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SPECTRAL STABILITY OF ELLIPTIC SOLUTIONS TO THE DEFOCUSING LAKSHMANAN-PORSEZIAN-DANIEL EQUATION*

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Abstract In this paper, by exploiting the integrability of the defocusing Lakshmanan-Porsezian-Daniel (LPD) equation, we establish the spectral stability for the elliptic solutions with respect to subharmonic perturbations. We achieve this goal by constructing explicitly the squared-eigenfunction connection between the linear stability problem and its Lax pair. Furthermore, based on the spectral stability results, the linear stability for subharmonic perturbations is obtained by applying directly the Skew-symmetric Composed with Self-adjoint (SCS) basis lemma. Although it is challenging to determine analytically the spectra for all operators except the simplest ones, this research provides a detailed analytical description of the stable Lax spectrum and the stability spectrum.

Keywords Spectral stability, LPD equation, elliptic solutions, subharmonic perturbations, integrability.

MSC(2010) 37K45, 35Q51, 33E05.

1. Introduction

The defocusing Lakshmanan-Porsezian-Daniel (LPD) equation [30] is given by

$$\mathrm{i}u_t + \alpha(-u_{xx} + 2|u|^2u) + \beta(-u_{xxxx} - 6u|u|^4 + 4|u_x|^2u + 6u_x^2u^* + 2u^2u_{xx}^* + 8u_{xx}|u|^2) = 0, \quad (1.1)$$

where $u = u(x,t)$ represents the complex envelope. Here, x and t denote the spatial and temporal coordinates respectively. The subscripts indicate the partial derivatives, and the asterisk $*$ designates the complex conjugation. The real-valued parameter α is the coefficient of the second-order dispersion, while $\beta \in \mathbb{R}$ signifies the strength of high-order linear and nonlinear effects. The LPD equation (1.1) is a higher-order nonlinear Schrödinger (NLS) model, which includes the well-known NLS equation corresponding to $\alpha = -1$, $\beta = 0$ and the fourth order nonlinear Schrödinger (4NLS) equation corresponding to $\alpha = 0$, $\beta = 1$ as two special cases.

The LPD equation, initially derived to model the nonlinear spin excitations in the one-dimensional (1D) isotropic biquadratic Heisenberg ferromagnetic spin chain [10,30,37], is one of

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the most crucial equations for characterizing the dynamics of soliton propagation. Notably, (1.1) is an extension of the NLS equation and describes the nonlinear effect more clearly. Hence, it can be used to model the ultrashort optical pulses propagating in long-distance and high-speed optical fibre transmission [5,11]. At the same time, as a more general model of the 4NLS equation (the fifth member in the integrable NLS hierarchy [2]), the LPD equation (1.1) is endowed with numerous remarkable features, e.g., an infinite number of conservation laws, complete integrability, and Hamiltonian structures [9, 51]. For integrable systems in general, the study of integrability, the Riemann-Hilbert problem and the Cauchy problem has received considerable attention [24, 26]. In particular, Hu et al. investigated the Riemann-Hilbert problem associated with both the matrix LPD system [25] and the vector LPD model [22]. Due to the significant characteristics and extensive applications of the LPD equation, many aspects of (1.1) have been explored. For example, the mathematical properties of the LPD equation and large classes of its exact solutions have been investigated by many researchers [20,28,30,33,35,37,45–48,50–52]. In addition, the dynamic behaviors of exact solutions for various types of the LPD equation have been discussed [3, 18, 23, 31, 32, 34, 36, 53]. Besides, well-posedness of the Cauchy problem for (1.1) with different nonlinearities has been studied [27,39,40]. In a recent study, Chen, Liu, and Wang [8] have established the sharp local well-posedness for the LPD equation in the Sobolev space $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$, which extends the results obtained by Huo and Jia [27].

It is well known that the stability of nonlinear partial differential equations (PDEs) is of great significance. This is because nonlinear equations often capture the essence of physically relevant systems through mathematical simplification, and stability analysis can offer in-depth insights into a given mathematical physics system for understanding the exact dynamics of nonlinear phenomena [19]. The modulational instability of solitons for different kinds of higher-order NLS equations with fourth-order dispersion has been investigated extensively. For instance, Saha and Sarma [38] derived bright and dark solitary wave solutions for the generalized normalized NLS equation with cubic-quintic nonlinearity and performed a modulation instability analysis in both the anomalous and normal dispersion regimes. Arshad, Seadawy, and Lu [4] constructed new periodic solitary wave elliptic function solutions for the higher-order NLS equation with cubic nonlinearity, and discussed the stability of these solutions via modulation instability analysis. Moreover, they employed the linear stability analysis to investigate the evolution of weak and time-dependent perturbations with respect to the propagation distance. Here, the elliptic function solutions (abbreviated as the elliptic solutions) are periodic solutions in terms of elliptic functions with the elliptic modulus $k \in [0, 1)$. Additionally, some authors [43] focused on the stationary solutions of the 1D NLS equation with a fourth-order diffraction and computed the linear stability spectrum numerically. In fact, the research on the stability of spatially-periodic stationary solutions to nonlinear wave equations has advanced both numerically and analytically over the past few years. For example, this includes numerical investigations on spectral stability [7, 13, 14], and analytical studies on spectral and orbital stability [6, 12, 17, 41, 42, 49]. However, the stability of spatially-periodic stationary solutions to the integrable LPD equation has attracted less attention than that of solitons. More importantly, the integrability of the LPD equation has not been adequately exploited to establish the stability in most stability studies. For the integrable equation (1.1), the Floquet discriminant [1, 29] can be applied to compute numerically the Lax spectrum for periodic potentials. Unfortunately, it is unable to provide a completely analytical description of the Lax spectrum. Different from the Floquet discriminant, using its integrability allows for an explicit analytical determination of the Lax spectrum and the stability spectrum.

In this research, using the integrability of the LPD equation (1.1), we attempt to investigate analytically the spectral and linear stability of elliptic solutions under subharmonic perturbations. The primary reasons for considering the stability of solutions under subharmonic perturbations are as follows. Firstly, compared to co-periodic perturbations, subharmonic perturbations are a larger class of periodic perturbations whose period is an integer multiple of the period of the underlying solution. For some PDEs [15–17], it is possible for elliptic solutions to be unstable for subharmonic perturbations yet stable for co-periodic perturbations. Secondly, it possesses greater physical significance, since the domain of perturbations considered in applications is often wider than the period of the underlying solution. Therefore, it is necessary to investigate the stability with respect to subharmonic perturbations, which are not restricted to co-periodic perturbations.

This paper is structured as follows. In Section 2, the elliptic solutions to (1.1) are obtained. In Section 3, we linearize the LPD equation (1.1) about the stationary solutions to derive the linear stability problem and the spectral problem. In Section 4, using square-eigenfunctions to connect the spectral problem and the Lax spectral problem, we establish the spectral stability for elliptic solutions of the LPD equation. Meanwhile, a complete and explicit description of the stable Lax spectrum is provided. Finally, Section 5 presents our conclusions.

