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CONTROL AND SYNCHRONIZATION OF FRACTAL BEHAVIORS OF THE DISCRETE FRACTIONAL T-S FUZZY PREY-PREDATOR MODEL*

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Abstract There are important significances to discuss the prey-predator model in various aspects such as biodiversity conservation, resource management, ecosystem services and so on. At first, to investigate the fractal behavior of the predator-prey model from a fractal perspective, the two-dimensional continuous prey-predator model is discretized, and its Caputo fractional-order form is obtained. Secondly, utilizing the sector nonlinear method, a Takagi-Sugeno (T-S) fuzzy model of the discrete fractional-order predator-prey model is established, and the Julia set of the model based on the T-S fuzzy model is introduced. Thirdly, a parallel distributed compensation (PDC) approach is employed to design the corresponding fuzzy controller, and sufficient conditions for the stability of the system are given in the form of linear matrix inequalities (LMIs). The controller gain parameters are obtained by solving LMIs, and control of the Julia set of the discrete fractional-order predator-prey model based on the T-S fuzzy model is carried out. Finally, using the exact linearization technique, synchronization of the Julia sets of the predator-prey models is achieved based on the T-S fuzzy model by the linear control method.

Keywords Prey-predator model, T-S fuzzy model, Julia set, synchronization.

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1. Introduction

Mathematical models play a crucial role in describing and understanding complex dynamics in nature. Particularly in the field of ecology, mathematical models are widely used to describe and predict the changes in biological populations over time. One of the most influential models is the Lotka-Volterra model, which was independently proposed by American chemist Alfred J. Lotka [22] and Italian mathematician Vito Volterra [38]. This model could describe the interaction between predator and prey populations through a set of nonlinear differential equations, becoming a classic example of interdisciplinary research between ecology and mathematics.

The Lotka-Volterra model is not only significant in ecology but also demonstrates rich dynamical characteristics in mathematical analysis. In recent years, researchers have used modern mathematical tools, such as stability analysis [36], bifurcation theory, and chaotic dynam-

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ics [6, 41], to conduct in-depth studies on the model. These studies have revealed the model's behavior under different parameter conditions, including the existence of periodic solutions, stability, and chaotic phenomena. As research progresses, Zhao studied a stochastic Lotka-Volterra system with infinite delay. A new concept of extinction, namely, the almost sure β-extinction was proposed and sufficient conditions for the solution to be almost sure β-extinction were obtained [48]. The global dynamics of a feasible discrete-time Lotka-Volterra model with two predators and one prey in three dimensions was studied in [18]. Pang et al. investigated the long time behavior of bounded solutions to a two-species time-periodic Lotka–Volterra reaction–diffusion system with strong competition [29].

Fractional-order calculus is a generalization of traditional integer-order calculus, which can effectively describe dynamic systems with memory effects and non-local properties by introducing non-integer order differential and integral operators. There are extensive applications in various fields such as physics, engineering [8], control theory [26, 32], organic chemistry [16, 19], neural networks [3, 15, 47] and so on. However, many real-world problems are inherently discrete, especially in the fields of digital computation, numerical simulation, and signal processing, making the study of fractional-order derivatives in discrete systems particularly important. To meet this demand, the discrete Caputo fractional-order derivative, as an effective discretization method, has become an important topic in recent research about the fractional-order calculus. The Caputo fractional-order derivative is one of the most commonly used definitions in fractional-order calculus, with the advantage of preserving the physical meaning of initial conditions and having good mathematical properties. Unlike the classical integer-order derivative, the Caputo fractional-order derivative defines a non-local derivative operator that can capture the historical dependence and lag effects in the system, which is very effective for modeling phenomena with memory and complex dynamical behaviors [30].

The fundamental idea of the discrete Caputo fractional-order derivative is to discretize the definition of the continuous Caputo fractional-order derivative so as to adapt it for use in discrete-time systems. Unlike traditional fractional-order derivatives, the discrete Caputo fractional-order derivative can be calculated at discrete time points and is widely applied in many areas such as digital signal processing, image processing, system identification, and control system design [12]. For discrete-time systems, the introduction of fractional-order derivatives can capture the long-term dependencies and memory effects of the system, which cannot be achieved by integer-order calculus [10]. The T-S fuzzy model, also known as the Takagi-Sugeno fuzzy model, is named after the Japanese scholars Takagi and Sugeno, who first proposed this model in 1985 [35]. This model decomposes complex nonlinear systems into multiple local linear subsystems and uses fuzzy rules for smooth transitions, thereby achieving effective modeling and control of complex systems. The main advantage of the T-S fuzzy model lies in its ability to combine the flexibility of fuzzy logic with the precise mathematical description of traditional control theory. It is not only suitable for dealing with uncertain, imprecise, or partially unknown system information but also shows significant superiority in the fields of system identification, prediction, control and so on [4, 23]. There are various design methods for T-S fuzzy controllers, which can be designed according to different requirements. For example, the Parallel Distributed Compensation (PDC) controller [2] which implements distributed control, can effectively handle system complexity. The stability and performance of T-S fuzzy control systems are usually analyzed and guaranteed using mathematical techniques such as Linear Matrix Inequalities (LMI) [37], providing a powerful framework for stability analysis and controller design, and ensuring robust stability and performance. In the field of inverted pendulum (RIP) control, suppressing disturbances is a key challenge. To this end, the authors in [28] developed a control strategy that integrates T-S fuzzy control and H∞ methods. By selecting variables appropriately, this approach enhances the stability and robustness of the control system, and heightens the disturbance rejection capability of the RIP. This strategy effectively deals with input and output constraints, ensuring that the pendulum angle position and control inputs remain within predefined limits, thus improving the safety and stability of the system. Additionally, constrained optimization techniques are employed to generate control signals that satisfy the imposed constraints and optimize the performance of the RIP system, ensuring effective control and operational reliability.

Fractals are a concept of great importance in both mathematics and nature, describing forms that exhibit self-similarity and complex structures. The characteristic of fractals is that their structures display similar shapes at different scales, meaning that fractals show similar forms regardless of whether they are viewed at a large or small scale. These structures often lack a simple geometric description but are ubiquitous in both nature and mathematics. Compared to the dimensions of traditional geometric shapes, the dimension of fractals is typically not an integer but a fractional dimension. With the development of fractal theory, there are extensive applications about fractals such as physics [21], computer science [27], biology [7, 17] and so on. In recent years, they have also been widely applied in network science, machine learning [31], and artificial intelligence.

