\left|\varphi(\eta)-\sum_{i=0}^{n}\varphi_i(\eta)\right| \leq \frac{\lambda^n}{1-\lambda}\max_{\eta\in\Delta}|\varphi_1|.$$

Proof. Using Eq. (5.9) above, we can arrive at:
$\|\omega_i-\omega_n\| \leq \frac{\lambda^n}{1-\lambda}\max_{\eta\in\Delta}|\varphi_1|$. So as $i\to\infty$, we have $\omega_i\to\varphi(\eta)$.

So,

$$\|\varphi(\eta)-\omega_n\| \leq \frac{\lambda^n}{1-\lambda}\max_{\eta\in\Delta}|\varphi_1(\eta)|.$$

Therefore, the maximum absolute truncation error for Δ is:

$$\max_{\eta\in\,\Delta}\left|\varphi(\eta)-\sum_{i=0}^{n}\varphi_i(\eta)\right| \leq \max_{\eta\in\,\Delta}\frac{\lambda^n}{1-\lambda}|\varphi_1(\eta)| = \frac{\lambda^n}{1-\lambda}\|\varphi_1(\eta)\|.$$

We've established Theorem 4.3. □

5. Application of $\mathcal{FJADM}$ method for NLPDEs

In this section, we present the methodology of the ($\mathcal{FJADM}$) for nonlinear fractional PDEs and then apply it to some applications of fractional differential equations.

Methodology for NLFPDEs. Consider the nonlinear fractional partial differential equation of the form:

$$ {}^{c}D_{\tau}^{\alpha}\nu(\rho,\tau)+\mathcal{K}\,\nu(\rho,\tau)+\mathcal{N}\,\nu(\rho,\tau)=w(\rho,\tau), \quad 0\leq\alpha\leq 1. \tag{5.1} $$

Along with I.C.:

$$ \nu(\rho,0)=h(\rho), \tag{5.2} $$

where ${}^{c}D_{\tau}^{\alpha}\nu(\rho,\tau)$ represents the Caputo fractional derivative of the function $\nu(\rho,\tau)$ and $\mathcal{K}$, $\mathcal{N}$ represent the linear and nonlinear differential operators, respectively, and $w(\rho,\tau)$ is the source term.
Applying the $\gimel$-transform and property 4 to Eq. (5.1), one concludes:

$$ \begin{aligned} &\gimel\left[D_{\tau}^{\alpha}\nu(\rho,\tau)\right]+\gimel\left[\mathcal{K}\nu(\rho,\tau)\right]+\gimel\left[\mathcal{N}\nu(\rho,\tau)\right]=\gimel\left[w(\rho,\tau)\right], \\ &\frac{s^{\alpha}}{u^{\alpha}}\gimel\left[\nu(\rho,\tau)\right]-u\sum_{k=0}^{\iota-1}\left(\frac{s}{u}\right)^{\alpha-k-1}\left[D^{(k)}\nu(\rho,\tau)\right]_{\tau=0}=\gimel\left[w(\rho,\tau)\right]-\gimel\left[\mathcal{K}\nu(\rho,\tau)+\mathcal{N}\nu(\rho,\tau)\right], \\ &\gimel\left[\nu(\rho,\tau)\right]=\frac{u^{\alpha+1}}{s^{\alpha}}\sum_{k=0}^{\iota-1}\left(\frac{s}{u}\right)^{\alpha-k-1}\left[D^{(k)}\nu(\rho,0)\right]+\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[w(\rho,\tau)\right]-\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\mathcal{K}\nu(\rho,\tau)+\mathcal{N}\nu(\rho,\tau)\right]. \end{aligned} \tag{5.3} $$

Substitute Eq. (5.2) into Eq. (5.3), to obtain:

$$ \gimel\left[\nu(\rho,\tau)\right]=\frac{u^{2}}{s}h(\rho)+\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[w(\rho,\tau)\right]-\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\mathcal{K}\nu(\rho,\tau)+\mathcal{N}\nu(\rho,\tau)\right]. \tag{5.4} $$

Applying the $\gimel^{-1}$ to Eq. (5.4), one concludes:

$$ \begin{aligned} \nu(\rho,\tau)&=\gimel^{-1}\left[\frac{u^{2}}{s}h(\rho)\right]+\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[w(\rho,\tau)\right]\right]+\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\mathcal{K}\nu(\rho,\tau)\right]\right] \\ &=\mathcal{G}(\rho,\tau)-\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\mathcal{K}\nu(\rho,\tau)+\mathcal{N}\nu(\rho,\tau)\right]\right]. \end{aligned} \tag{5.5} $$

Note that the non-homogeneous term and the given initial condition represented by $\mathcal{G}(\rho,\tau)$. Assume that we have a solution $\nu(\rho,\tau)$ as follows:

$$ \nu(\rho,\tau)=\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau). \tag{5.6} $$

Also, the nonlinear term can be represented by:

$$ \mathcal{N}\;\nu(\rho,\tau)=\sum_{\iota=0}^{\infty}A_{\iota}, \tag{5.7} $$

where A_{ι} are the Adomian polynomials of $\nu_0,\nu_1,...,\nu_{\iota}$, that can be computed by the following formula:

$$ A_{\iota}=\frac{1}{\iota\,!}\,\frac{d^{\iota}}{d\lambda^{\iota}}\left[F\left(\sum_{i=0}^{\iota}\lambda^{i}\nu_{i}\right)\right]_{\lambda=0}. \tag{5.8} $$

From Eq. (5.8), one can conclude:

$$\begin{aligned}
&A_0 = F(\nu_0),\\
&A_1 = \nu_1\, F'(\nu_0),\\
&A_2 = \nu_2\, F'(\nu_0) + \frac{1}{2!}\nu_1^2\, F''(\nu_0),\\
&A_3 = \nu_3\, F'(\nu_0) + \nu_1\, \nu_2\, F_1''(\nu_0) + \frac{1}{3!}\nu_1^3\, F'''(\nu_0).
\end{aligned} \tag{5.9}$$

We continue in this manner to get the other polynomials and compute all nonlinear terms in the same way.

