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STABILITY AND BIFURCATION ANALYSIS OF A PREDATOR-PREY SYSTEM UNVEILING THE ROLE OF PREY REFUGE AND COOPERATION WITH FADING MEMORY

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Abstract This paper explores the impact of prey group cooperation on predator-prey dynamics through a novel mathematical model incorporating a Caputo fractional derivative and gestation delay. Solutions' existence, uniqueness, and boundedness of solutions are verified within the framework. The stability analysis indicates that the coexistence equilibrium point is globally stable and that periodic oscillations are caused by the Hopf bifurcation. Our results reveal a critical link between model order, prey refuge rate, and cooperation level. As the model order decreases or the prey refuge rate and cooperation level diminish, the system transitions from unstable to stable behavior. These findings suggest that while strong memory (represented by a higher model order) hinders stable coexistence, weaker memory (lower order) can promote it. This study highlights the significance of incorporating memory effects and prey behavior into predator-prey models for a more comprehensive understanding of population dynamics.

Keywords Caputo derivative, stability, existence and uniqueness, time-delay, Hopf bifurcation.

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1. Introduction

There has been a lot of focus on ecological models in recent years to capture and explain an extensive spectrum of ecological phenomena by examining predator-prey relationships from many angles. Despite this, it is a well-documented ecological phenomenon that prey populations utilize varied defense tactics to evade predators [4,6]. With its prevalence in the literature [1,28]

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and recent prominence as a problem in the theoretical ecosystem [23, 46, 49], refuge stands out among these concepts. Preserving a consistent amount of prey from predation is an essential characteristic of a refuge. The predator-prey interactions may be stabilized by the behavior of prey concealment, which also protects prey refuges from predation. A refuge is a concept in ecology and biology that describes how an organism might avoid predators by covering an area that is either too remote or too difficult to reach. One key component of predator-prey interactions is the presence of safe havens for both species [24, 35, 36, 42]. Numerous empirical and theoretical studies concentrated on the consequences of shelter for prey [7, 8, 13, 20, 31].

Since it was shown many years ago that refuge plays a stabilizing role in predator-prey systems, Holling type II nonlinearities have been used to analyze different features of predator-prey systems, including refuge [16, 19]. Intra-specific rivalry within predators has a significant influence on the predator-prey scheme, according to the studies presented by Mapunda et al. [22] and Kumar et al. [19]. Saha and Samanta demonstrated the influence of predator cooperation on the predator-prey scheme by investigating cooperative hunting techniques in their research [37]. Sarwadi et al. [40, 41] carried out an analysis of the impact of interspecific rivalry within the predator populace in a three-species structure. Rihan et al. [33] examined the stabilizing impacts of shelter with cooperation within prey species and how they can evade the annihilation of prey. Because they will always have access to resources, predators are meant to live with one another until the population becomes too big for any individual to remain in the refuge.

The time lag is present in almost all biological processes. When compared to non-delayed systems, delayed ecological systems provide an additional accurate representation in mathematical modeling [5, 9, 38, 39]. The conversion of energy via predation in a predator-prey framework is not instantaneous; there is a gestation delay. Ma and Wang's [21] investigation explored the changing character of a predator-prey framework in a complex environment that was induced by a delay. Tripathi et al. [44] employed a density-dependent predator-prey framework, which includes the Beddington–DeAngelis form functional response and the discrete form gestation delay to examine the function of shelter for prey and the extent of concurrent interference within predators. Additionally, they have noticed the fact that the system experiences a Hopf bifurcation and loses stability whenever the latency parameter exceeds a particular level. Research on the effects of two separate time latencies in a predator-prey framework that depends on the ratio of the two latencies proved that they satisfied the conditions regarding the presence of Hopf bifurcation [9]. Different delay parameter values have been shown by Wang and Jiang [47] to either enhance or eliminate the chaotic behavior of the predator-prey framework through the predator's dormant state. Beretta and Kuang [5] established a geometric requirement within a delay differential equation to change stability. The aforementioned results emphasize the significance of time delay in reliability evaluation.

The idea of a fractional differential equation is relatively new and belongs to the field of abstract mathematics. As a modeling tool, fractional-order differential equations provide several benefits for predator-prey systems. The presence of time-based memory is a consequence of non-integer order frameworks, though it is non-existent in classical order frameworks. The non-integer order derivative is connected with the full-time province of biological development, whereas the integer order derivative signifies a specific change or characteristic at a given moment. To ensure accuracy about the order of differentiation ψ, the integer-order framework must be accurately transformed into a non-integer-order framework. However, the behavior of the solutions may vary significantly if ψ is changed even a little [29]. Although integer-order differential equations (IDEs) are technically ineffective in representing complicated biological processes, including non-linear behavior and long-term memory, fractional-order differential equations (FDEs) can do so. In recent years, there has been a shortage of literature on fractional order delay [17, 34, 43]. These studies inspire us to think about how computational biological models could benefit from the techniques of fractional differential calculus. Our goal in writing this article is to build a prey-predator model that uses the Caputo fractional derivative with delay. The capacity of this non-integer order derivative to retain and utilize the essential system information from the beginning to the desired time is a fundamental factor for this decision. Computational biology research involving dynamic systems may greatly benefit from this practical aspect. The current study aims to address the research void by investigating the consequences of non-integer order and cooperation among the prey population on the predator-prey system when they escape from predators through the refuge. The results indicate that the system dynamics attain stability rapidly, given that the prey individuals compete with one another for sanctuary in the reserve surroundings. Recently, many researchers investigated the complex dynamics of different types of prey-predator interaction involving multiple functional responses [2,3,14,25,45].

The study's motivation:

Biological relevance: Predator-prey models have been extensively investigated to comprehend complicated ecological dynamics. Memory effects, time delays, and environmental variability are commonly missed by standard integer-order models.

Realistic dynamics and fading memory: Natural systems are known to have memory effects, in which the system's current state is impacted by previous interactions and ambient elements in addition to the current circumstances. Classical models oversimplify ecological systems by assuming Markovian processes (no memory). The system's reliance on past states is taken into account by fractional-order derivatives, which better reflect memory effects.

The function of cooperation and prey refuge: Prey refuges mitigate predation pressure by offering prey concealment areas. This can stabilize the population and affect the intensity of predator-prey interaction. The competitive equilibrium is altered, and population dynamics are modified as cooperation among prey individuals enhances survival and growth rates.

The integration of memory effects: Fractional models provide a better representation of historical population densities, which are necessary for ecological processes like predator hunting success and prey avoidance.

Non-local dynamics: Models of integer order characterize local dynamics in which the state change relies solely on the current state. Fractional models illustrate non-local dynamics in which the state change is influenced by a weighted average of all previous states, effectively simulating authentic biological memory.

Fractional-order derivatives reflect biological complexity: As in nature, feedback processes in biology and the environment are best modeled using fractional derivatives, which show the cumulative effects of small, continuous changes over time.

The structure of the work is as outlined below: The mathematical framework is developed in Section 2 by employing (i) the coefficient of cooperation within the prey population that is attempting to escape threat from predators by seeking refuge, (ii) gestational delay, and (iii) fractional order derivative. A short description of the mathematical prerequisites of the non-integer order differential equation is also included in the same section. In Section 3, the existence, uniqueness, and boundedness of the delayed system are examined, while in Section 4, the local behavior of all equilibrium points and Hopf bifurcation criteria are covered. In the same part, the co-existing equilibria's global stability is also examined. In Section 5, comprehensive Matlab numerical simulations confirming the theoretical findings are presented. A detailed discussion of the findings is carried out in Section 6. Lastly, in Section 7, we provide a biological interpretation of our findings.

2. Mathematical model formulation

Here, we suggest the Lotka-Volterra predator-prey framework, which includes cooperation and proportionate prey refuge. The subsequent presumptions will be made with respect to this system as follows:

In the model, there are two populations: the amount of the prey populace represents $N(t)$, while $P(t)$ reflects the populace size of the predator for any given instant t. The prey's inherent degree of advancement and carrying capacity are denoted by r, K correspondingly. Predation pace is indicated by a, conversion factor (c) indicates the effectiveness of predators for each caught prey, and γ reflects the predator mortality rate without the presence of prey.

$\frac{a}{b}$ corresponds to the highest prey intake rate per unit of time, and $\frac{1}{b}$ indicates the half-saturation constant.

The predator's approach to prey is represented using the expression $\frac{aN}{1+bN}$. Here, the predator employs a Holling type II functional response as its reaction.

Consider the proportional shelter of prey with m against predators, where $0 < m < 1$ and m remains constant. The amount of Prey's $mN(t)$ resides inside the sanctuary environment, while its $(1-m)N(t)$ fraction is in exposed environments, making it available to predators. θ is the cooperation measure of the prey population, and they are attempting to escape the threat of predation by using the refuge.

By employing the earlier presumptions, it is possible to establish the following mathematical framework.