With the rapid development of nonlinear dynamics and chaos theory, mathematicians and physicists have expanded their research of complex systems across multiple disciplines, exploring phenomena that appear disordered yet possess inherent regularities. The Julia set [33], as an important set in fractals, has become one of the core focuses in this field due to its unique properties in complexity and self-similarity. It not only has rich mathematical connotations but also finds extensive applications in physics [40], engineering [46], computer graphics and so on. The formation of the Julia set originates from the iterative process of complex functions, generating patterns through the repeated application of a simple complex function (such as $f(z) = z^2 + c$, where c is a complex constant). Each specific value of c produces a different Julia set, and these sets exhibit astonishing geometric shapes and complex structures, displaying self-similarity and fractal characteristics. By adjusting parameters, one can even transform the patterns of the Julia set from regular to completely chaotic forms. In practical applications, the control [13, 45] and synchronization [34, 39] issues of the Julia sets have gradually attracted the attention of scholars, especially in dynamic systems with multiple interactions. “Control” refers to changing the behavior of a system, while “synchronization” means achieving a certain degree of coordination or consistency between multiple dynamic systems. This research direction not only helps us understand how complex systems manifest alternating phenomena of order and disorder at different levels but also provides profound insights into the design and optimization of various practical engineering systems.

In recent years, introducing fractional calculus into the classical generation mechanism of Julia sets to construct fractional-order Julia sets has become a frontier hotspot in the interdisciplinary research of nonlinear science and fractal theory. Fractional-order Julia sets not only inherit the infinite self-similarity and aesthetic complexity of classical Julia sets, but also their ‘memory effect’ during iteration endows the system with unique dynamical characteristics, causing the fractal structures to exhibit a far more sensitive dependence on initial conditions and system order, resulting in chaotic landscapes that are richer and more complex than those in the integer-order cases. Wang [42, 44] investigated the chaotic and fractal dynamics of fractional coupled logistic maps based on the Caputo fractional h-difference and proposed fractional quantum Julia sets based on a fractional-difference map. M. Mohammad [25] explored viscous flow dynamics around fractal geometries, specifically the Mandelbrot and Julia sets, using finite element simulations. Wang [43] proposed fractional Mandelbrot sets with impulses based on a fractional difference map and preliminarily investigated their fractal dynamic characteristics by numerical methods and graphical explorations. Manoj Kumar [24] examined the fractal behavior in a non-linear fractional disease model, and applied the fractal perspective to the discrete fractional tumor-immune model.

In this paper, control and synchronization of Julia set of the discrete fractional T-S prey-pradator model are discussed. Section 2 introduces the basic properties of the Caputo fractional-order derivative. Section 3 discusses the basic properties of Julia sets, discretizes the two-dimensional continuous prey-predator model, and then obtains its discrete Caputo fractional-order derivative. Based on this, the Julia set of the corresponding discrete fractional-order derivative model is obtained. In Section 4, a discrete fractional-order prey-predator model based on the T-S fuzzy model is established, and it is controlled using the Parallel Distributed Compensation (PDC) method. In Section 5, synchronization of the Julia sets of two prey-predator models is achieved through linear control based on the T-S fuzzy model.

2. Fundamental theory

2.1. Fractional-order derivative

There are many types of fractional derivatives, among which the Caputo fractional derivative has a wide range of applications in the modeling process of many practical problems. The definition of the Caputo fractional derivative is given below.

Definition 2.1. [20] The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is defined by the following formula

$$D_{0^+}^{\alpha} f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0, \infty)$.

Now, we introduce the definition of discrete fractional calculus, which has widespread applications in both theory and practice.

Definition 2.2. [5] Let u:$\mathbb{N}_a \to \mathbb{R}$ and $v > 0$ be given. Then the fractional sum of order is defined by

$$\triangle_a^{-v} u(t) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - \sigma(s))^{(v-a)} u(s), t \in \mathbb{N}_{n+a},$$

where a is the starting point, $\sigma(s) = s + 1$ and $t^{(v)}$ is the falling function defined as

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}.$$

Definition 2.3. [1] For $v > 0$, $v \notin \mathbb{N}$ and $u(t)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by

$${}^C \triangle_a^v u(t) := \triangle_a^{-(m-v)} \triangle^m u(t)$$

$$= \frac{1}{\Gamma(m-v)} \sum_{s=a}^{t-(m-v)} (t-\sigma(s))^{m-v-1} \vartriangle_s^m u(s).$$

Theorem 2.1. [9] *For the delta fraction difference equation*

$$\begin{aligned} &{}^C \vartriangle_a^v u(t) = f(t+v-1, u(t+v-1)),\\ &\vartriangle^k u(a) = u_k,\ m = [v]+1,\ k = 0,1,\cdots,m-1, \end{aligned}$$

the equivalent discrete integral equation can be obtained as

$$u(t) = u_0(t) + \frac{1}{\Gamma(v)} \sum_{s=a+m-v}^{t-v} (t-\sigma(s))^{(v-1)} f(s+v-1, u(s+v-1)),$$

where the initial iteration $u_0(t)$ is

$$u_0(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \vartriangle^k u(a).$$

Fix $a = 0$, and let $v \in (0,1]$, $s = k-(v-1)$, $t = n+1$, $u_n = u(n)$.
Then, from the above equation, we can obtain the fractional-order difference map

$$\begin{aligned} u_{n+1} &= u_0 + \frac{1}{\Gamma(v)} \sum_{k=0}^{n} (n-k-1+v)_{(v-1)} f(k, u_k)\\ &= u_0 + \frac{1}{\Gamma(v)} \sum_{k=0}^{n} \frac{\Gamma(n-k+v)}{\Gamma(n-k+1)} f(k, u_k). \end{aligned}$$

2.2. Julia set

In this section, some necessary definitions related to Julia set are given.

Assume that $f : \mathbb{C} \to \mathbb{C}$ is a continuous mapping, denote f^k as the kth iteration, which means $f^0(\omega) = \omega$, $f^1(\omega) = f(\omega)$, $f^2(\omega) = f(f(\omega))$, $\cdots$, $f^k(\omega) = f(f(\cdots(f(\omega))\cdots))$. The iterative function graph $\{f^k\}$ is called a discrete dynamical system.

ω is called a fixed point if ω satisfies $f(\omega) = \omega$. Meanwhile, ω is called a cycle point if there is an integer p such that $f^p(\omega) = \omega$, where $p \geq 1$. In addition, p is called the period of ω if there is a minimum p which can meet this equation $f^p(\omega) = \omega$.

Let ω be the period point, p be the period, and $(f^p)'(\omega) = \lambda$, then (1) ω is super-attractive, when $\lambda = 0$; (2) ω is attractive, when $0 \leq\mid \lambda \mid< 1$; (3) ω is neutral, when $\mid \lambda \mid= 1$; (4) ω is exclusive, when $\mid \lambda \mid> 1$.

The classical Julia set is defined to be the clouser of the exclusive period points in the complex plane.

For convenience, an equivalent definition of Julia set is introduced.