Using Eq. (5.6), along with Eq. (5.5), one can arrive at:

$$\begin{aligned}
\sum_{\iota=0}^{\infty}\nu_\iota(\rho,\tau) =&\mathcal{G}(\rho,\tau) - ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[\mathcal{K}\sum_{\iota=0}^{\infty}\nu_\iota(\rho,\tau)\right]\right]\\
&- ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[\sum_{\iota=0}^{\infty}A_\iota\right]\right].
\end{aligned} \tag{5.10}$$

Comparing the two sides of Eq. (5.7), one can conclude that:

$$\begin{aligned}
&\nu_0(\rho,\tau) = \mathcal{G}(\rho,\tau),\\
&\nu_1(\rho,\tau) = -ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[\mathcal{K}\,\nu_0(\rho,\tau)\right]\right] - ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[A_0\right]\right],\\
&\nu_2(\rho,\tau) = -ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[\mathcal{K}\,\nu_1(\rho,\tau)\right]\right] - ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[A_1\right]\right],\\
&\nu_3(\rho,\tau) = -ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[\mathcal{K}\,\nu_2(\rho,\tau)\right]\right] - ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[A_2\right]\right].
\end{aligned}$$

The general formula is given as follows:

$$\nu_{\iota+1}(\rho,\tau) = -ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[\mathcal{K}\,\nu_\iota(\rho,\tau)\right]\right] - ℸ^{-1}\left[\frac{u^\alpha}{s^\alpha}ℸ\left[A_\iota\right]\right],\ \iota \geq 0. \tag{5.11}$$

Hence, the exact solution is given by:

$$\nu(\rho,\tau) = \sum_{\iota=0}^{\infty}\nu_\iota(\rho,\tau). \tag{5.12}$$

Example 5.1. Consider the nonlinear time-fractional diffusion equation:

$${}^{c}D_\tau^\alpha(\nu(\rho,\tau)) + \nu_\rho(\rho,\tau)\,\nu(\rho,\tau) - \nu(\rho,\tau)(1-\nu(\rho,\tau)) = 0. \tag{5.13}$$

Accompanied by its I.C.:

$$\nu(\rho,0) = e^{-\rho}. \tag{5.14}$$

Solution. Applying the ℸ-transform to Eq. (5.13), we conclude:

$$ℸ\left[D_\tau^\alpha \nu(\rho,\tau)\right] = ℸ\left[\nu(\rho,\tau) - \nu^2(\rho,\tau) - \nu(\rho,\tau)\nu_\rho(\rho,\tau)\right]. \tag{5.15}$$

Using property 4, and Eq. (5.14), we get:

$$
\begin{aligned}
&\frac{s^{\alpha}}{u^{\alpha}}\daleth\left[\nu(\rho,\tau)\right]-\sum_{k=0}^{\iota-1}(u)\left(\frac{s}{u}\right)^{\alpha-k-1}\left[D_{\tau}^{k}(\nu(\rho,\tau))\right]_{\tau=0}=\daleth\left[\nu(\rho,\tau)-\nu^{2}(\rho,\tau)-\nu(\rho,\tau)\nu_{\rho}(\rho,\tau)\right],\\
&\daleth\left[\nu(\rho,\tau)\right]=\frac{u^{\alpha}}{s^{\alpha}}\left((u)\left(\frac{s}{u}\right)^{\alpha-1}\left[\nu(\rho,0)\right]\right)\\
&\qquad\qquad\quad+\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu(\rho,\tau)-\nu^{2}(\rho,\tau)-\nu(\rho,\tau)\,\nu_{\rho}(\rho,\tau)\right],\\
&\daleth\left[\nu(\rho,\tau)\right]=\frac{u^{2}}{s}e^{-\rho}+\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu(\rho,\tau)-\nu^{2}(\rho,\tau)-\nu(\rho,\tau)\,\nu_{\rho}(\rho,\tau)\right].
\end{aligned}
\tag{5.16}
$$

Applying the $\daleth^{-1}$ to Eq. (5.16), to obtain:

$$
\nu(\rho,\tau)=e^{-\rho}+\daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu(\rho,\tau)-\nu^{2}(\rho,\tau)-\nu(\rho,\tau)\,\nu_{\rho}(\rho,\tau)\right]\right].
\tag{5.17}
$$

Assume that the solution $\nu(\rho,\tau)$ has the form:

$$
\nu(\rho,\tau)=\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau).
\tag{5.18}
$$

Substitute Eq. (5.18) into Eq. (5.17) to arrive at:

$$
\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau)=e^{-\rho}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu(\rho,\tau)-\nu^{2}(\rho,\tau)-\nu(\rho,\tau)\,\nu_{\rho}(\rho,\tau)\right]\right].
\tag{5.19}
$$

Given that

$$
\nu(\rho,\tau)=\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau),\quad \nu^{2}(\rho,\tau)=\sum_{\iota=0}^{\infty}A_{\iota},\quad \nu(\rho,\tau)\nu_{\rho}(\rho,\tau)=\sum_{\iota=0}^{\infty}B_{\iota},
\tag{5.20}
$$

where:

$$
\begin{aligned}
&A_0=(\nu_0)^2, &&B_0=\nu_0\,(\nu_0)_{\rho},\\
&A_1=2\nu_0\,\nu_1, &&B_1=\nu_0\,(\nu_1)_{\rho}+\nu_1(\nu_0)_{\rho},\\
&A_2=2\nu_0\,\nu_2+(\nu_1)^2, &&B_2=\nu_0\,(\nu_2)_{\rho}+(\nu_1)_{\rho}\,\nu_1+\,\nu_2(\nu_0)_{\rho}.
\end{aligned}
$$

