HUNTING COOPERATION AND GESTATION DELAY IN A PREY-PREDATOR MODEL WITH FRACTIONAL DERIVATIVE

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Abstract Predator-prey dynamics are central to ecological modeling, with the Lotka-Volterra framework serving as a foundational tool for studying these interactions. In this study, we propose a novel fractional-order predator-prey model incorporating cooperative hunting and gestation delays to better capture the complexities of real-world ecosystems. The cooperative hunting mechanism enhances predator efficiency, while gestation delay accounts for the time required for biomass transfer from prey to predator reproduction. Additionally, we integrate fractional derivatives to introduce memory effects, allowing the system to retain past influences on population dynamics. We establish the dynamical analysis of the model. Through numerical simulations, we demonstrate the interplay between cooperation, delay, and memory effects, revealing rich dynamical behaviors.

Keywords Prey-predator, hunting cooperation, gestation delay, Caputo fractional derivative, Hopf bifurcation.

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1. Introduction

Mathematical biology and population dynamics have been extensively explored using different mathematical models. Among them, the Lotka-Volterra prey-predator model is a key framework built on a nonlinear system of ordinary differential equations that explains the interactions between two populations: Prey and predator. Over the years, various models have been developed and refined to capture diverse ecological phenomena, enhancing our understanding of ecological systems. Predator-prey interactions play a vital role in species' survival and biodiversity maintenance, making such models essential for studying species dynamics.

In predator-prey models, the functional response represents the relationship between the predator's consumption rate and the prey population density. This is a crucial aspect of ecological modeling, as it determines how effectively predators utilize their prey under varying environmental conditions. Over time, different forms of functional responses have been proposed to capture diverse ecological scenarios. Hunting cooperation, or foraging facilitation, is often incorporated into the functional response to reflect the idea that higher predator densities lead to increased foraging efficiency and, consequently, higher feeding rates for individuals in the group. Many species exhibit cooperative hunting behavior, including carnivores [20], aquatic organisms [5], and birds [14]. Berec [3] studied a prey-predator model involving cooperative hunting. In [23], Pal et al. introduce hunting cooperation for a prey-predator model with

fear effect. Yousef et al. [31] examined a multi-species fractional-order competition model with hunting cooperation. In this work, we build on the study by Alves and Hilker [1], where they used a classical prey-predator system to explore the effects of hunting cooperation mechanisms. In [7], the authors consider the fractional order version of this model to incorporate memory effects.

In real ecosystems, the interaction between prey and predator is not instantaneous; their past states influence it. Many researchers point out that time delays naturally occur in predator-prey dynamics because of lags in the reactions of both populations. This makes it essential to include time delays in such models. These delays can take various forms -temporal, maturation, gestation, or dispersal delays- but maturation and gestation delays are widespread and unavoidable. Recently, the authors of [11, 26] examined a prey-predator model with hunting cooperation and maturation delay introduced for the predator. In [11], they found that prolonged maturation delays can suppress predator population growth, potentially leading to predator extinction. Additionally, their study revealed that even brief hunting times could trigger significant shifts in ecological coexistence when coupled with substantial maturation delays. On the contrary, a prey-predator system introduced in [26] possesses delay-induced stabilization. In [29], a stage-structured predator-prey model involving refuge and cooperation is analyzed with multiple delays.

This study focuses on gestation delay, which affects the predator population's growth. It is well established that every organism needs a specific amount of time to produce offspring. [12]. Simply put, predators need time to reproduce after consuming prey, and we incorporate a discrete delay to reflect this lag in transferring biomass from prey to predator. Recently, the authors in [13] considered gestation delays in a prey-predator stage-structured food chain system. In [8], the authors examined a mathematical study of a scenario where the Holling type III predator shows gestation delay upon consumption of the prey. In [22], an epidemic model is considered with gestation delay and disease in the predator population.

Fractional calculus has been incorporated into various areas of science and engineering due to its ability to describe memory and hereditary properties of complex systems [25]. Many works have also been produced on fractional differential equations (FDEs) in broader mathematical contexts, including impulsive systems, fuzzy models, and boundary value problems involving different definitions of fractional derivatives [21, 27, 28]. Besides, in ecological modeling, FDEs have emerged as practical tools for capturing populations' memory and genetic influences across generations. Researchers have applied FDEs to predator-prey models to better represent the complexity of species interactions, which often exhibit long-term memory effects [2, 7, 31]. To incorporate memory-related effects in our ecological setting, we adopt the Caputo fractional derivative. This formulation possesses a non-local nature, where the influence of past states on current behavior intensifies as the fractional order α decreases. Compared to other definitions, such as the Riemann–Liouville and Grunwald–Letnikov derivatives, the Caputo derivative is more commonly used in modeling applications due to its suitability for initial value problems and relatively simpler initial condition structure.

Recent works have combined fractional derivatives with time delays in ecological models [16, 19, 30]. However, combining fractional derivatives with time delays in such models is still relatively uncommon. In this study, we integrate both. The gestation delay introduces a localized memory effect, while the fractional derivative adds a broader, system-wide memory effect. Integrating fractional-order derivatives with time delays in predator-prey models offers a more nuanced representation of ecological dynamics by capturing both historical influences and inherent time lags. This combination enhances the model's ability to reflect real-world scenarios

where species interactions are affected by past behaviors and delayed responses.

2. The model with hunting cooperation and gestation delay

In this section, we constitute the model starting from a classical prey-predator model in the form

$$\begin{cases} \dfrac{dX}{dT} = rX\left(1 - \dfrac{X}{K}\right) - \Phi(X,Y)\,Y, \\[3mm] \dfrac{dY}{dT} = c\,\Phi(X,Y)\,Y - m\,Y, \end{cases} \tag{2.1}$$

where X and Y are prey and predator densities at time T. Prey grows logistically with the per capita intrinsic growth rate r and carrying capacity K. Here, c is the conversion parameter, and m is the per capita death rate of a predator. The function $\Phi(X,Y)$ stands for the functional response of predators on prey. Generally, it is assumed solely on the prey population X, but there are also density-dependent functional responses. Here, the dependence of Φ on predator population Y based on hunting cooperation between predators. Following the work of [1], we assume that Φ is in the form

$$\Phi(X,Y) = (\lambda + a\,Y)\,X, \tag{2.2}$$

where $a\,Y$ is the cooperation term with $a \geq 0$ serving as the parameter for hunting cooperation. In [1], the authors consider a simplified hunting cooperation term based on the work of Berec [3]. In [3], it is also assumed that the functional response is Holling-type II without hunting cooperation. In our case, $a = 0$ gives $\Phi(X,Y) = \lambda X$, corresponding to a linear functional response with $\lambda > 0$ as the predator's attack rate. In [1], the authors show the existence of oscillatory behavior generated by cooperative hunting, and they focus on the ecological scenario in which predators are unable to survive without hunting cooperation. With the assumption (2.2) and by incorporating a time-discrete gestation delay, the model (2.1) turns into

$$\begin{cases} \dfrac{dX}{dT} = rX\left(1 - \dfrac{X}{K}\right) - (\lambda + a\,Y)\,X\,Y, \\[3mm] \dfrac{dX}{dT} = c(\lambda + a\,Y)\,X(t - \overline{\tau})\,Y(t - \overline{\tau}) - m\,Y, \end{cases} \tag{2.3}$$

where $\overline{\tau}$ represents the time lag between a predator consuming prey and the resulting increase in the predator population. Then, we get the following model by applying a non-dimensionalization process:

$$\begin{cases} \dfrac{dx}{dt} = \sigma x\left(1 - \dfrac{x}{\kappa}\right) - (1 + \beta\,y)\,x\,y, \\[3mm] \dfrac{dy}{dt} = (1 + \beta\,y)\,x(t - \tau)\,y(t - \tau) - y. \end{cases} \tag{2.4}$$

Here, the dimensionless variables are defined as $x = \frac{c\lambda}{m}X$, $y = \frac{\lambda}{m}Y$, $t = mT$, while the dimensionless parameters are $\sigma = \frac{r}{m}$, $\kappa = \frac{c\lambda K}{m}$, $\beta = \frac{am}{\lambda^2}$, and $\tau = m\overline{\tau}$. The gestation delay parameter τ is incorporated into the linear functional response term for predator growth, while we assume that hunting cooperation does not involve any time delay.