Definition 2.4. [11] The filled in Julia set of the function f is defined as

$$K(f) = \{z \in \mathbb{C} \mid f^k(z) \nrightarrow \infty, k \to \infty\},$$

where $\mathbb{C}$ is the complex space, $f^k(z)$ is kth iterate of function f and $K(f)$ denotes the filled Julia set.

The Julia set of the function f is defined to be the boundary of $K(f)$, i.e.,

$$J(f) = \partial K(f),$$

where $J(f)$ denotes the Julia set.

3. Fractional-order T-S fuzzy prey-predator model and its Julia set

The two-dimensional prey-predator Lotka-Volterra model is

$$\begin{cases} \dot{x}_1(t) = x_1(t)(a_1 - b_1 x_2(t)), \\ \dot{x}_2(t) = x_2(t)(-a_2 + b_2 x_1(t)), \end{cases} \tag{3.1}$$

where $x_1(t), x_2(t)$ $(x_1(t) > 0, x_2(t) > 0)$ denote the species density of the preys and the predators at time t, respectively. The coefficient $a_1 > 0$ denotes the birth rate of the preys, $a_2 > 0$ denotes the birth rate of the predators, the other two coefficients b_1 and b_2 (both positives) describe interactions between the species.

Julia set is generated from the iteration of the discrete form. Therefore, we first utilizes

$$\dot{x}_i \to \frac{x_i(t + \Delta t) - x_i(t)}{\Delta t}, \ i = 1, 2,$$

replace $x_1(t), x_2(t)$ with x_n, y_n, and replace $x_1(t + \Delta t), x_2(t + \Delta t)$ with x_{n+1}, y_{n+1}. The Δt is expressed by δ, then the discrete version of system is obtained

$$\begin{cases} x_{n+1} = x_n + a_1 \delta x_n - b_1 \delta x_n y_n, \\ y_{n+1} = y_n - a_2 \delta y_n + b_2 \delta x_n y_n. \end{cases}$$

Let $\varepsilon_1 = a_1\delta, \gamma_1 = b_1\delta, \varepsilon_2 = a_2\delta, \gamma_2 = b_2\delta$, then the above model can be changed to the following form

$$\begin{cases} \triangle\ x_{n+1} = \varepsilon_1 x_n - \gamma_1 x_n y_n, \\ \triangle\ y_{n+1} = -\varepsilon_2 y_n + \gamma_2 x_n y_n. \end{cases} \tag{3.2}$$

Transform the above equation into the μ-order fractional difference form in discrete fractional calculus, we get

$$\begin{cases} \triangle_a^\mu\ x(t) = \varepsilon_1 x(t + \mu - 1) - \gamma_1 x(t + \mu - 1) y(t + \mu - 1), \\ \triangle_a^\mu\ y(t) = -\varepsilon_2 y(t + \mu - 1) + \gamma_2 x(t + \mu - 1) y(t + \mu - 1). \end{cases} \tag{3.3}$$

When $a = 0, \mu \in (0, 1], t = n + 1$, denote $x_n = x(n), y_n = y(n)$, then we can obtain

$$\begin{cases} x_{n+1} = x_0 + \dfrac{1}{\Gamma(\mu)} \displaystyle\sum_{j=0}^{n} \dfrac{\Gamma(n - j + \mu)}{\Gamma(n - j + 1)} (\varepsilon_1 x_j - \gamma_1 x_j y_j), \\ y_{n+1} = y_0 + \dfrac{1}{\Gamma(\mu)} \displaystyle\sum_{j=0}^{n} \dfrac{\Gamma(n - j + \mu)}{\Gamma(n - j + 1)} (-\varepsilon_2 y_j + \gamma_2 x_j y_j). \end{cases} \tag{3.4}$$

Finally, we fuzzify the fractional-order prey-predator model based on the T-S fuzzy model.

Denote $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = X_{n+1}$, $\begin{bmatrix} x_j \\ y_j \end{bmatrix} = X_j$, then the above model can be transformed into the following form

$$X_{n+1} = X_0 + \frac{1}{\Gamma\mu} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} A_j X_j, \tag{3.5}$$

where $A_j = \begin{bmatrix} \varepsilon_1 - \gamma_1 y_j & 0 \\ 0 & -\varepsilon_2 + \gamma_2 x_j \end{bmatrix}$.

Let $g_{j1}(x,y) = \varepsilon_1 - \gamma_1 y_j$, $g_{j2}(x,y) = -\varepsilon_2 + \gamma_2 x_j$ be nonlinear functions defined on $[C_1, D_1] \times [C_2, D_2]$. Since $x_0, x_1, \cdots, x_n \in [C_1, D_1]$, $y_0, y_1, \cdots, y_n \in [C_2, D_2]$, the maximum and minimum values of $g_{j1}(x,y), g_{j2}(x,y)$ exist. Let $\max g_{j1}(x,y) = F_{j1}$, $\max g_{j2}(x,y) = F_{j2}$, $\min g_{j1}(x,y) = f_{j1}$, $\min g_{j2}(x,y) = f_{j2}$, therefore $F_{11} = F_{21} = \cdots = F_{n1} = F_1$, $F_{12} = F_{22} = \cdots = F_{n2} = F_2$, $f_{11} = f_{21} = \cdots = f_{n1} = f_1, f_{12} = f_{22} = \cdots = f_{n2} = f_2$.

Below, we will use the concept of the Takagi-Sugeno fuzzy model to fuzzify the aforementioned model.

$$\begin{aligned} &\text{Rule : IF } g_{j1}(x,y) \text{ is } \psi_{ij1} \text{ and } g_{j2}(x,y) \text{ is } \psi_{ij2}, \\ &\quad \text{THEN } X_{n+1} = X_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} A_{ij} X_j, \quad i = 1,2,3,4, \end{aligned}$$

where

$$\begin{aligned} &A_{1j} = \begin{bmatrix} F_{j1} & 0 \\ 0 & F_{j2} \end{bmatrix}, \ A_{2j} = \begin{bmatrix} F_{j1} & 0 \\ 0 & f_{j2} \end{bmatrix}, \ A_{3j} = \begin{bmatrix} f_{j1} & 0 \\ 0 & F_{j2} \end{bmatrix}, \ A_{4j} = \begin{bmatrix} f_{j1} & 0 \\ 0 & f_{j2} \end{bmatrix}, \\ &\psi_{1j1} = \frac{g_{j1}(x,y) - f_{j1}}{F_{j1} - f_{j1}}, \ \psi_{2j1} = \frac{F_{j1} - g_{j1}(x,y)}{F_{j1} - f_{j1}}, \\ &\psi_{1j2} = \frac{g_{j2}(x,y) - f_{j2}}{F_{j2} - f_{j2}}, \ \psi_{2j2} = \frac{F_{j2} - g_{j2}(x,y)}{F_{j2} - f_{j2}}. \end{aligned}$$