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right) - \frac{a(1-m)NP}{1+b(1-m)N} + \theta N^2, \\ \frac{dP}{dt} &= -\gamma P + \frac{ca(1-m)NP}{1+b(1-m)N}, \end{aligned} \tag{2.1}$$

under the initial biological circumstances $N(0)$ and $P(0)$ are both positive. In most ecosystems, a time-lapse τ that is required for assimilation and gestation after the consumption of prey controls the predator's reproductive action. Model (2.1) can be formulated as follows by combining the gestation delay:

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{K}\right) - \frac{a(1-m)NP}{1+b(1-m)N} + \theta N^2, \\ \frac{dP}{dt} &= -\gamma P + \frac{ca(1-m)N(t-\tau)P(t-\tau)}{1+b(1-m)N(t-\tau)}. \end{aligned} \tag{2.2}$$

To represent the influence of a species' memory from its life cycle, the aforementioned integer order scheme (2.2) is inadequate. Prey anti-predator actions cannot be constant throughout an ecological process; rather, they should be species-specific and influenced by their time memory, with resulting repercussions being carefully considered. A non-integer order differential equation in the Caputo sense has been derived from the aforementioned model system (2.2) as follows:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\psi} N(t) &= \tilde{r}N\left(1-\frac{N}{K}\right) - \frac{\tilde{a}(1-\tilde{m})NP}{1+\tilde{b}(1-\tilde{m})N} + \tilde{\theta}N^2, \\ {}_{t_0}^{C}D_t^{\psi} P(t) &= -\tilde{\gamma}P + \frac{\tilde{c}\tilde{a}(1-\tilde{m})N(t-\tau)P(t-\tau)}{1+\tilde{b}(1-\tilde{m})N(t-\tau)}, \end{aligned} \tag{2.3}$$

where ${}_{t_0}^{C}D_t^{\psi}$ stands for the Caputo fractional (non-integer) order derivative of order ψ $(0 < \psi < 1)$.

The initial challenge faced when transitioning from classic rates defined by first derivatives to fractional derivatives is that the units of the rate constants, utilized in linear models, shift from "per time" to "per time to the power of Psi", where Psi represents the order of the derivative indicated on the left-hand side of the fractional differential equation. A clear issue emerges when a single rate constant is present in multiple FDEs, each of varying order, as this results in the same parameter having different units across different sections of the system. This inconsistency indicates a fundamental issue with the system setup. It is appealing to address the issue by eliminating the units entirely, as Popovic et al. [30] have implemented through the normalization of these parameters; however, this adjustment alone fails to resolve the underlying problem. In this context, it is important to exercise caution when reporting mean values of estimates for rate constants expressed in units $(time)^{-\psi}$. In instances where individual values lack uniform units, as observed with the rate constants presented in Table 1 of [30], it becomes impossible to establish a mean value. To address the situation, we will make changes to the aforementioned model (2.3) in the following manner:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\psi}N\left(t\right) &= \tilde{r}^{\psi}N\left(1-\frac{N}{K}\right)-\frac{\tilde{a}\left(1-\tilde{m}^{\psi}\right)NP}{1+\tilde{b}^{\psi}\left(1-\tilde{m}^{\psi}\right)N}+\tilde{\theta}^{\psi}N^2,\\ {}_{t_0}^{C}D_t^{\psi}P\left(t\right) &= -\tilde{\gamma}^{\psi}P+\frac{\tilde{c}^{\psi}\tilde{a}\left(1-\tilde{m}^{\psi}\right)N\left(t-\tau\right)P\left(t-\tau\right)}{1+\tilde{b}^{\psi}\left(1-\tilde{m}^{\psi}\right)N\left(t-\tau\right)}. \end{aligned} \tag{2.4}$$

To facilitate the investigation, we will redefine the parameters as follows:

$$\tilde{r}^{\psi}=r,\ \tilde{a}^{\psi}=a,\ \tilde{m}^{\psi}=m,\ \tilde{\theta}^{\psi}=\theta,\ \tilde{\gamma}^{\psi}=\gamma,\ \tilde{c}^{\psi}=c,\ \tilde{b}^{\psi}=b.$$

Consequently, the revised system (2.4) can be finalized in the following manner:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\psi}N\left(t\right) &= rN\left(1-\frac{N}{K}\right)-\frac{a\left(1-m\right)NP}{1+b\left(1-m\right)N}+\theta N^2,\\ {}_{t_0}^{C}D_t^{\psi}P\left(t\right) &= -\gamma P+\frac{c\,a\left(1-m\right)N\left(t-\tau\right)P\left(t-\tau\right)}{1+b\left(1-m\right)N\left(t-\tau\right)}. \end{aligned} \tag{2.5}$$

The biological explanations of all parameters have been explored under the assumptions in model (2.1), with beginning settings $N\left(0\right)>0$ along with $P\left(t\right)=\phi\left(t\right)>0\left(t\in\left[-\tau,0\right]\right)$, where $\phi\left(t\right)$ denotes a continuous function. The impact of the time latency τ on the changing behavior of the framework (2.5) is the focus of this study.

2.1. Preliminaries of fractional calculus

Within this piece of work, we have reviewed some significant hypotheses that will aid us in our further examination. The foundation of our model relies on Caputo fractional differential equations. We revisited fundamental concepts and established stability requirements for equilibrium, which were later used in this study.

Definition 2.1. ([18,29]) The fractional derivative a continuous function $f\left(x\right)\in C^n\left(\left[t_0,+\infty\right),\mathbb{R}\right)$ with order $\psi>0$is defined as ${}_{t_0}^{C}D_t^{\psi}f\left(t\right)=\frac{1}{\Gamma(n-\psi)}\int_{t_0}^{t}\frac{f^n(\zeta)}{(t-\zeta)^{\psi-n+1}}d\zeta$, in which $\Gamma\left(.\right)$ is the gamma function, $n\in\mathbb{N}$, $\psi\in\left(n-1,n\right)$ and $t\geq t_0$. In the situation when $n=1$, the Caputo derivative is expressed in the way of ${}_{t_0}^{C}D_t^{\psi}f\left(t\right)=\frac{1}{\Gamma(1-\psi)}\int_{t_0}^{t}\frac{f'(\zeta)}{(t-\zeta)^{\psi}}d\zeta,\ 0<\psi<1$. The term ψ is commonly known as the non-integer derivative order.

Definition 2.2. ([18, 29]) Suppose $f\left(t\right)$ is n times continuously differentiable function and the Caputo derivative of $f\left(t\right)$, i.e., ${}_{t_0}^{C}D_t^{\psi}f\left(t\right)$ is piecewise continuous on $\left[t_0,\infty\right)$ where $\psi>0$ and $n-1<\psi<n\in\mathbb{N}$. Then, the Laplace transform of Caputo derivative is given by $L\left\{{}_{t_0}^{C}D_t^{\psi}f\left(t\right)\right\}=p^{\psi}F\left(p\right)-\sum\limits_{i=0}^{n-1}p^{\psi-i-1}f^{(i)}\left(t_0\right)$, in which $F\left(p\right)=L\left\{f\left(t\right)\right\}$. It is important to note that in order for the improper integral associated with the Laplace transformation to converge, real component of the complex number p has to be higher than or identical to zero.

Lemma 2.1. ([48]) *Assume the n-dimensional non-integer order framework using delay is* ${}_{t_0}^{C}D_t^{\psi}x_j\left(t\right)=f_j\left(x_1\left(t\right),...,x_n\left(t\right);\tau\right),j=1,2,\ldots,n$, *in which* $0<\psi<1$ *and time delay* $\tau\geq0$. *Under some circumstances, this system exhibits a Hopf bifurcation at an equilibrium*

point $x^* = (x_1^*, ..., x_n^*)$ *at* $\tau = 0$*:*
(i) In the linearized system with $\tau = 0$*, every latent value* λ_j *of the coefficient matrix A fulfils* $|\arg(\lambda_j)| > \frac{\psi\pi}{2}$.
(ii) The latent equation of the linearized system contains a couple of entirely fictitious roots $\pm\, i\omega_0$ *in the case of* $\tau = \tau_0$.
(iii) $\mathrm{Re}\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau=\tau_0,\,\omega=\omega_0)} > 0$*, in which* $\mathrm{Re}[.]$ *indicates the real component of the complex number.*

3. Existence, uniqueness, and boundedness

Here, an analysis is carried out to determine the presence and distinctiveness of a solution to the framework (2.5).

$$\begin{aligned} {}_{t_0}^{C}D_t^{\psi} N(t) &= rN\left(1-\frac{N}{K}\right) - \frac{a(1-m)NP}{1+b(1-m)N} + \theta N^2, \\ {}_{t_0}^{C}D_t^{\psi} P(t) &= -\gamma P + \frac{ca(1-m)N(t-\tau)P(t-\tau)}{1+b(1-m)N(t-\tau)},\ t \in [t_0,\, t_0+H]. \end{aligned} \tag{3.1}$$

$(N(t),\, P(t)) = \eta(t) := (\eta_1(t),\, \eta_2(t)),\ t \in [t_0-\tau,\, t_0]$, where $0 < \psi \le 1,\ t_0 \ge 0,\ \tau > 0,\ H > 0$, and the initial value function $\eta(t) \in C\left([t_0-\tau,\, t_0],\, R^2\right)$.