2. The elliptic solutions of the LPD equation

We begin by constructing the spatially-periodic stationary solutions to (1.1) in the form

$$u = e^{-\mathrm{i}\omega t}\phi(x),$$

where ω is a real constant and $\phi(x)$ is a real-valued function. In the following, we denote the higher-order derivative ϕ_{xxxx} as ϕ_{4x} for simplicity. Then $\phi(x)$ satisfies the ordinary differential equation

$$\omega\phi - \alpha(\phi_{xx} - 2\phi^3) + \beta(-\phi_{4x} + 10\phi^2\phi_{xx} - 6\phi^5 + 10\phi\phi_x^2) = 0. \tag{2.1}$$

Next, we reduce the order of (2.1) to obtain its elliptic solutions by introducing the transformation

$$\phi_{xx} = d_0\phi(x) + 2\phi(x)^3, \tag{2.2}$$

where d_0 is a real parameter. This second-order differential equation is equivalent to the following planar dynamical system:

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}x} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = d_0\phi + 2\phi^3. \end{cases} \tag{2.3}$$

Moreover, the first integral of (2.3) is given by

$$H(\phi, y) = \phi^4 + d_0\phi^2 - y^2 = h, \tag{2.4}$$

where h is a constant. From (2.4), it follows that

$$\phi_x^2 = \phi^4 + d_0\phi^2 - h.$$

Under the above transformation (2.2), (2.1) is rewritten as

$$\phi_x^2 = \phi^4 + d_0\phi^2 + \frac{\omega - \alpha d_0 - \beta d_0^2}{2\beta}. \tag{2.5}$$

According to the bifurcation theory of dynamical systems, when $d_0 < 0$ and $-\frac{\omega-\alpha d_0-\beta d_0^2}{2\beta} = h \in (-d_0^2/4, 0)$, there exists a family of periodic solutions for (2.5).

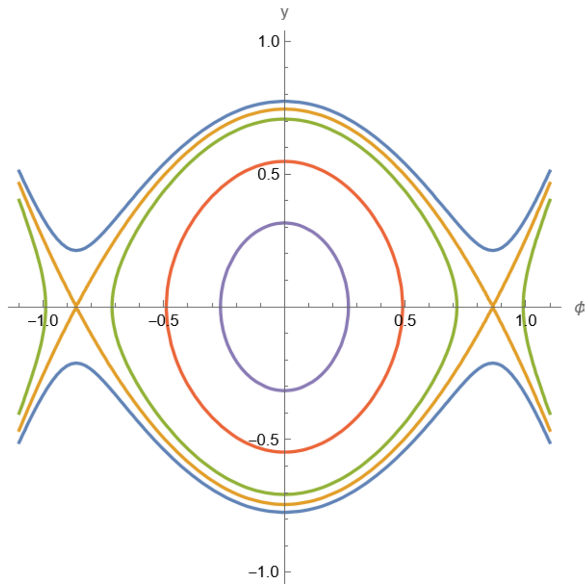


Figure 1. A family of periodic orbits of system (2.3) for $d_0 = -1.49$.

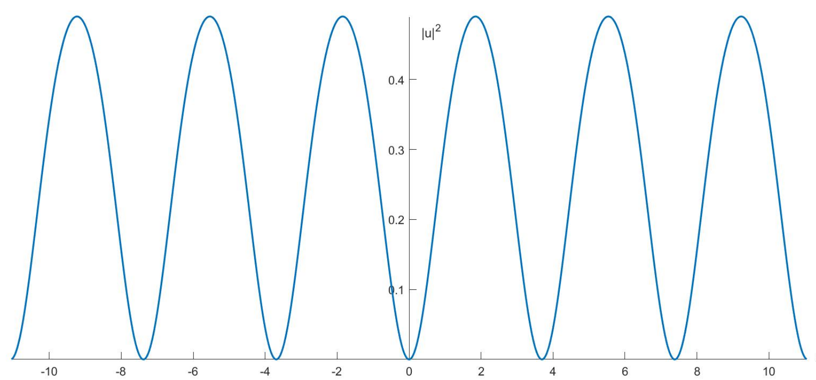


Figure 2. $|u|^2$ as a function of x with $m = 1$ and $k = 0.7$.

To solve the above equation (2.5), we will introduce the Jacobi elliptic functions. As one of its most essential functions, the Jacobi elliptic sine function $\mathrm{sn}(x, k)$, where x is the argument and $k \in [0, 1)$ is the elliptic modulus, satisfies the first-order nonlinear differential equation

$$f_x^2 = (1 - f^2)(1 - k^2 f^2). \tag{2.6}$$

Inspired by (2.6), the Jacobi elliptic function solutions of (2.5) are in the following form:

$$\phi(x) = \mathrm{g}\,\mathrm{sn}(mx, k), \tag{2.7}$$

where

$$\mathrm{g}^2 = k^2 m^2, \quad \omega = \beta(1 + 4k^2 + k^4)m^4 - \alpha(1 + k^2)m^2, \quad d_0 = -(1 + k^2)m^2. \tag{2.8}$$

From the expression (2.8), it is apparent that g, ω and d_0 are parametrized by real-valued parameters m and k. This means that g, ω and d_0 are all real parameters satisfying the conditions (2.8), and $\phi(x)$ is a periodic function with period $T(k) = \frac{4K(k)}{m}$. Here,

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}y}{\sqrt{1 - k^2 \sin^2(y)}}$$

is the complete elliptic integral of the first kind.

In terms of elliptic functions, the spatially-periodic solutions to (1.1) can be expressed as

$$u(x,t) = \mathrm{g}\, e^{-\mathrm{i}\omega t}\, \mathrm{sn}(mx, k), \tag{2.9}$$

with $\mathrm{g}^2 = k^2 m^2$ and $\omega = \beta(1 + 4k^2 + k^4)m^4 - \alpha(1 + k^2)m^2$. Taking $m = 1$ and $k = 0.7$, we show a family of periodic orbits of system (2.3) in Figures 1 and an example of solutions (2.9) in Figures 2.

Remark 2.1. When $\alpha = 0$ and $\beta = 1$, the LPD equation (1.1) simplifies to the 4NLS equation, and the conclusions in this paper can be translated into the corresponding results in [42]. Moreover, when $\alpha = -1/2$ and $\beta = 0$, (1.1) simplifies to the NLS equation, and our results align with the known conclusions [6] for elliptic solutions with trivial phase profiles.

3. The linear stability problem

Using the transformation $u = e^{-\mathrm{i}\omega t} q(x,t)$, we write (1.1) as

$$\omega q + \mathrm{i}q_t + \alpha(-q_{xx} + 2q|q|^2) + \beta(-q_{xxxx} + 8|q|^2 q_{xx} - 6q|q|^4 + 4q|q_x|^2 + 6q_x^2 q^* + 2q^2 q_{xx}^*) = 0. \tag{3.1}$$

The linear stability of elliptic solutions to (3.1) is examined by considering

$$q(x,t) = \phi(x) + \epsilon\mu(x,t) + \mathrm{i}\epsilon\nu(x,t) + \mathcal{O}(\epsilon^2), \tag{3.2}$$

where $\mu(x,t)$ and $\nu(x,t)$ are all real-valued functions, and ϵ is a small parameter. By ignoring the higher-order terms in ϵ, we are focusing on the linear stability problem. To be specific, substituting (3.2) into (3.1), ignoring higher-than-first-order terms in ϵ, and separating the real and imaginary parts, we can derive the linear stability problem as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix}\mu\\ \nu\end{pmatrix} = J\mathcal{L}\begin{pmatrix}\mu\\ \nu\end{pmatrix} = J\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix}, \tag{3.3}$$

where

$$J = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

In this context, $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as

$$\mathcal{L}_1 = \alpha(\partial_{xx} - 6\phi^2) - \omega + \beta(-10\phi_x^2 - 20\phi\phi_{xx} + 30\phi^4 - 20\phi\phi_x\partial_x - 10\phi^2\partial_{xx} + \partial_{4x}),$$
$$\mathcal{L}_2 = \alpha(\partial_{xx} - 2\phi^2) - \omega + \beta(2\phi_x^2 - 4\phi\phi_{xx} + 6\phi^4 - 12\phi\phi_x\partial_x - 6\phi^2\partial_{xx} + \partial_{4x}).$$

Since $\phi(x)$ solves the LPD equation, the zero-order terms in ϵ disappear.