Using the singleton fuzzifier, product interference, and defuzzification process, the model can be represented as

$$X_{n+1} = X_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} \sum_{i}^{4} h_{ij}(g(x,y)) A_{ij} X_j, \tag{3.6}$$

where

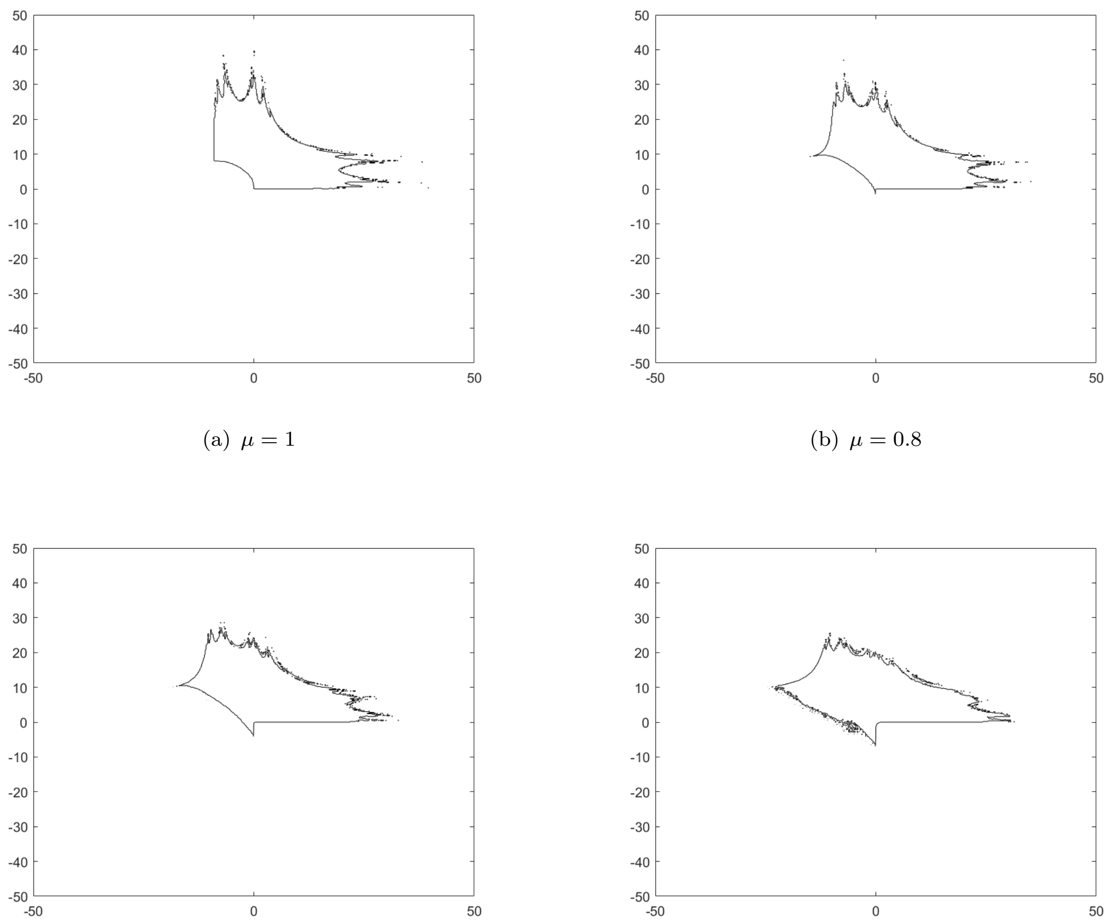
$$\begin{aligned} &h_{1j}(g(x,y)) = \psi_{1j1}\psi_{1j2}, \ h_{2j}(g(x,y)) = \psi_{1j1}\psi 2j2, \\ &h_{3j}(g(x,y)) = \psi_{2j1}\psi 1j2, \ h_{4j}(g(x,y)) = \psi_{2j1}\psi 2j2, \\ &\sum_{i}^{4} h_{ij}(g(x,y)) = 1. \end{aligned}$$

Since $F_{11} = F_{21} = \cdots = F_{n1} = F_1,\ F_{12} = F_{22} = \cdots = F_{n2} = F_2,\ f_{11} = f_{21} = \cdots = f_{n1} = f_1,\ f_{12} = f_{22} = \cdots = f_{n2} = f_2$, it follows that $A_{ij} = A,\ i = 1, 2, 3, 4,\ j = 1, 2, \cdots, n$. Then we have

$$X_{n+1} = X_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} \sum_{i=1}^{4} h_{ij}(g(x,y))A_i X_j. \tag{3.7}$$

Definition 3.1. The filled Julia set of the system (3.7) denoted by K is the set $K = \{(x_1, x_2) \in R^2 \mid \text{the iteration } \{x_1(k), x_2(k)\}_{k=1}^{\infty} \text{ remains bounded with initial points } (x_1, x_2)\}$. The Julia set of the system (3.7) denoted by J is the boundary of the filled Julia set K, that is $J = \partial K$.

Figure 1 is the Julia set with different orders of the system (3.7) when the parameters are taken to be $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1$.



(a) $\mu = 1$ (b) $\mu = 0.8$

(c) $\mu = 0.6$ (d) $\mu = 0.4$

Figure 1. Julia sets of system (3.7) when the parameters are taken as $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1$, by changing the order of the fractional derivative.

4. Control of the Julia set of fractional-order prey-predator model based on the T-S fuzzy model

The characteristic of the T-S model is that the fuzzy consequents are described using linear equations, which facilitates the design of related controllers using traditional linear control strategies. By applying control to the aforementioned model, we obtain

$$\begin{cases} x_{n+1} = x_0 + \dfrac{1}{\Gamma(\mu)} \displaystyle\sum_{j=0}^{n} \dfrac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\varepsilon_1 x_j - \gamma_1 x_j y_j) + u_1(k), \\ y_{n+1} = y_0 + \dfrac{1}{\Gamma(\mu)} \displaystyle\sum_{j=0}^{n} \dfrac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (-\varepsilon_2 y_j + \gamma_2 x_j y_j) + u_2(k), \end{cases} \tag{4.1}$$

where $u_1(k), u_2(k)$ are the control variables, representing the applications of control to the two-dimensional prey-predator model in fractional-order form.

Through the T-S fuzzy model, the nonlinear system (4.1) can be represented as

$$\begin{aligned} &\text{Rule : IF } g_{j1}(x,y) \text{ is } \psi_{ij1} \text{ and } g_{j2}(x,y) \text{ is } \psi_{ij2}, \\ &\text{THEN } X_{n+1} = X_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} A_{ij} X_j \\ &\qquad + B_{ij} u(k), \quad i = 1,2,3,4, \end{aligned}$$

where $B_{1j} = B_{2j} = B_{3j} = B_{4j} = B = \mathrm{diag}(1,1)$ and $u(k) = [u_1(k), u_2(k)]^T$.