Define
$$V(t) = (N(t),\, P(t)),\ g(V(t)) = (g_1(V(t)), g_2(V(t))),$$
in which
$$\begin{aligned} g_1(V(t)) &= rN\left(1-\frac{N}{K}\right) - \frac{a(1-m)NP}{1+b(1-m)N} + \theta N^2, \\ g_2(V(t)) &= -\gamma P + \frac{ca(1-m)N(t-\tau)P(t-\tau)}{1+b(1-m)N(t-\tau)}. \end{aligned} \tag{3.2}$$

For $V = (N,\, P) \in \mathbb{R}^2$, take the norm $\|V\| = |N| + |P|$. Take $\sigma = C\left([t_0-\tau,\, t_0+H],\, \mathbb{R}^2\right)$, and define the norm$\|V\|_\sigma = \max_{t\in[t_0-\tau,\, t_0+H]} \|V(t)\|$ for $V(t) = (N(t),\, P(t)) \in \sigma$.

Consider
$$\begin{aligned} W = \{&V \in \sigma : V(t) = \eta(t) \text{ for } t \in [t_0-\tau,\, t_0], \\ &\text{and} \max\nolimits_{t\in[t_0,\, t_0+H]} \|V(t) - \eta(t_0)\| \le D\}\ (D > 0). \end{aligned}$$

Obviously, for any $V(t) \in W$, we have $\|V\|_\sigma \le M := \max\left\{\max_{t\in[t_0-\tau,\, t_0]} \|\eta(t)\|,\, \|\eta(t_0)\| + D\right\}$.

Therefore, for any
$$V(t) = (N(t),\, P(t)),\quad \overline{V}(t) = \left(\overline{N}(t),\, \overline{P}(t)\right) \in W,\ t \in [t_0,\ t_0+H].$$

We have
$$\begin{aligned} &\left\|g(V(t)) - g\left(\overline{V}(t)\right)\right\| \\ =&\left|g_1(V(t)) - g_1\left(\overline{V}(t)\right)\right| + \left|g_2(V(t)) - g_2\left(\overline{V}(t)\right)\right| \\ =&\left|rN(t)\left(1-\frac{N(t)}{K}\right) - \frac{a(1-m)N(t)P(t)}{1+b(1-m)N(t)} + \theta N(t)^2 - r\bar{N}(t)\left(1-\frac{\bar{N}(t)}{K}\right)\right. \\ &\left.+\frac{a(1-m)\bar{N}(t)\bar{P}(t)}{1+b(1-m)\bar{N}(t)} - \theta\bar{N}(t)^2\right| \end{aligned}$$

$$\begin{aligned}
&+\left|-\gamma P(t)+\frac{ca(1-m)N(t-\tau)P(t-\tau)}{1+b(1-m)N(t-\tau)}+\gamma\bar{P}(t)-\frac{ca(1-m)\bar{N}(t-\tau)\bar{P}(t-\tau)}{1+b(1-m)\bar{N}(t-\tau)}\right|\\
\leq&\left(r+\frac{2rM}{K}+2\theta M+a(1-m)MM_1^2\right)\left|\left(N(t)-\bar{N}(t)\right)\right|\\
&+(\gamma+a(1-m)MM_1)\left|\left(P(t)-\bar{P}(t)\right)\right|+ca(1-m)MM_1^2\left|N(t-\tau)-\bar{N}(t-\tau)\right|\\
&+cab(1-m)^2M^2M_1^2\left|P(t-\tau)-\bar{P}(t-\tau)\right|\\
\leq& L\left(\left\|V(t)-\overline{V}(t)\right\|+\left\|V(t-\tau)-\overline{V}(t-\tau)\right\|\right)
\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}
L:=\max\Bigg\{&\left(r+\frac{2rM}{K}+2\theta M+a(1-m)MM_1^2\right),(\gamma+a(1-m)MM_1),\\
&ca(1-m)MM_1^2,\ cab(1-m)^2M^2M_1^2\Bigg\}
\end{aligned}$$

and $M_1=\frac{1}{(1+b(1-m)M)}$.

In the same way, for any $V(t)\in W,\ t\in[t_0,\ t_0+H]$, we have

$$\begin{aligned}
&\|g(V(t))\|\\
=&|g_1(V(t))|+|g_2(V(t))|\\
=&\left|rN\left(1-\frac{N}{K}\right)-\frac{a(1-m)NP}{1+b(1-m)N}+\theta N^2\right|+\left|-\gamma P+\frac{ca(1-m)N(t-\tau)P(t-\tau)}{1+b(1-m)N(t-\tau)}\right|\\
\leq&\left(r+\frac{rM}{K}+(\theta+(1+c)M_1)M\right)|N(t)|+\gamma|P(t)|\\
\leq& L\|V(t)\|.
\end{aligned}\tag{3.4}$$

Subsequently, employing the non-integer integral operator to system (3.1) results in a reconfiguration of the system into an analogous Volterra equation of type two:

$$\begin{aligned}
&V(t)=\eta(t_0)+\frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}g(V(\vartheta))\,d\vartheta,\ t\in[t_0,\ t_0+H],\\
&V(t)=\eta(t)=(\eta_1(t),\eta_2(t)),\ t\in[t_0-\tau,\ t_0].
\end{aligned}$$

Define the operator $\Upsilon:\ W\to W$, in a manner that ensures the following:

$$\begin{aligned}
&\Upsilon V(t):=\eta(t_0)+\frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}g(V(\vartheta))\,d\vartheta,\ t\in[t_0,\ t_0+H],\\
&\Upsilon V(t):=\eta(t)=(\eta_1(t),\eta_2(t)),\ t\in[t_0-\tau,\ t_0].
\end{aligned}\tag{3.5}$$

Then Υ possesses only one fixed point in W, it indicates that problem (3.1) encounters at most a single solution.

By (3.3) and (3.5), for any

$$V(t)=(N(t),P(t)),\quad \overline{V}(t)=\left(\overline{N}(t),\overline{P}(t)\right)\in W,\ t\in[t_0,\ t_0+H],$$

we have

$$\begin{aligned}
&\left\|\Upsilon V(t)-\Upsilon\overline{V}(t)\right\|\\
\leq&\frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\left\|g(V(\vartheta))-g\left(\overline{V}(\vartheta)\right)\right\|\,d\vartheta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\left(\left\|V(\vartheta)-\overline{V}(\vartheta)\right\|+\left\|V(\vartheta-\tau)-\overline{V}(\vartheta-\tau)\right\|\right)d\vartheta\\
&\leq \frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\Bigg(\max_{\vartheta\in[t_0,\,t_0+H]}\left\|V(\vartheta)-\overline{V}(\vartheta)\right\|\\
&\quad+\max\left\{\max_{\vartheta\in[t_0-\tau,\,t_0]}\left\|V(\vartheta)-\overline{V}(\vartheta)\right\|,\max_{\vartheta\in[t_0,\,t_0+H]}\left\|V(\vartheta)-\overline{V}(\vartheta)\right\|\right\}\Bigg)d\vartheta\\
&\leq \frac{2L}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\left(\max_{\vartheta\in[t_0,\,t_0+H]}\left\|V(\vartheta)-\overline{V}(\vartheta)\right\|\right)d\vartheta\\
&\leq \frac{2LH^{\psi}}{\Gamma(\psi+1)}\left\|V-\overline{V}\right\|_{\sigma}.
\end{aligned}$$

Thus, $\left\|\Upsilon V(.)-\Upsilon\overline{V}(.)\right\|_{\sigma}\leq\frac{2LH^{\psi}}{\Gamma(\psi+1)}\left\|V-\overline{V}\right\|_{\sigma}$, it suggests that Υ is a contraction operator if $H<\left(\frac{\Gamma(\psi+1)}{2L}\right)^{\frac{1}{\psi}}$. Given that$V(t)\in W,\ t\in[t_0,\,t_0+H]$, we may deduce from equations (3.3) and (3.4), that

$$\begin{aligned}
\left\|\Upsilon(V(t)-\eta(t_0))\right\| &\leq \frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\left\|V(\vartheta)\right\|d\vartheta\\
&\leq \frac{1}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\left\|g(V(\vartheta))\right\|d\vartheta\\
&\leq \frac{L}{\Gamma(\psi)}\int_{t_0}^{t}(t-\vartheta)^{\psi-1}\max_{\vartheta\in[t_0,\,t_0+H]}\left\|V(\vartheta)\right\|d\vartheta\\
&\leq \frac{LH^{\psi}}{\Gamma(\psi+1)}\max_{\vartheta\in[t_0,\,t_0+H]}\left\|V(\vartheta)\right\|\\
&\leq \frac{LH^{\psi}M}{\Gamma(\psi+1)}. \qquad (3.6)
\end{aligned}$$

When $H<\left(\frac{\Gamma(\psi+1)}{LM}\right)^{\frac{1}{\psi}}$, therefore (3.6) states that $\max\limits_{\vartheta\in[t_0,\,t_0+H]}\left\|\Upsilon(V(t)-\eta(t_0))\right\|\leq D$, which means that$\Upsilon(V(t))\in W$, for any $V(t)\in W$.

Υ holds a unique fixed point in W if $H<\min\left\{\left(\frac{\Gamma(\psi+1)D}{LM}\right)^{1/\psi},\left(\frac{\Gamma(\psi+1)}{2L}\right)^{1/\psi}\right\}$, as shown by the Banach contraction principle. On the basis of the above study, it is possible to deduce the subsequent result.