Given that (3.3) is autonomous with respect to t, we may separate the variables as follows:

$$\begin{pmatrix} \mu(x,t) \\ \nu(x,t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \mu_1(x,\lambda) \\ \nu_1(x,\lambda) \end{pmatrix}.$$

Therefore, the spectral problem could be stated as

$$\lambda \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} = J\mathcal{L} \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} = J \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix}. \tag{3.4}$$

Before we further investigate the above spectral problem, a few definitions need to be introduced.

Definition 3.1. The stability spectrum $\sigma_{J\mathcal{L}}$ of the operator $J\mathcal{L}$ is the set

$$\sigma_{J\mathcal{L}} = \Big\{ \lambda \in \mathbb{C} : \sup_{x\in\mathbb{R}} \big(|\mu_1(x)|, |\nu_1(x)|\big) < \infty \Big\}.$$

Definition 3.2. The solutions $\phi(x)$ are spectrally stable if the open right-half of the complex λ-plane is not intersected by the spectrum $\sigma_{J\mathcal{L}}$. In particular, the solutions of the LPD equation (1.1) are spectrally stable only if $\sigma_{J\mathcal{L}}$ is a subset of the imaginary axis, i.e., $\sigma_{J\mathcal{L}} \subset \mathrm{i}\mathbb{R}$, since (1.1) is Hamiltonian.

Definition 3.3. The elliptic solutions $\phi(x)$ are linearly stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|\mu(x,0) + \mathrm{i}\nu(x,0)\| < \delta$, then $\|\mu(x,t) + \mathrm{i}\nu(x,t)\| < \varepsilon$ for all $t > 0$. One should keep in mind that this definition relies on the selection of the norm $\|\cdot\|$ of the perturbations, which will be discussed later.

Particularly, if the eigenfunctions corresponding to the stability spectrum $\sigma_{J\mathcal{L}}$ are complete in the space determined by the norm $\|\cdot\|$, then the spectral stability of an elliptic solution indicates its linear stability. In this particular case, one may obtain all solutions of (3.3) through the linear combinations of the solutions to (3.4).

Definition 3.4. A P-subharmonic perturbation of a solution is a periodic perturbation whose period is an integer multiple P of the solution's period.

Based on the above definitions, to establish the spectral stability of elliptic solutions analytically, we intend to determine the stability spectrum $\sigma_{J\mathcal{L}}$ and related eigenfunctions. For this purpose, we study the eigenfunctions first.

From the definition of the operator $\mathcal{L}$, it is obvious that $\mathcal{L}$ has periodic coefficients. By applying the Floquet–Bloch decomposition, the eigenfunctions can be written in the following form:

$$\begin{pmatrix} \mu_1(x) \\ \nu_1(x) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\rho x} \begin{pmatrix} \hat{\mu}(x) \\ \hat{\nu}(x) \end{pmatrix}, \tag{3.5}$$

with $\rho \in \big[-\pi/2T(k), \pi/2T(k)\big)$. Here, $\hat{\mu}\big(x + T(k)\big) = \hat{\mu}(x)$ and $\hat{\nu}\big(x + T(k)\big) = \hat{\nu}(x)$. According to Floquet's Theorem, all bounded solutions of (3.4) follow this form. Here, boundedness means that $\max_{x\in\mathbb{R}} \big\{|\mu_1(x)|, |\nu_1(x)|\big\}$ is finite, implying

$$\mu_1(x), \nu_1(x) \in C_b^0(\mathbb{R}).$$

Also, since the exponential factor in (3.5) vanishes in the calculation of the L^2-norm, we have

$$\mu_1(x), \nu_1(x) \in L^2_{per}\big([-T(k)/2, T(k)/2]\big),$$

i.e., the space of square-integrable functions with period $T(k)$. As a result, we can conclude that

$$\mu_1(x), \nu_1(x) \in C_b^0(\mathbb{R}) \cap L^2_{per}\big([-T(k)/2, T(k)/2]\big).$$

Importantly, this choice ensures that perturbations considered in our investigations have an arbitrary period which is an integer multiple of $T(k)$. In the case of P-subharmonic perturbations,

$$\rho = n\frac{2\pi}{PT(k)}, \quad n = 0, \ldots, P-1.$$

By employing the Floquet–Bloch decomposition, one can define the subharmonic stability spectrum associated with the parameter ρ as follows:

$$\sigma_\rho = \Big\{\lambda \in \mathbb{C} : \hat{\mu}_\rho, \hat{\nu}_\rho \in C_b^0(\mathbb{R}) \cap L^2_{\text{per}}\big([-T(k)/2, T(k)/2]\big)\Big\}.$$

4. The Lax pair and squared-eigenfunction connection

For many integrable systems [6, 12, 15–17, 42, 49], a squared-eigenfunction connection between the linear stability problem and its Lax pair has been established to prove the spectral stability. In this section, we aim to construct the squared-eigenfunction connection between the eigenfunctions of the linear stability problem and those of its Lax pair. With the squared-eigenfunction connection, an analytical representation for both the stability spectrum and its associated eigenfunctions can be derived. As demonstrated by Upsal and Deconinck [44], a purely real Lax spectrum indicates spectral stability. Therefore, we study the Lax spectrum first. The Lax pair of (1.1) is given as follows:

$$\Psi_x = X_1\Psi, \quad \Psi_t = T_1\Psi, \tag{4.1}$$

where

$$X_1 = \begin{pmatrix} -\mathrm{i}\xi & u(x,t) \\ u^*(x,t) & \mathrm{i}\xi \end{pmatrix}, \quad T_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix}, \tag{4.2}$$

with

$$\begin{aligned} A_1 =& -2\beta\xi(uu_x^* - u_xu^*) - \mathrm{i}\beta|u_x|^2 + \mathrm{i}\beta(u_{xx}u^* + uu_{xx}^*) + (2\mathrm{i}\alpha - 4\mathrm{i}\beta|u|^2)\xi^2 - 3\mathrm{i}\beta|u|^4 \\ & - 8\mathrm{i}\beta\xi^4 + \mathrm{i}\alpha|u|^2, \\ B_1 =& -\xi(2\beta u_{xx} - 4\beta|u|^2u + 2\alpha u) + 6\mathrm{i}\beta|u|^2u_x + 4\mathrm{i}\beta u_x\xi^2 - \mathrm{i}\beta u_{xxx} + 8\beta u\xi^3 - \mathrm{i}\alpha u_x, \\ C_1 =& -\xi(-4\beta|u|^2u^* + 2\beta u_{xx}^* + 2\alpha u^*) - 6\mathrm{i}\beta|u|^2u_x^* - 4\mathrm{i}\beta u_x^*\xi^2 + \mathrm{i}\beta u_{xxx}^* + 8\beta u^*\xi^3 + \mathrm{i}\alpha u_x^*. \end{aligned}$$

The first equation of Lax pair (4.1) could be rewritten as a Lax spectral problem with the spectral parameter ξ, i.e.,

$$\begin{pmatrix} \mathrm{i}\partial_x & -\mathrm{i}u \\ \mathrm{i}u^* & -\mathrm{i}\partial_x \end{pmatrix}\Psi = \xi\Psi. \tag{4.3}$$

Note that (4.3) is a self-adjoint spectral problem, so the Lax spectrum σ_L is a subset of the real line and defined as

$$\sigma_L := \big\{\xi \in \mathbb{C} : \sup_{x\in\mathbb{R}} \big(|\Psi_1|, |\Psi_2|\big) < \infty\big\} \subset \mathbb{R}.$$