The rule based in the T-S model can be utilized for controller design, especially when the controlled object is also described by a T-S model. Such a controller design scheme is called Parallel Distributed Compensation (PDC). The PDC method is a widely applied approach. The essence of the PDC algorithm is to divide the entire controlled nonlinear system into several fuzzy subsystems, and then design a fuzzy controller for each fuzzy subsystem. The global fuzzy controller can be obtained through the fuzzy weighting of these local fuzzy controllers. The controller parameters in the control statements of each fuzzy rule consequent do not affect each other and are independently designed. Therefore, this method of designing controllers is called Parallel Distributed Compensation.

The controller based on the PDC method is as follows

$$\begin{aligned} &\text{Rule : IF } g_{j1}(x,y) \text{ is } \psi_{ij1} \text{ and } g_{j2}(x,y) \text{ is } \psi_{ij2}, \\ &\text{THEN } u(n) = K_i X_j,\ i = 1,2,3,4. \end{aligned}$$

The control law for the entire system is the weighted sum of the local feedback controls of each subsystem, that is,

$$X_{n+1} = X_0 + \frac{1}{\Gamma(\mu)} \sum_{i=1}^{n} \sum_{m=1}^{4} h_{mj}(g(x,y)) h_{ij}(g(x,y)) (A_{ij} + BK_m) X_j. \tag{4.2}$$

Theorem 4.1. [14] *Discrete closed-loop fuzzy system* (4.2) *is asymptotically stable at the fixed point* $(0,0)$, *if there is a common positive definite matrix* $Q > 0$ *and real matrix* $M_i(i =$

$1, 2, \cdots, m)$, *for any* i *and* j $(i, j = 1, 2, \cdots, n, i \leq j,$ *except for* $h_i(g(x, y))h_j(g(x, y)) = 0)$, *satisfying the following inequalities*

$$\begin{bmatrix} -4Q & \Phi^T \\ \Phi & -Q \end{bmatrix} < 0, \ (i, j = 1, 2, \cdots, n, \ i \leq j), \tag{4.3}$$

where $\Phi = (A_i + A_j)Q + B_iM_j + B_jM_i$. *And the controller parameter can be taken as* $K_i = M_iQ^{-1}$.

From the above theorem, it is known that the key to design this controller is to find a common positive definite matrix Q and a real matrix $M_i (i = 1, 2, \cdots, m)$. We can use the matrix toolbox in MATLAB to obtain the numerical solution of the above matrices, ensuring that the discrete closed-loop fuzzy system is asymptotically stable at the zero equilibrium point.

Based on linear matrix inequalities, using the MATLAB toolbox, the parameters of the positive definite matrix Q and the real matrix $M_i (i = 1, 2, \cdots, m)$ can be obtained as follows

$$M_1 = \begin{bmatrix} -0.4504 & 0.0000 \\ 0.0000 & 10.2912 \end{bmatrix}, \ M_2 = \begin{bmatrix} -0.4504 & 0.0000 \\ 0.0000 & -2.4931 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 16.7554 & 0.0000 \\ 0.0000 & 10.2912 \end{bmatrix}, \ M_4 = \begin{bmatrix} 16.7554 & 0.0000 \\ 0.0000 & -2.4931 \end{bmatrix},$$

$$Q = \begin{bmatrix} 3.0556 & 0.0000 \\ 0.0000 & 3.0556 \end{bmatrix}.$$

Accordingly, the controller parameter K_i can be obtained as follows

$$K_1 = \begin{bmatrix} -0.1474 & 0.0000 \\ 0.0000 & 3.3680 \end{bmatrix}, \ K_2 = \begin{bmatrix} -0.1474 & 0.0000 \\ 0.0000 & -0.8159 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 5.4835 & 0.0000 \\ 0.0000 & 3.3680 \end{bmatrix}, \ K_4 = \begin{bmatrix} 5.4835 & 0.0000 \\ 0.0000 & -0.8159 \end{bmatrix}.$$

By substituting the controller K_i derived in the above theorem, we obtain the following Julia set.

Next, by adjusting the controller parameters, we can achieve a more pronounced control effect. Figure 3 depicts the Julia sets of model (4.2) when the parameters are taken as $\varepsilon_1 = 0.2$, $\gamma_1 = 0.15$, $\varepsilon_2 = 0.1$, $\gamma_2 = 0.1$, with the order $\mu = 0.6$. As the parameters increasingly approximate the controller parameters obtained using the MATLAB toolbox, the Julia set of model (4.2) also grows larger.

In Figure 3, K^a denotes setting $K_1^a = [-0.1, 0; 0, 0.1]$, $K_2^a = [-0.1, 0; 0, -0.2]$, $K_3^a = [1, 0; 0, 0.5]$, $K_4^a = [1, 0; 0, -0.2]$, K^b denotes setting $K_1^b = [-0.1, 0; 0, 0.5]$, $K_2^b = [-0.1, 0; 0, -0.3]$, $K_3^b = [2, 0; 0, 0.5]$, $K_4^b = [2, 0; 0, -0.3]$, K^c denotes setting $K_1^c = [-0.1, 0; 0, 1]$, $K_2^c = [-0.1, 0; 0, -0.35]$, $K_3^c = [2, 0; 0, 1]$, $K_4^c = [2, 0; 0, -0.35]$, K^d denotes setting $K_1^d = [-0.1, 0; 0, 1.5]$, $K_2^d = [-0.1, 0; 0, -0.4]$, $K_3^d = [2.5, 0; 0, 1.5]$, $K_4^d = [2.5, 0; 0, -0.4]$.

From Figure 3, it can be observed that as the control parameters increase, Julia sets of the prey-predator model also enlarge.

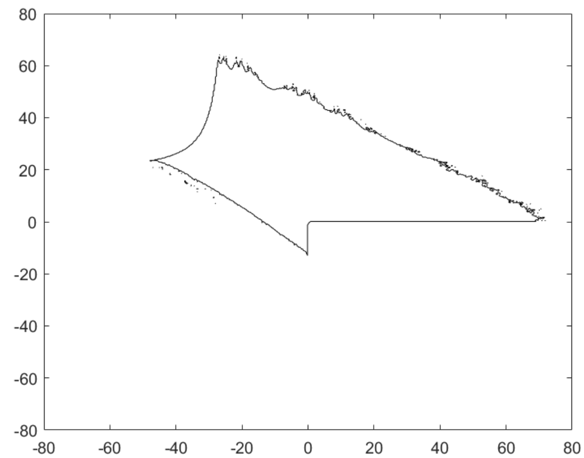


Figure 2. Julia set of model (4.2) when $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1,\ \mu = 0.9$.