Theorem 3.1. *If* $H<\min\left\{\left(\frac{\Gamma(\psi+1)D}{LM}\right)^{1/\psi},\left(\frac{\Gamma(\psi+1)}{2L}\right)^{1/\psi}\right\}$, *then there is only one solution to the initial value problem* (3.1).

4. Stability analysis and Hopf bifurcation

4.1. Equilibrium points

In this part, we'll look at the parametric criteria for systems to have equilibrium points, together with the examination of their resilience and the occurrence of bifurcation near these equilibrium points. A framework (2.5) usually contains a prey-predator-free equilibrium point $E_0\,(0,\,0)$ and predator-free equilibrium point $E_1\left(\frac{Kr}{r-K\theta},\,0\right)$. Due to the fact that an increased amount of mortality is detrimental to every population and leads to its extinction, which is biologically intuitive, it is presumed that $\theta < \dfrac{r}{K}$ for the duration of this manuscript. The co-existence equilibrium point $E_2\left(N^*,\, P^*\right)$ in the interior of the first quadrant concurrently fulfils the ensuing two non-trivial nullclines for prey and predator.

$$\begin{aligned} &N\left[r\left(1-\frac{N}{K}\right)-\frac{a\,(1-m)\,P}{1+b\,(1-m)\,N}+\theta N\right]=0,\\ &P\left[-\gamma+\frac{c\,a\,(1-m)\,N}{1+b\,(1-m)\,N}\right]=0. \end{aligned} \tag{4.1}$$

By solving the second equation, we see that $N = \frac{\gamma}{(1-m)(c\,a-\gamma b)}$. Substituting the value of N into the first equation yields $P = \frac{ca[rK(1-m)(ca-\gamma b)-\gamma(r-K\theta)]}{K(1-m)^2(ca-\gamma b)^2}$. These are positive if $c\,a > \gamma b$ and $m < 1$.

4.2. Stability analysis

In view of the gestation time lag τ, the robustness criteria for the delayed system (2.5) are modified. We have found Hopf bifurcations within a particular interval of τ, and solutions are asymptotically stable outside of this range. Analyzing the delayed non-integer order framework (2.5) requires the following lemma.

Lemma 4.1. [11] *The n- dimensional linear non-integer order delayed frame work in the Caputo sense is as follows:*

$$\begin{aligned} &{}_{t_0}^{C}D_t^{\psi}(u_1(t)) = l_{11}u_1(t-\tau_{11}) + l_{12}u_2(t-\tau_{12}) + \ldots + l_{1n}u_n(t-\tau_{1n}),\\ &{}_{t_0}^{C}D_t^{\psi}(u_2(t)) = l_{21}u_1(t-\tau_{21}) + l_{22}u_2(t-\tau_{22}) + \ldots + l_{2n}u_n(t-\tau_{2n}),\\ &\ldots\ldots\\ &{}_{t_0}^{C}D_t^{\psi}(u_n(t)) = l_{n1}u_1(t-\tau_{n1}) + l_{n2}u_2(t-\tau_{n2}) + \ldots + l_{nn}u_n(t-\tau_{nn}). \end{aligned} \tag{4.2}$$

Where the order of derivative $\psi \in (0,\,1)$, the initial settings $u_i = \chi_i\,(t)$ that are taken into account for $t_0 - \tau \leq t \leq t_0$ with $i,\, j = 1,\, 2,\, \ldots,\, n,\ t_0 \geq 0$ and $\tau = \max_{i,\,j}\,(\tau_{ij})$. In system (4.2), $\delta = (\tau_{ij}) \in (R^+)_{n\times n}$symbolizestime-delay matrix, $E = (l_{ij})_{n\times n}$coefficient matrix, $u_i\,(t)\,,\ u_i\,(t-\tau_{ij}) \in R$ state variables and $\chi_i\,(t) \in C^0\,[t_0-\tau,\ t_0]$. Now, we define a matrix as

$$\Omega\,(s) = \begin{pmatrix} s^{\psi} - l_{11}e^{-s\tau_{11}} & -l_{12}e^{-s\tau_{12}} & \ldots & -l_{1n}e^{-s\tau_{1n}} \\ -l_{21}e^{-s\tau_{21}} & s^{\psi} - l_{22}e^{-s\tau_{22}} & \ldots & -l_{2n}e^{-s\tau_{2n}} \\ \ldots & \ldots & \ldots & \ldots \\ -l_{n1}e^{-s\tau_{n1}} & -l_{n2}e^{-s\tau_{n2}} & \ldots & s^{\psi} - l_{nn}e^{-s\tau_{nn}} \end{pmatrix}.$$

In this case, system (4.2) has an asymptotically stable zero solution if every root of the latent equation $\det(\Omega\,(s)) = 0$ has negative real components.

In this part, our objective is to determine the resilience criteria for the system (2.5). Furthermore, we have determined the criteria for the presence of Hopf bifurcation in the framework (2.5) by employing time delay τ as the bifurcation parameter. To assess the resilience of distinct equilibria, it is imperative to linearize the framework (2.5) with respect to the specific equilibrium point and subsequently apply Lemma 2.1.

Let's explore the shift in position. The variables $z_1 = N(t) - N^*$ and $z_2 = P(t) - P^*$ represent minor fluctuations about the equilibrium point $(N^*,\, P^*)$. The system (2.5) may be turned into the linearized system (4.3) by disregarding the higher-order components.

$$\begin{aligned}
&{}_{t_0}^{C}D_t^{\psi}\left(z_1(t)\right) = l_{11}\, z_1(t) + l_{12}\, z_2(t),\\
&{}_{t_0}^{C}D_t^{\psi}\left(z_2(t)\right) = l_{21}\, z_1(t-\tau) + l_{22}\, z_2(t),\\
&\Omega(s) = \begin{pmatrix} s^{\psi} - l_{11} & -l_{12} \\ -l_{21}e^{-s\tau} & s^{\psi} - l_{22}e^{-s\tau} \end{pmatrix}, \qquad (4.3)\\
&l_{11} = r - \frac{2rN}{K} - \frac{a(1-m)P}{\left(1+b(1-m)N\right)^2} + 2\theta N,\ l_{12} = -\frac{a(1-m)N}{1+b(1-m)N},\\
&l_{21} = \frac{ac(1-m)P}{\left(1+b(1-m)N\right)^2}e^{-s\tau},\ l_{22} = -\gamma + \frac{ac(1-m)N}{1+b(1-m)N}e^{-s\tau}.
\end{aligned}$$

Theorem 4.1. *The trivial equilibrium point E_0 remains constantly a saddle point.*

Proof. At E_0, the Jacobian matrix is given by

$$\Omega_0(s) = \begin{pmatrix} s^{\psi} - r & 0 \\ 0 & s^{\psi} + \gamma \end{pmatrix}. \qquad (4.4)$$

The matrix above has the characteristic equation $\left(s^{\psi} - r\right)\left(s^{\psi} + \gamma\right) = 0$, where $s_1 = r$ and $s_2 = -\gamma$ are the eigenvalues. One of the eigenvalues is positive,with $|\arg(s_1)| = 0 < \frac{\psi\pi}{2}$, and the other is negative, with $|\arg(s_2)| = \pi > \frac{\psi\pi}{2}$, $\forall\psi \in (0,\, 1]$. Thus E_0 is a saddle point. □

Theorem 4.2. *If, as stated in Lemma 2.1, criteria (i) is true for system* (2.5)*, then if $\theta < \frac{r}{K}$ and $\frac{Kac(1-m)r}{(r-K\theta)+Kb(1-m)r} < \gamma < \gamma + r$, then the auxiliary equilibrium point $E_1 = (N,\, 0)$ at $\tau = 0$ is asymptotically stable for $\tau \in [0,\ \tau^*)$ as well as framework* (2.5) *experiences a Hopf bifurcation at equilibrium E_1 when $\tau = \tau^*$. Subsequently,* $\mathrm{Re}\left[\frac{ds}{d\tau}\right]_{(\phi=\phi_0,\,\tau=\tau_*)} \neq 0$ *is held by the transversality requirement (iii) of (Lemma 2.1).*

Proof. It is possible to get the community matrix at E_1 by

$$\Omega_1(s) = \begin{pmatrix} s^{\psi} - r + \dfrac{2rN}{K} - 2\theta N & \dfrac{a(1-m)N}{1+b(1-m)N} \\ 0 & s^{\psi} + \gamma - \dfrac{ac(1-m)N}{1+b(1-m)N}e^{-s\tau} \end{pmatrix}. \qquad (4.5)$$

The latent equation is

$$s^{2\psi} + L_1 s^{\psi} + L_2 - e^{-s\tau}\left(L_3 s^{\psi} + L_4\right) = 0, \qquad (4.6)$$

where $L_1 = \gamma + \frac{2rN}{K} - r - 2\theta N$, $L_2 = -\gamma\left(r - \frac{2rN}{K} + 2\theta N\right)$, $L_3 = \frac{ac(1-m)N}{1+b(1-m)N}$ and $L_4 = -\left(r - \frac{2rN}{K} + 2\theta N\right)\frac{ac(1-m)N}{1+b(1-m)N}$. When $\tau = 0$,

$$s^{2\psi} + (L_1 - L_3)\, s^{\psi} + (L_2 - L_4) = 0. \qquad (4.7)$$

Equation (4.7) makes it clear that for $L_1 - L_3 > 0$ and $L_2 - L_4 > 0$ if $\frac{Kac(1-m)r}{(r-K\theta)+Kb(1-m)r} < \gamma < \gamma + r$ and $\theta < \frac{r}{K}$. Consequently, the equilibrium point E_1 is asymptotically stable when $\frac{Kac(1-m)r}{(r-K\theta)+Kb(1-m)r} < \gamma < \gamma + r$ and $\theta < \frac{r}{K}$ with $\tau = 0$ satisfy $|\arg(s_i)| > \frac{\psi\pi}{2}$.