3. Fractional version and well-posedness

In this section, we introduce the fractional formulation of the model (2.4), followed by essential preliminaries on fractional calculus. We then establish the boundedness of solutions and prove

their existence and uniqueness. The fractional counterpart of the dimensionless model (2.4) is derived by substituting the left-hand side derivatives with fractional-order derivatives, as shown below:

$$\begin{cases} D_t^\alpha x(t) = \sigma x \left(1 - \dfrac{x}{\kappa}\right) - (1 + \beta\, y)\, x\, y, \\ D_t^\alpha y(t) = (1 + \beta\, y)\, x(t - \tau)\, y(t - \tau) - y, \end{cases} \tag{3.1}$$

where $x(0) > 0$, $y(0) > 0$, σ, κ, β are positive parameters with time delay $\tau > 0$, and D_t^α represents Caputo fractional derivative of order α with $0 < \alpha \le 1$.

Definition 3.1. [25] The Caputo fractional order derivative of order α can be defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\rho - \alpha)} \int_0^t (t - s)^{\rho - \alpha - 1} f^{(\rho)}(s) ds,$$

where $\rho - 1 < \alpha \le \rho \in Z^+$, $\Gamma(\cdot)$ represents the Gamma function defined as $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$.

Applying the Laplace transform rule, we also have that

$$L\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{\rho - 1} s^{\alpha - k - 1} f^{(k)}(0),\ \rho - 1 < \alpha \le \rho \in Z^+,$$

where $F(s) = L\{f(t); s\}$.

Lemma 3.1. [15] *Let $u(t)$ be a continuous function on $[0, \infty)$ and satisfying*

$$\begin{cases} D_t^\alpha u(t) \le -\lambda u(t) + \mu, \\ u(0) = u_0, \end{cases}$$

where $0 < \alpha < 1$, $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda \ne 0$, and $t = 0$ is the initial time. Then

$$u(t) \le \left(u_0 - \frac{\mu}{\lambda}\right) E_\alpha\left[-\lambda\, t^\alpha\right] + \frac{\mu}{\delta}, \tag{3.2}$$

where $E_\alpha(\cdot)$ is the Mittag–Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Lemma 3.2. [18] *Consider the system*

$$D_t^\alpha x(t) = f(t, x),\ t > 0 \tag{3.3}$$

with initial condition x_0, $\alpha \in (0, 1]$, $f : [0, \infty) \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$. If $f(t, x)$ satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of (3.3) on $[0, \infty) \times \Omega$.

Theorem 3.1. [24] *Consider the fractional system*

$$D_t^\alpha x(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n,\ \alpha \in (0, 1) \tag{3.4}$$

with $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ and $f : [f_1, \ldots, f_n] : \mathbb{R}^n \to \mathbb{R}^n$. Let

$$J_{E^*} = \frac{\partial f}{\partial x} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$$

be the Jacobian matrix at equilibrium point E^* of the system (3.4). If the eigenvalues λ_i ($i = 1, \ldots, n$) of J_{E^*} meet $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$ for $i \in \{1, 2, \ldots, n\}$, then E^* is locally asymptotically stable (LAS).

Lemma 3.3. [9] *Consider the following Caputo fractional delayed system:*

$$D^\alpha x(t) = A\,x(t) + B\,x(t - \tau),\ x(t) = \phi(t),\ t \in [-\tau, 0] \tag{3.5}$$

with $\alpha \in (0, 1]$, $x \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$, and $\tau \in \mathbb{R}^{+(n \times n)}$. The characteristic equation of the system is $\det |\,r^\alpha I - A - B\,e^{-r\tau}\,| = 0$. If all of the roots of the characteristic equation have negative real parts, then the zero solution of the fractional delayed system (3.5) is LAS.

3.1. Boundness of solutions

Theorem 3.2. *The solutions of system (3.1) are uniformly bounded for $\tau = 0$.*

Proof. Define $W(t) = x(t) + y(t)$. Taking Caputo fractional derivative of both sides gives

$$
\begin{aligned}
D^\alpha W(t) &= D^\alpha x(t) + D^\alpha y(t) \\
&= \sigma x \left(1 - \frac{x}{\kappa}\right) - y \\
&= \sigma x \left(1 - \frac{x}{\kappa}\right) - (W - x).
\end{aligned}
$$

Then, we can write

$$
\begin{aligned}
D^\alpha W(t) + W &= \sigma x \left(1 - \frac{x}{\kappa}\right) + x \\
&= -\frac{\sigma x^2}{\kappa} + (\sigma + 1)x \\
&\le \frac{-\sigma}{k}\left(x - \frac{k(\sigma + 1)}{2\sigma}\right)^2 + \frac{k(\sigma + 1)^2}{2\sigma} \\
&\le \frac{k(\sigma + 1)^2}{2\sigma}.
\end{aligned}
$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
W(t) &\le W(t_0) - \frac{k(\sigma + 1)^2}{2\sigma} E_\alpha[-t^\alpha] + \frac{k(\sigma + 1)^2}{2\sigma} \\
&\le U(t_0)E_\alpha[-t^\alpha] + \frac{k(\sigma + 1)^2}{2\sigma}(1 - E_\alpha[-t^\alpha]).
\end{aligned}
$$

We have $U(t) \le \frac{k(\sigma+1)^2}{2\sigma}$ as $t \to \infty$. Hence, all solutions of system (3.1) initiating from $\mathbb{R}^2_+$ confined in the following positively invariant region $D \subseteq \mathbb{R}^2_+$ defined as

$$D = \left\{ (x(t), y(t)) \in \mathbb{R}^2_+ \,|\, x(t) + y(t) \le \frac{k(\sigma + 1)^2}{2\sigma} + \psi, \psi > 0 \right\}.$$

$\square$

3.2. Existence-uniqueness

Theorem 3.3. *Define the set $\Lambda = \{(x,y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < \mathcal{N}\}$ where $\mathcal{N}$ is a positive constant. For each $(x_0, y_0) \in \Lambda$, system (3.1) under the initial state (x_0, y_0) admits a unique solution $U = (x, y) \in \Lambda$.*

Proof.   Define the map $f(U) = (f_1(U), f_2(U))$ where

$$f_1(U) = \sigma x \left(1 - \frac{x}{\kappa}\right) - (1 + \beta y) \, x \, y,$$
$$f_2(U) = (1 + \beta y) \, x(t - \tau) \, y(t - \tau) - y.$$