Comparing both sides of Eq. (5.19):

$$
\begin{aligned}
\nu_0(\rho,\tau)&=e^{-\rho},\\
\nu_1(\rho,\tau)&=\daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu_0(\rho,\tau)-A_0-B_0\right]\right],\\
\nu_2(\rho,\tau)&=\daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu_1(\rho,\tau)-A_1-B_1\right]\right],\\
\nu_3(\rho,\tau)&=\daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu_2(\rho,\tau)-A_2-B_2\right]\right].
\end{aligned}
$$

Continue in the same manner to arrive at

$$\nu_{\iota+1}(\rho,\tau) = \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_{\iota}(\rho,\tau) - A_{\iota} - B_{\iota}\right]\right], \quad \iota \geq 0. \tag{5.21}$$

Calculating the remaining terms using Eq. (5.21), one can conclude:

$$\begin{aligned}
\nu_1(\rho,\tau) &= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_0(\rho,\tau) - A_0 - B_0\right]\right] \\
&= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[e^{-\rho} - e^{-2\rho} - e^{-\rho}(-e^{-\rho})\right]\right] \\
&= \gimel^{-1}\left[\frac{(e^{-\rho})u^{\alpha+2}}{s^{\alpha+1}}\right] \\
&= e^{-\rho}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

And,

$$\begin{aligned}
\nu_2(\rho,\tau) &= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_1(\rho,\tau) - A_1 - B_1\right]\right] \\
&= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_1 - 2\nu_0\nu_1 - (\nu_0\ (\nu_1)_{\rho} + (\nu_0)_{\rho}\ \nu_1)\right]\right] \\
&= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[e^{-\rho}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right]\right] \\
&= e^{-\rho}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

Also,

$$\begin{aligned}
\nu_3(\rho,\tau) &= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_2(\rho,\tau) - A_2 - B_2\right]\right] \\
&= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_2 - \left(\nu_0\nu_2 + (\nu_1)^2\right) - (\nu_0(\nu_2)_{\rho} + \nu_1(\nu_1)_{\rho} + \nu_2(\nu_0)_{\rho})\right]\right] \\
&= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[e^{-\rho}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right] \\
&= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\left[e^{-\rho}\frac{u^{2\alpha+2}}{s^{2\alpha+1}}\right]\right] \\
&= e^{-\rho}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}, \\
\nu_4(\rho,\tau) &= \gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\nu_3(\rho,\tau) - A_3 - B_3\right]\right] \\
&= e^{-\rho}\frac{\tau^{4\alpha}}{\Gamma(4\alpha+1)}.
\end{aligned}$$

Hence, the exact solution is given by:

$$\nu(\rho,\tau) = \sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau)$$

$$
\begin{aligned}
&= \nu_0(\rho,\tau) + \nu_1(\rho,\tau) + \nu_2(\rho,\tau) + \nu_3(\rho,\tau) + \nu_4(\rho,\tau) + \ldots \\
&= e^{-\rho} + e^{-\rho}\frac{\tau^{\alpha}}{\Gamma(\alpha+1)} + e^{-\rho}\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + e^{-\rho}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \\
&= e^{-\rho}E_{\alpha}(\tau^{\alpha}). \qquad (5.22)
\end{aligned}
$$

Substituting $\alpha = 1$ in Eq. (5.22), results in the exact solution:

$$\nu(\rho,\tau) = e^{\tau-\rho}. \qquad (5.23)$$

Remark 5.1. Figures 1, 2 and 3 demonstrate the numerical results for various values of α. It is evident that the curves are affected differently by the values of α, and all of these curves have no cusps. Thus, this illustrates that as α increases, the approximate solutions are indeed converging to the exact solution, which indicates the effectiveness of the $\mathcal{FJADM}$. Table 1 presents numerical values, to Example 5.1 for different values of α, τ, and ρ.

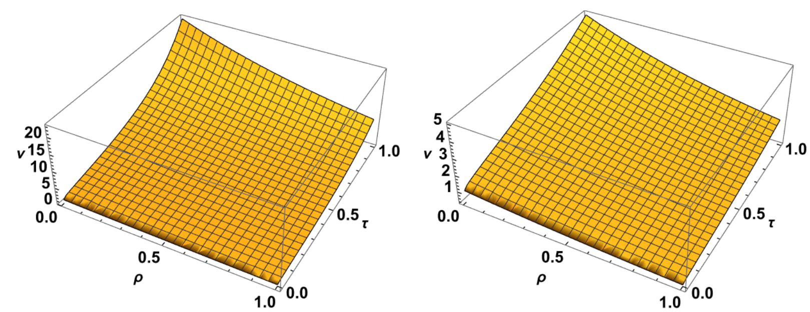

Figure 1. Plot of the exact solution for $\alpha = 0.25,\ \alpha = 0.50$, respectively.

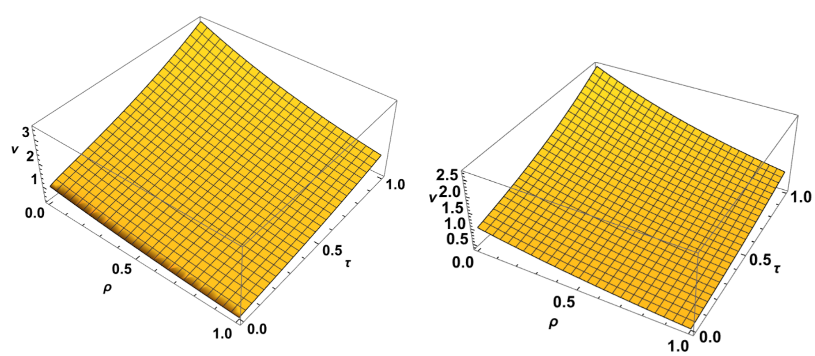

Figure 2. Plot of the exact solution for $\alpha = 0.75,\ \alpha = 1$, respectively.