We suppose providing the solution $s = i\xi$ to (4.6) is valid when $\tau > 0$,

$$\begin{aligned}
&(i\xi)^{2\psi} + L_1(i\xi)^{\psi} + L_2 - e^{-i\xi\tau}\Big(L_3(i\xi)^{\psi} + L_4\Big) = 0,\\
&-\xi^{2\psi} + iL_1\xi^{\psi} + L_2 - (\cos\xi\tau - i\sin\xi\tau)\Big(iL_3\xi^{\psi} + L_4\Big) = 0.
\end{aligned}$$

In order to separate the real and imaginary components, the subsequent equations can be obtained:

$$L_3\xi^{\psi}\sin\xi\tau + L_4\cos\xi\tau = L_2 - \xi^{2\psi}, \tag{4.8}$$

$$-L_3\xi^{\psi}\cos\xi\tau + L_4\sin\xi\tau = L_1\xi^{\psi}. \tag{4.9}$$

Solving (4.8) and (4.9), we get

$$\xi^{4\psi} + \left(L_1^2 - 2L_2 - L_3^2\right)\xi^{2\psi} + \left(L_2^2 - L_4^2\right) = 0. \tag{4.10}$$

Positive real does not meet equation (4.10) if both ξ does not meet equation (4.10) if both $\left(L_1^2 - 2L_2 - L_3^2\right) > 0$ and $(L_2 - L_4) > 0$. On the other hand, when $(L_2 - L_4) < 0$ then (4.10) contains one positive root with ξ_0, while the latent (4.6) having a pair of roots are entirely fictitious $\pm i\xi_0$. Considering $s(\tau) = \omega(\tau) + i\xi(\tau)$ is the eigenvalue of (4.11) such that $\omega(\tau^*) = 0$ and $\xi(\tau^*) = \phi_0$. From (4.8) and (4.9), we have,

$$\tau^* = \frac{1}{\phi_0}\arccos\left[\frac{L_2L_4 - (L_4 + L_1L_3)\,\phi_0^{2\psi}}{L_4^2 + L_3^2\phi_0^{2\psi}}\right] + \frac{2j\psi\pi}{\phi_0^{\psi}},$$

and from (4.10)

$${\phi_0}^{2\psi} = \frac{1}{2}\left(L_3^2 + 2L_2 - L_1^2\right) + \frac{1}{2}\sqrt{\left(L_3^2 + 2L_2 - L_1^2\right) - 4\left(L_2^2 - L_4^2\right)} < 0.$$

To satisfy the permanence condition of the delayed framework, it is necessary to confirm the transversality requirement in the following manner. It is possible to express (4.6) as

$$\alpha_1(s) + \alpha_2(s)\,e^{-s\tau} = 0. \tag{4.11}$$

Differentiating (4.11) with respect to τ, we get

$$\left[\alpha'_1(s) + \alpha'_2(s)\,e^{-s\tau} - \tau\alpha_2(s)\,e^{-s\tau}\right]\frac{ds}{d\tau} = s\alpha_2(s)\,e^{-s\tau}.$$

From above, we have

$$\frac{ds}{d\tau} = \frac{G_1(s)}{G_2(s)} = \frac{G_1(s)\,\overline{G}_2(s)}{|G_2(s)|^2}, \tag{4.12}$$

where $G_1(s) = s\alpha_2(s)\,e^{-s\tau}$, $G_2(s) = \alpha'_1(s) + \alpha'_2(s)\,e^{-s\tau} - \tau\alpha_2(s)\,e^{-s\tau}$, $G_1(i\phi_0) = G_{11} + iG_{12}$ and $G_2(i\phi_0) = G_{21} + iG_{22}$. Separation of the real component from each side of (4.12)

$$\operatorname{Re}\left[\frac{ds}{d\tau}\right]_{(\phi=\phi_0,\,\tau=\tau_*)} = \frac{G_{11}G_{21} + G_{12}G_{22}}{G_{21}^2 + G_{22}^2}, \tag{4.13}$$

where

$$\begin{aligned}
&G_{11} = \phi_0\left({\alpha_2}^{\mathrm{Im}}\cos\phi_0\tau - {\alpha_2}^{\mathrm{Re}}\sin\phi_0\tau\right),\ G_{12} = \phi_0\left(-{\alpha_2}^{\mathrm{Re}}\cos\phi_0\tau - {\alpha_2}^{\mathrm{Im}}\sin\phi_0\tau\right),\\
&G_{21} = {\alpha'_1}^{\mathrm{Re}} - {\alpha'_2}^{\mathrm{Re}}\cos\phi_0\tau + \tau\alpha_2^{\mathrm{Re}}\cos\phi_0\tau + \tau\alpha_2^{\mathrm{Im}}\sin\phi_0\tau,\\
&G_{22} = {\alpha'_1}^{\mathrm{Im}} + {\alpha'_2}^{\mathrm{Re}}\sin\phi_0\tau - \tau\alpha_2^{\mathrm{Re}}\sin\phi_0\tau + \tau\alpha_2^{\mathrm{Im}}\cos\phi_0\tau.
\end{aligned}$$

From (4.13), the transversality requirement is true if $\frac{G_{11}G_{21}+G_{12}G_{22}}{G_{21}^2+G_{22}^2} \neq 0$. As a result, the auxiliary equilibrium point Lemma 2.1 is proven. □

Theorem 4.3. *Assuming that Lemma 2.1, criterion (i) for framework* (2.5) *holds true, the coexistence equilibrium point* $E_2 = (N^*,\ P^*)$ *is asymptotically stable at* $\tau = 0$ *when* $\theta > \frac{r}{K}$*, and* $P < \frac{(1+b(1-m)N)^2(r-K\theta)}{Kab(1-m)^2}$ *then the coexistence equilibrium point is asymptotically stable* $\tau \in [0,\ \tau^*)$ *and framework* (2.5) *exhibits a Hopf bifurcation at the coexistence equilibrium while* $\tau = \tau^*$*. This means that the transversality requirement (iii) of Lemma 2.1 meets* $\mathrm{Re}\left[\frac{ds}{d\tau}\right]_{(\phi=\phi_0,\,\tau=\tau_*)} \neq 0$.

Proof. At positive equilibrium point E_2, the community matrix is provided by

$$\Omega_2(s) = \begin{pmatrix} s^{\psi} - a_{11} & -a_{12} \\ -a_{21}e^{-s\tau} & s^{\psi} - a_{22}e^{-s\tau} \end{pmatrix}, \tag{4.14}$$

where

$$\begin{aligned}
a_{11} &= \frac{ab(1-m)^2NP}{(1+b(1-m)N)^2} - \frac{rN}{K} + \theta N,\ \ a_{12} = -\frac{a(1-m)N}{1+b(1-m)N},\\
a_{21} &= \frac{ac(1-m)P}{(1+b(1-m)N)^2}e^{-s\tau},\ \ a_{22} = \frac{ac(1-m)N}{1+b(1-m)N}\left(e^{-s\tau}-1\right).
\end{aligned}$$

The latent equation is now $\det(\Omega_2(s)) = 0$, this is same as

$$s^{2\psi} + T_1 s^{\psi} + T_2 + e^{-s\tau}\left(T_3 s^{\psi} + T_4\right) = 0, \tag{4.15}$$

where

$$\begin{aligned}
T_1 &= -\left[\frac{ab(1-m)^2NP}{(1+b(1-m)N)^2} - \frac{rN}{K} + \theta N - \frac{ac(1-m)N}{1+b(1-m)N}\right],\\
T_2 &= -\frac{ab(1-m)^2NP}{(1+b(1-m)N)^2} + \frac{rN}{K} - \theta N,\\
T_3 &= -\frac{ac(1-m)N}{1+b(1-m)N},\\
T_4 &= \frac{a^2bc(1-m)^3N^2P}{(1+b(1-m)N)^3} - \frac{ac(1-m)N^2r}{K(1+b(1-m)N)} + \frac{ac\theta(1-m)N^2}{1+b(1-m)N} + \frac{a^2c(1-m)^2N^2P}{(1+b(1-m)N)^3}.
\end{aligned}$$

When $\tau = 0$,

$$s^{2\psi} + (T_1 + T_3)s^{\psi} + (T_2 + T_4) = 0. \tag{4.16}$$

Here, $T_1 + T_3 > 0$ if $P < \frac{(1+b(1-m)N)^2(r-K\theta)}{Kab(1-m)^2}$, and $T_2 + T_4 > 0$ if $P < \frac{(1+b(1-m)N)^2(r-K\theta)}{Kab(1-m)^2}$ and $\theta > \dfrac{r}{K}$. Thus, by Routh-Hurwitz criteria, equations (4.16) have negative roots and the system (2.5) is stable at $E_2 = (N^*,\ P^*)$ for latency is equal to zero.