Let $U = (x, y)$, $\tilde{U} = (\tilde{x}, \tilde{y}) \in \Lambda$. It follows that

$$
\begin{aligned}
||f(U) - f(\tilde{U})|| = & |f_1(U) - f_1(\tilde{U})| + |f_2(U) - f_2(\tilde{U})| \\
= & |\sigma x \left(1 - \frac{x}{\kappa}\right) - (1 + \beta y)xy - \sigma \tilde{x}\left(1 - \frac{\tilde{x}}{\kappa}\right) - (1 + \beta \tilde{y})\tilde{x}\tilde{y}| \\
& + |(1 + \beta y)x(t-\tau)y(t-\tau) - y - (1 + \beta\tilde{y})\tilde{x}(t-\tau)\tilde{y}(t-\tau) - \tilde{y}| \\
\leq & \sigma|x - \tilde{x}| + \frac{\sigma}{\kappa}|x^2 - \tilde{x}^2| + (x + \beta xy + \beta x\tilde{y})|y - \tilde{y}| + (\tilde{y} + \beta\tilde{y}^2)|x - \tilde{x}| \\
& + (1 + x + \beta xy + \beta x\tilde{y})|y - \tilde{y}| + (\tilde{y} + \beta\tilde{y}^2)|x - \tilde{x}| \\
\leq & (\sigma + \frac{2\sigma\mathcal{N}}{\kappa} + 2(\mathcal{N} + \beta\mathcal{N}^2))|x - \tilde{x}| + (1 + 2\mathcal{N} + 4\beta\mathcal{N}^2)|y - \tilde{y}| \\
\leq & \rho_1|x - \tilde{x}| + \rho_2|y - \tilde{y}| \\
\leq & \rho|U - \tilde{U}|,
\end{aligned}
$$

where $\rho = \max\{\rho_1, \rho_2\}$ with $\rho_1 = \sigma + \frac{2\sigma\mathcal{N}}{\kappa} + 2(\mathcal{N} + \beta\mathcal{N}^2)$ and $\rho_2 = 1 + 2\mathcal{N} + 4\beta\mathcal{N}^2$. Therefore, $f(U)$ meets the Lipschitz condition with respect to U, and the theorem follows directly from Lemma 3.2.   □

4. Dynamical analysis of the model

In this section, after determining equilibrium points of the system (3.1), we make the stability and bifurcation analysis around equilibrium points with considering $\tau = 0$ and $\tau > 0$ in order.

4.1. Equilibria analysis

While the equilibrium points of the system (3.1) are consistent with those of the corresponding ODE model analyzed in [1], our work provides an explicit analytical derivation of the existence conditions for positive equilibria, which were not stated in [1], where the analysis was mainly based on numerical phase plane evaluations.

   i. The trivial equilibrium point $E_0 = (0, 0)$.

  ii. The boundary equilibrium point $E_\kappa = (\kappa, 0)$.

 iii. The coexistence equilibrium $E^* = (x^*, y^*)$ where

$$x^* = \frac{1}{1 + \beta y^*}, \tag{4.1}$$

and the predator coordinate y^* of E^* should satisfy

$$f(y) = \beta^2 \kappa y^3 + 2\beta \kappa y^2 + \kappa(1 - \sigma\beta)y + \sigma(1 - k) = 0. \tag{4.2}$$

Depending on the equation (4.2), we have the following sub-cases. Firstly, if If $k > 1$, the equation (4.2) allows only one positive root y^* and the coexistence equilibrium $E^* = (x^*, y^*)$ is unique [1]. For $k = 1$, we have the coexistence equilibrium E^* only if $\sigma\beta > 1$. Now, assume that $k < 1$ and $\sigma\beta > 1$. In this case, only the linear coefficient of equation (4.2) is negative, leading to two sign changes. As a result, the equation can have either zero or two positive roots. Now, for the cubic equation (4.2), we have $f(0) = \sigma(1-k) > 0$, $f(y) \to \infty$ as $y \to \infty$, and the negative linear term forces a downward slope initially for $y = 0$. For a positive root to exist, $f(y)$ must become negative for some positive y. Now, we make a derivative analysis to locate extrema of $f(y)$. To obtain critical points we check the equation

$$f'(y) = 3\kappa\beta^2 x^2 + 4\kappa\beta x + \kappa(1 - \sigma\beta) = 0 \tag{4.3}$$

with discriminant $\Delta = 4\kappa^2\beta^2(1 + 3\sigma\beta) > 0$, and real roots

$$y_\pm = \frac{-2 \pm \sqrt{1 + 3\sigma\beta}}{3\beta}, \tag{4.4}$$

where $y_- < 0$ and $y_+ > 0$. For the function $f(y)$ to cross the horizontal axis for positive y values, the local minimum $f(y_+)$ at $y_+ > 0$ must be negative. We can rewrite this condition $f(y_+) < 0$ as

$$\sigma(\kappa - 1) > \frac{\kappa\varphi^3 + 6\kappa\varphi^2 + 9\kappa(1 - \sigma\beta)\varphi}{27a}, \tag{4.5}$$

where $\varphi = -2 + \sqrt{1 + 3\sigma\beta}$. If we have the equality for the above equation, $y = 0$ will be a double root and we again have unique coexistence equilibrium. For all other cases, the equation $f(y)$ does not allow positive roots. Below, we summarize our findings:

a. If $k > 1$, the coexistence equilibrium $E^* = (x^*, y^*)$ is unique.
b. If $k = 1$ and $\sigma\beta > 1$, the coexistence equilibrium $E^* = (x^*, y^*)$ is unique.
c. If $k < 1$, $\sigma\beta > 1$, and we have a equality at (4.5), the coexistence equilibrium $E^* = (x^*, y^*)$ is unique.
d. If $k < 1$, $\sigma\beta > 1$ and the condition (4.5) is satisfied, there exist two coexistence equilibrium.
e. Otherwise, coexistence equilibria are not feasible.

Note that the equilibrium and phase plane analysis presented in [1] is consistent with our findings. The work [6] extends the same model by allowing both prey and predators to disperse linearly in the environment, while [7] examines the fractional version of the model. In both studies [6,7], the existence conditions for two coexistence equilibria were given as

$$\frac{-1 + \sqrt{1 + 3\sigma\beta}}{\sigma\beta} < k \le 1 \text{ and } \sigma > 1/\beta. \tag{4.6}$$

However, this condition is incorrect. For instance, even though the parameter values $\sigma = 3$, $\kappa = 0.8$, $\beta = 1$ satisfy the condition above, the system (3.1) does not admit any positive equilibria. Therefore, the correct version of the existence condition for the case $\kappa < 1$ and $\sigma\beta > 1$ is provided in (4.5).

4.2. Stability analysis without delay

Firstly, we assume $\tau = 0$ and derive the stability of the system (3.1) around equilibrium points. We get the following results by applying Theorem 3.1.

Theorem 4.1. *Assume* $\tau = 0$. *The trivial equilibrium* E_0 *is a saddle point. The boundary equilibrium is LAS if* $\kappa < 1$; *otherwise, it is a saddle.*

Proof. The Jacobian matric evaluated at E_0 has eigenvalues $\lambda_1 = -1$, $\lambda_2 = \sigma > 0$ and λ_2 fails to satisfy LAS condition. The Jacobian matric evaluated at E_κ has eigenvalues $\lambda_1 = -1 + \kappa$, $\lambda_2 = -\sigma$. If $\kappa < 1$, both eigenvalues satisfies the LAS condition $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$. The results follow from Theorem 3.1.   □

For $\tau = 0$, the Jacobian matrix J_{E^*} at the coexistence equilibrium point E^* is calculated as

$$J_{E^*} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with

$$a_{11} = -y^*(1 + \beta y^*) + \frac{(-2 + \kappa + \beta\kappa y^*)\sigma}{\kappa(1 + \beta y^*)}, \quad a_{12} = -1 - \frac{\beta y^*}{1 + \beta y^*},$$

$$a_{21} = y^*(1 + \beta y^*), \quad a_{22} = \frac{\beta y^*}{1 + \beta y^*}.$$

The characteristic equation can be easily obtained as $\lambda^2 + p_1\lambda + p_0 = 0$ with roots $\lambda_{1,2} = \frac{-p_1 \pm \sqrt{\Delta}}{2}$ where $p_0 = a_{11}a_{22} - a_{12}a_{21}$, $p_1 = -(a_{11} + a_{22})$, $\Delta = p_1^2 - 4p_0$. Theorem 3.1 leads us to the following result:

Theorem 4.2. *Assume that* $\tau = 0$, $\alpha \in (0,1)$, *and the existence conditions of* $E^* = (x^*, y^*)$ *of the system* (3.1) *are satisfied.*

  i. $E^* = (x^*, y^*)$ *is LAS if one of the followings holds:*

  (a) $p_1 \geq 0$, $p_0 > 0$,
  (b) $p_1 \leq 0$, $\Delta < 0$, $\left| \tan^{-1}\left(\frac{\sqrt{-\Delta}}{p_1} \right) \right| > \frac{\alpha\pi}{2}$.

  ii. There exists a Hopf bifurcation around E^* *for the system* (3.1) *as* α *passes through the value* $\alpha_h = \frac{2}{\pi} \left| \tan^{-1}\left(\frac{\sqrt{-\Delta}}{p_1} \right) \right|$.