Evaluating (4.1) at $u(x,t) = e^{-\mathrm{i}\omega t} q(x,t)$, we have

$$\Psi_x = X_2\Psi, \quad \Psi_t = T_2\Psi, \tag{4.4}$$

where

$$\begin{aligned} X_2 &= \begin{pmatrix} -\mathrm{i}\xi & q(x,t) \\ q^*(x,t) & \mathrm{i}\xi \end{pmatrix}, \\ T_2 &= \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix} + \begin{pmatrix} \frac{\mathrm{i}\omega}{2} & 0 \\ 0 & -\frac{\mathrm{i}\omega}{2} \end{pmatrix}, \end{aligned} \tag{4.5}$$

with

$$\begin{aligned} A_2 =& -2\beta\xi\big(qq_x^* - q_xq^*\big) - \mathrm{i}\beta|q_x|^2 + \mathrm{i}\beta\big(q_{xx}q^* + qq_{xx}^*\big) + \big(2\mathrm{i}\alpha - 4\mathrm{i}\beta|q|^2\big)\xi^2 - 3\mathrm{i}\beta|q|^4 \\ & - 8\mathrm{i}\beta\xi^4 + \mathrm{i}\alpha|q|^2, \\ B_2 =& -\xi\big(2\beta q_{xx} - 4\beta|q|^2q + 2\alpha q\big) + 6\mathrm{i}\beta|q|^2q_x + 4\mathrm{i}\beta q_x\xi^2 - \mathrm{i}\beta q_{xxx} + 8\beta q\xi^3 - \mathrm{i}\alpha q_x, \\ C_2 =& -\xi\big(-4\beta|q|^2q^* + 2\beta q_{xx}^* + 2\alpha q^*\big) - 6\mathrm{i}\beta|q|^2q_x^* - 4\mathrm{i}\beta q_x^*\xi^2 + \mathrm{i}\beta q_{xxx}^* + 8\beta q^*\xi^3 + \mathrm{i}\alpha q_x^*. \end{aligned}$$

In this case, the compatibility condition $\Psi_{xt} = \Psi_{tx}$ is equivalent to (3.1). And then, we restrict the Lax pair (4.4) to the elliptic solutions $\phi(x)$. Therefore, the t-part of Lax pair (4.4) is rewritten as

$$\Psi_t = T_3\Psi = \begin{pmatrix} A_3 & B_3 \\ C_3 & -A_3 \end{pmatrix}\Psi, \tag{4.6}$$

where

$$\begin{aligned} A_3 =& -\mathrm{i}\beta\phi_x^2 + 2\mathrm{i}\beta\phi\phi_{xx} + \big(2\mathrm{i}\alpha - 4\mathrm{i}\beta\phi^2\big)\xi^2 - 3\mathrm{i}\beta\phi^4 - 8\mathrm{i}\beta\xi^4 + \mathrm{i}\alpha\phi^2 + \frac{\mathrm{i}\omega}{2}, \\ B_3 =& -\xi\big(2\beta\phi_{xx} - 4\beta\phi^3 + 2\alpha\phi\big) + 6\mathrm{i}\beta\phi^2\phi_x + 4\mathrm{i}\beta\phi_x\xi^2 - \mathrm{i}\beta\phi_{xxx} + 8\beta\phi\xi^3 - \mathrm{i}\alpha\phi_x, \\ C_3 =& -\xi\big(-4\beta\phi^3 + 2\beta\phi_{xx} + 2\alpha\phi\big) - 6\mathrm{i}\beta\phi^2\phi_x - 4\mathrm{i}\beta\phi_x\xi^2 + \mathrm{i}\beta\phi_{xxx} + 8\beta\phi\xi^3 + \mathrm{i}\alpha\phi_x. \end{aligned}$$

Since A_3, B_3 and C_3 are independent of t, we may separate variables. Let

$$\Psi(x,t) = e^{\Omega t}\varphi(x), \tag{4.7}$$

where Ω is independent of t and x, to be determined later. Substituting (4.7) into the t-part of the Lax pair and canceling the exponential term, we can get

$$\begin{pmatrix} A_3 - \Omega & B_3 \\ C_3 & -A_3 - \Omega \end{pmatrix}\varphi(x) = \mathbf{0}. \tag{4.8}$$

The existence of nontrivial solutions of (4.8) requires

$$\begin{aligned}\Omega^2 &= (A_3)^2 + B_3C_3\\ &= -\frac{1}{4\beta}\big(\alpha+\beta(d_0-4\xi^2)\big)^2\big(3d_0^2\beta+16\beta\xi^4+2d_0(\alpha+4\beta\xi^2)-2\omega\big),\end{aligned}\tag{4.9}$$

which is obtained by using the expressions of ϕ_x^2, ϕ_{xx} and ϕ_{xxx}.

With the help of (2.8), (4.9) is rewritten as

$$\Omega^2 = -64\beta(\xi-\xi_1)(\xi+\xi_1)(\xi-\xi_2)(\xi+\xi_2)(\xi-\xi_3)^2(\xi+\xi_3)^2,\tag{4.10}$$

where

$$\xi_1 = \frac{1}{2}(1-k)m,\quad \xi_2 = \frac{1}{2}(1+k)m,\quad \xi_3 = \sqrt{\frac{\alpha-\beta m^2(k^2+1)}{4\beta}}.$$

Since (4.10) establishes an explicit connection between Ω and ξ, it is expected to determine the eigenvector $\varphi(x)$. From (4.8), the eigenvectors corresponding to the eigenvalue Ω are

$$\varphi(x) = \begin{pmatrix} -B_3 \\ A_3-\Omega \end{pmatrix}\gamma(x),\tag{4.11}$$

where $\gamma(x)$ represents an undetermined scalar function. Actually, due to the vector part of (4.11), $\Psi(x,t)$ satisfies (4.6), i.e., the t-part of the Lax pair. Then, substituting (4.11) into the x-part of the Lax pair, we can derive

$$\begin{aligned}\gamma(x) =&\gamma_0\exp\left(\int\frac{\mathrm{i}\xi B_3+\phi(A_3-\Omega)+B_{3,x}}{-B_3}\,\mathrm{d}x\right)\\ =&\gamma_1\exp\left(\int\frac{-\phi^*B_3+\mathrm{i}\xi(A_3-\Omega)-A_{3,x}}{A_3-\Omega}\,\mathrm{d}x\right).\end{aligned}\tag{4.12}$$

For almost all $\xi\in\mathbb{C}$, we have determined two linearly independent solutions of (4.4). Specifically, for all $\xi\in\mathbb{C}$ such that $\Omega\neq 0$, we can obtain two linearly independent solutions of (4.4) by combining (4.11) and (4.12). This is because for each such ξ, there are two distinct solutions corresponding to different signs of Ω. On the other hand, for ξ such that $\Omega=0$, we can construct only one solution directly and find the second solution by employing the reduction of order approach.

Based on the definition of Lax spectrum and the above analysis, σ_L is equivalent to the set of all ξ such that (4.11) is bounded for all x. Given that the vector part of (4.11) is bounded with respect to x, we aim to determine the set of all ξ for which the scalar function $\gamma(x)$ is bounded. According to the expression (4.12), the boundedness of $\gamma(x)$ requires the following necessary and sufficient condition:

$$\left\langle\Re\left(-\frac{\mathrm{i}\xi B_3+\phi(A_3-\Omega)+B_{3,x}}{B_3}\right)\right\rangle = \left\langle\Re\left(\frac{-\phi^*B_3+\mathrm{i}\xi(A_3-\Omega)-A_{3,x}}{A_3-\Omega}\right)\right\rangle = 0,$$

where $\langle\cdot\rangle = \frac{1}{T(k)}\int_0^{T(k)}\cdot\,\mathrm{d}x$ means the average over a period and $\Re$ denotes the real part. Using the above condition, we can further establish a connection between Ω and σ_L.

Since $\xi \in \sigma_L$ implies $\xi \in \mathbb{R}$, we can infer from (4.10) that Ω is either real or purely imaginary. Now we prove that the Lax spectrum σ_L is composed of all ξ values such that $\Omega^2 \leq 0$.