We suppose that the solution to (4.16) $s = i\varepsilon$ must fulfil,

$$\begin{aligned}
&(i\varepsilon)^{2\psi} + T_1(i\varepsilon)^{\psi} + T_2 + e^{-i\varepsilon\tau}\Big(T_3(i\varepsilon)^{\psi} + T_4\Big) = 0 \text{ when } \tau > 0,\\
&\varepsilon^{2\psi} - T_2 = T_3\varepsilon^{\psi}\sin\varepsilon\tau + T_4\cos\varepsilon\tau + iT_1\varepsilon^{\psi} + i\Big(T_3\varepsilon^{\psi}\cos\varepsilon\tau - T_4\sin\varepsilon\tau\Big).
\end{aligned}$$

The result of separating the real and imaginary portions is:

$$T_3\varepsilon^{\psi}\sin\varepsilon\tau + T_4\cos\varepsilon\tau = \varepsilon^{2\psi} - T_2, \tag{4.17}$$

$$T_3\varepsilon^{\psi}\cos\varepsilon\tau - T_4\sin\varepsilon\tau = -T_1\varepsilon^{\psi}. \tag{4.18}$$

The expression that is the result of squaring and adding equations (4.17) and (4.18) is as follows:

$$\varepsilon^{4\psi} + \left(T_1^2 - 2T_2 - T_3^2\right)\varepsilon^{2\psi} + \left(T_2^2 - T_4^2\right) = 0. \tag{4.19}$$

We can immediately establish that $\left(T_1^2 - 2T_2 - T_3^2\right) > 0$ and $(T_2 - T_4) > 0$, which implies that there is no positive real that satisfies (4.19). Consequently, the roots of (4.15) are nonpositive. Conversely, for $(T_2 - T_4) < 0$ then (4.19) contains just a single positive root, given by ε_0, as well as (4.15) holds a pair of entirely fictitious roots $\pm i\varepsilon_0$. Considering $s(\tau) = \nu(\tau) + i\varepsilon(\tau)$ is the eigenvalue of (4.15) such that $\nu(\tau^*) = 0$ and $\varepsilon(\tau^*) = \varepsilon_0$. From (4.17) and (4.18), we have $\tau^* = \frac{1}{\varepsilon_0}\arccos\left[\frac{(T_4+T_1T_3)\varepsilon_0^{2\psi} - T_2T_4}{T_4^2+T_3^2\varepsilon_0^{2\psi}}\right] + \frac{2j\psi\pi}{\varepsilon_0^{\psi}}$, from (4.19)

$$\varepsilon_0^{2\psi} = \frac{1}{2}\left(T_3^2 + 2T_2 - T_1^2\right) + \frac{1}{2}\sqrt{\left(T_3^2 + 2T_2 - T_1^2\right)^2 - 4\left(T_2^2 - T_4^2\right)} < 0. \tag{4.20}$$

The following transversality criteria must be fulfilled in order to meet the delayed framework's permanence requirement. It is possible to express the characteristic (4.15) as $\alpha_1(s) + \alpha_2(s)e^{-s\tau} = 0$. The similar approaches used in Theorem 4.2 lead us to,

$$\operatorname{Re}\left[\frac{ds}{d\tau}\right]_{(\varepsilon=\varepsilon_0,\,\tau=\tau_*)} = \frac{G_{11}G_{21} + G_{12}G_{22}}{G_{21}^2 + G_{22}^2}, \tag{4.21}$$

where

$$\begin{aligned}
&G_{11} = \varepsilon_0\left({\alpha_2}^{\mathrm{Re}}\sin\,\varepsilon_0\tau - {\alpha_2}^{\mathrm{Im}}\cos\varepsilon_0\tau\right),\ G_{12} = \varepsilon_0\left({\alpha_2}^{\mathrm{Re}}\cos\varepsilon_0\tau + {\alpha_2}^{\mathrm{Im}}\sin\varepsilon_0\tau\right),\\
&G_{21} = {\alpha'_1}^{\mathrm{Re}} + {\alpha'_2}^{\mathrm{Re}}\cos\,\varepsilon_0\tau - \tau\alpha_2^{\mathrm{Re}}\cos\varepsilon_0\tau - \tau\alpha_2^{\mathrm{Im}}\sin\varepsilon_0\tau,\\
&G_{22} = {\alpha'_1}^{\mathrm{Im}} - {\alpha'_2}^{\mathrm{Re}}\sin\varepsilon_0\tau + \tau\alpha_2^{\mathrm{Re}}\sin\varepsilon_0\tau - \tau\alpha_2^{\mathrm{Im}}\cos\varepsilon_0\tau.
\end{aligned}$$

From (4.21), the transversality criterion is satisfied if $\frac{G_{11}G_{21}+G_{12}G_{22}}{G_{21}^2+G_{22}^2} \neq 0$. The coexistence equilibrium point is therefore proven by Lemma 2.1. □

4.3. Global stability analysis

Here, we further develop the study to investigate the global stability criteria of the non-integer order delay differential framework [11,12]. We transform the framework into a linearized structure to analyze the global resilience of the equation (2.5)'s equilibrium points.

$$\begin{aligned}
&D^{\psi}N(t) = a_1N(t) + a_2P(t),\\
&D^{\psi}P(t) = b_1N(t-\tau) + b_2P(t) + b_3P(t-\tau).
\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
&a_1 = r - \frac{2rN^*}{K} + 2\theta N^* - \frac{a\left(1-m\right)P^*}{1+b\left(1-m\right)N^*},\ a_2 = -\frac{a\left(1-m\right)P^*}{1+b\left(1-m\right)N^*},\\
&b_1 = \frac{ca\left(1-m\right)P^*}{1+b\left(1-m\right)N^*},\ b_2 = -\gamma,\ b_3 = \frac{ca\left(1-m\right)N^*}{1+b\left(1-m\right)N^*}.
\end{aligned}$$

When the linear non-integer differential equation contains a nonzero equilibrium point, it is possible to relocate the point to the origin. In order to get the resultant equation (4.23), substitute $\bar{N}\left(t\right) = N\left(t\right) - N^*$ and $\bar{P}\left(t\right) = P\left(t\right) - P^*$ into equation (4.22):

$$\begin{aligned}
&D^{\psi}\bar{N}\left(t\right) = a_1\bar{N}\left(t\right) + a_2\bar{P}\left(t\right),\\
&D^{\psi}\bar{P}\left(t\right) = b_1\bar{N}\left(t-\tau\right) + b_2\bar{P}\left(t\right) + b_3\bar{P}\left(t-\tau\right).
\end{aligned} \tag{4.23}$$

System (2.5)'s stability is investigated by applying the Laplace transform [18] to either side of (4.23). Subsequently, we obtain

$$\begin{aligned}
&\left(s^{\psi} - a_1\right)\ N\left(s\right) - a_2P\left(s\right)\\
&= s^{\psi-1}\rho_1\left(0\right) - b_1e^{-s\tau}N\left(s\right) + \left(s^{\psi} - b_2 - b_3e^{-s\tau}\right)\ P\left(s\right)\\
&= s^{\psi-1}\rho_2\left(0\right) + b_1e^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}\rho_1\left(t\right)\ dt + b_3e^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}\rho_2\left(t\right)\ dt.
\end{aligned} \tag{4.24}$$

It is imperative to specify that the initial values $\bar{N}\left(t\right)$ and $\bar{P}\left(t\right)$ are $\rho_1\left(t\right)$ and $\rho_2\left(t\right)$ for $t \in [-\tau,\ 0]$. Furthermore $N\left(s\right) = L\left(\bar{N}\left(t\right)\right)$ and $P\left(s\right) = L\left(\bar{P}\left(t\right)\right)$ are the Laplace transform of $\bar{N}\left(t\right)$ and $\bar{P}\left(t\right)$, respectively. A possible alternative to system (4.24) is as stated below:

$$\Lambda\left(s\right)\begin{pmatrix} N\left(s\right) \\ P\left(s\right) \end{pmatrix} = \begin{pmatrix} f_1\left(s\right) \\ f_2\left(s\right) \end{pmatrix}, \tag{4.25}$$

where

$$\Lambda\left(s\right) = \begin{pmatrix} s^{\psi} - a_1 & -a_2 \\ -b_1e^{-s\tau} & s^{\psi} - b_2 - b_3e^{-s\tau} \end{pmatrix},$$

and

$$f_1\left(s\right) = s^{\psi-1}\rho_1\left(0\right),\ f_2\left(s\right) = s^{\psi-1}\rho_2\left(0\right) + b_1e^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}\rho_1\left(t\right)\ dt + b_3e^{-s\tau}\int_{-\tau}^{0} e^{-s\tau}\rho_2\left(t\right)\ dt.$$

