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CRITICAL TRAVELING WAVES IN A DELAYED DIFFUSIVE EPIDEMIC SYSTEM

Jingdong Wei[1], Han Jiang[1], Zaili Zhen[1,†] and Jiangbo Zhou[1]

Abstract In this paper, we investigate the existence of critical traveling wave solutions in a diffusive epidemic system with delay. The existence of super-critical traveling wave solutions is well-known. By constructing suitable upper-lower solutions of wave system and applying Schauder's fixed point theorem coupled with delicate analysis, we derive the existence of non-trivial positive bounded critical traveling wave solution for the first time. Moreover, if the transmission rate equals to the removed rate, then the nonexistence of traveling wave solutions with any positive wave velocity is obtained.

Keywords Diffusive epidemic system, critical traveling wave, time delay.

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1. Introduction and main results

The investigation of existence of super-critical and critical traveling wave solutions for diffusive epidemic systems have been paid much attention since they can describe the spatial transmission patterns of disease. We refer to some classical results by Diekmann [4, 5], Ducrot et al. [7, 8], Hosono and Ilyas [11], Li and Zou [12], Wang and Wu [19], Wang et al. [18] and other related references [1–3, 6, 17, 20–30, 32, 33, 35]. The reaction-diffusion system

$$\begin{cases} \partial_t S(x,t) = d_1 \partial_{xx} S(x,t) - \dfrac{\beta S(x,t) I(x,t)}{S(x,t)+I(x,t)}, \\ \partial_t I(x,t) = d_2 \partial_{xx} I(x,t) + \dfrac{\beta S(x,t) I(x,t)}{S(x,t)+I(x,t)} - \gamma I(x,t), \\ \partial_t R(x,t) = d_3 \partial_{xx} R(x,t) + \gamma I(x,t), \end{cases} \tag{1.1}$$

was suggested by Wang et al. [18] as a mathematical model to describe the disease transmission into the susceptible individuals from the initial equilibrium to a final equilibrium. In (1.1), $S(x,t)$, $I(x,t)$ and $R(x,t)$ denote the densities of susceptible, infected and removed individuals at location x and time t, respectively. The transmission coefficient β, removed rate γ, spatial motility of each class d_i $(i = 1, 2, 3)$ are positive constants. Since $R(x,t)$ does not appear in the first two equations in (1.1), they just considered the subsystem for S-component and I-component. With the aid of upper-lower solutions method and Schauder's fixed point theorem, they showed that if $\beta > \gamma$ and the wave velocity $c > c^* = 2\sqrt{d_2(\beta - \gamma)}$ (c^* is the critical wave

[†]The corresponding author.
[1]School of Mathematical Sciences, Jiangsu University, Zhenjiang, Jiangsu 212013, China
Email: weijingdong@ujs.edu.cn(J. Wei), 18018321165@163.com(H. Jiang),
zhenzaili@ujs.edu.cn(Z. Zhen), ujszjb@126.com(J. Zhou)

velocity), then (1.1) has a non-trivial and non-negative traveling wave solution $(S, I)(x+ct)$ satisfying

$$\begin{cases} cS' = d_1 S'' - \beta SI/(S+I), \\ cI' = d_2 I'' + \beta SI/(S+I) - \gamma I, \\ S(-\infty) = S_0,\ S(\infty) \in [0, S_0),\ I(\pm\infty) = 0, \end{cases} \tag{1.2}$$

where $S_0 > 0$ is a given constant. Utilizing the bilateral Laplace transform, they proved that (1.1) has no non-trivial and non-negative traveling wave solutions for $\beta \le \gamma$ or $c < c^*$. In the Section 5.1 of [18], they listed two open problems. One is the exact value of $S(\infty)$, the other is the existence or nonexistence of the critical traveling wave solution. In our recent work [34], motivated by [9,13,14,18], we established the existence of the critical traveling wave solution in (1.1), which satisfies

$$(S, I)(-\infty) = (S_0, 0), \quad (S, I)(\infty) = (0, 0),$$

for a given constant $S_0 > 0$. Moreover, we proved that the critical traveling wave solution is positive on the real line; $S(z)$ is strictly decreasing on $\mathbb{R}$; $I(z)$ is a unimodal function on $\mathbb{R}$. The nonlocal delayed version of (1.1)

$$\begin{cases} \partial_t S(x,t) = d_1 \partial_{xx} S(x,t) - \dfrac{\beta S(x,t) J * I(x,t)}{S(x,t) + J * I(x,t)}, \\ \partial_t I(x,t) = d_2 \partial_{xx} I(x,t) + \dfrac{\beta S(x,t) J * I(x,t)}{S(x,t) + J * I(x,t)} - \gamma I(x,t), \\ \partial_t R(x,t) = d_3 \partial_{xx} R(x,t) + \gamma I(x,t), \end{cases} \tag{1.3}$$

where

$$J * I(x,t) = \int_0^\infty \int_{\mathbb{R}} J(y,s) I(x-y, t-s)\, dy ds,$$

has been investigated by Li et al. [13]. Note that the third equation in (1.3) is relatively dependent, they only studied the first two equations. Under certain assumptions on kernel function $J(y,s)$, they obtained that if $\beta > \gamma$ and the wave velocity $c > c^*$ ($c^* > 0$ is the critical wave velocity), then (1.3) has a positive traveling wave solution satisfying

$$(S, I)(-\infty) = (S_0, 0), \quad (S, I)(\infty) = (0, 0),$$

for a given constant $S_0 > 0$. On the other hand, they derived that if $\beta < \gamma$ and $c > 0$ or $\beta > \gamma$ and $c < c^*$, then (1.3) has no non-trivial positive bounded traveling wave solutions. In view of these results, there exist two open problems.

Does the traveling wave solutions of (1.3) exist if $\beta > \gamma$ and $c = c^*$ or $\beta = \gamma$ and $c > 0$?

How the number of change of removed individuals in (1.3)?

The aim of the present paper is to solve above problems for (1.3) with $J(y,s) = \delta(y)\delta(s-\tau)$ and $\delta(\cdot)$ is the Dirac-delta function. In other words, we consider these problems in the following functional differential system

$$\begin{cases} \partial_t S(x,t) = d_1 \partial_{xx} S(x,t) - \dfrac{\beta S(x,t) I(x,t-\tau)}{S(x,t) + I(x,t-\tau)}, \\ \partial_t I(x,t) = d_2 \partial_{xx} I(x,t) + \dfrac{\beta S(x,t) I(x,t-\tau)}{S(x,t) + I(x,t-\tau)} - \gamma I(x,t), \\ \partial_t R(x,t) = d_3 \partial_{xx} R(x,t) + \gamma I(x,t), \end{cases} \tag{1.4}$$

where the constant $\tau > 0$ is the time delay. We should mention that an infinite domain is slightly easier to handle than a finite one because, in the latter case, the individuals, as well as moving around, may have been interacting with the domain's boundaries. Meanwhile, we assume the individuals are performing an unbiased random walk, so that the motion can be modelled in terms of Laplacian diffusion with one-dimensional infinite domain. Herein, for (1.4), we shall work on the infinite one-dimension spatial domain $x \in \mathbb{R}$. For our purpose, we state the following lemma.