5. Synchronization of Julia sets of the discrete fractional-order T-S fuzzy prey-predator models

For system (3.7), when the parameters are taken as $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1$, and the order μ is 0.9, the Julia set of system (3.7) is shown in Figure 4. The system under these conditions is taken as the drive system. When the parameters are set to $\varepsilon_3 = 0.01,\ \gamma_3 = 0.2,\ \varepsilon_4 = 1,\ \gamma_4 = 0.3$, and the order μ is set to 0.9, the Julia set of system (3.7) is shown in Figure 5. Under this conditions, the system is taken as the response system.

The fractional-order model of the response system is

$$Y_{n+1} = Y_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} D_j Y_j. \tag{5.1}$$

The model established based on the T-S fuzzy model can be obtained using the aforementioned method

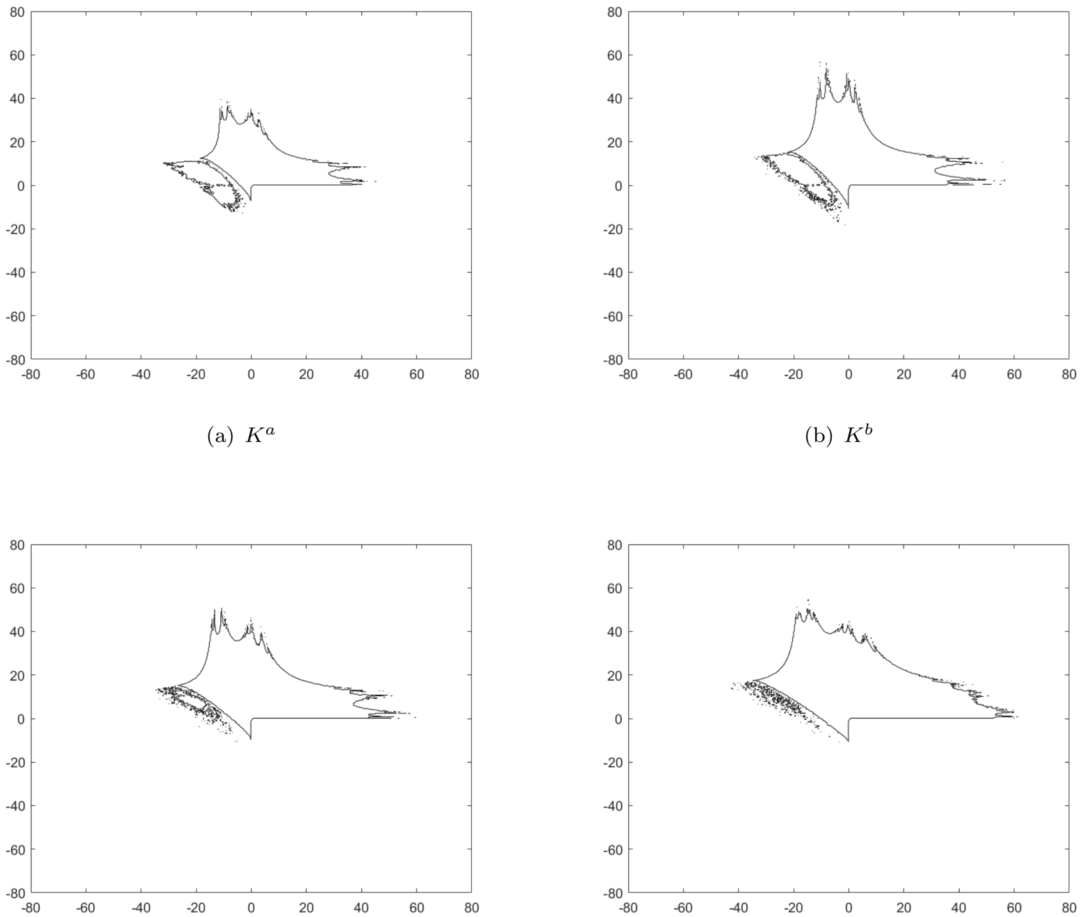
$$Y_{n+1} = Y_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} \sum_{i}^{4} l_{ij}(g(x,y)) D_{ij} Y_j. \tag{5.2}$$

Denote $\begin{bmatrix} \bar{x}_{n+1} \\ \bar{y}_{n+1} \end{bmatrix} = Y_{n+1},\ \begin{bmatrix} \bar{x}_j \\ \bar{y}_j \end{bmatrix} = Y_j$. Let the controller be $u(n)$. Similar to the previous analysis, after applying control to the drive system, it becomes

$$Y_{n+1} = Y_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\sum_{i}^{4} l_{ij}(g(x,y)) D_{ij} Y_j + Bu(n)). \tag{5.3}$$

Let $u(n) = u_l(n) + u_a(n)$ be the control input applied to the response system, where the form of the controller $u_l(n)$ is

$$\text{Rule}: \text{IF } g_{j1}(\bar{x},\bar{y}) \text{ is } \varphi_{ij1} \text{ and } g_{j2}(\bar{x},\bar{y}) \text{ is } \varphi_{ij2},$$
$$\text{Then } u_1(n) = -W_i Y_j, i = 1,2,3,4,$$



(a) K^a (b) K^b

(c) K^c (d) K^d

Figure 3. Julia sets of system (4.2) with varying controller parameters, when the parameters are taken as $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1$, and the order μ is 0.6.

and the form of the controller $u_a(n)$ is

$$\begin{aligned}\text{Rule : IF } g_{j1}(x,y) \text{ is } \psi_{ij1} \text{ and } g_{j2}(x,y) \text{ is } \psi_{ij2},\\ \text{Then } u_a(n) = Z_i X_j, i = 1,2,3,4.\end{aligned}$$

Then the total control input can be written as

$$u(n) = -\sum_{i=1}^{4} l_{ij}(g(x,y))W_i Y_j + \sum_{i=1}^{4} h_{ij}(g(x,y))Z_i X_j. \tag{5.4}$$

Substituting (5.4) into (5.3), we have

$$\begin{aligned}Y_{n+1} =& Y_0 + \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\sum_{i=1}^{4} l_{ij}(g(x,y))(D_i - BW_i)Y_j \\ &+ \sum_{i=1}^{4} h_{ij}(g(x,y))BZ_i Xj).\end{aligned} \tag{5.5}$$

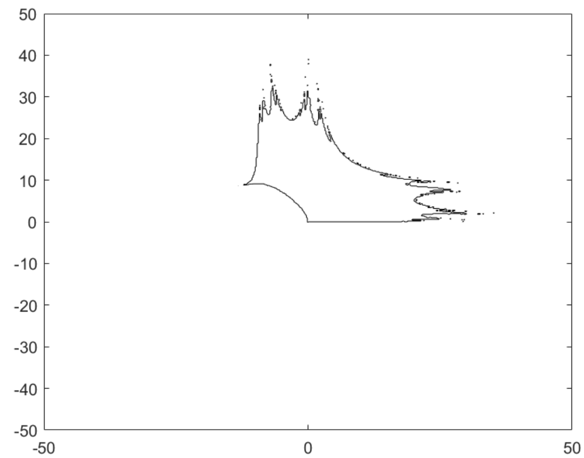


Figure 4. Julia set of model (3.7) when $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1,\ \mu = 0.9$.