• Case I. When $\Omega \in \mathrm{i}\mathbb{R}$ (including the special case $\Omega = 0$), with the help of $\phi_{xx} = d_0\phi(x) + 2\phi(x)^3$, we have

$$\left\langle \Re\left(\frac{-\phi^* B_3 + \mathrm{i}\xi(A_3 - \Omega) - A_{3,x}}{A_3 - \Omega}\right)\right\rangle = \left\langle \Re\left(\frac{-\phi^* B_3 - A_{3,x}}{A_3 - \Omega}\right)\right\rangle = \frac{1}{T(k)} \int_0^{T(k)} -\frac{\mathrm{i}\phi\phi_x(\beta d_0 - 4\beta\xi^2 + \alpha)}{A_3 - \Omega}\, \mathrm{d}x,$$

which is a total derivative. As a result, its average value over a period equals zero and every ξ value such that $\Omega^2 \leq 0$ belongs to the Lax spectrum σ_L.

• Case II. When $\Omega \in \mathbb{R}$ and $\Omega \neq 0$, we have

$$\left\langle \Re\left(\frac{-\phi^* B_3 + \mathrm{i}\xi(A_3 - \Omega) - A_{3,x}}{A_3 - \Omega}\right)\right\rangle = \left\langle \Re\left(\frac{-\phi^* B_3 - A_{3,x}}{A_3 - \Omega}\right)\right\rangle = \left\langle \frac{-2\xi\Omega(\beta d_0 - 4\beta\xi^2 + \alpha)\phi^2}{\Omega^2 + \big(\mathrm{Im}(A_3)\big)^2} \right\rangle.$$

Here, $\mathrm{Im}(A_3)$ means the imaginary part of A_3. When $\Omega \in \mathbb{R}$ and $\Omega \neq 0$, it is obvious that

$$\frac{1}{T(k)} \int_0^{T(k)} \frac{-2\xi\Omega\big(\beta d_0 - 4\beta\xi^2 + \alpha\big)\phi^2}{\Omega^2 + (\mathrm{Im}(A_3))^2}\, \mathrm{d}x \neq 0,$$

which means that every ξ value such that $\Omega^2 > 0$ does not belong to the Lax spectrum σ_L.

From the above analysis, we have demonstrated that the Lax spectrum is composed of all ξ such that $\Omega^2 \leq 0$. Further, to present the set of Lax spectrum σ_L more explicitly, one needs to make full use of (4.9) and (4.10). Without loss of generality, we assume $m > 0$ and $\beta > 0$. Note that for other cases, e.g., $m < 0$ and $\beta < 0$, similar results can be obtained. Under the assumption that $m > 0$ and $\beta > 0$, the following cases should be considered.

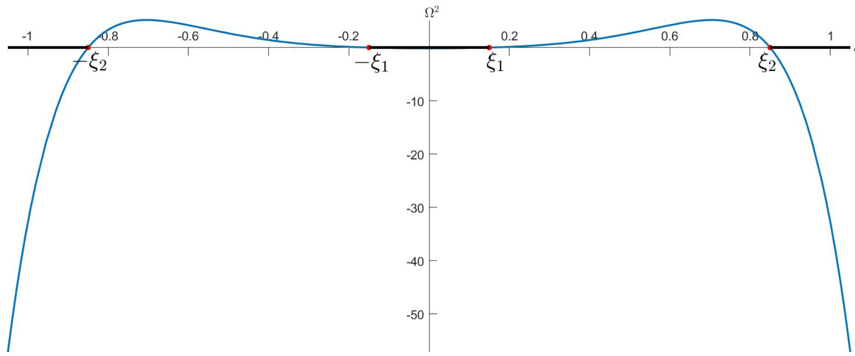
(1) When $\alpha \leq \beta(k^2 + 1)m^2$, we have $\xi_3 \in \mathrm{i}\mathbb{R}$ or $\xi_3 = 0$. Consequently, as shown in Figure 3(a), the set of Lax spectrum is given by

$$\sigma_L = (-\infty, -\xi_2] \cup [-\xi_1, \xi_1] \cup [\xi_2, \infty).$$

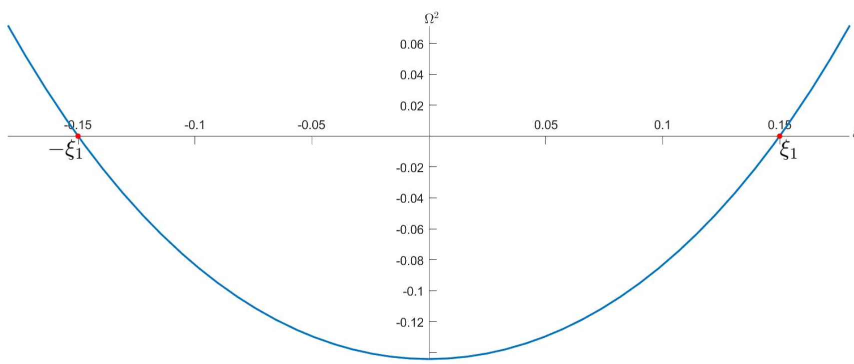
It is noted that $\xi \in \sigma_L$ implies $\Omega \in \mathrm{i}\mathbb{R}$. Specifically, when $\xi \in (-\infty, -\xi_2] \cup [\xi_2, \infty)$, Ω covers the whole imaginary axis twice since Ω^2 takes on all negative values twice. Besides, when $\xi \in [-\xi_1, \xi_1]$, Ω covers all imaginary values in $\left[-\mathrm{i}\sqrt{|\Omega^2(0)|}, \mathrm{i}\sqrt{|\Omega^2(0)|}\right]$ twice since Ω^2 takes on all negative values in $\left[\Omega^2(0), 0\right]$ twice. Here, $\Omega^2(0)$ is the local minimum value for $\xi \in [-\xi_1, \xi_1]$. Therefore, we conclude

$$\Omega \in (\mathrm{i}\mathbb{R})^2 \cup \left[-\mathrm{i}\sqrt{|\Omega^2(0)|}, \mathrm{i}\sqrt{|\Omega^2(0)|}\right]^2,$$

where the exponents represent multiplicities (see Figure 3).



(a) The plot of $\Omega^2(\xi)$ over the interval $\xi \in (-1.05, 1.05)$. The union of the bold line intervals constitutes the set of Lax spectrum σ_L.



(b) A clearer display of (a) over the interval $\xi \in (-\xi_1, \xi_1)$.

Figure 3. Ω^2 as a function of ξ with $m = 1$, $k = 0.7$, $\alpha = 0$, and $\beta = 1$ in case (1).

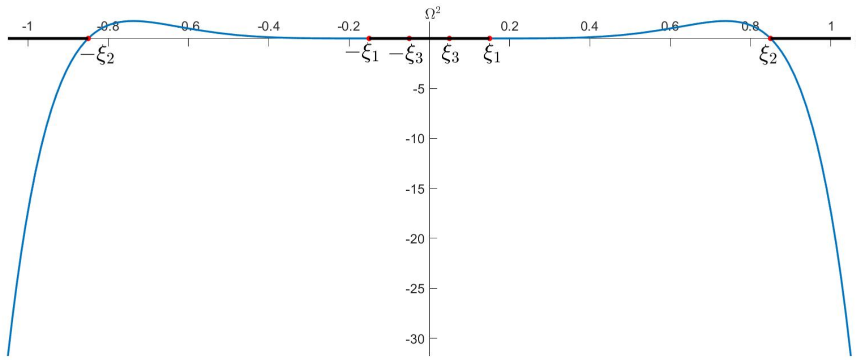
(2) When $\beta(k^2 + 1)m^2 < \alpha < 2\beta m^2(k^2 - k + 1)$, we have $\xi_3 \in \mathbb{R}$ and $0 < \xi_3 < \xi_1 < \xi_2$. Consequently, as shown in Figure 4(a), the set of Lax spectrum is given by

$$\sigma_L = (-\infty, -\xi_2] \cup [-\xi_1, -\xi_3] \cup [-\xi_3, \xi_3] \cup [\xi_3, \xi_1] \cup [\xi_2, \infty).$$