Lemma 1.1. *Assume $\beta > \gamma$ and let*

$$F(\lambda, c) := d_2\lambda^2 - c\lambda + \beta e^{-\lambda c\tau} - \gamma,$$

then there exists a positive pair (λ^*, c^*) such that

$$F(\lambda^*, c^*) = F_\lambda(\lambda^*, c^*) = 0. \tag{1.5}$$

Proof. This lemma can be proved similarly as that in [10, Lemma 2.1]. For the sake of completeness, we still provide a detail proof. Due to $\beta > \gamma$, we get that $F(\lambda, 0) = d_2\lambda^2+\beta-\gamma > 0$ and $F(0, c) = \beta - \gamma > 0$. For every fixed $\lambda > 0$, it follows that $\lim_{c\to\infty} F(\lambda, c) = -\infty$ and $F_c(\lambda, c) = -\lambda - \beta\lambda\tau e^{-\lambda c\tau} < 0$. For every fixed $c > 0$, we deduce that $\lim_{\lambda\to\infty} F(\lambda, c) = \infty$ and $F_\lambda(0, c) = -c-\beta c\tau < 0$. Notice that $F_{\lambda\lambda}(\lambda, c) = 2d_2 + \beta c^2\tau^2 e^{-\lambda c\tau} > 0$ for all $(\lambda, c) \in \mathbb{R}\times\mathbb{R}$. By the intermediate value theorem, monotonicity and convexity properties, there exists a positive pair (λ^*, c^*) such that $F(\lambda^*, c^*) = F_\lambda(\lambda^*, c^*) = 0$. The proof is finished. □

A critical traveling wave solution of (1.4) is a special solution in the form of $(S(x, t), I(x, t), R(x, t)) = (S(z), I(z), R(z))$, where $z := x + c^*t$ is the moving coordinate and c^* is the critical wave velocity. Meanwhile, $(S(z), I(z), R(z)) \in C^2(\mathbb{R}, \mathbb{R}^3)$ is the wave profile that propagates in the one-dimension spatial domain at the constant critical wave velocity and connects the initial equilibrium to the final equilibrium. Inserting $(S(x, t), I(x, t), R(x, t)) = (S(z), I(z), R(z))$, $z = x + c^*t$, into (1.4) yields

$$\begin{cases} d_1S''(z) - c^*S'(z) - \dfrac{\beta S(z)I(z - c^*\tau)}{S(z) + I(z - c^*\tau)} = 0, \\ d_2I''(z) - c^*I'(z) + \dfrac{\beta S(z)I(z - c^*\tau)}{S(z) + I(z - c^*\tau)} - \gamma I(z) = 0, \\ d_3R''(z) - c^*R'(z) + \gamma I(z) = 0. \end{cases} \tag{1.6}$$

Then our first task is to establish the existence of positive solutions of (1.6) satisfying

$$(S, I, R)(-\infty) = (S_0, 0, 0), \quad (S, I, R)(\infty) = (0, 0, S_0) \quad \text{for} \quad \beta > \gamma, \tag{1.7}$$

where $S_0 > 0$ is a given constant. Moreover, our second task is to show the nonexistence of positive solutions of (1.6) satisfying (1.7) if $\beta = \gamma$ and $c > 0$.

Now we state our results.

Theorem 1.1. *If $\beta > \gamma$ and $c = c^*$, then system (1.4) admits a critical traveling wave solution $(S(z), I(z), R(z))$ satisfying asymptotic boundary (1.7). Furthermore,*

(i) $S'(z)$, $I'(z)$, $R'(z)$, $S''(z)$, $I''(z)$, $R''(z) \to 0$ *as* $z \to \pm\infty$;

(ii) $0 < I(z) < \frac{\beta-\gamma}{\gamma}S_0$ *and* $I(z) = O(-ze^{\lambda^* z})$ *as* $z \to -\infty$;

(iii) $S(z)$ *is strictly decreasing on* $\mathbb{R}$ *and* $0 < S(z) < S_0$*;*

(iv) $R(z)$ *is strictly increasing on* $\mathbb{R}$ *and* $0 < R(z) < S_0$*;*

(v) $\gamma \int_{\mathbb{R}} I(z)dz = \beta \int_{\mathbb{R}} \frac{S(z)I(z-c^*\tau)}{S(z)+I(z-c^*\tau)}dz = c^* S_0$.

Theorem 1.2. *If* $\beta = \gamma$ *and* $c > 0$*, then system (1.4) admits no positive traveling wave solutions* $(S(z), I(z), R(z))$ *satisfying asymptotic boundary (1.7).*

Remark 1.1. Theorem 1.1 shows that (1.4) has a non-trivial positive bounded critical traveling wave solution and gives a explicit description for the number of change of removed individuals, i.e., $R(-\infty) = 0$ and $R(\infty) = S_0$. By the monotonicity and asymptotic boundary of $S(z)$ and $R(z)$, we obtain that S-component and R-component look like front-type solution. From the positiveness and asymptotic boundary of $I(z)$, we have that I-component seems pulse-type solution. Theorem 1.2 reflects that the restriction on the infection force (i.e., $\beta = \gamma$) in model (1.4) leads to the non-existence of traveling waves in (1.4), which indicates that this restriction can cut off the spread of real-life disease.

The rest of the paper is organized as follows. In Section 2, we obtain the existence of critical traveling wave solution by Schauder's fixed point theorem coupled with upper-lower solutions method. In Section 3, we prove the positiveness, boundedness, non-triviality of critical traveling wave solution by subtle analysis. In Section 4, using contradictory argument we establish the nonexistence of non-trivial positive bounded traveling wave solutions for $\beta = \gamma$ and $c > 0$.

2. Existence of critical traveling wave solution

Motivated by [9, 13, 31, 34], we define the following functions on $\mathbb{R}$.

$$S_+(z) := S_0, \qquad I_+(z) := \begin{cases} -L_1 z e^{\lambda^* z}, \ z < z_1, \\ \bar{I}, \ z \ge z_1, \end{cases}$$

$$S_-(z) := \begin{cases} S_0 - q e^{\lambda_1 z}, \ z < z_2, \\ \epsilon_1 e^{-\lambda_2 z}, \ z \ge z_2, \end{cases} \qquad I_-(z) := \begin{cases} [-L_1 z - L_2(-z)^{\frac{1}{2}}] e^{\lambda^* z}, \ z < z_3, \\ 0, \ z \ge z_3, \end{cases}$$

$$R_+(z) := L_3 e^{\epsilon_2 z}, \qquad R_-(z) := 0,$$

where λ^* is defined in Lemma 1.1,

$$\bar{I} = \frac{\beta - \gamma}{\gamma} S_0, \ L_1 = e\lambda^* \bar{I}, \ z_1 = -\frac{1}{\lambda^*}, \ z_3 = -\frac{L_2^2}{L_1^2}, \ \lambda_1 = \min\left\{\frac{\lambda^*}{2}, \frac{c^*}{2d_1}\right\}, \ \lambda_2 = \frac{\beta}{c^*},$$

$z_2, q, \epsilon_1, \epsilon_2, L_2, L_3 \in \mathbb{R}$ are constants to be determined later.

Lemma 2.1. *For given small enough* $\epsilon_2 > 0$ *and large enough* $L_2 > 0$*,* $L_3 > 0$*,* $q > S_0$*, there exist* $z_2 \in \mathbb{R}$*,* $\epsilon_1 > 0$ *such that* $S_-(z)$ *is continuous and*

$$d_1 S''_+(z) - c^* S'_+(z) - \frac{\beta S_+(z) I_-(z - c^*\tau)}{S_+(z) + I_-(z - c^*\tau)} \le 0, \quad z \in \mathbb{R}, \tag{2.1}$$

$$d_2 I''_+(z) - c^* I'_+(z) + \frac{\beta S_+(z) I_+(z - c^*\tau)}{S_+(z) + I_+(z - c^*\tau)} - \gamma I_+(z) \le 0, \quad z \ne z_1, \tag{2.2}$$

$$d_3R''_+(z) - c^*R'_+(z) + \gamma I_+(z) \le 0, \quad z \in \mathbb{R}, \tag{2.3}$$

$$d_1S''_-(z) - c^*S'_-(z) - \frac{\beta S_-(z)I_+(z-c^*\tau)}{S_-(z)+I_+(z-c^*\tau)} \ge 0, \quad z \ne z_2, \tag{2.4}$$

$$d_2I''_-(z) - c^*I'_-(z) + \frac{\beta S_-(z)I_-(z-c^*\tau)}{S_-(z)+I_-(z-c^*\tau)} - \gamma I_-(z) \ge 0, \quad z \ne z_3, \tag{2.5}$$

$$d_3R''_-(z) - c^*R'_-(z) + \gamma I_-(z) \ge 0, \quad z \in \mathbb{R}. \tag{2.6}$$

Proof. By the explicit expressions of $S_+(z)$, $I_-(z)$ and $R_-(z)$ on $\mathbb{R}$, one can easily obtain that (2.1) and (2.6) hold trivially. In the following, we give the detailed proofs for (2.2)-(2.5).