The matrix $\Lambda\left(s\right)$ is known as the latent matrix of framework (2.5), while the determinant of $\Lambda\left(s\right)$ is referred to as its latent polynomial. Thus, the dissemination of the latent values of the characteristic polynomial establishes the permanency of the framework (2.5). Thus, if there are negative parts in every one of the roots of the latent equation, the equilibrium of the non-integer order system described above is Lyapunov globally asymptotically stable, provided the equilibrium exists [10]. Multiplying (4.25) by s both sides yield the following result:

$$\Lambda\left(s\right)\begin{pmatrix} sN\left(s\right) \\ sP\left(s\right) \end{pmatrix} = \begin{pmatrix} sf_1\left(s\right) \\ sf_2\left(s\right) \end{pmatrix}. \tag{4.26}$$

As a result, when each of the roots in the transcendental equation $\det(\Lambda(s)) = 0$ are located in the open left complex plane, which is $\mathrm{Re}(s) < 0$, subsequently we consider (4.26) in $\mathrm{Re}(s) \geq 0$. Within this specific region, there exists only one solution to system (4.26) with respect to $(sN(s), sP(s))$ in which $\lim\limits_{s\to 0,\,\mathrm{Re}(s)\geq 0} s\,N(s) = 0$, $\lim\limits_{s\to 0,\,\mathrm{Re}(s)\geq 0} s\,P(s) = 0$. By making the assumption that every solution of the latent equation $\det(\Lambda(s)) = 0$, and using the final-value theorem of Laplace transform [18], we may conclude that

$$\lim_{t\to+\infty} \bar{N}(t) \equiv \lim_{s\to 0,\,\mathrm{Re}(s)\geq 0} s\,N(s) = 0, \text{ and } \lim_{t\to+\infty} \bar{P}(t) \equiv \lim_{s\to 0,\,\mathrm{Re}(s)\geq 0} s\,P(s) = 0.$$

The statement indicates the trivial solution of the non-integer order framework is Lyapunov globally asymptotic. As a result, we have the ensuing outcome.

Theorem 4.4. *The positive equilibrium points* (N^*, P^*) *of framework* (2.5) *are Lyapunov globally asymptotically stable, provided each of the roots of the characteristic equation det*$(\Lambda(s)) = 0$ *hold negative real portions.*

5. Numerical simulations

The numerical simulations that confirm the theoretical conclusions from the preceding sections were carried out using the Adams-Bashforth-Moulton Predictor-corrector P(EC)mE technique [15, 26, 27, 32] and the system devised by Ivo Petras using the fde12 solver through MATLAB R2024a. Furthermore, how non-integer order, time delay, cooperation, and prey refuge affect the stability of the equilibrium points has been detailed. The stability of coexistence equilibrium E_2 is of greater concern to us than that of trivial equilibrium E_0 and axial equilibrium E_1. Parameter values (provided in graph captions) must be chosen hypothetically as there is no presently accessible data that applies to our suggested model.

As shown in Figures 1 and 2, we have now created the phase portraits and time series graphs of the framework (2.5) about E_2 given $\psi = 0.95$ with three different values of $\tau = 0.2$, 2, and 5. Based on these figures, system (2.5) goes from being stable to unstable as soon as τ goes over $\tau^* = 1.5$. For $\tau = 1.5 > \tau^*$, an unstable source can be observed at the nontrivial equilibrium point E_2. Theorem 4.3 establishes that E_2 remains stable when it acts like a sink at $\tau = 1.5 < \tau^*$. The oscillatory motion is dampened by the derivative of non-integer order, as observed in Figure 3. The results indicate that the system is not stable when $\psi = 0.98$, but it is stable when $\psi = 0.85$. As our results show, the non-integer order of the derivative has a major impact on the framework's behavior.

Finally, for both constant delays and fractional orders, we will show how the prey refuge and cooperation coefficient affect each population density. When $m = 0$ (as well as the remaining values are specified in the description of Figure 4), the unstable source is seen in Figure 4. This means that the unstable source may exist if the refuge is disregarded. However, when $m = 0.7$ is used, the unstable source vanishes and the coexisting equilibrium turns into a steady focal point.

The system is stable in Figure 6 when $m = 0.6$ and $\theta = 0$ are used, indicating that the prey population is not cooperating to identify a refuge habitat. The system becomes unstable at $\theta = 0.07$ when we raise the amount of θ, which means that the prey members cooperate more to locate a safe haven (the remaining parameters are shown in the caption of Figure 4). This leads us to the conclusion that θ has an influence on the stable steady state's instability. Figures 5 and 7 show the bifurcation illustrations of system (2.5) for the parameters refuge and cooperation coefficient, respectively, for $\psi = 1$ and $\tau = 0$.

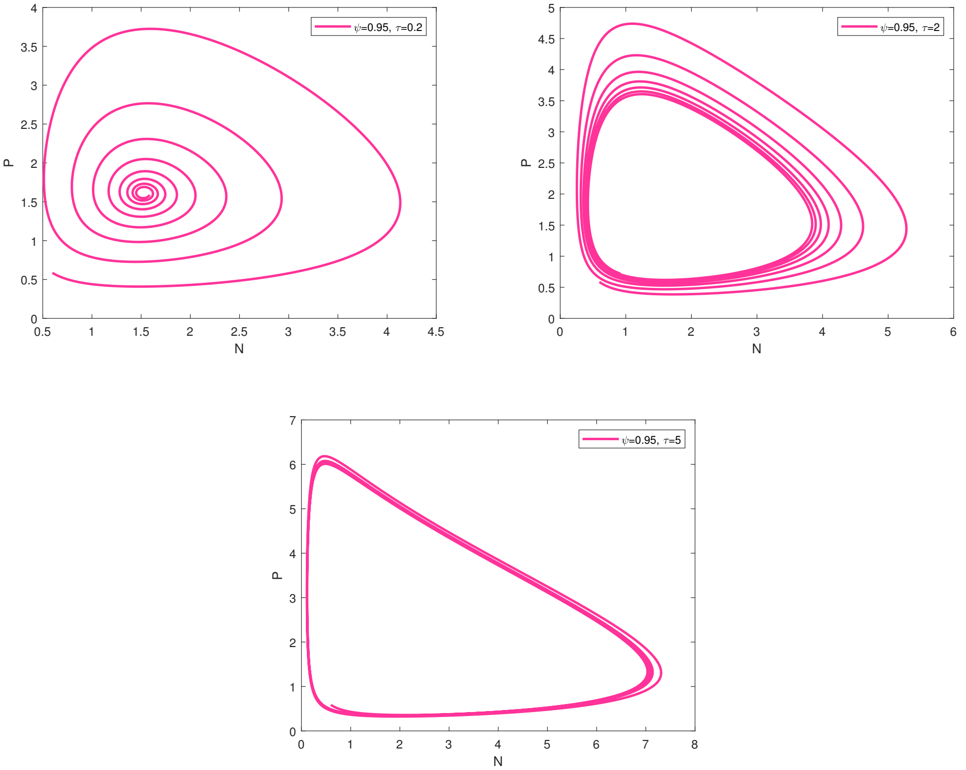

Figure 1. Phase portraits of model (2.5) in various time delays with variables such as $r = 0.1489,\ k = 0.973,\ m = 0.621,\ a = 0.249,\ b = 0.219,\ \theta = 0.1439,\ c = 0.948,\ \gamma = 0.121$ which exhibits recurrent outbreak as a result of Hopf bifurcation.

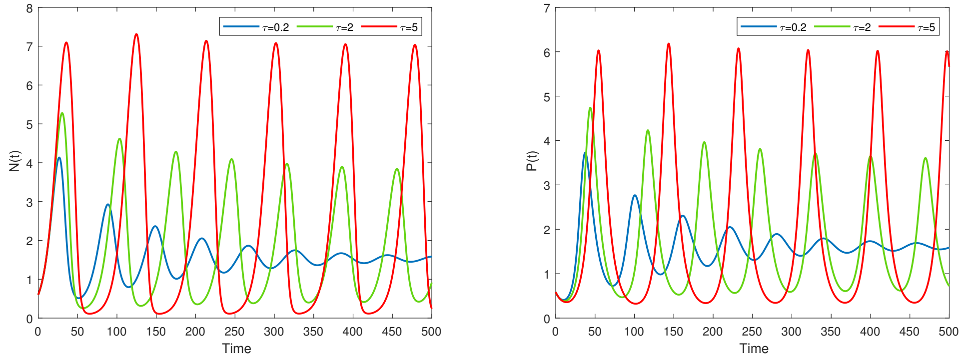

Figure 2. Time series solutions of the model (2.5) using the identical values as shown in Figure 1.

6. Discussion

A fundamental predator-prey framework is characterized by the effect of certain components that include predation, reproduction, and competition for resources on the population dynamics of predators and prey. Adding a non-integer order, prey species cooperation, and refuge in the presence of a gestation delay complicates the model. In this research, we assumed that the predator response function is Holling type II, and we investigated a predator-prey system with a prey refuge. In addition, the prey population's cooperation in seeking refuge from predator

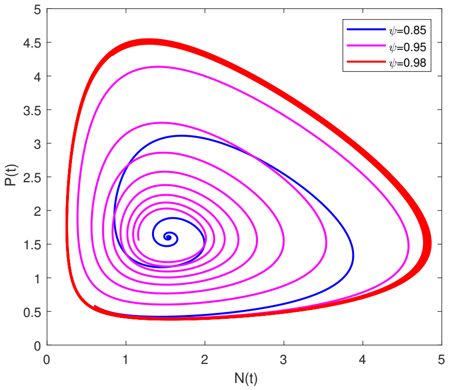

Figure 3. Using identical values as shown in Figure 1, model (2.5) exhibits a various non-integer order ($0 < \psi \leq 1$). The non-integer order derivative damps the oscillation behavior.