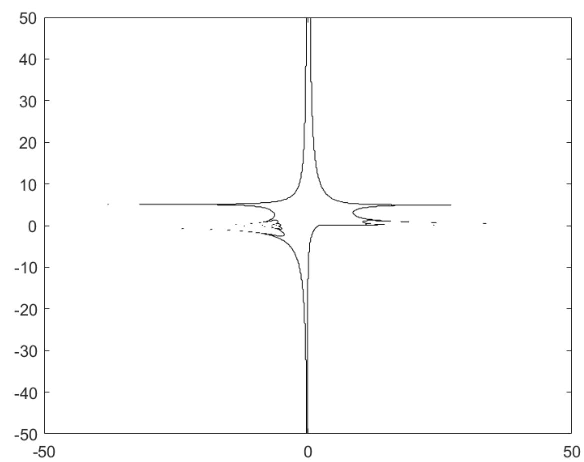


Figure 5. Julia set of model (3.7) when $\varepsilon_3 = 0.01,\ \gamma_3 = 0.2,\ \varepsilon_4 = 1,\ \gamma_4 = 0.3,\ \mu = 0.9$.

Define the error as

$$e(n+1) = Y_{n+1} - X_{n+1}. \tag{5.6}$$

Substituting equations (3.7) and (5.5) into equation (5.6), we have

$$\begin{aligned} e(n+1) =& Y_0 - X_0 \\ &+ \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\sum_{i=1}^{4} l_{ij}(g(x,y))(D_i - BW_i)Y_j \\ &- \sum_{i=1}^{4} h_{ij}(g(x,y))(A_i - BZ_i)Xj). \end{aligned} \tag{5.7}$$

Thus, the synchronization problem of the aforementioned system is transformed into the asymptotic stability problem of the equilibrium point of the error system.

Let $Y_0 = X_0$, then equation (5.7) is transformed into the following form

$$\begin{aligned} e(n+1) =& \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\sum_{i=1}^{4} l_{ij}(g(x,y))(D_i - BW_i)Y_j \\ & - \sum_{i=1}^{4} h_{ij}(g(x,y))(A_i - BZ_i)Xj). \end{aligned} \tag{5.8}$$

Theorem 5.1. *The error system* (5.8) *represented by the fuzzy system* (3.7) *and* (5.2) *is exactly linearized via the fuzzy controller Eq.* (5.4) *if there exists the feedback gains W_i and Z_j such that*

$$\{(D_1 - BW_1) - (D_i - BW_i)\}^T \cdot \{(D_1 - BW_1) - (D_i - BW_i)\} = 0, \ i = 2, 3, 4, \tag{5.9}$$

$$\{(D_1 - BW_1) - (A_i - BZ_i)\}^T \cdot \{(D_1 - BW_1) - (A_i - BZ_i)\} = 0, \ i = 1, 2, 3, 4. \tag{5.10}$$

Then, the error fuzzy system (5.8) *can be exactly linearized to $e(n+1) = 2Ge(n)$, where $G = D_1 - BW_1 = D_i - BW_i = A_i - BZ_i$.*

Proof. If equations (5.9) and (5.10) hold, system (5.8) can be expressed as

$$\begin{aligned} e(n+1) =& \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (\sum_{i=0}^{4} l_{ij}(g(x,y))GY_j \\ & - \sum_{i=1}^{4} h_{ij}(g(x,y))GX_j). \end{aligned}$$

Since $\sum_{i=0}^{4} l_{ij}(g(x,y)) = 1, \ \sum_{i=0}^{4} h_{ij}(g(x,y)) = 1$, we have

$$\begin{aligned} e(n+1) &= \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (GY_j - GX_j) \\ &= \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (GY_j - GX_j) \\ &\quad + \frac{1}{\Gamma(\mu)} \frac{\Gamma(n-n+\mu)}{\Gamma(n-n+1)} (GY_n - GX_n) \\ &= G[\frac{1}{\Gamma(\mu)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (Y_j - X_j) + (Y_n - X_n)]. \end{aligned}$$

Also, since

$$e(n) = \frac{1}{\Gamma(\mu)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\mu)}{\Gamma(n-j+1)} (Y_j - X_j) = Y_n - X_n,$$

it follows that

$$e(n+1) = G[e(n) + e(n)] = 2Ge(n).$$

□

Theorem 5.2. *Given the system* (5.2)*, another system with different structures can be controlled by the PDC method to make the closed-loop error system asymptotically stable. That is* $x_n - y_n \to 0 (n \to \infty)$ *if the following conditions should be met*

$$\{(D_1 - BW_1) - (D_i - BW_i)\}^T \cdot \{(D_1 - BW_1) - (D_i - BW_i)\} = 0, \ i = 2, 3, 4,$$

$$\{(D_1 - BW_1) - (A_i - BZ_i)\}^T \cdot \{(D_1 - BW_1) - (A_i - BZ_i)\} = 0, \ i = 1, 2, 3, 4,$$

$$4(D_i - BW_i)^T P(D_i - BW_i) - P < 0, \ i = 1, 2, 3, 4, \tag{5.11}$$

$$4(A_j - BZ_j)^T P(A_j - BZ_j) - P < 0, \ i = 1, 2, 3, 4. \tag{5.12}$$

Proof. From theorem 3, it is assumed that the error system can be exactly linearized, i.e. $G = D_1 - BW_1 = D_i - BW_i = A_i - BZ_i$. Take the Lyapunov function $V(n) = e(n)^T Pe(n)$, then the time derivative of $V(n)$ along the trajectory of the exact linearized system (5.8) is written as

$$\begin{aligned}\Delta V(n) &= V(n+1) - V(n) \\ &= e(n+1)^T Pe(n+1) - e(n)^T Pe(n) \\ &= e(n)^T (4G^T PG - P)e(n) \\ &= e(n)^T (4(D_i - BW_i)^T P(D_i - BW_i) - P)e(n) \\ &= e(n)^T (4(A_j - BZ_j)^T P(A_j - BZ_j) - P)e(n).\end{aligned}$$

If $4(D_i - BW_i)^T P(D_i - BW_i) - P < 0$ and $4(A_j - BZ_j)^T P(A_j - BZ_j) - P < 0$, then $\Delta V(n) < 0$. So the closed-loop error system is asymptotically stable, and when $t \to \infty, e \to 0$, that is $X \to Y$. Thus, the drive system and the response system are synchronized. □