Proof of (2.2). If $z < z_1$, then $I_+(z) = -L_1 z e^{\lambda^* z}$ and $I_+(z-c^*\tau) = -L_1(z-c^*\tau)e^{\lambda^*(z-c^*\tau)}$. By (1.5), we get

$$\begin{aligned} &d_2I''_+(z) - c^*I'_+(z) + \frac{\beta S_+(z)I_+(z-c^*\tau)}{S_+(z)+I_+(z-c^*\tau)} - \gamma I_+(z) \\ \le &d_2I''_+(z) - c^*I'_+(z) + \beta I_+(z-c^*\tau) - \gamma I_+(z) \\ = &- L_1 e^{\lambda^* z}\left[F(\lambda^*, c^*)z + F_\lambda(\lambda^*, c^*)\right] \\ = &0. \end{aligned}$$

If $z > z_1$, then $I_+(z-c^*\tau) \le I_+(z) = \bar{I} = \frac{\beta-\gamma}{\gamma}S_0$ and $S_+(z) = S_0$. It follows that

$$d_2I''_+(z) - c^*I'_+(z) + \frac{\beta S_+(z)I_+(z-c^*\tau)}{S_+(z)+I_+(z-c^*\tau)} - \gamma I_+(z) \le \frac{\beta S_0 \bar{I}}{S_0+\bar{I}} - \gamma \bar{I} = 0.$$

Proof of (2.3). Choose large enough $L_3 > 0$ and $\epsilon_2 \in (0, \min\{c^*/d_3, \lambda^*\})$ such that

$$d_3\epsilon_2^2 - c^*\epsilon_2 - \frac{\gamma L_1}{L_3} z e^{(\lambda^*-\epsilon_2)z} \le 0, \quad z < z_1, \tag{2.7}$$

and

$$d_3\epsilon_2^2 - c^*\epsilon_2 + \frac{\gamma \bar{I}}{L_3} e^{-\epsilon_2 z} \le 0, \quad z \ge z_1. \tag{2.8}$$

If $z < z_1$, then $R_+(z) = L_3 e^{\epsilon_2 z}$ and $I_+(z) = -L_1 z e^{\lambda^* z}$. We have from (2.7) that

$$\begin{aligned} &d_3R''_+(z) - c^*R'_+(z) + \gamma I_+(z) \\ = &d_3L_3\epsilon_2^2 e^{\epsilon_2 z} - c^*L_3\epsilon_2 e^{\epsilon_2 z} - \gamma L_1 z e^{\lambda^* z} \\ = &L_3 e^{\epsilon_2 z}\left[d_3\epsilon_2^2 - c^*\epsilon_2 - \frac{\gamma L_1}{L_3} z e^{(\lambda^*-\epsilon_2)z}\right] \\ \le &0, \quad z < z_1. \end{aligned}$$

If $z \ge z_1$, then $R_+ = L_3 e^{\epsilon_2 z}$ and $I_+(z) = \bar{I}$. We infer from (2.8) that

$$\begin{aligned} &d_3R''_+(z) - c^*R'_+(z) + \gamma I_+(z) \\ = &d_3L_3\epsilon_2^2 e^{\epsilon_2 z} - c^*L_3\epsilon_2 e^{\epsilon_2 z} + \gamma \bar{I} \\ = &L_3 e^{\epsilon_2 z}\left[d_3\epsilon_2^2 - c^*\epsilon_2 + \frac{\gamma \bar{I}}{L_3} e^{-\epsilon_2 z}\right] \\ \le &0, \quad z \ge z_1. \end{aligned}$$

Proof of (2.4). Let $q > S_0$ be large enough and $\epsilon_1 > 0$ such that $S_0 - qe^{\lambda_1 z} = \epsilon_1 e^{-\lambda_2 z}$ admits two negative real roots and we select z_2 as the bigger one. Again let $q > S_0$ be large enough such that

$$q\lambda_1(c^* - d_1\lambda_1) + \beta L_1(z - c^*\tau)e^{(\lambda^*-\lambda_1)z-\lambda^* c^*\tau} > 0, \quad z < z_2. \tag{2.9}$$

If $z < z_2$, then $S_-(z) = S_0 - qe^{\lambda_1 z} > 0$. From (2.9), we compute that

$$\begin{aligned} & d_1 S''_-(z) - c^* S'_-(z) - \frac{\beta S_-(z) I_+(z - c^*\tau)}{S_-(z) + I_+(z - c^*\tau)} \\ \geq & d_1 S''_-(z) - c^* S'_-(z) - \beta I_+(z - c^*\tau) \\ \geq & - d_1 q\lambda_1^2 e^{\lambda_1 z} + c^*\lambda_1 q e^{\lambda_1 z} + \beta L_1(z - c^*\tau)e^{\lambda^*(z-c^*\tau)} \\ = & e^{\lambda_1 z}\left[q\lambda_1(c^* - d_1\lambda_1) + \beta L_1(z - c^*\tau)e^{(\lambda^*-\lambda_1)z-\lambda^* c^*\tau}\right] \\ \geq & 0, \quad z < z_2. \end{aligned}$$

If $z > z_2$, then $S_-(z) = \epsilon_1 e^{-\lambda_2 z}$. Using the definition of λ_2, we have that

$$\begin{aligned} & d_1 S''_-(z) - c^* S'_-(z) - \frac{\beta S_-(z) I_+(z - c^*\tau)}{S_-(z) + I_+(z - c^*\tau)} \\ \geq & \ d_1 S''_-(z) - c^* S'_-(z) - \beta S_-(z) \\ \geq & \ d_1\epsilon_1\lambda_2^2 e^{-\lambda_2 z} + c^*\epsilon_1\lambda_2 e^{-\lambda_2 z} - \beta\epsilon_1 e^{-\lambda_2 z} \\ = & \ \epsilon_1 e^{-\lambda_2 z}(d_1\lambda_2^2 + c^*\lambda_2 - \beta) \\ \geq & \ 0, \quad z > z_2. \end{aligned}$$

Proof of (2.5). Select large enough $L_2 > 0$ such that $z_3 < z_2$, $z_3 < z_1$, $S_0 - qe^{\lambda_1 z_3} \geq S_0/2$ and

$$2S_0^{-1} L_1^2 (-z)^{\frac{3}{2}} (-z + c^*\tau)^2 e^{\lambda^*(z-c^*\tau)} < \frac{1}{16} L_2 (c^*)^2 \tau^2, \quad z < z_3. \tag{2.10}$$

Then we get for $z < z_3$ that

$$I_-(z) = I_+(z) - L_2(-z)^{\frac{1}{2}} e^{\lambda^* z}, \quad S_-(z) \geq S_0/2. \tag{2.11}$$

An elementary computation gives

$$c^* I'_-(z) = c^* I'_+(z) + c^* L_2 e^{\lambda^* z}\left[\frac{1}{2}(-z)^{-\frac{1}{2}} - \lambda^*(-z)^{\frac{1}{2}}\right], \tag{2.12}$$

and

$$\begin{aligned} d_2 I''_-(z) &= d_2 I''_+(z) + d_2 L_2 e^{\lambda^* z}\left[\frac{1}{4}(-z)^{-\frac{3}{2}} + \lambda^*(-z)^{-\frac{1}{2}} - (\lambda^*)^2(-z)^{\frac{1}{2}}\right] \\ &\geq d_2 I''_+(z) + d_2 L_2 e^{\lambda^* z}\left[\lambda^*(-z)^{-\frac{1}{2}} - (\lambda^*)^2(-z)^{\frac{1}{2}}\right], \quad z < z_3. \end{aligned} \tag{2.13}$$

By Taylor's theorem, we obtain for $z < z_3$ that

$$(-z + c^*\tau)^{\frac{1}{2}} \leq (-z)^{\frac{1}{2}} + \frac{1}{2}c^*\tau(-z)^{-\frac{1}{2}} - \frac{1}{8}(c^*)^2\tau^2(-z)^{-\frac{3}{2}} + \frac{1}{16}(c^*)^3\tau^3(-z)^{-\frac{5}{2}}. \tag{2.14}$$

From (2.11), we have that

$$\begin{aligned} & -\beta I_-(z-c^*\tau)+\frac{\beta S_-(z)I_-(z-c^*\tau)}{S_-(z)+I_-(z-c^*\tau)} \\ =& -\frac{\beta I_-^2(z-c^*\tau)}{S_-(z)+I_-(z-c^*\tau)} \\ \geq & -\frac{\beta I_+^2(z-c^*\tau)}{S_-(z)} \\ \geq & -2S_0^{-1}\beta L_1^2(-z+c^*\tau)^2 e^{2\lambda^*(z-c^*\tau)}, \quad z<z_3. \end{aligned} \tag{2.15}$$