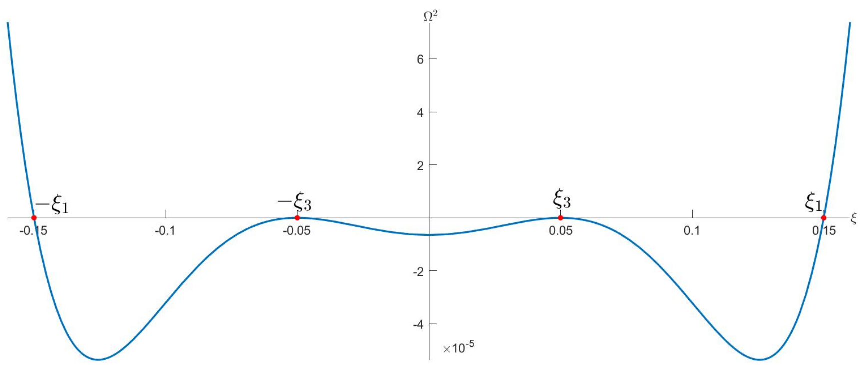
It is noted that $\xi \in \sigma_L$ implies $\Omega \in \mathrm{i}\mathbb{R}$. Specifically, when $\xi \in (-\infty, -\xi_2] \cup [\xi_2, \infty)$, Ω covers the whole imaginary axis twice since Ω^2 takes on all negative values twice. Besides, when $\xi \in [-\xi_3, \xi_3]$, Ω covers all imaginary values in $\left[-\mathrm{i}\sqrt{|\Omega^2(0)|}, \mathrm{i}\sqrt{|\Omega^2(0)|}\right]$ twice since Ω^2 takes on all negative values in $\left[\Omega^2(0), 0\right]$ twice. Here, $\Omega^2(0)$ is the local minimum value for $\xi \in [-\xi_3, \xi_3]$. Similar to the above analysis, for $\xi \in [-\xi_1, -\xi_3] \cup [\xi_3, \xi_1]$, Ω covers all imaginary values in $\left[-\mathrm{i}\sqrt{|\Omega^2(\zeta^*)|}, \mathrm{i}\sqrt{|\Omega^2(\zeta^*)|}\right]$ four times since Ω^2 takes on all negative values in $\left[\Omega^2(\zeta^*), 0\right]$ four times. Here, ζ^* and $-\zeta^*$ are the local minimums for $\xi \in [\xi_3, \xi_1]$ and $\xi \in [-\xi_1, -\xi_3]$, respectively. From (4.10), it is evident that $\Omega^2(\zeta^*) = \Omega^2(-\zeta^*)$. Thus we conclude

$$\Omega \in (\mathrm{i}\mathbb{R})^2 \cup \left[-\mathrm{i}\sqrt{|\Omega^2(0)|}, \mathrm{i}\sqrt{|\Omega^2(0)|}\right]^2 \cup \left[-\mathrm{i}\sqrt{|\Omega^2(\zeta^*)|}, \mathrm{i}\sqrt{|\Omega^2(\zeta^*)|}\right]^4,$$

where the exponents represent multiplicities (see Figure 4).



(a) The plot of $\Omega^2(\xi)$ over the interval $\xi \in (-1.05, 1.05)$. The union of the bold line intervals constitutes the set of Lax spectrum σ_L.



(b) A clearer display of (a) over the interval $\xi \in (-\xi_1, \xi_1)$.

Figure 4. Ω^2 as a function of ξ with $m = 1, k = 0.7, \alpha = 1.5$, and $\beta = 1$ in case (2).

(3) When $2\beta m^2(k^2 - k + 1) < \alpha < 2\beta m^2(k^2 + k + 1)$, we have $\xi_3 \in \mathbb{R}$ and $0 < \xi_1 < \xi_3 < \xi_2$. Consequently, as shown in Figure 5(a), the set of Lax spectrum is given by

$$\sigma_L = (-\infty, -\xi_2] \cup [-\xi_1, \xi_1] \cup [\xi_2, \infty).$$

It is noted that $\xi \in \sigma_L$ implies $\Omega \in \mathrm{i}\mathbb{R}$. In this case, Ω can be analyzed similarly to case (1). Thus we conclude

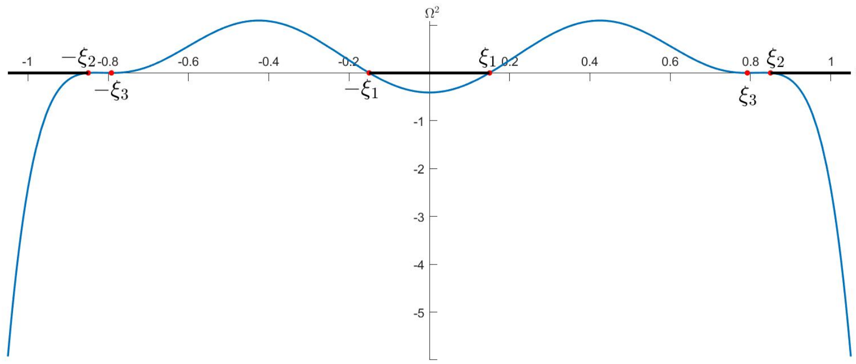
$$\Omega \in (\mathrm{i}\mathbb{R})^2 \cup \left[- \mathrm{i}\sqrt{\left|\Omega^2(0)\right|}, \mathrm{i}\sqrt{\left|\Omega^2(0)\right|} \right]^2,$$

where the exponents represent multiplicities (see Figure 5).

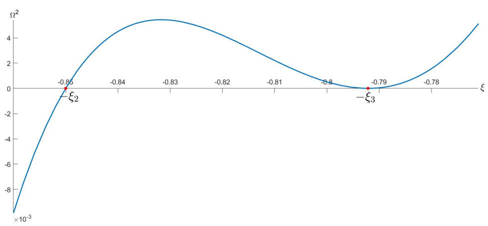
(4) When $\alpha > 2\beta m^2(k^2 + k + 1)$, we have $\xi_3 \in \mathbb{R}$ and $0 < \xi_1 < \xi_2 < \xi_3$. Consequently, as shown in Figure 6(a), the set of Lax spectrum is given by

$$\sigma_L = (-\infty, -\xi_3] \cup [-\xi_3, -\xi_2] \cup [-\xi_1, \xi_1] \cup [\xi_2, \xi_3] \cup [\xi_3, \infty).$$

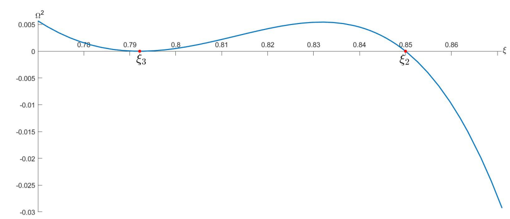
It is noted that $\xi \in \sigma_L$ implies $\Omega \in \mathrm{i}\mathbb{R}$. Specifically, when $\xi \in (-\infty, -\xi_3] \cup [\xi_3, \infty)$, Ω covers the whole imaginary axis twice since Ω^2 takes on all negative values twice. Besides,



(a) The plot of $\Omega^2(\xi)$ over the interval $\xi \in (-1.05, 1.05)$. The union of the bold line intervals constitutes the set of Lax spectrum σ_L.



(b) A clearer display of (a) over the interval $\xi \in (-\xi_2, -\xi_3)$.



(c) A clearer display of (a) over the interval $\xi \in (\xi_3, \xi_2)$.

Figure 5. Ω^2 as a function of ξ with $m = 1, k = 0.7, \alpha = 4$, and $\beta = 1$ in case (3).