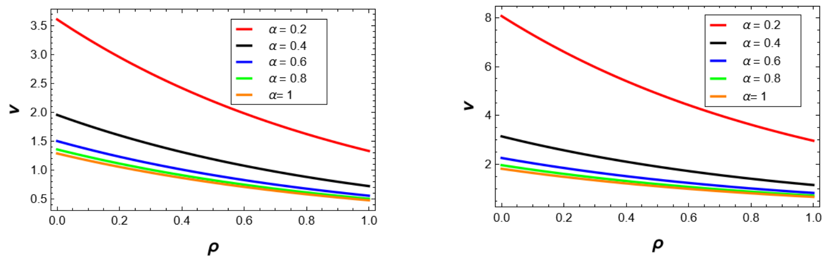

Figure 3. Plot for Example 5.1 for multiple values of α at $\tau = 0.25$ and $\tau = 0.6$.

Table 1. The results obtained for Example 5.1, for multiple values of α.

ρ	τ	$\alpha = 0.10$	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 1$
0.2	0.02	2.66921	0.96767	0.85829	0.83527
	0.05	3.33895	1.07431	0.90370	0.86070
	0.08	3.85391	1.16261	0.94601	0.88692
0.4	0.02	2.18539	0.79226	0.70271	0.68386
	0.05	2.73377	0.87957	0.73981	0.70468
	0.08	3.15532	0.95186	0.77453	0.72614
0.8	0.02	1.46491	0.53107	0.47104	0.45840
	0.05	1.83246	0.58959	0.49596	0.47236
	0.08	2.11507	0.63805	0.51918	0.48675

Example 5.2. Consider the nonlinear time-fractional Fisher's equation:

$$ {}^{c}D_{\tau}^{\alpha}(\nu(\rho,\tau)) = \nu_{\rho\rho}(\rho,\tau) + 6\,\nu(\rho,\tau)(1-\nu(\rho,\tau)), \rho \in \mathbb{R},\ \tau > 0,\ \ 0 < \alpha \leq 1. \tag{5.24}$$

Accompanied by its I.C.:

$$\nu(\rho,0) = \frac{1}{(1+e^{\rho})^2}. \tag{5.25}$$

Solution: Apply the $\daleth$-transform to Eq. (5.24), we get:

$$\daleth\left[{}^{c}D_{\tau}^{\alpha}(\nu(\rho,\tau))\right] = \daleth\left[\nu_{\rho\rho}(\rho,\tau) + 6\nu(\rho,\tau)(1-\nu(\rho,\tau))\right]. \tag{5.26}$$

Using property 4, and Eq. (5.25), we get:

$$\daleth\left[\nu(\rho,\tau)\right] - \sum_{k=0}^{\iota-1}\frac{u^{k+2}}{s^{k+1}}\left[D_{\tau}^{k}(\nu(\rho,0))\right] = \frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu_{\rho\rho}(\rho,\tau) + 6\,\nu(\rho,\tau)(1-\nu(\rho,\tau))\right]. \tag{5.27}$$

Substituting Eq. (5.25) into Eq. (5.27) to arrive at:

$$\daleth\left[\nu(\rho,\tau)\right] = \frac{u^2}{s}\,\frac{1}{(1+e^{\rho})^2} + \frac{u^{\alpha}}{s^{\alpha}}\,\daleth\left[\nu_{\rho\rho}(\rho,\tau) + 6(1-\nu(\rho,\tau))\nu(\rho,\tau)\right]. \tag{5.28}$$

Applying the $\daleth^{-1}$ to Eq. (5.28), one can conclude:

$$\nu(\rho,\tau) = \frac{1}{(1+e^{\rho})^2} + \daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\,\daleth\left[\nu_{\rho\rho}(\rho,\tau) + 6\nu(\rho,\tau) - 6\,\nu^2(\rho,\tau)\right]\right]. \tag{5.29}$$

Assume we have an infinite series solution of the form:

$$\nu(\rho,\tau)=\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau). \tag{5.30}$$

And,

$$\nu^{2}(\rho,\tau)=\sum_{\iota=0}^{\infty}A_{\iota}, \tag{5.31}$$

where,

$$\begin{aligned}
A_0&=\nu_0^2(\rho,\tau),\\
A_1&=2(\nu_0(\rho,\tau))\,(\nu_1(\rho,\tau)),\\
A_2&=2(\nu_0(\rho,\tau))\,(\nu_2(\rho,\tau))+\nu_1^2(\rho,\tau),\\
A_3&=2(\nu_1(\rho,\tau))\,(\nu_2(\rho,\tau))+2(\nu_0(\rho,\tau))\,(\nu_3(\rho,\tau)).
\end{aligned}$$

Substituting Eq. (5.30) into Eq. (5.29), we arrive at

$$\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau)=\frac{1}{(1+e^{\rho})^{2}}+\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[\sum_{\iota=0}^{\infty}(\nu_{\iota})_{\rho\rho}(\rho,\tau)+6\sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau)-6\sum_{\iota=0}^{\infty}A_{\iota}\right]\right]. \tag{5.32}$$

Comparing both sides of Eq. (5.32), to arrive at:

$$\begin{aligned}
\nu_0(\rho,\tau)&=\frac{1}{(1+e^{\rho})^{2}},\\
\nu_1(\rho,\tau)&=\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[(\nu_0)_{\rho\rho}(\rho,\tau)+6\nu_0(\rho,\tau)-6A_0\right]\right],\\
\nu_2(\rho,\tau)&=\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[(\nu_1)_{\rho\rho}(\rho,\tau)+6\nu_1(\rho,\tau)-6A_1\right]\right],\\
\nu_3(\rho,\tau)&=\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[(\nu_2)_{\rho\rho}(\rho,\tau)+6\nu_2(\rho,\tau)-6A_2\right]\right].
\end{aligned}$$