By (1.5) and (2.10)-(2.15), we derive for $z<z_3$ that

$$\begin{aligned} & d_2I_-''(z)-c^*I_-'(z)+\frac{\beta S_-(z)I_-(z-c^*\tau)}{S_-(z)+I_-(z-c^*\tau)}-\gamma I_-(z) \\ =& d_2I_-''(z)-c^*I_-'(z)+\beta I_-(z-c^*\tau)-\gamma I_-(z) \\ & -\beta I_-(z-c^*\tau)+\frac{\beta S_-(z)I_-(z-c^*\tau)}{S_-(z)+I_-(z-c^*\tau)} \\ \geq & d_2I_+''(z)-c^*I_+'(z)+\beta I_+(z-c^*\tau)-\gamma I_+(z) \\ & +d_2L_2e^{\lambda^*z}\left[\lambda^*(-z)^{-\frac{1}{2}}-(\lambda^*)^2(-z)^{\frac{1}{2}}\right]-c^*L_2e^{\lambda^*z}\left[\frac{1}{2}(-z)^{-\frac{1}{2}}-\lambda^*(-z)^{\frac{1}{2}}\right] \\ & -\beta L_2(-z+c^*\tau)^{\frac{1}{2}}e^{\lambda^*(z-c^*\tau)}+\gamma L_2(-z)^{\frac{1}{2}}e^{\lambda^*z} \\ & -2S_0^{-1}\beta L_1^2(-z+c^*\tau)^2e^{2\lambda^*(z-c^*\tau)} \\ \geq & -L_1e^{\lambda^*z}\left[F(\lambda^*,c^*)z+F_\lambda(\lambda^*,c^*)\right]+d_2L_2e^{\lambda^*z}\left[\lambda^*(-z)^{-\frac{1}{2}}-(\lambda^*)^2(-z)^{\frac{1}{2}}\right] \\ & -c^*L_2e^{\lambda^*z}\left[\frac{1}{2}(-z)^{-\frac{1}{2}}-\lambda^*(-z)^{\frac{1}{2}}\right]+\gamma L_2(-z)^{\frac{1}{2}}e^{\lambda^*z} \\ & -2S_0^{-1}\beta L_1^2(-z+c^*\tau)^2e^{2\lambda^*(z-c^*\tau)}-\beta L_2\left[(-z)^{\frac{1}{2}}+\frac{1}{2}c^*\tau(-z)^{-\frac{1}{2}}\right. \\ & \left.-\frac{1}{8}(c^*)^2\tau^2(-z)^{-\frac{3}{2}}+\frac{1}{16}(c^*)^3\tau^3(-z)^{-\frac{5}{2}}\right]e^{\lambda^*(z-c^*\tau)} \\ =& L_2e^{\lambda^*z}(-z)^{-\frac{1}{2}}\left[\frac{1}{2}F_\lambda(\lambda^*,c^*)+F(\lambda^*,c^*)z\right] \\ & +\beta L_2e^{\lambda^*(z-c^*\tau)}\left[\frac{1}{8}(c^*)^2\tau^2(-z)^{-\frac{3}{2}}-\frac{1}{16}(c^*)^3\tau^3(-z)^{-\frac{5}{2}}\right] \\ & -2S_0^{-1}\beta L_1^2(-z+c^*\tau)^2e^{2\lambda^*(z-c^*\tau)} \\ =& \beta(-z)^{-\frac{3}{2}}e^{\lambda^*(z-c^*\tau)}\left[\frac{1}{16}L_2(c^*)^2\tau^2-2S_0^{-1}L_1^2(-z)^{\frac{3}{2}}(-z+c^*\tau)^2e^{\lambda^*(z-c^*\tau)}\right] \\ & +\frac{1}{16}\beta L_2(c^*)^2\tau^2(-z)^{-\frac{3}{2}}e^{\lambda^*(z-c^*\tau)}\left(1+\frac{c^*\tau}{z}\right) \\ \geq & 0. \end{aligned}$$

If $z>z_3$, then $I_-(z)=0$ and inequality (2.5) holds trivially. □

Introduce a functional space

$$B_\mu(\mathbb{R},\mathbb{R}^3) := \Big\{\varphi(z) = (\varphi_1(z),\varphi_2(z),\varphi_3(z)) \in C(\mathbb{R},\mathbb{R}^3) : \sup_{z\in\mathbb{R}} |\varphi_i(z)|e^{-\mu|z|} < \infty, \quad i = 1,2,3\Big\},$$

endowed with the norm $|\varphi|_\mu := \max\big\{\sup_{z\in\mathbb{R}} |\varphi_i(z)|e^{-\mu|z|},\ i = 1,2,3\big\}$, where $\mu \in (\epsilon_2, \mu_0)$ is a constant and μ_0 will be specified later. Define a cone by

$$\mathcal{S} := \left\{ (S,I,R)(z) \in B_\mu(\mathbb{R},\mathbb{R}^3) \;\middle|\; \begin{array}{l} S_-(z) \le S(z) \le S_+(z), \\ I_-(z) \le I(z) \le I_+(z), \\ R_-(z) \le R(z) \le R_+(z) \end{array} \right\}.$$

It is not difficult to verify that $\mathcal{S}$ is non-empty, bounded, closed and convex in $B_\mu(\mathbb{R},\mathbb{R}^3)$. Let $\alpha > \beta$ be a constant and recall that $\beta > \gamma$, then one can see that:

$$H_1[S,I,R](z) := \alpha S(z) - \frac{\beta S(z)I(z-c^*\tau)}{S(z)+I(z-c^*\tau)}$$

is increasing with respect to S and decreasing with respect to I;

$$H_2[S,I,R](z) := \frac{\beta S(z)I(z-c^*\tau)}{S(z)+I(z-c^*\tau)} + (\alpha-\gamma)I(z)$$

is increasing in both S and I;

$$H_3[S,I,R](z) := \alpha R(z) + \gamma I(z)$$

is increasing in both I and R. For any $(S,I,R) \in \mathcal{S}$, define a nonlinear map $\mathcal{M} := (\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)$ on the space $B_\mu(\mathbb{R},\mathbb{R}^3)$ by

$$\mathcal{M}_i[S,I,R](z) := \frac{1}{\Lambda_i}\Big\{\int_{-\infty}^{z} e^{\sigma_{i1}(z-\eta)}H_i[S,I,R](\eta)d\eta + \int_{z}^{\infty} e^{\sigma_{i2}(z-\eta)}H_i[S,I,R](\eta)d\eta\Big\},$$

where

$$\sigma_{i1} = \frac{c^* - \sqrt{(c^*)^2+4d_i\alpha}}{2d_i},\ \sigma_{i2} = \frac{c^* + \sqrt{(c^*)^2+4d_i\alpha}}{2d_i},\ \Lambda_i = d_i(\sigma_{i2}-\sigma_{i1}),\ i = 1,2,3.$$

Lemma 2.2. *$\mathcal{M}(\mathcal{S}) \subset \mathcal{S}$.*

Proof. Obviously, $(\mathcal{M}_1[S,I,R](z),\mathcal{M}_2[S,I,R](z),\mathcal{M}_3[S,I,R](z)) \in B_\mu(\mathbb{R},\mathbb{R}^3)$ for any $(S,I,R) \in \mathcal{S}$. Then by the monotonicity of H_i $(i = 1,2,3)$, we need to show that

$$S_-(z) \le \mathcal{M}_1[S_-,I_+,R](z) \le \mathcal{M}_1[S,I,R](z) \le \mathcal{M}_1[S_+,I_-,R](z) \le S_+(z), \tag{2.16}$$

$$I_-(z) \le \mathcal{M}_2[S_-,I_-,R](z) \le \mathcal{M}_2[S,I,R](z) \le \mathcal{M}_2[S_+,I_+,R](z) \le I_+(z), \tag{2.17}$$

$$R_-(z) \le \mathcal{M}_3[S,I_-,R_-](z) \le \mathcal{M}_3[S,I,R](z) \le \mathcal{M}_3[S,I_+,R_+](z) \le R_+(z) \tag{2.18}$$

for any $(S,I,R) \in \mathcal{S}$.