Let $W_1 = \begin{bmatrix} 0.5 & -0.3 \\ 0.4 & -0.9 \end{bmatrix}$. According to (5.9), (5.10), (5.11), (5.12), by using MATLAB Mosek toolbox, we can obtain the feedback gain matrix as follows

$$P = \begin{bmatrix} -0.0422 & -0.0229 \\ -0.0229 & -0.1338 \end{bmatrix}, \ W_2 = \begin{bmatrix} 0.4998 & -0.3274 \\ 0.4007 & 26.0435 \end{bmatrix},$$

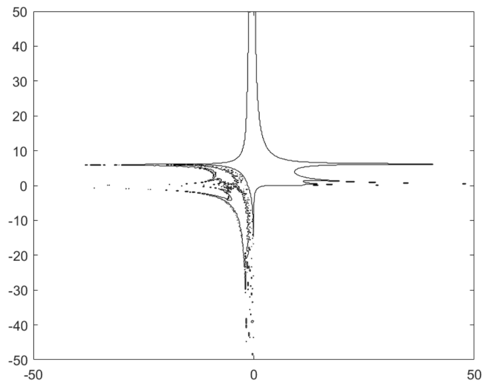
$$W_3 = \begin{bmatrix} -47.1137 & -0.2993 \\ 0.3646 & -0.9003 \end{bmatrix}, \ W_4 = \begin{bmatrix} -47.1166 & -0.3027 \\ 0.3953 & 26.0376 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} -14.6675 & -0.3029 \\ 0.3953 & 6.6932 \end{bmatrix}, \ Z_2 = \begin{bmatrix} -14.6553 & -0.2902 \\ 0.4137 & 10.9602 \end{bmatrix},$$

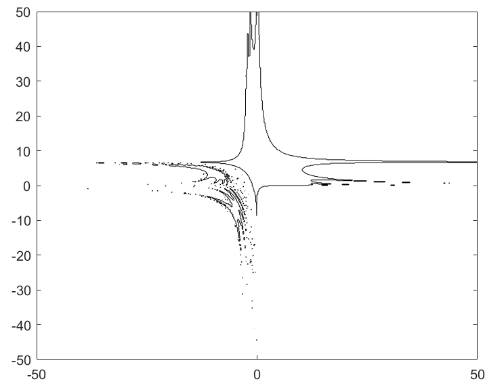
$$Z_3 = \begin{bmatrix} -20.3463 & -0.3051 \\ 0.3859 & 6.6891 \end{bmatrix}, \ Z_4 = \begin{bmatrix} -20.3002 & -0.3109 \\ 0.3784 & 10.9370 \end{bmatrix}.$$

By simultaneously adjusting the coefficients of the control parameters W and Z, the synchronization process of the two models can be achieved as shown in Figure 6.

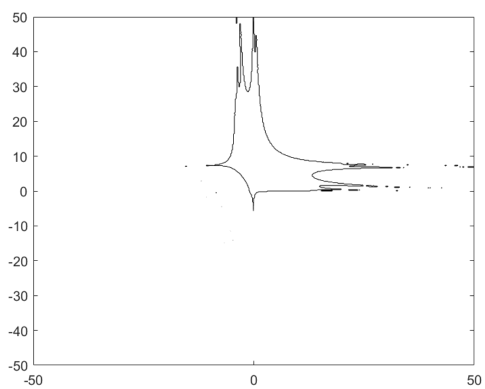
From Figure 6, it can be seen that when the control parameters are set to $0.3W$ and $0.3Z$, the shape of the Julia set of the system is relatively similar to that of the response system. As the parameters gradually approach W and Z, the Julia set of the response system increasingly resembles that of the driving system, ultimately achieving synchronization between the two systems.



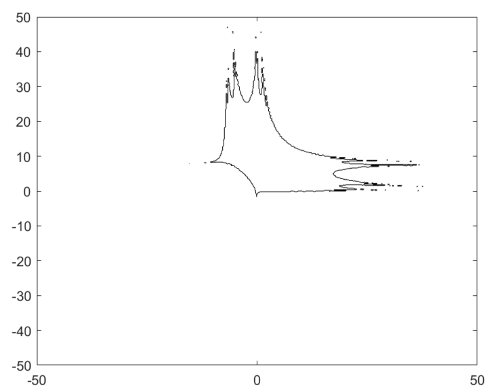
(a) $0.3W, 0.3Z$



(b) $0.5W, 0.5Z$



(c) $0.7W, 0.7Z$



(d) $0.9W, 0.9Z$

Figure 6. Synchronization of Julia set of model (3.7) by changing the coefficients of W and Z, when $\varepsilon_1 = 0.2,\ \gamma_1 = 0.15,\ \varepsilon_2 = 0.1,\ \gamma_2 = 0.1,\ \varepsilon_3 = 0.01,\ \gamma_3 = 0.2,\ \varepsilon_4 = 1,\ \gamma_4 = 0.3,\ \mu = 0.9$.

6. Conclusion

This paper proposes a control and synchronization methods for the Julia sets of discrete fractional-order predator-prey models based on the T-S fuzzy model. Firstly, the predator-prey model is discretized to obtain its discrete fractional-order model, considering the memory effects and long-time delay behavior of the system. Then, based on the T-S fuzzy model, the nonlinear behavior of the system is decomposed into a weighted combination of multiple linear subsystems for modeling. On this basis, a controller is designed for the discrete fractional-order predator-prey model by using the PDC method. And by adjusting the controller parameters, control over the model's Julia set is achieved. Finally, linear control is utilized to realize the synchronization of the Julia sets of the discrete fractional-order predator-prey model based on the T-S fuzzy model.

The predator-prey model is a classic model in ecology. Studying the control and synchronization of its fractional-order form helps to more accurately describe and predict the population dynamics of species in ecosystems, thereby providing theoretical support for ecological conservation and resource management. The application of the T-S fuzzy model and the PDC method in the control of complex nonlinear systems offers new ideas and methods for solving similar problems, especially for systems that have uncertainties and complex dynamic behaviors. This research integrates ecology, control theory, and fractal theory, demonstrating the advantages of interdisciplinary research and providing references for cross-domain studies.

Despite considering memory effects and long-time delay behaviors, the predator-prey model still has certain simplifying assumptions. For example, it only considers the interaction between a single predator and prey, ignoring the influence of other ecological factors. In practice, the parameters of ecological systems are often uncertain, while this study assumes that the parameters are known or can be estimated in some way, which may be difficult to achieve in reality.

Moving forward, we will further improve the predator-prey model by incorporating more ecological factors (such as environmental changes and species diversity) and validate the model's effectiveness through real-world data. We will also investigate how to control and synchronize the Julia set when parameters are unknown and develop adaptive control strategies to enhance the robustness of the system. Additionally, we will explore the synchronization of multiple predator-prey models and their applications in complex ecological networks, studying the coupling mechanisms between different models.

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