We continue in the same manner to obtain:

$$\begin{aligned}
\nu_{\iota+1}(\rho,\tau)&=\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[(\nu_{\iota})_{\rho\rho}(\rho,\tau)+6\nu_{\iota}(\rho,\tau)-6\sum_{\iota=0}^{\infty}A_{\iota}\right]\right],\quad \iota\geq 0, \qquad (5.33)\\
A_0&=\nu_0^2(\rho,\tau)=\frac{1}{(1+e^{\rho})^{4}}.
\end{aligned}$$

From Eq. (5.33), we conclude:

$$\begin{aligned}
\nu_1(\rho,\tau)&=\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left[(\nu_0)_{\rho\rho}(\rho,\tau)+6\nu_0(\eta,\tau)-6A_0\right]\right]\\
&=\gimel^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\gimel\left(\frac{-2e^{\rho}(-2e^{\rho}+1)}{(1+e^{\rho})^{4}}+\frac{6}{(1+e^{\rho})^{2}}-\frac{6}{(1+e^{\rho})^{4}}\right)\right]\\
&=\gimel^{-1}\left[\frac{u^{\alpha+2}}{s^{\alpha+1}}\left(\frac{10e^{\rho}}{(e^{\rho}+1)^{3}}\right)\right]
\end{aligned}$$

$$= \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{10e^{\rho}}{(e^{\rho}+1)^3}\right).$$

And,

$$\begin{aligned} A_1 &= 2\,\nu_0(\rho,\tau)\,\nu_1(\rho,\tau) \\ &= 2\left(\frac{1}{(1+e^{\rho})^2}\right)\left(\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{10e^{\rho}}{(e^{\rho}+1)^3}\right)\right) \\ &= \frac{2\tau^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{10e^{\rho}}{(e^{\rho}+1)^5}\right). \end{aligned}$$

Thus,

$$\begin{aligned} \nu_2(\rho,\tau) &= \daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[(\nu_1)_{\rho\rho}(\rho,\tau)+6\nu_1(\rho,\tau)-6A_1\right]\right] \\ &= \daleth^{-1}\left[\frac{u^{\alpha+2}}{s^{\alpha+1}}\left(\frac{10e^{\rho}(-7e^{\rho}+4^{2\rho}+1)}{(1+e^{\rho})^5}+\frac{60e^{\rho}}{(1+e^{\rho})^3}-\frac{120e^{\rho}}{(1+e^{\rho})^5}\right)\right] \\ &= \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{100e^{2\rho}-50e^{\rho}}{(e^{\rho}+1)^4}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \nu_3(\rho,\tau) &= \daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[(\nu_2)_{\rho\rho}(\rho,\tau)+6\nu_2(\rho,\tau)-6[A_2]\right]\right] \\ &= \daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{50e^{\rho}(18e^{\rho}+8e^{3\rho}-33e^{2\rho}-1)}{(e^{\rho}+1)^6}\right)\right]\right] \\ &\quad + \daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[6\nu_2(\rho,\tau)-12\,\nu_0(\rho,\tau)\,(\nu_2(\rho,\tau)+6\,\nu_1(\rho,\tau)^2\right]\right] \\ &= \frac{50e^{\rho}\,\tau^{3\alpha}(\Gamma(\alpha+1)^2\,(5e^{2\rho}(4e^{\rho}-3)-6e^{\rho}+5)-12\Gamma(1+2\alpha)e^{\rho})}{(\Gamma(1+\alpha))^2\,\Gamma(3\alpha+1)(1+e^{\rho})^6}. \end{aligned}$$

The exact solution of $\nu(\rho,\tau)$ is given by

$$\begin{aligned} \nu(\rho,\tau) &= \sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau) \\ &= \nu_0(\rho,\tau)+\nu_1(\rho,\tau)+\nu_2(\rho,\tau)+\nu_3(\rho,\tau)+\nu_4(\rho,\tau)+... \\ &= \frac{1}{(1+e^{\rho})^2}+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{10e^{\rho}}{(e^{\rho}+1)^3}\right)+\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\left(\frac{100e^{2\rho}-50e^{\rho}}{(e^{\rho}+1)^4}\right) \\ &\quad + \frac{50e^{\rho}\,\tau^{3\alpha}(\Gamma(\alpha+1))^2\,(5e^{2\rho}(4e^{\rho}-3)-6e^{\rho}+5)-12\Gamma(1+2\alpha)e^{\rho})}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)(1+e^{\rho})^6}+\dots. \end{aligned} \tag{5.34}$$

Substituting $\alpha = 1$ in Eq. (5.34), the exact solution is given by:

$$\begin{aligned} \nu(\rho,\tau) &= \frac{1}{(1+e^{\rho})^2}+\frac{10e^{\rho}\,\tau}{(1+e^{\rho})^3}+\frac{25e^{\rho}(2e^{\rho}-1)\,\tau^2}{(1+e^{\rho})^4} \\ &\quad + \frac{25e^{\rho}(5-30e^{\eta}-15e^{2\eta}+20e^{3\rho})\,\tau^3}{3(1+e^{\rho})^6}+\dots \end{aligned}$$

$$= \frac{1}{\left(1 + e^{\rho - 5\tau}\right)^2}. \tag{5.35}$$

This is the exact solution of Fisher's equation in the standard case, i.e., of integer order.

Remark 5.2. According to Figures 4 and 5, and the numerical results in Table 2, the approximate solutions decrease as both ρ and τ increase. Moreover, the absolute error for Example 5.2 when $\alpha = 0.75$ is presented in Figure 6. Finally, Table 2 presents numerical values, to Example 5.2 for different values of α, τ, and ρ.