Proof of (2.16). Using (2.1) and $S_+(z) = S_0$, we have that

$$\begin{aligned}\mathcal{M}_1[S_+, I_-, R](z) &= \frac{1}{\Lambda_1}\bigg\{\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)} H_1[S_+, I_-, R](\eta)d\eta \\ &\quad + \int_{z}^{\infty} e^{\sigma_{12}(z-\eta)} H_1[S_+, I_-, R](\eta)d\eta\bigg\} \\ &\leq \frac{1}{\Lambda_1}\bigg\{\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)}[\alpha S_+(\eta) + c^* S_+'(\eta) - d_1 S_+''(\eta)]d\eta \\ &\quad + \int_{z}^{\infty} e^{\sigma_{12}(z-\eta)}[\alpha S_+(\eta) + c^* S_+'(\eta) - d_1 S_+''(\eta)]d\eta\bigg\} \\ &= \frac{\alpha S_0}{\Lambda_1}\bigg[\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)}d\eta + \int_{z}^{\infty} e^{\sigma_{12}(z-\eta)}d\eta\bigg] \\ &= S_0, \quad z \in \mathbb{R}.\end{aligned}$$

It follows from (2.4) that

$$\begin{aligned}\mathcal{M}_1[S_-, I_+, R](z) &= \frac{1}{\Lambda_1}\bigg\{\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)} H_1[S_-, I_+, R](\eta)d\eta \\ &\quad + \int_{z}^{\infty} e^{\sigma_{12}(z-\eta)} H_1[S_-, I_+, R](\eta)d\eta\bigg\} \\ &\geq \frac{1}{\Lambda_1}\bigg\{\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)}[\alpha S_-(\eta) + c^* S_-'(\eta) - d_1 S_-''(\eta)]d\eta \\ &\quad + \int_{z}^{\infty} e^{\sigma_{12}(z-\eta)}[\alpha S_-(\eta) + c^* S_-'(\eta) - d_1 S_-''(\eta)]d\eta\bigg\} \\ &= S_-(z) + \frac{e^{\sigma_{11}(z-z_2)}[S_-'(z_2+0) - S_-'(z_2-0)]}{\sigma_{12} - \sigma_{11}} \\ &\geq S_-(z), \quad z \neq z_2.\end{aligned}$$

By the continuity of both $\mathcal{M}_1[S_-, I_+, R](z)$ and $S_-(z)$, we obtain that

$$\mathcal{M}_1[S_-, I_+, R](z) \geq S_-(z), \quad z \in \mathbb{R}.$$

Proof of (2.17). From (2.2) and (2.5), we get that

$$\begin{aligned}\mathcal{M}_2[S_+, I_+, R](z) &= \frac{1}{\Lambda_2}\bigg\{\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)} H_2[S_+, I_+, R](\eta)d\eta \\ &\quad + \int_{z}^{\infty} e^{\sigma_{22}(z-\eta)} H_2[S_+, I_+, R](\eta)d\eta\bigg\} \\ &\leq \frac{1}{\Lambda_2}\bigg\{\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}[\alpha I_+(\eta) + c^* I_+'(\eta) - d_2 I_+''(\eta)]d\eta \\ &\quad + \int_{z}^{\infty} e^{\sigma_{22}(z-\eta)}[\alpha I_+(\eta) + c^* I_+'(\eta) - d_2 I_+''(\eta)]d\eta\bigg\} \\ &= I_+(z) + \frac{e^{\sigma_{21}(z-z_1)}[I_+'(z_1+0) - I_+'(z_1-0)]}{\sigma_{22} - \sigma_{21}} \\ &\leq I_+(z), \quad z \neq z_1,\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_2[S_-,I_-,R](z) &= \frac{1}{\Lambda_2}\Big\{\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}H_2[S_-,I_-,R](\eta)d\eta \\
&\quad + \int_{z}^{\infty} e^{\sigma_{22}(z-\eta)}H_2[S_-,I_-,R](\eta)d\eta\Big\} \\
&\geq \frac{1}{\Lambda_2}\Big\{\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}[\alpha I_-(\eta)+c^*I_-'(\eta)-d_2I_-''(\eta)]d\eta \\
&\quad + \int_{z}^{z_3} e^{\sigma_{22}(z-\eta)}[\alpha I_-(\eta)+c^*I_-'(\eta)-d_2I_-''(\eta)]d\eta\Big\} \\
&= \frac{1}{\Lambda_2}\Big\{\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}\Big[(d_2(\lambda^*)^2-c^*\lambda^*-\alpha)L_1\eta e^{\lambda^*\eta} \\
&\quad + (d_2(\lambda^*)^2-c^*\lambda^*-\alpha)L_2(-\eta)^{\frac{1}{2}}e^{\lambda^*\eta}-\frac{1}{4}d_2L_2(-\eta)^{-\frac{3}{2}}e^{\lambda^*\eta}\Big]d\eta \\
&\quad + \int_{z}^{z_3} e^{\sigma_{22}(z-\eta)}\Big[(d_2(\lambda^*)^2-c^*\lambda^*-\alpha)L_1\eta e^{\lambda^*\eta} \\
&\quad + (d_2(\lambda^*)^2-c^*\lambda^*-\alpha)L_2(-\eta)^{\frac{1}{2}}e^{\lambda^*\eta}-\frac{1}{4}d_2L_2(-\eta)^{-\frac{3}{2}}e^{\lambda^*\eta}\Big]d\eta\Big\} \\
&= [-L_1z-L_2(-z)^{\frac{1}{2}}]e^{\lambda^*z}+\frac{L_1d_2}{2\Lambda_2}e^{\lambda^*z_3}e^{\sigma_{22}(z-z_3)} \\
&\geq I_-(z), \quad z<z_3,
\end{aligned}$$

and

$$\mathcal{M}_2[S_-,I_-,R](z) = \frac{1}{\Lambda_2}\int_{-\infty}^{z_3} e^{\sigma_{21}(z-\eta)}H_2[S_-,I_-,R](\eta)d\eta \geq I_-(z), \quad z>z_3.$$

Using the continuity of both $\mathcal{M}_2[S_\pm,I_\pm,R](z)$ and $I_\pm(z)$, we have that

$$\mathcal{M}_2[S_+,I_+,R](z)\leq I_+(z), \quad \mathcal{M}_2[S_-,I_-,R](z)\geq I_-(z), \quad z\in\mathbb{R}.$$

Proof of (2.18). From (2.3), (2.6) and the expressions of $R_\pm(z)$, we deduce that

$$\begin{aligned}
\mathcal{M}_3[S,I_+,R_+](z) &= \frac{1}{\Lambda_3}\Big\{\int_{-\infty}^{z} e^{\sigma_{31}(z-\eta)}H_3[S,I_+,R_+](\eta)d\eta \\
&\quad + \int_{z}^{\infty} e^{\sigma_{32}(z-\eta)}H_3[S,I_+,R_+](\eta)d\eta\Big\} \\
&\leq \frac{1}{\Lambda_3}\Big\{\int_{-\infty}^{z} e^{\sigma_{31}(z-\eta)}[\alpha R_+(\eta)+c^*R_+'(\eta)-d_3R_+''(\eta)]d\eta \\
&\quad + \int_{z}^{\infty} e^{\sigma_{32}(z-\eta)}[\alpha R_+(\eta)+c^*R_+'(\eta)-d_3R_+''(\eta)]d\eta\Big\} \\
&= R_+(z), \quad z\in\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_3[S,I_-,R_-](z) &= \frac{1}{\Lambda_3}\Big\{\int_{-\infty}^{z} e^{\sigma_{31}(z-\eta)}H_3[S,I_-,R_-](\eta)d\eta \\
&\quad + \int_{z}^{\infty} e^{\sigma_{32}(z-\eta)}H_3[S,I_-,R_-](\eta)d\eta\Big\}
\end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{\Lambda_3}\bigg\{\int_{-\infty}^{z} e^{\sigma_{31}(z-\eta)}[\alpha R_-(\eta)+c^*R_-'(\eta)-d_3R_-''(\eta)]d\eta\\&\quad+\int_z^{\infty} e^{\sigma_{32}(z-\eta)}[\alpha R_-(\eta)+c^*R_-'(\eta)-d_3R_-''(\eta)]d\eta\bigg\}\\&=R_-(z),\quad z\in\mathbb{R}.\end{aligned}$$

The proof is finished. □

Lemma 2.3. *The map* $\mathcal{M}:=(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)$ *is completely continuous with respect to the norm* $|\cdot|_\mu$ *in* $B_\mu(\mathbb{R},\mathbb{R}^3)$*.*