Proof. The first part of the theorem is a direct consequence of Theorem 3.1. The second part of the proof follows from Theorem 3.1 of [17]. For more detailed proof, we refer to [7].   □

4.3. Stability analysis with delay

We linearize the system about the positive equilibrium point $E^* = (x^*, y^*)$ and the following system is obtained:

$$\begin{cases} {}_{t_0}^{C}D_t^\zeta x(t) = a_{11}\, x(t) + a_{12}\, y(t), \\ {}_{t_0}^{C}D_t^\zeta y(t) = a_{21}\, x(t - \tau) + a_{22}\, y(t) + y(t - \tau), \end{cases} \tag{4.7}$$

where the equality (4.1) is applied and the coefficients a_{ij} are defined as

$$a_{11} = \sigma - \frac{2\sigma x^*}{\kappa} - y^* - \beta(y^*)^2, \qquad a_{12} = -x^*(1 + 2\beta y^*),$$

$$a_{21} = y^*(1 + \beta y^*) = \frac{y^*}{x^*}, \qquad a_{22} = -1 + \beta x^* y^*.$$

From Theorem 3.2, we get the characteristic equation corresponding to (4.7) as

$$\det \begin{pmatrix} r^\alpha - a_{11} & -a_{12} \\ -e^{-r\tau} a_{21} & r^\alpha - a_{22} - e^{-r\tau} \end{pmatrix} = 0. \tag{4.8}$$

This equation can be written as

$$r^{2\alpha} + r^\alpha (-a_{11} - a_{22}) + a_{11}a_{22} + e^{-r\tau}(-r^\alpha + a_{11} - a_{12}a_{21}) = 0. \tag{4.9}$$

Assume that the equation (4.9) has a pair of pure imaginary root $r_{1,2} = \omega \left(\cos \frac{\Pi}{2} \pm i \sin \frac{\pi}{2} \right)$ with $\omega > 0$. Plug $r_1 = \omega \left(\cos \frac{\pi}{2} + i \sin \frac{\Pi}{2} \right)$ into the equation (4.9) and separating real and imaginary parts yields

$$\begin{cases} C_1 \cos(\omega\tau) + C_2 \sin(\omega\tau) = C_3, \\ D_1 \cos(\omega\tau) + D_2 \sin(\omega\tau) = D_3, \end{cases} \tag{4.10}$$

where

$$C_1 = a_{11} - a_{12}a_{21} - \omega^\alpha \cos \frac{\alpha\pi}{2}, \qquad C_2 = -\omega^\alpha \sin \frac{\alpha\pi}{2},$$

$$C_3 = \omega^{2\alpha} \cos \alpha\pi - \omega^\alpha \cos \frac{\alpha\pi}{2}(a_{11} + a_{22}) + a_{11}a_{22},$$

$$D_1 = C_2, \qquad D_2 = -C_1,$$

$$D_3 = \omega^{2\alpha} \sin \alpha\pi - \omega^\alpha \sin \frac{\alpha\pi}{2}(a_{11} + a_{22}).$$

Solving the system (4.10) brings that

$$\begin{cases} \cos \omega\tau = \frac{E_1(\omega)}{E_3(\omega)}, \\ \sin \omega\tau = \frac{E_2(\omega)}{E_3(\omega)}, \end{cases} \tag{4.11}$$

where $E_1(\omega) = C_3 D_2 - D_2 C_3$, $E_2(\omega) = C_1 D_3 - C_3 D_1$, and $E_3(\omega) = C_1 D_2 - C_2 D_1$. Now, by squaring and adding both sides of the (4.11) and by using the equality $\cos^2 \omega\tau + \sin^2 \omega\tau = 1$, we get the following equation:

$$E_3^2(\omega) = E_1^2(\omega) + E_2^2(\omega). \tag{4.12}$$

Defining $H(\omega) = E_3^2(\omega) - E_1^2(\omega) - E_2^2(\omega)$, $f_1 = a_{11} - a_{12}a_{21}$, $f_2 = a_{11} + a_{22}$, $f_3 = a_{11}a_{22}$, we rewrite the equation (4.12) as

$$H(\omega) = \omega^{6\alpha} + \zeta_1 \omega^{5\alpha} + \zeta_2 \omega^{4\alpha} + \zeta_3 \omega^{3\alpha} + \zeta_4 \omega^{2\alpha} + \zeta_5 \omega^\alpha + \zeta_6 = 0, \tag{4.13}$$

where

$$\zeta_1 = -2(f_1 + f_2) \cos \frac{\alpha\pi}{2},$$

$$\zeta_2 = -1 + f_1^2 + f_2^2 + 4f_1 f_2 \cos^2 \frac{\alpha\pi}{2} + 2f_3 \cos \alpha\pi,$$

$$\zeta_3 = 4f_1 \cos \frac{\alpha\pi}{2} - 2f_1 f_2 \cos \frac{\alpha\pi}{2} (f_1 \cos \alpha\pi + f_2) - 2f_3 \cos \frac{\alpha\pi}{2} (2f_1 \cos \alpha\pi + 2f_2),$$

$$\zeta_4 = -4f_1^2 \cos^2 \frac{\alpha\pi}{2} + f_1^2(f_2^2 - 2) + 2f_1^2 f_3 \cos \alpha\pi + 4 \cos^2 \frac{\alpha\pi}{2} f_1 f_2 f_3 + f_3^2,$$

$$\zeta_5 = (4f_1^3 - 2f_1^2 f_2 f_3 - 2f_1 f_3^2) \cos \frac{\alpha\pi}{2},$$

$$\zeta_6 = -f_1^4 + f_1^2 f_3^2.$$

Here, we make the following assumption:

(H1) There exists positive real roots for the equation (4.13).

Then it follows from the first equation of (4.11) that

$$\tau^{(k)} = \frac{1}{\omega} \left[\arccos \left(\frac{E_1(\omega)}{E_3(\omega)} \right) + 2k\pi \right], \ k \in \{0, 1, 2, \dots\}. \tag{4.14}$$

Now, we set the critical value as

$$\tau_0 = \min_k \{\tau^{(k)}\}. \tag{4.15}$$

Next, we check the transversality condition at $\tau = \tau_0$. Rewrite equation (4.9) as

$$Q_1(r) + Q_2(r)e^{-r\tau} = 0, \tag{4.16}$$

where $Q_1(r) = r^{2\alpha} + r^\alpha (-a_{11} - a_{22}) + a_{11}a_{22}$ and $Q_2(r) = -r^\alpha + a_{11} - a_{12}a_{21}$.