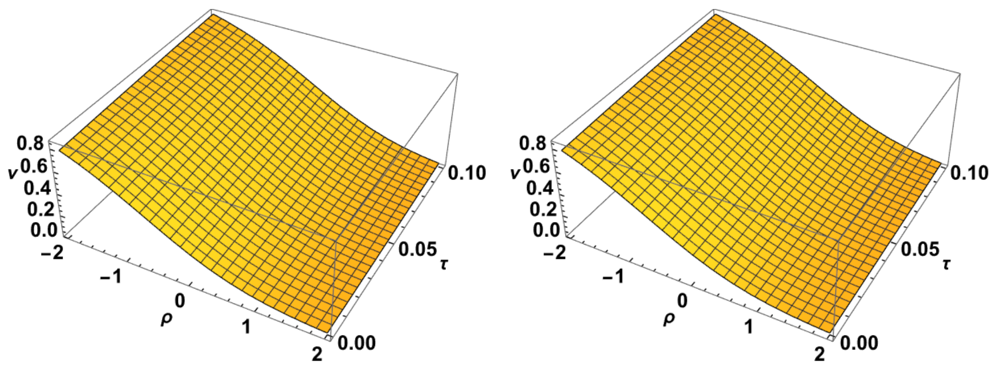

Figure 4. Plot of the exact solution and approximate for $\alpha = 1$.

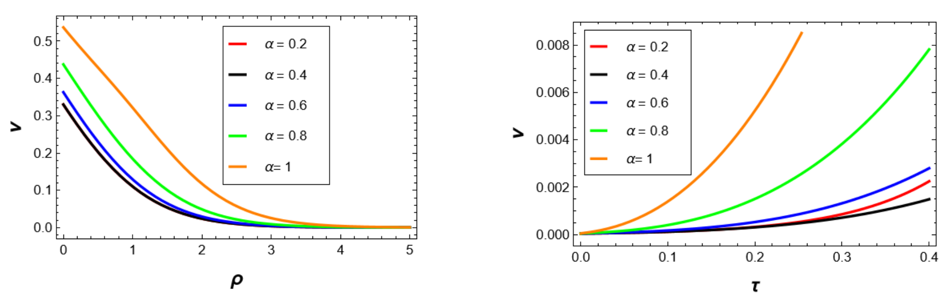

Figure 5. Plot of solutions for different values of α at $\rho = 0.06$ and $\tau = 5$, respectively.

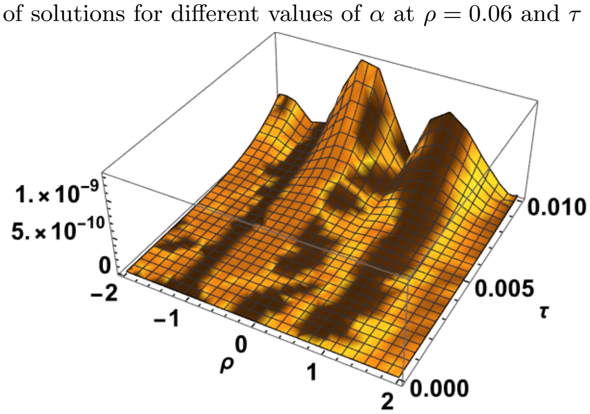

Figure 6. Plot of the absolute error of Example 5.2 for $\alpha = 1$ at ρ and τ.

Table 2. Outcomes of Example 5.2, for various α values with $n = 5$.

ρ	τ	$\alpha = 0.60$	$\alpha = 0.75$	$\alpha = 0.90$	$\alpha = 1$
-5	0.002	0.98827	0.98731	0.98691	0.98679
	0.004	0.98900	0.98774	0.98712	0.98692
	0.008	0.98997	0.98841	0.98751	0.98717
3	0.002	0.002945	0.002483	0.002333	0.002292
	0.004	0.00342	0.00266	0.00240	0.002336
	0.008	0.004347	0.002994	0.002558	0.002427
5	0.002	0.0000594	0.0000496	0.0000465	0.0000457
	0.004	0.0000697	0.00005341	0.0000481	0.0000466
	0.008	0.0000901	0.0000604	0.0000512	0.0000485

Example 5.3. Consider the nonlinear time-fractional Harry Dym equation of the form:

$$ {}^{c}D_{\tau}^{\alpha}\nu(\rho,\tau) = \nu^{3}(\rho,\tau)\ \nu_{\rho\rho\rho}(\rho,\tau),\ \rho \in \mathbb{R},\quad \tau > 0,\quad 0 < \alpha \leq 1. \tag{5.36}$$

Accompanied by its I.C.:

$$\nu(\rho,0) = \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{2/3}, \qquad \text{where } a \text{ and } b \text{ are constants.} \tag{5.37}$$

Solution: Apply the $\daleth$-transform to Eq. (5.36), we get:

$$\daleth\left[{}^{c}D_{\tau}^{\alpha}(\nu(\rho,\tau))\right] = \daleth\left[\nu^{3}(\rho,\tau)\nu_{\rho\rho\rho}(\rho,\tau)\right]. \tag{5.38}$$

Using property 4, and Eq. (5.37), we get:

$$\frac{s^{\alpha}}{u^{\alpha}}\daleth\left[\nu(\rho,\tau)\right] - \sum_{k=0}^{\iota-1}(u)\left(\frac{s}{u}\right)^{\alpha-k-1}\left[D_{\tau}^{k}(\nu(\rho,\tau))\right]_{\tau=0} = \daleth\left[\nu^{3}(\rho,\tau)\nu_{\rho\rho\rho}(\rho,\tau)\right]. \tag{5.39}$$