Proof. For any $\Psi_1=(S_1,I_1,R_1)\in\mathcal{S}$ and $\Psi_2=(S_2,I_2,R_2)\in\mathcal{S}$, we derive that

$$\begin{aligned}&|H_1(S_1,I_1,R_1)(z)-H_1(S_2,I_2,R_2)(z)|e^{-\mu|z|}\\\leq&(\alpha+\beta)|S_1(z)-S_2(z)|e^{-\mu|z|}+\beta|I_1(z-c^*\tau)-I_2(z-c^*\tau)|e^{-\mu|z|}\\\leq&(\alpha+\beta)|S_1-S_2|_\mu+\beta e^{\mu c^*\tau}|I_1-I_2|_\mu\\\leq& l|\Psi_1-\Psi_2|_\mu,\\&|H_2(S_1,I_1,R_1)(z)-H_2(S_2,I_2,R_2)(z)|e^{-\mu|z|}\\\leq&\beta|S_1-S_2|_\mu+(\alpha+\beta e^{\mu c^*\tau}+\gamma)|I_1-I_2|_\mu\\\leq& l|\Psi_1-\Psi_2|_\mu,\end{aligned}$$

and

$$\begin{aligned}&|H_3(S_1,I_1,R_1)(z)-H_3(S_2,I_2,R_2)(z)|e^{-\mu|z|}\\\leq&\alpha|R_1-R_2|_\mu+\gamma|I_1-I_2|_\mu\\\leq& l|\Psi_1-\Psi_2|_\mu,\end{aligned}$$

where $l=\alpha+\beta+\gamma+\beta e^{\mu c^*\tau}$. Then choosing $\mu\in(\epsilon_2,-\sigma_{i1})$, we have that

$$\begin{aligned}&|\mathcal{M}_i[S_1,I_1,R_1](z)-\mathcal{M}_i[S_2,I_2,R_2](z)|e^{-\mu|z|}\\\leq&\frac{1}{\Lambda_i}|H_i(S_1,I_1,R_1)-H_i(S_2,I_2,R_2)|_\mu\bigg[\int_{-\infty}^{z}e^{\sigma_{i1}(z-\eta)}e^{\mu|\eta|-\mu|z|}d\eta\\&+\int_z^{\infty}e^{\sigma_{i2}(z-\eta)}e^{\mu|\eta|-\mu|z|}d\eta\bigg]\\\leq&\frac{l}{\Lambda_i}|\Psi_1-\Psi_2|_\mu\bigg[\int_{-\infty}^{z}e^{\sigma_{i1}(z-\eta)}e^{\mu|\eta-z|}d\eta+\int_z^{\infty}e^{\sigma_{i2}(z-\eta)}e^{\mu|\eta-z|}d\eta\bigg]\\=&\frac{l(2\mu+\sigma_{i1}-\sigma_{i2})}{d_i(\sigma_{i2}-\sigma_{i1})(\sigma_{i2}-\mu)(\sigma_{i1}+\mu)}|\Psi_1-\Psi_2|_\mu,\quad i=1,2,3,\end{aligned}$$

which implies that $\mathcal{M}$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$.

For any $(S,I,R)\in\mathcal{S}$, we deduce for $z\in\mathbb{R}$ that

$$\begin{aligned}\left|\frac{d\mathcal{M}_1[S,I,R](z)}{dz}\right|=&\bigg|\frac{\sigma_{11}}{\Lambda_1}\int_{-\infty}^{z}e^{\sigma_{11}(z-\eta)}H_1[S,I,R](\eta)d\eta\\&+\frac{\sigma_{12}}{\Lambda_1}\int_z^{\infty}e^{\sigma_{12}(z-\eta)}H_1[S,I,R](\eta)d\eta\bigg|\end{aligned}$$

$$\begin{aligned}
&\leq -\frac{\sigma_{11}}{\Lambda_1}\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)}H_1[S,I,R](\eta)d\eta \\
&\quad + \frac{\sigma_{12}}{\Lambda_1}\int_{z}^{\infty} e^{\sigma_{12}(z-\eta)}H_1[S,I,R](\eta)d\eta \\
&\leq -\frac{\sigma_{11}\alpha S_0}{\Lambda_1}\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)}d\eta + \frac{\sigma_{12}\alpha S_0}{\Lambda_1}\int_{z}^{\infty} e^{\sigma_{12}(z-\eta)}d\eta \\
&= \frac{2\alpha S_0}{\Lambda_1}, \qquad (2.19)
\end{aligned}$$

$$\begin{aligned}
\left|\frac{d\mathcal{M}_2[S,I,R](z)}{dz}\right| &= \left|\frac{\sigma_{21}}{\Lambda_2}\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}H_2[S,I,R](\eta)d\eta \right.\\
&\quad \left. + \frac{\sigma_{22}}{\Lambda_2}\int_{z}^{\infty} e^{\sigma_{22}(z-\eta)}H_2[S,I,R](\eta)d\eta\right| \\
&\leq -\frac{\sigma_{21}}{\Lambda_2}\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}H_2[S,I,R](\eta)d\eta \\
&\quad + \frac{\sigma_{22}}{\Lambda_2}\int_{z}^{\infty} e^{\sigma_{22}(z-\eta)}H_2[S,I,R](\eta)d\eta \\
&\leq -\frac{\sigma_{21}(\alpha+\beta-\gamma)\bar{I}}{\Lambda_2}\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)}d\eta \\
&\quad + \frac{\sigma_{22}(\alpha+\beta-\gamma)\bar{I}}{\Lambda_2}\int_{z}^{\infty} e^{\sigma_{22}(z-\eta)}d\eta \\
&= \frac{2(\alpha+\beta-\gamma)\bar{I}}{\Lambda_2}, \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{d\mathcal{M}_3[S,I,R](z)}{dz}\right| &= \left|\frac{\sigma_{31}}{\Lambda_3}\int_{-\infty}^{z} e^{\sigma_{31}(z-\eta)}H_3[S,I,R](\eta)d\eta \right.\\
&\quad \left. + \frac{\sigma_{32}}{\Lambda_3}\int_{z}^{\infty} e^{\sigma_{32}(z-\eta)}H_3[S,I,R](\eta)d\eta\right| \\
&\leq -\frac{\sigma_{31}}{\Lambda_3}\int_{-\infty}^{z} e^{\sigma_{31}(z-\eta)}(\alpha L_3 e^{\epsilon_2\eta}+\gamma\bar{I})d\eta \qquad (2.21)\\
&\quad + \frac{\sigma_{32}}{\Lambda_3}\int_{z}^{\infty} e^{\sigma_{32}(z-\eta)}(\alpha L_3 e^{\epsilon_2\eta}+\gamma\bar{I})d\eta \\
&\leq \left|\frac{\alpha L_3[2\sigma_{31}\sigma_{32}-\epsilon_2(\sigma_{31}+\sigma_{32})]}{\Lambda_3(\epsilon_2-\sigma_{31})(\epsilon_2-\sigma_{32})}\right| e^{\epsilon_2 z} + \frac{2\gamma\bar{I}}{\Lambda_3}.
\end{aligned}$$

By Lemma 2.2, we get that $\big|\mathcal{M}_1[S,I,R](z)\big|+\big|\mathcal{M}_2[S,I,R](z)\big|+\big|\mathcal{M}_3[S,I,R](z)\big| \leq S_0+\bar{I}+L_3e^{\epsilon_2 z}$ on $\mathbb{R}$. Recall that $\mu > \epsilon_2$. Then for any $\varepsilon > 0$, there is a sufficiently large number $N > 0$ such that

$$\begin{aligned}
&\Big\{\big|\mathcal{M}_1[S,I,R](z)\big| + \big|\mathcal{M}_2[S,I,R](z)\big| + \big|\mathcal{M}_3[S,I,R](z)\big|\Big\}e^{-\mu|z|} \\
&\leq (S_0+\bar{I}+L_3e^{\epsilon_2 z})e^{-\mu|z|} \qquad (2.22)\\
&< (S_0+\bar{I})e^{-\mu N} + L_3 e^{(\epsilon_2-\mu)N} \\
&< \varepsilon, \quad |z| > N.
\end{aligned}$$

Applying (2.19)-(2.21) and Arzerà-Ascoli theorem, one can choose finite elements in $\mathcal{M}(\mathcal{S})$ such that they are a finite ε-net of $\mathcal{M}(\mathcal{S})(z)$ on $[-N, N]$ with the supremum norm, which is also a finite ε-net of $\mathcal{M}(\mathcal{S})(z)$ on $\mathbb{R}$ with the decay norm $|\cdot|_\mu$ (see (2.22)). Hence $\mathcal{M}$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^3)$. □

Utilizing Lemmas 2.2, 2.3 and Schauder's fixed point theorem, we obtain that $\mathcal{M}$ has a fixed point $(S, I, R)(z) \in \mathcal{S}$, which is a solution of

$$\begin{cases} d_1 S''(z) - c^* S'(z) - \dfrac{\beta S(z) I(z - c^*\tau)}{S(z) + I(z - c^*\tau)} = 0, \\ d_2 I''(z) - c^* I'(z) + \dfrac{\beta S(z) I(z - c^*\tau)}{S(z) + I(z - c^*\tau)} - \gamma I(z) = 0, \\ d_3 R''(z) - c^* R'(z) + \gamma I(z) = 0. \end{cases} \tag{2.23}$$

Then the following existence result holds.