By differentiating (4.16) with respect to τ, we obtain

$$\left(\frac{dr}{d\tau} \right)^{-1} = \frac{Q_1'(r) + Q_2'(r)}{r \, Q_2(r) \, e^{-r\tau}} - \frac{\tau}{r}.$$

Let $r(\tau) = \mu_1(\tau) + i\mu_2(\tau)$ be the root of equation (4.9) near $\tau = \tau_0$ satisfying $\mu_1(\tau_0) = 0$ and $\mu_2(\tau_0) = \omega_0$. Then, we can write

$$\left(\frac{dr}{d\tau} \right)^{-1} \bigg|_{\omega=\omega_0, \tau=\tau_0} = \frac{M_1 + iM_2}{N_1 + iN_2} - \frac{\tau_0}{i\omega_0},$$

where

$$M_1 = Re[Q_1'(i\omega_0)] + Re[Q_2'(i\omega_0)],$$

$$M_2 = Im[Q_1'(i\omega_0)] + Im[Q_2'(i\omega_0)],$$

$$N_1 = -Re[Q_2(i\omega_0)]\omega_0 \sin(-\omega_0\tau_0) - Im[Q_2(i\omega_0)]\omega_0 \cos(-\omega_0\tau_0),$$

$$N_2 = Re[Q_2(i\omega_0)]\omega_0 \cos(-\omega_0\tau_0) - Im[Q_2(i\omega_0)]\omega_0 \sin(-\omega_0\tau_0).$$

It follows that

$$Re \left[\frac{dr}{d\tau} \right] \bigg|_{\omega=\omega_0, \tau=\tau_0} = \frac{M_1 N_1 + M_2 N_2}{N_1^2 + N_2^2}.$$

We present the next assumption to guarantee the transversality condition.

(H2) $\dfrac{M_1 N_1 + M_2 N_2}{N_1^2 + N_2^2} \neq 0.$

Theorem 4.3. *Suppose that the existence conditions of E^* and the LAS conditions (for $\tau = 0$) given in Theorem 4.2 are satisfied, and assumptions **(H1)-(H2)** hold. Then, the coexistence equilibrium E^* of the system* (3.1) *is LAS for $\tau < \tau_0$. Moreover, the system* (3.1) *undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$, i.e., it has a branch of periodic solutions bifurcating from E^* near $\tau = \tau_0$.*

Bifurcations in ecological systems occur at critical thresholds where slight variations in system parameters can lead to qualitative changes in population dynamics. In predator-prey interactions, Hopf bifurcation is crucial in determining the transition between stable coexistence and sustained oscillations. The emergence of periodic solutions through Hopf bifurcation signifies predator-prey cycles driven by intrinsic ecological factors.

5. Numerical examples

In this section, we present numerical examples to validate our analytical findings. To approximate solutions of FDEs, we employ the predictor-corrector method introduced in [10] and its extension for delayed FDEs [4]. The selection of parameter values is guided by previous studies [1,7]. Our numerical analysis particularly focuses on the effects of the delay parameter τ and the fractional-order parameter α. Additionally, we compare our results with existing studies to highlight the impact of hunting cooperation, gestation delay, and memory effects on system dynamics. All figures are interpreted within a biological context, emphasizing the role of hunting cooperation, which remains a key aspect of this study.
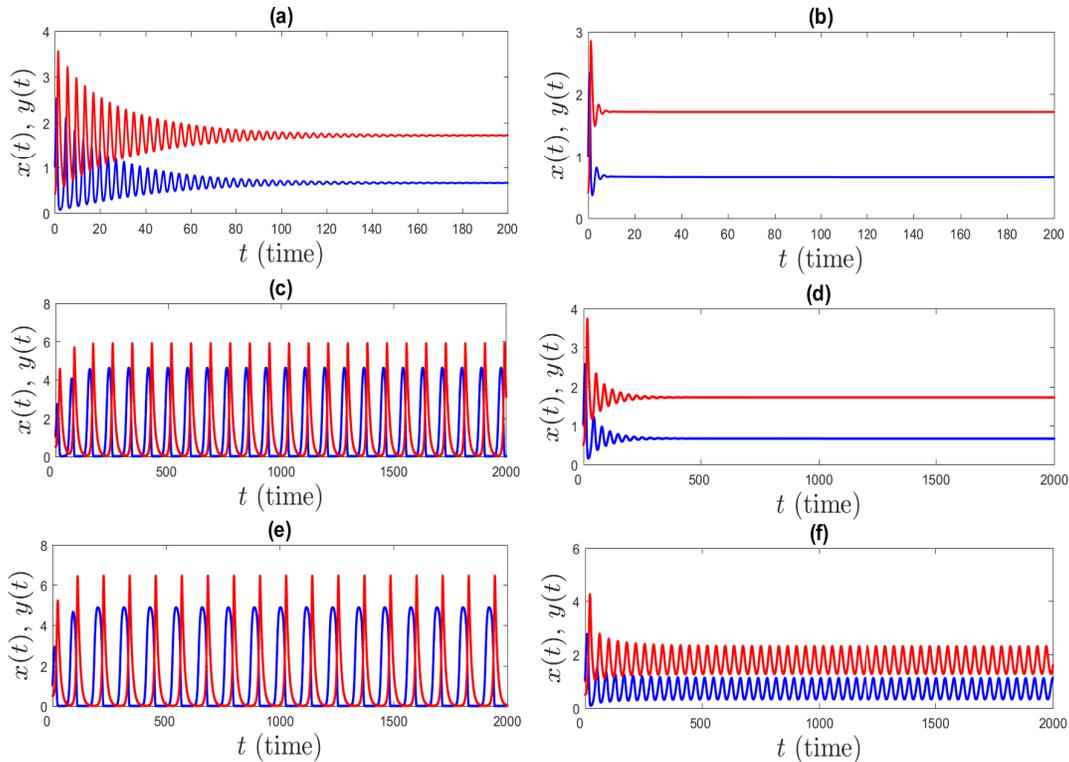

Figure 1. Prey (blue) and predator (red) populations with respect to time where the fractional order $\alpha = 1$ at left and $\alpha = 0.85$ at right; $\tau = 0$ at top, $\tau = 0.20$ at middle, and $\tau = 0.33$ at bottom.

Firstly, we set the parameter values of the system (3.1) as $\sigma = 3$, $\kappa = 5$, and $\beta = 0.3$. In [1], the authors discuss the effect of the hunting cooperation term on the existence of a coexistence equilibrium point. However, as the existence conditions suggest, this effect is most significant when $\kappa < 1$. Figure 1 presents time-series solutions of the system for two different values of the fractional order. On the left, we consider the integer-order system with $\alpha = 1$, while on the right, we take $\alpha = 0.85$. Additionally, we analyze three different values of the gestation delay parameter τ.
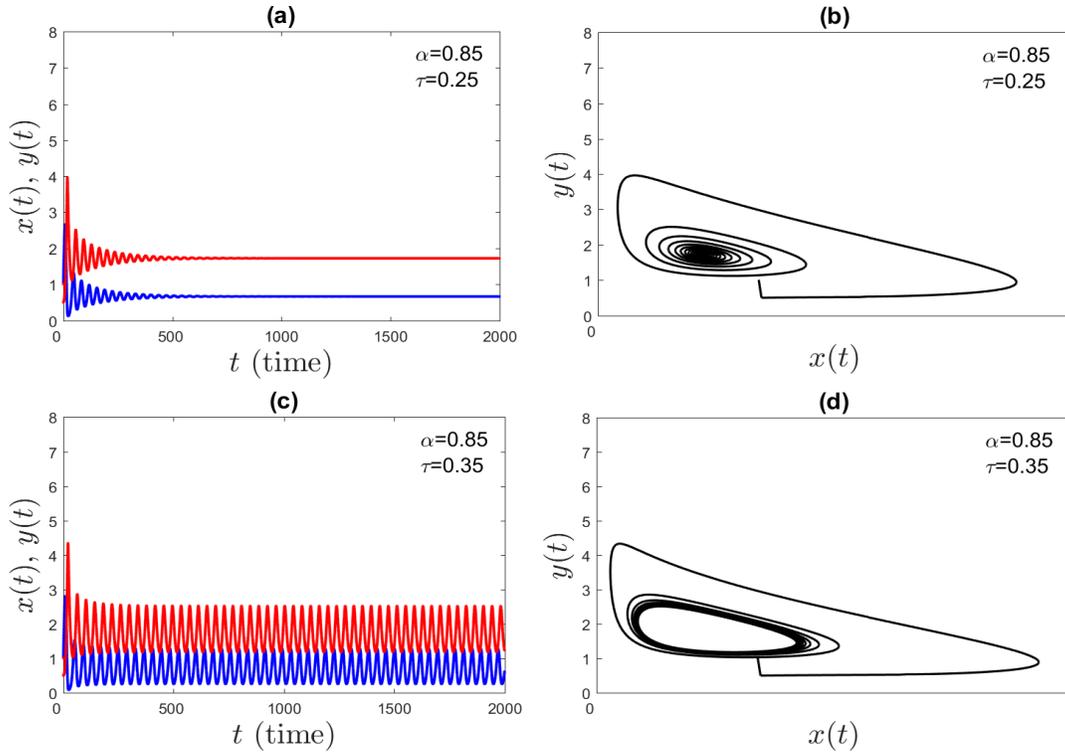
Figure 2. Time-series solutions and corresponding phase diagrams.