Substitute Eq. (5.37) into Eq. (5.39) to obtain:

$$\daleth\left[\nu(\rho,\tau)\right] = \frac{u^{2}}{s}\left(a - \frac{3\sqrt{b}}{2}\rho\right)^{2/3} + \frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu^{3}(\rho,\tau)\ \nu_{\rho\rho\rho}(\rho,\tau)\right]. \tag{5.40}$$

Applying $\daleth^{-1}$ to Eq. (5.40), to obtain:

$$\nu(\rho,\tau) = \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{2/3} + \daleth^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\daleth\left[\nu^{3}(\rho,\tau)\ \nu_{\rho\rho\rho}(\rho,\tau)\right]\right]. \tag{5.41}$$

Assume we have an infinite series solution of the form:

$$\nu(\rho,\tau) = \sum_{\iota=0}^{\infty}\nu_{\iota}(\rho,\tau). \tag{5.42}$$

And,

$$\nu_{\rho\rho\rho}^{3}(\rho,\tau) = \sum_{\iota=0}^{\infty}A_{\iota}. \tag{5.43}$$

Substituting Eq. (5.42), into Eq. (5.41), we arrive at:

$$\sum_{\iota=0}^{\infty}\nu_\iota(\rho,\tau)=\left(a-\frac{3\sqrt{b}}{2}\rho\right)^{2/3}+\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth\left[\sum_{\iota=0}^{\infty}A_\iota\right]\right]. \tag{5.44}$$

Comparing both sides of Eq. (5.44) to conclude:

$$\begin{aligned}
\nu_0(\rho,\tau)&=\left(a-\frac{3\sqrt{b}}{2}\rho\right)^{2/3},\\
\nu_1(\rho,\tau)&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_0]\right],\\
\nu_2(\rho,\tau)&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_1]\right],\\
\nu_3(\rho,\tau)&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_2]\right].
\end{aligned}$$

We continue in a similar manner to obtain:

$$\nu_{\iota+1}(\rho,\tau)=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_\iota]\right],\quad \iota\geq 0. \tag{5.45}$$

But,

$$A_0=\nu_0^2(\rho,\tau)=\frac{1}{(1+e^\rho)^4}.$$

From Eq. (5.45), we can conclude:

$$\begin{aligned}
\nu_1(\rho,\tau)&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_0]\right]\\
&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[\nu_0^3\ (\nu_0)^3_{\rho\rho\rho}]\right]\\
&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth\left[-b^{3/2}\left(a-\frac{3\sqrt{b}}{2}\rho\right)^{-1/3}\right]\right]\\
&=\frac{-\tau^\alpha}{\Gamma(\alpha+1)}\left(a-\frac{3\sqrt{b}}{2}\rho\right)^{-1/3}\ b^{3/2}.
\end{aligned}$$

And,

$$\begin{aligned}
\nu_2(\rho,\tau)&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_1]\right]\\
&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[3\ \nu_0^2\ \nu_1\ (\nu_0)_{\rho\rho\rho}+\nu_0^3\ (\nu_1)_{\rho\rho\rho}]\right]\\
&=\daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth\left[-b^3\left(a-\frac{3\sqrt{b}}{2}\rho\right)^{-4/3}\frac{\tau^\alpha}{2\Gamma(\alpha+1)}\right]\right]
\end{aligned}$$

$$= \frac{-b^3\ \tau^{2\alpha}}{2\Gamma(2\alpha+1)} \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-4/3}.$$

Similarly,

$$\begin{aligned}
\nu_3(\rho,\tau) &= \daleth^{-1}\left[\frac{u^\alpha}{s^\alpha}\daleth[A_2]\right] \\
&= \frac{b^{9/2}\ \tau^{3\alpha}}{\Gamma(3\alpha+1)} \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-7/3} \left(\frac{15\Gamma(2\alpha+1)}{4(\Gamma(\alpha+1))^2} - 16\right).
\end{aligned}$$

Thus, the approximate solution is given by:

$$\begin{aligned}
\nu(\rho,\tau) &= \sum_{\iota=0}^{\infty} \nu_\iota(\rho,\tau) \\
&= \nu_0(\rho,\tau) + \nu_1(\rho,\tau) + \nu_2(\rho,\tau) + \nu_3(\rho,\tau) + ... \\
&= \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{2/3} - \frac{\tau^\alpha}{\Gamma(\alpha+1)} \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-1/3} b^{3/2} \\
&\quad - \frac{b^3\ \tau^{2\alpha}}{2\Gamma(2\alpha+1)} \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-4/3} \\
&\quad + b^{9/2}\frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} \left(\frac{15\Gamma(2\alpha+1)}{4(\Gamma(\alpha+1)^2)} - 16\right) \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-7/3} - \dots.
\end{aligned} \tag{5.46}$$

Substituting $\alpha = 1$ in Eq. (5.46), the approximate solution is:

$$\begin{aligned}
\nu(\rho,\tau) &= \sum_{\iota=0}^{\infty} \nu_\iota(\rho,\tau) \\
&= \nu_0(\rho,\tau) + \nu_1(\rho,\tau) + \nu_2(\rho,\tau) + \nu_3(\rho,\tau) + ... \\
&= \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{2/3} - b^{3/2} \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-1/3} \tau \\
&\quad - \frac{b^3}{4}\left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-4/3} \tau^2 - \tau^3\ \frac{17\ b^{9/2}}{12} \left(a - \frac{3\sqrt{b}}{2}\rho\right)^{-7/3} + \cdots \\
&= \left(a - \frac{3\sqrt{b}}{2}(\rho + b\tau)\right)^{2/3}.
\end{aligned} \tag{5.47}$$

This is the exact solution for the Harry Dym equation in the standard case, i.e., integer order derivative. As a result, the approximate solution converges quickly to the exact solution.