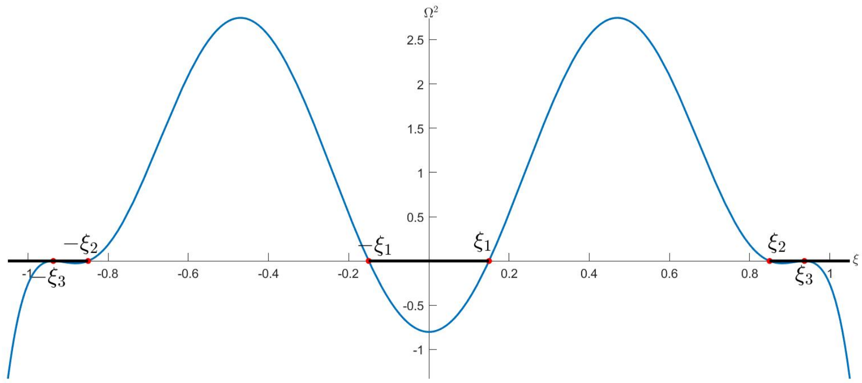
when $\xi \in [-\xi_1, \xi_1]$, Ω covers all imaginary values in $\left[-\mathrm{i}\sqrt{|\Omega^2(0)|}, \mathrm{i}\sqrt{|\Omega^2(0)|}\right]$ twice since Ω^2 takes on all negative values in $\left[\Omega^2(0), 0\right]$ twice. Here, $\Omega^2(0)$ is the local minimum value for $\xi \in [-\xi_1, \xi_1]$. Similar to the above analysis, for $\xi \in [-\xi_3, -\xi_2] \cup [\xi_2, \xi_3]$, Ω covers all imaginary values in $\left[-\mathrm{i}\sqrt{|\Omega^2(\xi^*)|}, \mathrm{i}\sqrt{|\Omega^2(\xi^*)|}\right]$ four times since Ω^2 takes on all negative values in $[\Omega^2(\xi^*), 0]$ four times. Here, ξ^* and $-\xi^*$ are the local minimums for $\xi \in [\xi_2, \xi_3]$ and $\xi \in [-\xi_3, -\xi_2]$, respectively. From (4.10), it is evident that $\Omega^2(\xi^*) = \Omega^2(-\xi^*)$. Thus we conclude

$$\Omega \in (\mathrm{i}\mathbb{R})^2 \cup \left[-\mathrm{i}\sqrt{|\Omega^2(0)|}, \mathrm{i}\sqrt{|\Omega^2(0)|}\right]^2 \cup \left[-\mathrm{i}\sqrt{|\Omega^2(\xi^*)|}, \mathrm{i}\sqrt{|\Omega^2(\xi^*)|}\right]^4,$$

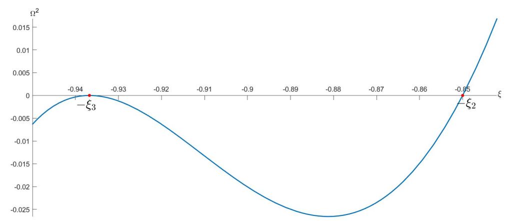
where the exponents represent multiplicities (see Figure 6).

Remark 4.1. In this paper, we consider α and β to be arbitrary real numbers. For further analysis of the set of Lax spectrum σ_L, without loss of generality, we assume that $\beta > 0$. Especially, when $\alpha = 0$ and $\beta = 1$, we obtain $\xi_3 = \frac{1}{2}\mathrm{i}\sqrt{k^2+1}m \in \mathrm{i}\mathbb{R}$ and the above four cases reduce to case (1), which is consistent with the known conclusions of the 4NLS equation [42].

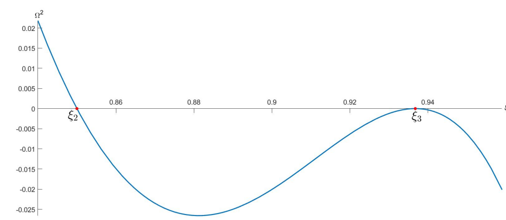
As the Lax spectrum has been determined explicitly, to establish the spectral stability, we are required to construct a squared-eigenfunction connection between the linear stability problem and its Lax pair. In order to show this connection, we first present the following theorem.



(a) The plot of $\Omega^2(\xi)$ over the interval $\xi \in (-1.05, 1.05)$. The union of the bold line intervals constitutes the set of Lax spectrum σ_L.



(b) A clearer display of (a) over the interval $\xi \in (-\xi_3, -\xi_2)$.



(c) A clearer display of (a) over the interval $\xi \in (\xi_2, \xi_3)$.

Figure 6. Ω^2 as a function of ξ with $m = 1, k = 0.7, \alpha = 5$, and $\beta = 1$ in case (4).

Theorem 4.1. *Let* $\Psi = (\Psi_1, \Psi_2)^T$ *be any solution of the Lax pair* (4.4) *restricted to the elliptic solution* $\phi(x)$*. The vector*

$$\begin{pmatrix} \mu(x,t) \\ \nu(x,t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} \begin{pmatrix} \Psi_1^2 \\ \Psi_2^2 \end{pmatrix} \tag{4.13}$$

satisfies the linear stability problem (3.3)*.*

Proof. The proof can be completed by direct calculation. More specifically, one can calculate $(\mu_t, \nu_t)^T$ either by applying the product rule along with the second equation of (4.4), or by substituting (4.13) into (3.3). Therefore, two different expressions for $(\mu_t, \nu_t)^T$ can be derived. Then, using the first equation of (4.4) to eliminate the x-derivatives of μ and ν, it is easy to find that the two expressions are equal. □

Remark 4.2. For any solution of (3.1), which is not necessarily an elliptic solution, this proof can be repeated.

With the above theorem established, we now turn our attention to constructing the squared-eigenfunction connection between the linear stability problem and its Lax pair. Accordingly, we substitute (4.7) and (4.13) into (3.4), which leads to

$$\mathrm{e}^{\lambda t} \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} = \mathrm{e}^{2\Omega t} \begin{pmatrix} \varphi_1^2 + \varphi_2^2 \\ -\mathrm{i}\varphi_1^2 + \mathrm{i}\varphi_2^2 \end{pmatrix}.$$

Thus we can obtain

$$\lambda = 2\Omega(\xi)$$

and

$$\begin{pmatrix}\mu_1\\ \nu_1\end{pmatrix} = \begin{pmatrix}\varphi_1^2+\varphi_2^2\\ -\mathrm{i}\varphi_1^2+\mathrm{i}\varphi_2^2\end{pmatrix}. \tag{4.14}$$

Given that $\xi \in \sigma_L$ implies $\Omega \in \mathrm{i}\mathbb{R}$, we can conclude that the stability spectrum $\sigma_{J\mathcal{L}}$ is determined by

$$\sigma_{J\mathcal{L}} = \mathrm{i}\mathbb{R}.$$

To establish the spectral stability for the elliptic solutions of (1.1), it is necessary to prove that all bounded solutions $(\mu_1, \nu_1)^T$ of (3.4) are constructed via the squared-eigenfunction connection (4.14). Therefore, we state the following theorem.

Theorem 4.2. *All but twelve solutions of* (3.4) *can be constructed through the squared-eigenfunction connection* (4.14).