In [7], the authors studied the model (3.1) without gestation delays and emphasized the stabilizing effect of memory resulting from the fractional derivative. Our results show that this stabilizing effect is even more pronounced in suppressing oscillations induced by gestation delay. Without gestation delay, the coexistence equilibrium is locally asymptotically stable (LAS) for all considered values of α. However, for $\alpha = 0.85$, the system converges to the coexistence equilibrium $E^* = (0.659851, 1.71831)$ more rapidly (see Figure 1(a)(b)). When gestation delay is incorporated, stability is highly sensitive to the order of the derivative. For the integer-order case ($\alpha = 1$), stability is lost when τ exceeds the critical value $\tau_0 = 0.0194939$, beyond which high-amplitude oscillations emerge (Figure 1(c)(e)). In contrast, for the fractional case with $\alpha = 0.85$, the critical delay value is significantly higher at $\tau_0 = 0.323097$, where the system undergoes a Hopf bifurcation around E^*. Below this threshold, the introduction of gestation delay leads to damped oscillations, and the system eventually stabilizes around E^* (Figure 1(d), Figure 2(a)(b)). However, when τ exceeds $\tau_0 = 0.323097$, the oscillations persist around the coexistence state. Notably, even in this oscillatory regime, the amplitudes remain significantly lower than in the integer-order case (Figure 1(e)(f), Figure 2(c)(d)).
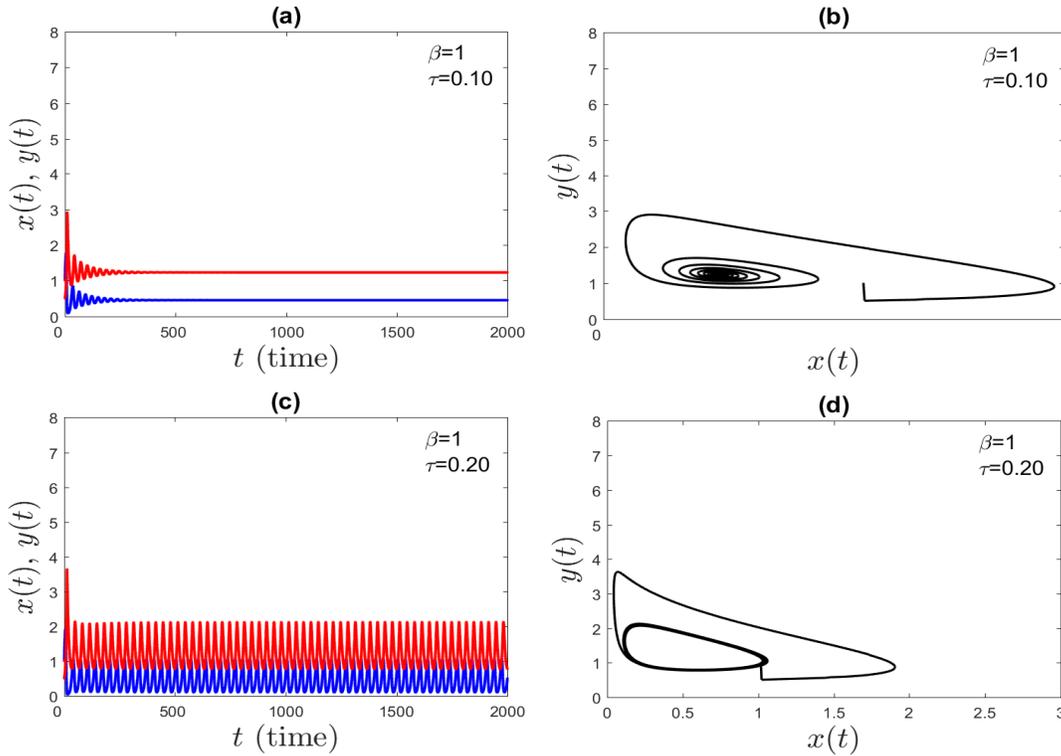
Figure 3. Time-series solutions and corresponding phase diagrams.

Then, we increase the hunting cooperation parameter β while keeping other parameters fixed at $\sigma = 3$, $\kappa = 5$, and $\alpha = 0.85$. For $\beta = 1$, the critical value of the gestation delay parameter decreases to $\tau_0 = 0.16453$, indicating that higher hunting cooperation lowers the threshold at which the system transitions to oscillatory behavior (compare Figure 2(a)(b) with Figure 3(c)(d)). This suggests that both hunting cooperation and gestation delay contribute to increased instability in the system. Additionally, the increase in β leads to a lower equilibrium population for both prey and predators, with the coexistence equilibrium shifting to $E^* = (0.449153, 1.22641)$. Biologically, while greater hunting cooperation enhances the predator's ability to capture prey, it can also destabilize the system. Initially, higher hunting efficiency may result in a temporary increase in predator numbers. However, over time, excessive predation depletes prey resources, ultimately leading to a decline in the predator population as well. However, introducing memory effects, modeled through fractional derivatives, mitigates these destabilizing effects. Decreasing the fractional order (i.e., increasing the memory effect) helps stabilize the system by dampening oscillations and enabling the populations to reach a stable coexistence state. From a biological perspective, this implies that species benefit from past experiences within the ecosystem, adjusting their behaviors over time to establish a more resilient population balance.