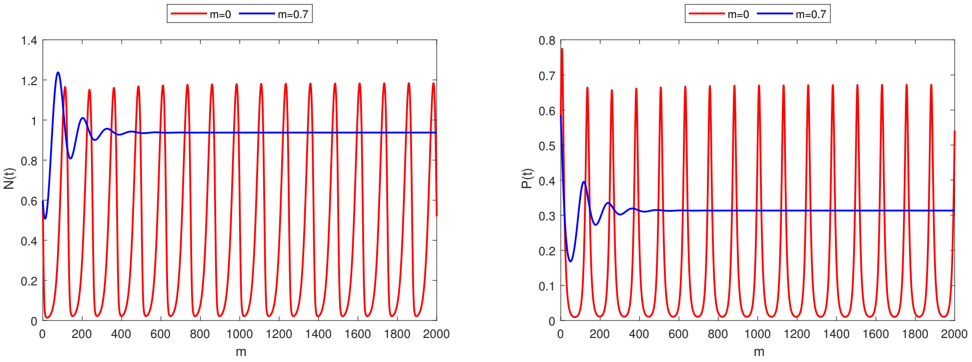

Figure 4. We can see how refuge affects the prey and predator populations in the system using the parameters $r = 0.078,\ k = 0.932,\ m = 0.7,\ a = 0.571,\ \theta = 0.046,\ b = 0.918,\ \tau = 3,\ c = 0.948,\ \gamma = 0.121,\ \psi = 0.98.$

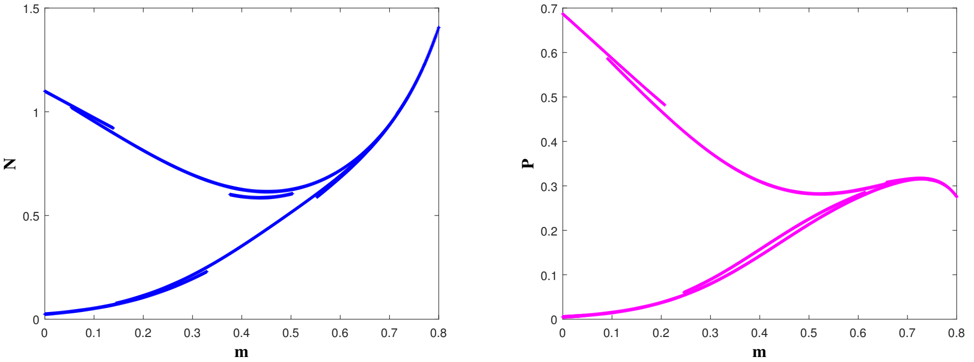

Figure 5. When $\psi = 1$ and $\tau = 0$, the bifurcation factor is the refuge (m), and all other factors are identical to those in Figure 4.

pressure is included in the system to determine whether a mechanism (such as cooperation, refuge, memory, or time delay) leads to improved population stabilization. We presented the suggested model's existence, uniqueness, and boundedness. We also addressed the local and global permanency findings for every feasible equilibrium state that the system may reach. We

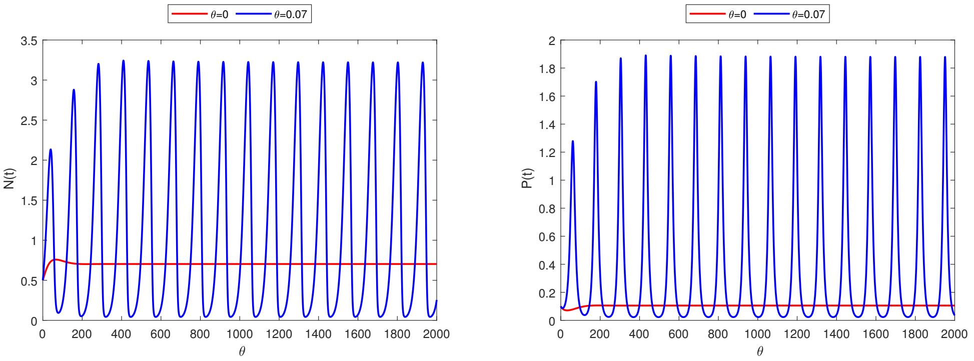

Figure 6. The effect of cooperation coefficient on the system's populations is shown with refuge $m = 0.6$, and all other factors are identical to those in Figure 4.

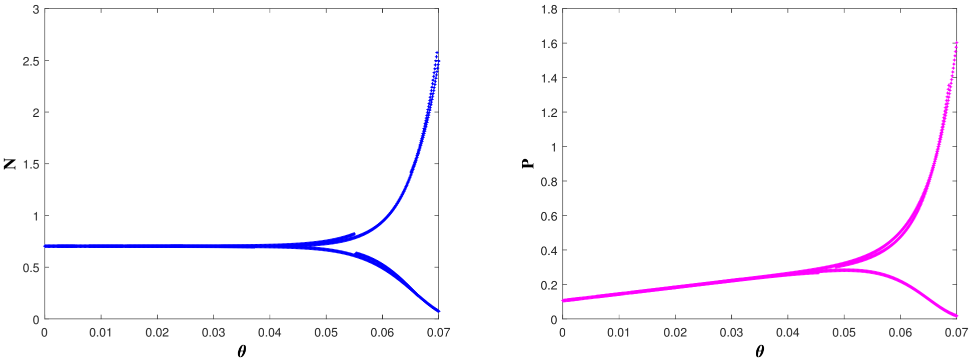

Figure 7. When $\psi = 1$ and $\tau = 0$, the bifurcation factor is the cooperation coefficient (θ), and all other factors are identical to those in Figure 4.

explore some novel and intriguing adequate criteria for the local asymptotic permanency findings of the non-integer order predator-prey framework. Adding non-integer order to a predator-prey system may make stability findings better and, in certain cases, reduce solution oscillation behaviors. As the model's time lag increases, a stable equilibrium might become unstable and vice versa. When the delay time reaches a critical number, a Hopf bifurcation occurs. The order of the derivative is significantly related to the time delay. Due to the fact that non-integer order is associated with memory, a higher order indexing can be associated with diminishing memory, while a lower order indexing can be associated with robust memory. Consequently, our research suggests that diminishing memory can contribute to the improvement of the predator-prey system's stable coexistence, whereas a robust memory can deteriorate it.

Furthermore, we looked at the bifurcation diagram in terms of cooperation (θ) and refuge (m). Predator pressure is lessened, and the prey population is shielded from predators by the refuge. Predators that overuse their prey population drive it into a condition of divergent oscillation. But the prey population's refuge measures assist in changing this precarious situation into a stable one. The presence of a consistent number of predators is dependent on the stability of the prey population. It is evident that cooperation destabilises the system dynamics during refuge since the stable equilibrium returns to an unstable state when the prey population decides to seek shelter together. From the perspective of the prey, a cooperative effort to reach the refuge increases the likelihood that as many individuals as possible will be able to escape the predator. In contrast, predators have a lower chance of success when preying on prey, and in a strictly linear system, they will eventually die out altogether because they have no other choice but to prey. The prey population's survival is enhanced by the fact that a regulated proportion of it is able to evade predators through refuge. Conversely, the predator population is never threatened with extinction, as it never experiences a food shortage. This balanced interaction expedites the time it takes for the system dynamics to achieve equilibrium. Finally, we apply our mathematical results to an ecological context by saying that prey species have a long memory of the effects of external factors on their life cycles, and this causes the system to have a large number of prey refuges. Therefore, there is a better possibility of sustained cohabitation in any ecological system where memory-fading species are present.

In summary, the presented research showed that the cooperative and prey refuge effects, together with the existence of fractional order and time delay, might all be very helpful in maintaining the biodiversity of the ecological system. Many scholars are working to enhance the subject of fractional order modelling in recent times, and they have produced some impressive theoretical and numerical findings.

7. Conclusion with future directions

This work investigated a mathematical model that explores the relationship between predators and prey. The model takes into account the cooperation among prey, the presence of a safe refuge, and the use of fractional order derivatives. The results of our study suggest that when prey cooperates, it can lead to instability in the system, whereas seeking shelter can help stabilize it. The presence of a fractional order derivative, which characterizes a system's memory, has an impact on its stability. We established critical thresholds for collaboration and sanctuary and analyzed the effect of time delay on the system's behavior. The findings indicate that a well-balanced environment, characterized by moderate cooperation, adequate refuge, and optimal memory levels, can support the long-term coexistence of species. Subsequent investigations will concentrate on determining the most effective control mechanisms for models with non-integer orders.

Our research enhances the understanding of predator-prey interactions and provides valuable insights for conservation efforts by taking into account these intricate ecological aspects. The creation of a non-integer order model with optimum control is our next goal.

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