Proof. For any given value of $\lambda \in \mathbb{C}$, the spectral problem (3.4) can be regarded as an eight-dimensional ordinary differential equation of the first-order. Thus, it has eight linearly independent solutions. For each such λ, the relation $\lambda = 2\Omega$ yields one value of Ω. We define

$$\begin{aligned} F(\Omega,\xi) :=& \Omega^2 + \frac{1}{4\beta}\big(\alpha+\beta(d_0-4\xi^2)\big)^2\big(3d_0^2\beta+16\beta\xi^4+2d_0(\alpha+4\beta\xi^2)-2\omega\big)\\ :=& \Omega^2 - Q_8(\xi), \end{aligned}$$

and let

$$\mathcal{B} := \big\{\lambda \in \mathbb{C}: \text{ the discriminant of } F(\Omega,\xi) \text{ with respect to } \xi \text{ is zero}\big\}.$$

(1) Consider λ such that the discriminant of $F(\Omega,\xi)$ with respect to ξ is non-zero. When $\lambda \in \mathbb{C}\backslash\mathcal{B}$, the zeros of $F(\Omega,\xi)$ give eight values of $\xi \in \mathbb{C}$. In this counting discussion, we no longer restrict ξ to the Lax spectrum σ_L for the reason that the boundedness of the solutions is not a matter of concern currently. That is to say, it is not necessary for each of these eight values of ξ to be in the Lax spectrum. For each of the eight $\xi \in \mathbb{C}$, the squared-eigenfunction connection (4.14) provides one solution to (3.4). Consequently, for each $\lambda \in \mathbb{C}\backslash\mathcal{B}$, (4.14) yields eight solutions of (3.4). In the following, we will prove that these eight solutions are linearly independent. Firstly, using the expressions of ϕ_x^2, ϕ_{xx} and ϕ_{xxx}, we rewrite A_3 and B_3 as

$$\begin{aligned} A_3 &= \frac{1}{2}\mathrm{i}\Big(-\alpha+\big(1+k^2\big)m^2\beta+4\beta\xi^2\Big)\Big(\big(1+k^2\big)m^2-4\xi^2-2\phi^2\Big),\\ B_3 &= -\bigg(\alpha-\beta\Big(\big(1+k^2\big)m^2+4\xi^2\Big)\bigg)\big(2\xi\phi+\mathrm{i}\phi_x\big), \end{aligned} \tag{4.15}$$

which are derived through the relation (2.8). From (4.15), it can be checked that

$$B_{3,x} = -2\phi(x)A_3 - 2\mathrm{i}\xi B_3. \tag{4.16}$$

With the help of (4.16), we rewrite (4.12) as

$$\gamma(x) = \gamma_0 \exp\left(-\int\left(\frac{B_{3,x}}{2B_3}-\frac{\phi\Omega}{B_3}\right)\mathrm{d}x\right).$$

Then the eigenfunctions (4.7) are expressed as

$$\begin{aligned}\Psi(x,t) &= e^{\Omega t}\begin{pmatrix}-B_3\\ A_3-\Omega\end{pmatrix}\gamma_0 \exp\left(-\int\left(\frac{B_{3,x}}{2B_3}-\frac{\phi\Omega}{B_3}\right)\mathrm{d}x\right)\\ &= e^{\Omega t}\begin{pmatrix}-B_3\\ A_3-\Omega\end{pmatrix}\frac{\gamma_0}{(B_3)^{1/2}}\exp\left(\int\frac{\phi\Omega}{B_3}\,\mathrm{d}x\right).\end{aligned}$$

When $\Omega \neq 0$, for these eight different values of ξ, the above derivation gives eight eigenfunctions, which exhibit singularities at the zeros of B_3. Due to the fact that the zeros of B_3 are fixed by ξ, these eight eigenfunctions corresponding to eight values of ξ exhibit different singularities in the complex x-plane. When $\Omega = 0$, the eigenfunctions (4.7) can be simplified to

$$\Psi(x,t) = \begin{pmatrix}-B_3\\ A_3\end{pmatrix}\frac{\gamma_0}{(B_3)^{1/2}},$$

which indicates that just one bounded solution is constructed through the squared-eigenfunction connection (4.14).

(2) Consider λ such that the discriminant of $F(\Omega,\xi)$ with respect to ξ is zero. When $\lambda \in \mathcal{B}$, the following cases should be considered.

(i) For

$$\Omega^2 = -\frac{1}{4}(k^2-1)^2 m^4(-\alpha+m^2\beta+k^2m^2\beta)^2,$$

we have

$$F(\Omega,\xi) = \xi^2(\xi^2-a_1^2)(\xi^2-a_2^2)(\xi^2-a_3^2),$$

which indicates that seven linearly independent solutions have been constructed.

(ii) For

$$\begin{aligned}\Omega^2 = -\frac{1}{128\beta^2}\Big(&\alpha^4-8(1+k^2)\alpha^3\beta m^2+16(1-16k^2-42k^4-16k^6+k^8)\beta^4m^8\\ &+8(3-4k^2+3k^4)\alpha^2\beta^2m^4-32(1+k^2)(1-8k^2+k^4)\alpha\beta^3m^6-(\alpha-2m^2\beta\\ &-2k^2m^2\beta)\big(\alpha^2-4m^2\alpha\beta-4k^2m^2\alpha\beta+4m^4\beta^2+40k^2m^4\beta^2+4k^4m^4\beta^2\big)^{\frac{3}{2}}\Big),\end{aligned}$$

we have

$$F(\Omega,\xi) = (\xi^2-b_1^2)^2(\xi^2-b_2^2)(\xi^2-b_3^2),$$

which indicates that six linearly independent solutions have been constructed.

(iii) For

$$\begin{aligned}\Omega^2 = -\frac{1}{128\beta^2}\Big(&\alpha^4-8(1+k^2)\alpha^3\beta m^2+16(1-16k^2-42k^4-16k^6+k^8)\beta^4m^8\\ &+8(3-4k^2+3k^4)\alpha^2\beta^2m^4-32(1+k^2)(1-8k^2+k^4)\alpha\beta^3m^6+(\alpha-2m^2\beta\\ &-2k^2m^2\beta)\big(\alpha^2-4m^2\alpha\beta-4k^2m^2\alpha\beta+4m^4\beta^2+40k^2m^4\beta^2+4k^4m^4\beta^2\big)^{\frac{3}{2}}\Big),\end{aligned}$$

we have

$$F(\Omega,\xi) = (\xi^2 - c_1^2)^2(\xi^2 - c_2^2)(\xi^2 - c_3^2),$$

which indicates that six linearly independent solutions have been constructed.

□

Therefore, the following theorem is established.

Theorem 4.3. *The elliptic solutions of the integrable defocusing LPD equation are spectrally stable. Furthermore, the spectrum of the operator $J\mathcal{L}$ related to the linear stability problem is determined explicitly by $\sigma_{J\mathcal{L}} = \mathrm{i}\mathbb{R}$.*

Through a direct application of the SCS basis lemma [21], we can infer that for any integer N, the eigenfunctions are complete in $L^2_{\mathrm{per}}([-N\frac{T}{2}, N\frac{T}{2}])$. As a result, the linear stability under subharmonic perturbations can be obtained directly.

Remark 4.3. In fact, the orbital stability of many integrable systems [6, 12, 17, 41, 42, 49] has been proved with a suitable Lyapunov functional. Based on our linear stability results and following the idea presented in [42], one can prove the orbital stability for elliptic solutions of the defocusing LPD equation under subharmonic perturbations. However, due to the substantial computational work involved, the orbital stability of the elliptic solutions to (1.1) will not be further discussed in this paper.

5. Conclusion

We have proved that the elliptic solutions of the defocusing LPD equation are spectrally and linearly stable under subharmonic perturbations. Specifically, applying the integrability method to the integrable equation (1.1), we have explicitly constructed the squared-eigenfunction connection to relate the eigenfunctions of the linear stability problem and those of its Lax pair. By utilizing the squared-eigenfunction connection, we have demonstrated the spectral stability of elliptic solutions. Meanwhile, the stability spectrum and the stable Lax spectrum are described analytically in detail. Finally, based on the spectral stability results and the SCS basis lemma, the linear stability for subharmonic perturbations is established, providing a basis for investigating the orbital stability of elliptic solutions to the LPD equation.

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Conflict of interest

The authors declare that there are no competing interests.

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