Proposition 2.1. *If $\beta > \gamma$ and $c = c^*$, then system (1.4) admits a traveling wave solution $(S, I, R)(z)$ such that*

$$\begin{aligned} &S_-(z) \le S(z) \le S_+(z),\ I_-(z) \le I(z) \le I_+(z), \\ &R_-(z) \le R(z) \le R_+(z),\ z \in \mathbb{R}. \end{aligned} \tag{2.24}$$

3. Properties of critical traveling wave solution

In this section, we focus on some properties of critical traveling wave solution of (1.4).

Proposition 3.1. *Let $(S, I, R)(z)$ be the critical traveling wave solution of (1.4) satisfying (2.24). Then*

(1) $S(z) > 0$, $I(z) > 0$ and $R(z) > 0$ on $\mathbb{R}$;

(2) $S(-\infty) = S_0$, $I(-\infty) = 0$, $R(-\infty) = 0$ and $I(z) = O(-ze^{\lambda^* z})$ as $z \to -\infty$;

(3) $S(z)$ is strictly decreasing and $R(z)$ is strictly increasing on $\mathbb{R}$; $S'(z)$, $I'(z)$, $R'(z)$, $S''(z)$, $I''(z)$, $R''(z) \to 0$ as $z \to \pm\infty$; $S(\infty) = 0$, $I(\infty) = 0$ and $R(\infty) = S_0$; $\gamma \int_{\mathbb{R}} I(z)dz = \beta \int_{\mathbb{R}} \frac{S(z)I(z-c^*\tau)}{S(z)+I(z-c^*\tau)} dz = c^* S_0$;

(4) $S(z) < S_0$, $I(z) < \frac{\beta-\gamma}{\gamma} S_0$ and $R(z) < S_0$ on $\mathbb{R}$.

Proof. (1) From (2.24), we have that $S(z) > 0$ on $\mathbb{R}$. Suppose that $I(\hat{z}) = 0$ for some $\hat{z} \in \mathbb{R}$. Then there are two constants $a, b \in \mathbb{R}$ such that $a < z_3 \le b$ and $a < \hat{z} < b$, which implies that $I(z)$ attains its minimum in (a, b). It follows from the second equation in (2.23) that $-d_2 I''(z) + c^* I'(z) + \gamma I(z) \ge 0$ for $z \in [a, b]$. By the strong maximum principle, we deduce that $I(z) \equiv 0$ for $z \in [a, b]$, which contradicts the fact that $I(z) \ge I_-(z) > 0$ for $z \in [a, z_3)$. Thus $I(z) > 0$ on $\mathbb{R}$. Assume that $R(\tilde{z}) = 0$ for some $\tilde{z} \in \mathbb{R}$, then $R'(\tilde{z}) = 0$ and $R''(\tilde{z}) \ge 0$. We infer from the third equation in (2.23) that $I(\tilde{z}) \le 0$, which contradicts the positiveness of $I(z)$ on $\mathbb{R}$. This implies that $R(z) > 0$ on $\mathbb{R}$.

(2) From (2.24), we get that

$$\begin{aligned} &S_0 - qe^{\lambda_1 z} \le S(z) \le S_0, \quad [-L_1 z - L_2(-z)^{\frac{1}{2}}]e^{\lambda^* z} \le I(z) \le -L_1 z e^{\lambda^* z}, \\ &0 \le R(z) \le L_3 e^{\epsilon_2 z}, \quad z \in \mathbb{R}. \end{aligned}$$

Then using sandwich rule yields that

$$S(-\infty) = S_0,\ I(-\infty) = 0,\ R(-\infty) = 0 \text{ and } I(z) = O(-ze^{\lambda^* z}) \text{ as } z \to -\infty. \tag{3.1}$$

(3) Since $S(z)$ and $I(z)$ are uniformly bounded on $\mathbb{R}$, we have from the first two equations in (2.23) that

$$\begin{cases} S(z) = \dfrac{1}{\Lambda_1}\Big\{ \displaystyle\int_{-\infty}^{z} e^{\sigma_{11}(z-\eta)} H_1[S,I,R](\eta)d\eta + \int_{z}^{\infty} e^{\sigma_{12}(z-\eta)} H_1[S,I,R](\eta)d\eta \Big\}, \\ I(z) = \dfrac{1}{\Lambda_2}\Big\{ \displaystyle\int_{-\infty}^{z} e^{\sigma_{21}(z-\eta)} H_2[S,I,R](\eta)d\eta + \int_{z}^{\infty} e^{\sigma_{22}(z-\eta)} H_2[S,I,R](\eta)d\eta \Big\}, \end{cases} \tag{3.2}$$

where

$$H_1[S,I,R](\eta) = \alpha S(\eta) - \frac{\beta S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)},$$

and

$$H_2[S,I,R](\eta) = \frac{\beta S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} + (\alpha - \gamma) I(\eta).$$

Using L'Hôpital rule in (3.2) gives

$$S'(\pm\infty) = 0 \quad \text{and} \quad I'(\pm\infty) = 0. \tag{3.3}$$

Integrating the first equation in (2.23) from $-\infty$ to z, using (3.1) and (3.3), we have that

$$\begin{aligned} \beta \int_{-\infty}^{z} \frac{S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} d\eta &= -c^*[S(z) - S_0] + d_1 S'(z) \\ &\le c^* S_0 + d_1 S'(z), \quad z \in \mathbb{R}. \end{aligned}$$

Again integrating the second equation in (2.23) from $-\infty$ to z, utilizing (3.1) and (3.3), we get that

$$\begin{aligned} \int_{-\infty}^{z} I(\eta)d\eta &= \frac{d_2}{\gamma} I'(z) - \frac{c^*}{\gamma} I(z) + \frac{\beta}{\gamma} \int_{-\infty}^{z} \frac{S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} d\eta \\ &\le \frac{d_2}{\gamma} I'(z) + \frac{\beta}{\gamma} \int_{-\infty}^{z} \frac{S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} d\eta \\ &\le \frac{d_2}{\gamma} I'(z) + \frac{c^*}{\gamma} S_0 + \frac{d_1}{\gamma} S'(z), \quad z \in \mathbb{R}. \end{aligned}$$

Then by the virtue of (3.3), we further obtain $\int_{\mathbb{R}} I(z)dz < \infty$, which together with the boundedness of $I'(z)$ on $\mathbb{R}$ (see (3.3)) implies that

$$I(\infty) = 0. \tag{3.4}$$

Notice from the first equation in (2.23) that

$$\Big[e^{-\frac{c^*}{d_1} z} S'(z) \Big]' = \frac{\beta}{d_1} e^{-\frac{c^*}{d_1} z} \frac{S(z) I(z - c^*\tau)}{S(z) + I(z - c^*\tau)}. \tag{3.5}$$