Remark 5.3. Figure 7 represents the numerical results for the exact solution provided in [23] and the approximate solution produced by $\mathcal{FJADM}$ for various values of ρ, τ, and α for $a = 4, b = 1$. Figure 7 shows the approximate solution for $\alpha = 1$ and the exact solution. The approximate solutions decreases as both ρ and τ increase in values. Additionally, the numerical results for the exact and approximate solutions produced by $\mathcal{FJADM}$ for various values of ρ,

τ, and α are displayed in Figure 8. Moreover, the absolute error for Example 5.3 when $\alpha = 0.75$ is presented in Figure 9. Finally, Table 3 presents numerical values, to Example 5.3 for multiple values of α, τ, and ρ.

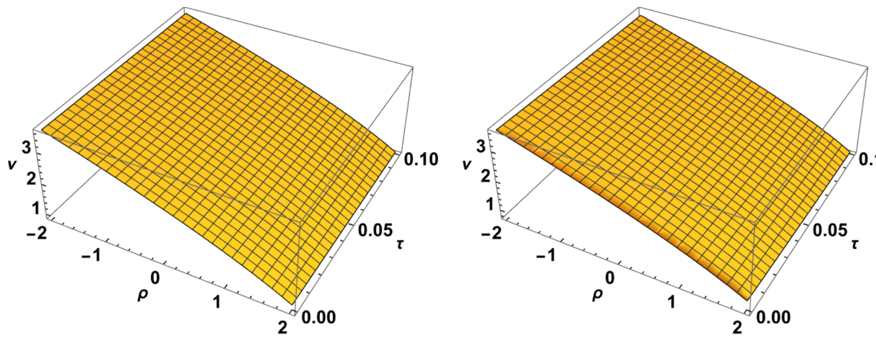

Figure 7. Plot of the exact solution and approximate for $\alpha = 1$, respectively.

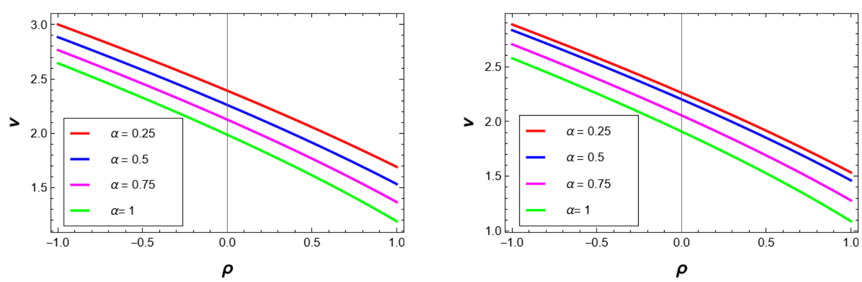

Figure 8. Plot of solutions for different values of α and τ.

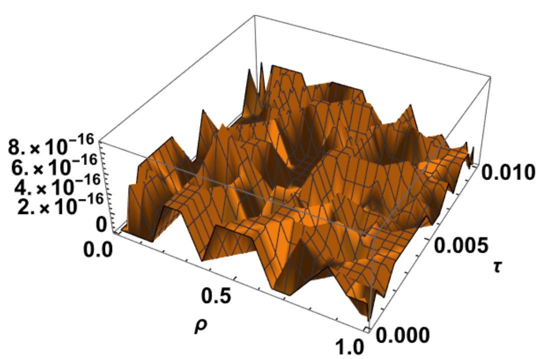

Figure 9. Plot of the absolute error of Example 5.3 for $\alpha = 1$ at ρ and τ.

Table 3. Results obtained for Example 5.3, for multiple values of α with $n = 5$.

ρ	τ	$\alpha = 0.60$	$\alpha = 0.75$	$\alpha = 0.90$	$\alpha = 1$
-5	0.002	0.98827	0.98731	0.98691	0.98679
	0.004	0.98900	0.98774	0.98712	0.98692
	0.008	0.98997	0.98841	0.98751	0.98717
3	0.002	0.002945	0.002483	0.002333	0.002292
	0.004	0.00342	0.00266	0.00240	0.002336
	0.008	0.004347	0.002994	0.002558	0.002427
5	0.002	0.0000594	0.0000496	0.0000465	0.0000457
	0.004	0.0000697	0.00005341	0.0000481	0.0000466
	0.008	0.0000901	0.0000604	0.0000512	0.0000485

6. Concluding remarks

The diffusion, Harry-Dym, and Fisher equations play a very crucial role in science and engineering. In this research work, we used a new method to find approximate and exact solutions for time-fractional differential equations, such as nonlinear time-fractional diffusion, Harry-Dym, and Fisher equations. We successfully gave detailed proofs to the existence, uniqueness, and error estimate applied to nonlinear ODEs of fractional order using the $\mathcal{FJADM}$. The current technique's simplicity and efficiency led us to believe that the method has demonstrated a great degree of improvement over other approaches that exist in the literature. The suggested new technique was used to experiment with various aspects of fractional Caputo derivatives, including their properties. In conclusion, we can apply the method more successfully for solving further fractional systems that are commonly seen in geometry and mathematical physics because this method has proven to be very successful in finding exact and approximate solutions.

Acknowledgment. The reviewers' time and effort in reviewing and improving our work in the present manuscript is greatly appreciated.

Declarations.

Funding. There is currently no funding available for the study project.

Compliance with ethical standards.

Conflicts of interest. The authors declare that they have no conflict of interest concerning the publication of this manuscript.

Availability of data and materials. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Code availability. Not applicable.

Ethics approval. Not applicable.

Consent to participate. participants is aware that they can contact the Jordan University of Science and Technology Ethics Officer if they have any concerns or complaints regarding the way in which the research is or has been conducted.

Consent for publication. Not applicable.

Availability of data and materials. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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Received April 2025; Accepted July 2025; Available online July 2025.