Next, we set the parameter values as $\sigma = 3$, $\kappa = 0.8$, and $\beta = 10$. In [1], the case $\kappa < 1$ referred to as the predator extinction case. As indicated by the existence conditions of the coexistence equilibrium for the case of $\kappa < 1$, to mediate coexistence both the carrying capacity parameter κ and the hunting cooperation parameter β must exceed certain threshold values. If the prey population remains too small to support predators, even in the presence of hunting cooperation, the predator population declines toward extinction. However, when hunting
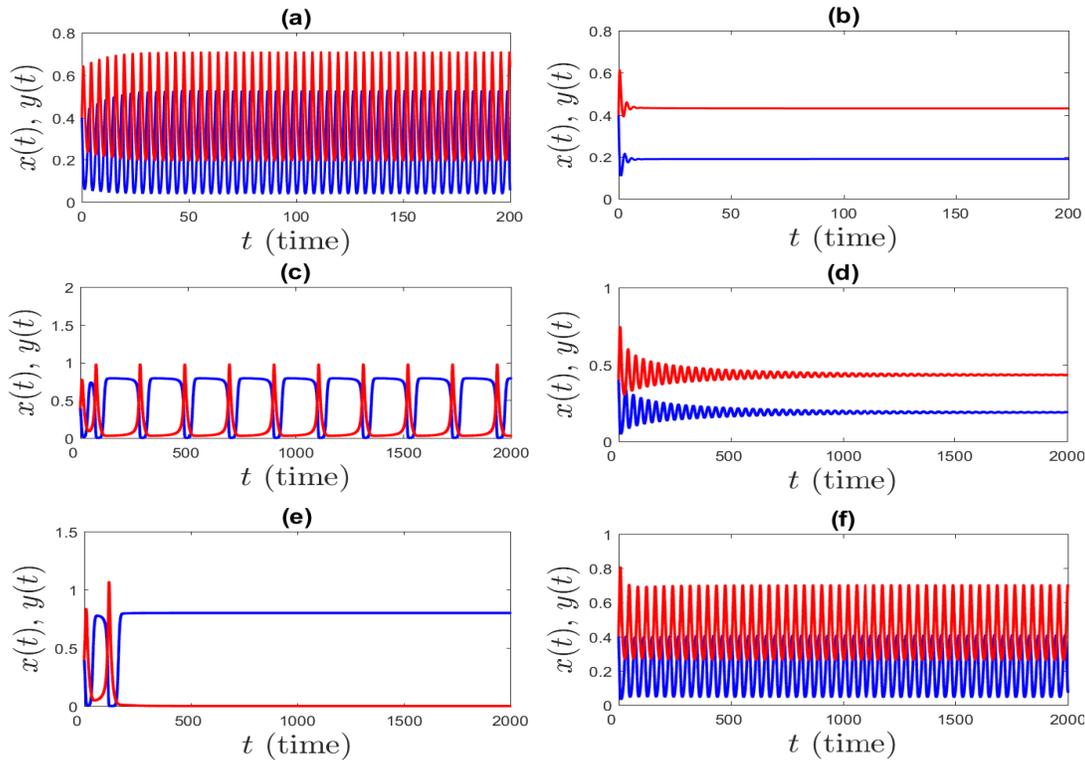
Figure 4. Time-series solutions where the fractional order $\alpha = 1$ at left and $\alpha = 0.85$ at right; $\tau = 0$ at top, $\tau = 0.20$ at middle, and $\tau = 0.25$ at bottom.

cooperation surpasses a critical threshold, cooperative hunting becomes efficient enough to sustain predator survival. In [1], the authors also discuss the presence of a strong Allee effect in the predator population under these conditions. Since the hunting cooperation term is directly proportional to predator density, the Allee threshold represents the critical predator population level below which predators face extinction and above which they persist.

Figure 4 presents time-series solutions of the system for two different values of the fractional order. On the left, we consider the integer-order case with $\alpha = 1$, while on the right, we take $\alpha = 0.85$. Additionally, we examine the impact of three different values of the gestation delay parameter τ. For $\sigma = 3$, $\kappa = 0.8$, and $\beta = 10$, the system (3.1) allows the coexistence equilibriums $E^* = (0.188103, 0.431624)$, $E_1^* = (0.791099, 0.0264065)$ though the latter is always unstable. At the top of the Figure (4), we consider the case without gestation delay ($\tau = 0$). Under these conditions, the equilibrium E^* is unstable for the integer order system (Figure 4(a)), whereas for $\alpha = 0.85$, the memory effect is strong enough to make E^* LAS (Figure 4(b)).

Incorporating gestation delay introduces oscillatory behavior in both cases. For the integer-order system, these oscillations drive the predator population dangerously close to extinction at $\tau = 0.20$ (Figure 4(c)), and for $\tau = 0.25$, the predator population completely collapses (Figure 4(e)). However, when the memory effect is included through fractional derivatives, the system stabilizes to a coexistence state with damped oscillations, provided that the gestation delay remains below the critical threshold $\tau_0 = 0.232849$ (Figure 4(e), Figure 5(a)(b)). For $\tau > \tau_0$, both populations exhibit persistent oscillations with consistent amplitudes, indicating the emergence of a limit cycle (Figure 4(f), Figure 5(c)(d)).

Figure 6 presents time-series solutions with different initial conditions for the case $\kappa < 1$.
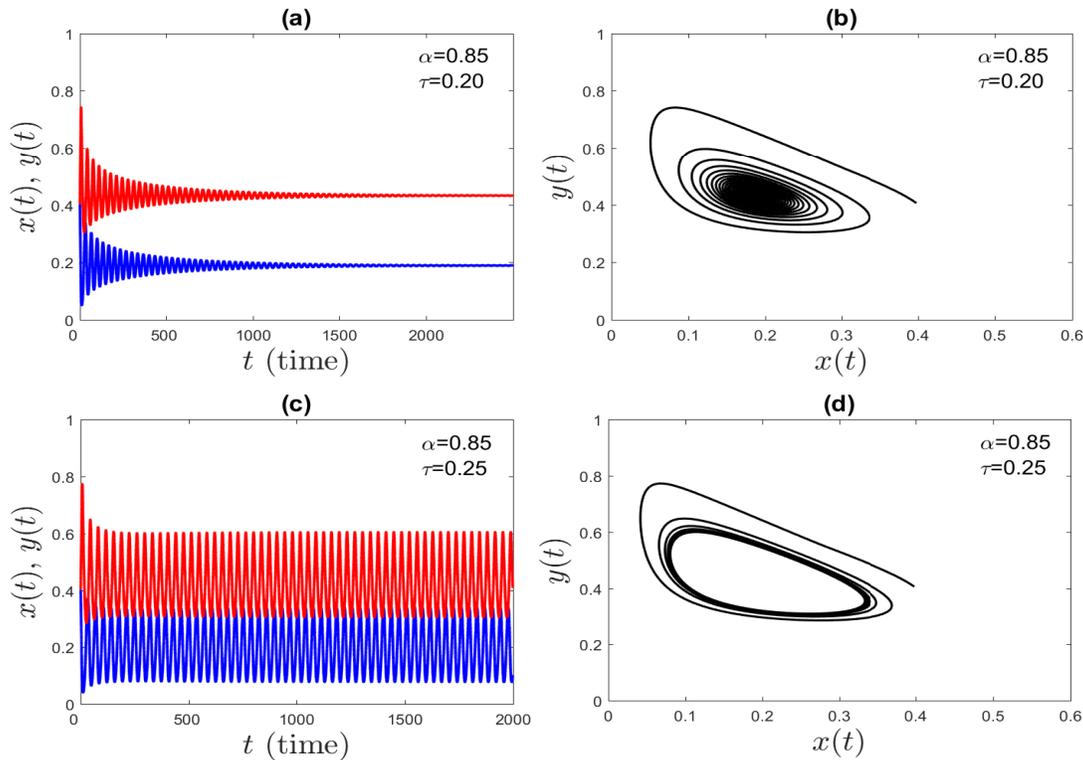
Figure 5. Time-series solutions and corresponding phase diagrams.

As mentioned in [1], we observe that the system is bistable. It also supports the existence of the Allee effect for predator population that is predators can either go extinct because of strong Allee effects or persist depending on initial conditions [1] (Figure 6).