Integrating (3.5) from z to ∞, utilizing $S'(\infty) = 0$ and $S(z), I(z) > 0$ on $\mathbb{R}$, we deduce

$$S'(z) = -\frac{\beta}{d_1} \int_{z}^{\infty} e^{\frac{c^*}{d_1}(z-\eta)} \frac{S(\eta) I(\eta - c^*\tau)}{S(\eta) + I(\eta - c^*\tau)} d\eta < 0, \tag{3.6}$$

which means that $S(z)$ is strictly decreasing on $\mathbb{R}$. This together with $S(z) > 0$ on $\mathbb{R}$ gives that the limit $S(\infty)$ exists. Moreover, inspired by [13, 14], we claim that $S(\infty) = 0$. If not, we suppose that $S(\infty) > 0$. Using the monotonicity of $S(z)$ on $\mathbb{R}$, we have that $I(x,t) = I(x + c^*t)$ satisfies

$$\begin{cases} I_t(x,t) \geq d_2 I_{xx}(x,t) + \dfrac{\beta S(\infty) I(x,t-\tau)}{S(\infty) + I(x,t-\tau)} - \gamma I(x,t), \ (x,t) \in \mathbb{R} \times (0,\infty), \\ I(x,s) = I(x + c^*s) > 0, \ (x,s) \in \mathbb{R} \times [-\tau, 0]. \end{cases} \tag{3.7}$$

From the comparison principle (Theorem 2.2 in [15]), we get that $I(x,t)$ is an upper solution of the following initial value problem

$$\begin{cases} v_t(x,t) = d_2 v_{xx}(x,t) + \dfrac{\beta S(\infty) v(x,t-\tau)}{S(\infty) + v(x,t-\tau)} - \gamma v(x,t), \ (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,s) = I(x + c^*s) > 0, \ (x,s) \in \mathbb{R} \times [-\tau, 0]. \end{cases} \tag{3.8}$$

Applying the theory of asymptotic spreading (Theorem 2.5 in [16]) yields

$$\liminf_{t\to\infty} v(0,t) = \frac{\beta - \gamma}{\gamma} S(\infty) > 0. \tag{3.9}$$

Thus

$$\liminf_{t\to\infty} I(0,t) \geq \liminf_{t\to\infty} v(0,t) > 0. \tag{3.10}$$

By the invariant form of $I(z)$ and (3.10), we get

$$\liminf_{t\to\infty} I(0,t) = \liminf_{z\to\infty} I(z) > 0,$$

which contradicts (3.4). Thus

$$S(\infty) = 0. \tag{3.11}$$

Moreover, an integration of the first equation in (2.23) over $\mathbb{R}$ gives

$$\beta \int_{\mathbb{R}} \frac{S(z)I(z - c^*\tau)}{S(z) + I(z - c^*\tau)} dz = c^* S_0, \tag{3.12}$$

where we have used (3.1), (3.3) and (3.11). Another integration of the second equation in (2.23) over $\mathbb{R}$ yields

$$\gamma \int_{\mathbb{R}} I(z)dz = \beta \int_{\mathbb{R}} \frac{S(z)I(z - c^*\tau)}{S(z) + I(z - c^*\tau)} dz, \tag{3.13}$$

since $I(\pm\infty) = I'(\pm\infty) = 0$. Solving the third equation in (2.23) and using $R(-\infty) = 0$ lead to

$$R(z) = Ce^{\frac{c^*}{d_3}z} + \frac{\gamma}{c^*} \int_{-\infty}^{z} I(\eta)d\eta + \frac{\gamma}{c^*} \int_{z}^{0} e^{\frac{c^*}{d_3}(z-\eta)} I(\eta)d\eta,$$

where C is a constant of integration. Since $R(z) \leq L_3 e^{\epsilon_2 z}$ and $\epsilon_2 < c^*/d_3$ (see the proof of Lemma 2.1), we obtain

$$R(z) = \frac{\gamma}{c^*} \int_{-\infty}^{z} I(\eta)d\eta + \frac{\gamma}{c^*} \int_{z}^{\infty} e^{\frac{c^*}{d_3}(z-\eta)} I(\eta)d\eta. \tag{3.14}$$

We infer from (3.12)-(3.14) and L'Hôpital's rule that

$$R(\infty) = \frac{\gamma}{c^*}\int_{\mathbb{R}} I(z)dz = S_0. \tag{3.15}$$

Differentiating (3.14) with respect to z and using $I(z) > 0$ on $\mathbb{R}$, we have

$$R'(z) = \frac{\gamma}{d_3}\int_z^{\infty} e^{\frac{c^*}{d_3}(z-\eta)} I(\eta)d\eta > 0, \tag{3.16}$$

which means that $R(z)$ is strictly increasing on $\mathbb{R}$. Combining (3.16), $I(\pm\infty) = 0$ and L'Hôpital's rule yields

$$R'(\pm\infty) = 0. \tag{3.17}$$

Note from (3.1), (3.3), (3.4), (3.11), (3.15), (3.17) and (2.23) that

$$S''(\pm\infty) = 0, \quad I''(\pm\infty) = 0 \quad \text{and} \quad R''(\pm\infty) = 0. \tag{3.18}$$

(4) Since $S(z)$ is strictly decreasing and $R(z)$ is strictly increasing on $\mathbb{R}$, we obtain $S(z) < S_0$ and $R(z) < S_0$ for $z \in \mathbb{R}$. Now we claim that $I(z) < \frac{\beta-\gamma}{\gamma}S_0$ on $\mathbb{R}$. For contradiction, we assume that $I(\acute{z}) = \frac{\beta-\gamma}{\gamma}S_0$ for some $\acute{z} \in \mathbb{R}$. Then $I'(\acute{z}) = 0$ and $I''(\acute{z}) \le 0$. By the second equation in (2.23) and $S(\acute{z}) < S_0$, we deduce that

$$\begin{aligned} 0 &= d_2 I''(\acute{z}) - c^* I'(\acute{z}) + \frac{\beta S(\acute{z}) I(\acute{z} - c^*\tau)}{S(\acute{z}) + I(\acute{z} - c^*\tau)} - \gamma I(\acute{z}) \\ &\le \frac{\beta S(\acute{z}) I(\acute{z} - c^*\tau)}{S(\acute{z}) + I(\acute{z} - c^*\tau)} - \gamma I(\acute{z}) \\ &< \frac{\beta S_0 \frac{\beta-\gamma}{\gamma} S_0}{S_0 + \frac{\beta-\gamma}{\gamma} S_0} - \gamma \frac{\beta-\gamma}{\gamma} S_0 \\ &= 0, \end{aligned}$$

a contradiction occurs. Thus $I(z) < \frac{\beta-\gamma}{\gamma}S_0$ on $\mathbb{R}$. □

4. Proof of Theorem 1.2

The proof is based on the reduction to absurdity. Suppose that the continuous positive function pair $(S(z), I(z), R(z))$ is a solution of (1.6) and (1.7). This implies that

$$\begin{aligned} I(z) =& \frac{\beta}{d_2(\lambda^+ - \lambda^-)}\Big[\int_{-\infty}^{z} e^{\lambda^-(z-\eta)} \frac{S(\eta)I(\eta - c\tau)}{S(\eta) + I(\eta - c\tau)} d\eta \\ &+ \int_z^{\infty} e^{\lambda^+(z-\eta)} \frac{S(\eta)I(\eta - c\tau)}{S(\eta) + I(\eta - c\tau)} d\eta\Big], \end{aligned} \tag{4.1}$$

where

$$\lambda^- = \frac{c - \sqrt{c^2 + 4d_2\gamma}}{2d_2} \quad \text{and} \quad \lambda^+ = \frac{c + \sqrt{c^2 + 4d_2\gamma}}{2d_2}.$$

Applying L'Hôpital rule in (4.1) yields $I'(\pm\infty) = 0$. Then integrating the second equation in (1.6) over $\mathbb{R}$ and using $\beta = \gamma$, we obtain

$$\int_{\mathbb{R}} I(z)dz = \int_{\mathbb{R}} \frac{S(z)I(z - c\tau)}{S(z) + I(z - c\tau)} dz. \tag{4.2}$$

By $\int_{\mathbb{R}} I(z)dz = \int_{\mathbb{R}} I(z-c\tau)dz$ and (4.2), we have

$$\int_{\mathbb{R}} \left[\frac{S(z)}{S(z)+I(z-c\tau)} - 1\right] I(z-c\tau)dz = 0, \tag{4.3}$$

which together with the positiveness of $I(z)$ and the continuities of $S(z), I(z)$ on $\mathbb{R}$ imply that

$$\frac{S(z)}{S(z)+I(z-c\tau)} \equiv 1 \Rightarrow I(z) \equiv 0.$$

A contradiction appears. The proof is finished.

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