Figure 7 illustrates time-series and phase portraits for various values of the hunting cooperation parameter β while keeping $\sigma = 3$ and $\kappa = 0.8$. As β increases, the predator's foraging efficiency improves, leading to initially enhanced predation. However, once β exceeds a certain threshold, the system undergoes a qualitative change: The previously stable coexistence equilibrium loses stability, giving rise to sustained oscillations in both prey and predator populations. This reflects a Hopf bifurcation driven by strong cooperative behavior. Notably, the memory effect associated with the fractional order mitigates extreme fluctuations and helps maintain bounded dynamics, preventing extinction-level drops seen in the integer-order case. The results suggest that while moderate levels of hunting cooperation support predator survival, excessively strong cooperation destabilizes the system, particularly under limited carrying capacity. These findings reinforce the ecological insight that cooperative hunting, if unchecked, may trigger long-term population cycles rather than promoting stable coexistence.

6. Concluding remarks

In this study, we proposed and analyzed a fractional-order predator-prey model incorporating hunting cooperation and gestation delay to better capture real-world ecological interactions. Our model extends classical prey-predator dynamics by introducing memory effects through the Caputo fractional derivative, allowing past states to influence present population behaviors. We
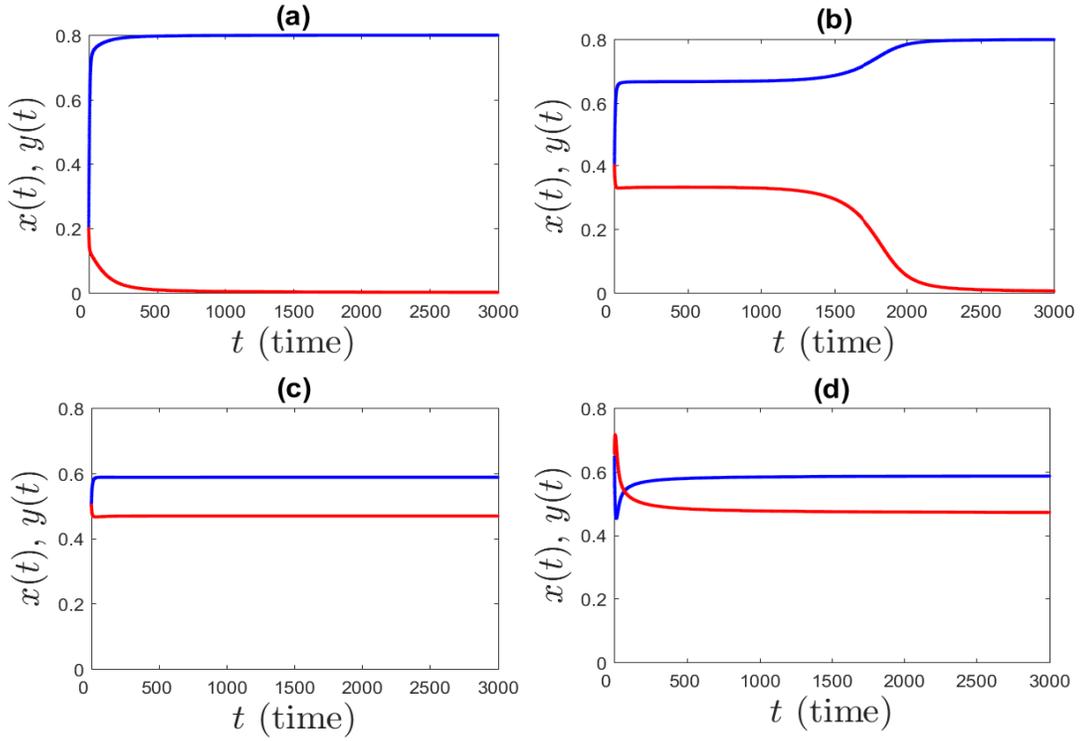
Figure 6. Time-series solutions of the system (3.1) for different initial conditions where $\sigma = 3$, $\kappa = 0.8$, $\beta = 1.5$, $\tau = 0.09$, $\alpha = 0.85$.
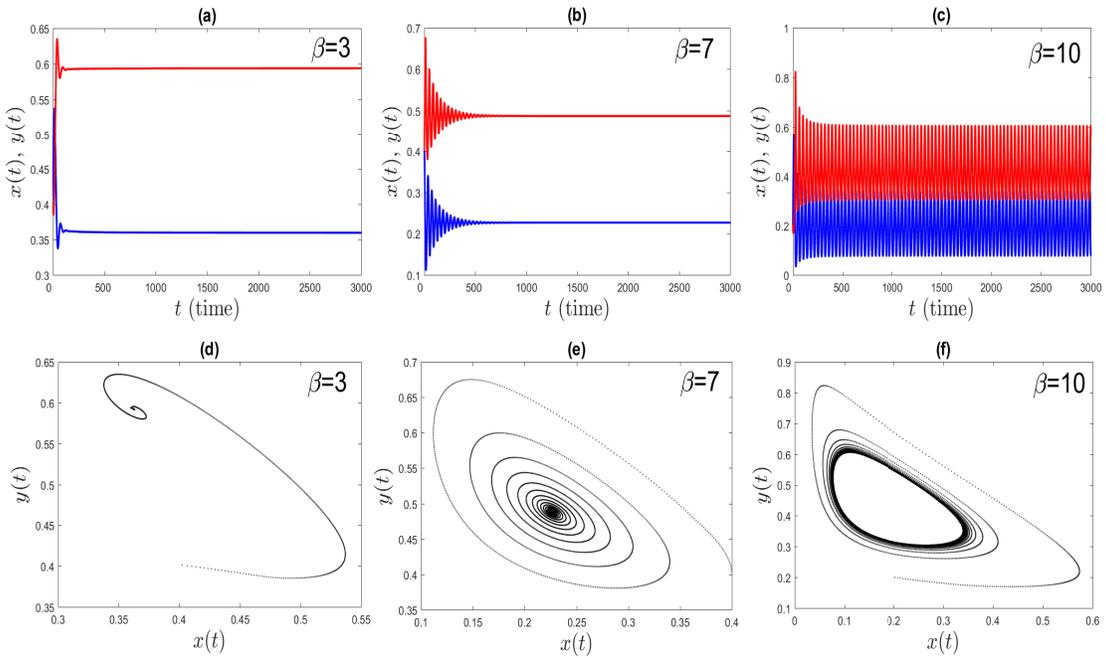

Figure 7. Time-series solutions and corresponding phase diagrams for different values of hunting cooperation parameter β where $\tau = 0.25$, $\alpha = 0.85$.

established the well-posedness of the system, proved the existence and uniqueness of solutions, and performed stability and bifurcation analyses to understand the impact of key parameters on system dynamics. The incorporation of hunting cooperation is more significant for lower values of carrying capacity in terms of the coexistence state. We also established the correct existence conditions for the case where $k < 1$, $\sigma\beta > 1$.

Through numerical simulations, we demonstrated that hunting cooperation enhances predator efficiency but can also destabilize the system if unregulated. Gestation delay introduces oscillatory behaviors, and its effect is more pronounced in integer-order systems. Our findings confirm that fractional-order derivatives help suppress oscillations, increasing system stability by allowing species to retain and adapt based on past ecological conditions. We also observed Hopf bifurcations, where increasing delay beyond a critical threshold led to persistent population cycles. Furthermore, we highlighted the opposing effects of memory effects and hunting cooperation, showing how their interplay determines population persistence or extinction.

Our findings suggest incorporating memory effects via fractional derivatives provides a more biologically realistic representation of predator-prey interactions. This framework is particularly valuable for ecological systems where historical dependencies and reproductive delays significantly influence population dynamics. Potential improvements could include incorporating of more complex functional responses, stage structures, or density-dependent gestation mechanisms. Moreover, extending the model to account for environmental variability, stochastic effects, or spatial heterogeneity may yield deeper ecological insights. Given the rising interest in fractional modeling within applied mathematics and theoretical biology, we believe this study offers a relevant and timely contribution that can serve as a foundation for future interdisciplinary research.

Author contributions. The authors contributed equally to all processes of this study.

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