Journal of Applied Analysis and Computation
Volume 16, Number 2, April 2026, 1095–1108
Website:http://www.jaac-online.com
DOI:10.11948/20250159

GROUND STATE TO A NEW CLASS OF KIRCHHOFF-TYPE EQUATION WITH DOUBLY CRITICAL HARTREE-TYPE NONLINEARITIES*

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Abstract In this paper, we consider the following class of Kirchhoff-type equation with doubly critical Hartree-type nonlinearities

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u+V(x)u=(I_\alpha*|u|^p)|u|^{p-2}u+(I_\alpha*|u|^q)|u|^{q-2}u, \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $a>0$, $b\geq 0$, $N\geq 3$, $\alpha\in(N-2,N)$, $p=\frac{N+\alpha}{N-2}$, $q=\frac{N+\alpha}{N}$, $V:\mathbb{R}^N\to\mathbb{R}$ is a potential function and I_α is a Riesz potential of order $\alpha\in(N-2,N)$. Under certain assumptions on potential function $V(x)$, we prove that the equation has at least a ground state solution by variational methods.

Keywords Kirchhoff equation, ground state solutions, Pohozaev identity, Nehari manifold.

MSC(2010) 35J60, 35J35, 35A15.

1. Introduction

In this article, we study the following Kirchhoff-type equation

$$\begin{cases} -(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)\Delta u+V(x)u=(I_\alpha*|u|^p)|u|^{p-2}u+(I_\alpha*|u|^q)|u|^{q-2}u, \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a>0$, $b\geq 0$, $N\geq 3$, $\alpha\in(N-2,N)$, $p=\frac{N+\alpha}{N-2}$, $q=\frac{N+\alpha}{N}$, $V:\mathbb{R}^N\to\mathbb{R}$ is a potential function and I_α is a Riesz potential whose order is $\alpha\in(N-2,N)$ defined by $I_\alpha=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}}$ and $V(x):\mathbb{R}^N\to\mathbb{R}$ is a potential function which satisfies:

$(V1)$ $V\in C^1(\mathbb{R}^N)\bigcap L^\infty$ and there exists a constant $A\in(0,a)$ such that

$$|(\nabla V(x),x)|\leq\frac{A}{2|x|^2}$$

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*The authors were supported by Scientific research start-up fund (Grant No. F701108M03), National Natural Science Foundation of China (11771198, 11901276) and Science and Technology project of Education Department of Jiangxi Province (Grant No. GJJ2403004).
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for all $x \in \mathbb{R}^N \setminus \{0\}$,

$(V2)$ there exists a constant $V_\infty > 0$ such that for all $x \in \mathbb{R}^N$,

$$0 \le V(x) \le \liminf_{|y|\to+\infty} V(y) = V_\infty < +\infty.$$

In recent years, many scholars have been studying various related problems of the solutions for the Kirchhoff-type equation:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = g(x,u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where $a > 0$, $b \ge 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Problem (1.2) is a nonlocal problem because of the presence of the term $b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$, which provokes some mathematical difficulties, but on the other hand, makes the research of this problem particular interesting. Besides, this problem has a very important physical background. In fact, if we set $V(x) = 0$ and replace $\mathbb{R}^3$ by a bounded domain $\Omega \subset \mathbb{R}^3$ in (1.2), then we get the following Kirchhoff Dirichlet problem

$$\begin{cases} -(a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x)\Delta u = g(x,u), & x \in \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

It has relation to the stationary analogue of the equation

$$u_{tt} - (a + b\int_{\Omega} |\nabla u|^2 \mathrm{d}x)\Delta u = g(x,u). \tag{1.4}$$

Such a hyperbolic equation is a general version of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - (\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|\mathrm{d}x)\frac{\partial^2 u}{\partial x^2} = 0,$$

which was first proposed by G. Kirchhoff as an extension of classical D'Alembert's wave equations for free vibration of elastic strings. Soon after, J. L. Lions [15] finished the previous work and introduced a functional analysis approach.

After that, more and more researchers have paid much attention to (1.3). Many of them focused on the Kirchhoff type problem defined on the whole space $\mathbb{R}^3$ (even $\mathbb{R}^N$), i.e., problem (1.2). However, most of their results need assume V verifies

$$\inf_{x\in\mathbb{R}^3} V(x) := V_0 > 0$$

and g satisfies classical Ambrosetti-Rabinowitz condition

$(A - R)$ condition: There exists $\mu > 2$ such that

$$0 < \mu G(x,s) \le sg(x,s)$$

for all $s > 0$. It is worth mentioning that in [8], Guo studied the following Kirchhoff-type problem:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = f(u), \text{ in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.5}$$

He obtained the existence of positive ground states to (1.5) and in his paper. There is no (A-R) type condition. He defined a new manifold which could be called the Nehari-Pohozaev manifold. Besides he used Jeanjean's monotonicity trick [9] in his paper. We must point out that $f \in C^1$ and the forth assumption about f are very important in Guo [8]. Actually, under above assumptions, $\mathcal{M}$ is a C^1 manifold and the Implicit Function Theorem can be used.

Most remarkably, as early as 2006, Ruiz [29] first proposed the prototype of this Nehari-Pohozaev manifold in his study of the Schrödinger-Poisson equation, which is a very great work. Besides, readers can see [4,20] for recent work.

But unfortunately, there still are very few results of existence of ground state solution to (1.2) with Hartree-type nonlinearities.

Actually, when $a = 1$, $b = 0$, $g = (I_\alpha * |u|^p)|u|^{p-2}u$, the equation (1.2) becomes

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u, \tag{1.6}$$

which is usually called nonlinear Choquard type equation. It also has several interesting physical origins. In the physical case $V(x) = 1$, $\alpha = 2$ and $p = 2$, (1.6) becomes

$$\begin{cases} -\Delta u + u = (I_2 * |u|^2)u, \ \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.7}$$

which is also known as the stationary Hartree equation or the nonlinear Schrödinger-Newton equation [22]. It goes back to the description of the quantum mechanics of a polaron at rest by S. I. Pekar in 1954 [28]. In 1976, P. Choquard used (1.7) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [13]. In 1996, R. Penrose proposed (1.7) as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [22]. The existence of solutions for (1.7) was proved by variational methods by E. H. Lieb, P. L. Lions and G. Menzala [13,16,21] and also by ordinary differential equations techniques [6,22,31].

It is well known that (1.6) have a solution if and only if $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$[[19], Theorem 1] (see also [[7], Lemma 2.7]). Under some cretain assumptions when $p \geq 2$, Ma and Zhao [19] proved that every positive solution for (1.6) is radially symmetric and monotone decreasing about some point. Moreover, for the optimal range of parameters, Moroz and Van Schaftingen [23] showed the regularity, radial symmetry and decay character of ground state solutions for (1.6).

Besides, readers can see [2,5,10,11,24,26,27,30,33] for recent achievements. Especially in [24] and [17], the authors studied (1.6) under the general Berestycki-Lions type conditions.

Inspired by the above works, especially by [3,8,10,17], we now research problem (1.2) in $\mathbb{R}^N$ with Hartree-type nonlinearities $g(x, u) = (I_\alpha * |u|^p)|u|^{p-2}u + (I_\alpha * |u|^q)|u|^{q-2}u$ which may be regarded as a Kirchhoff-type perturbation to (1.6). As we all know, there are very few results to (1.2) with Hartree-type nonlinearities with critical growth.

Our main results is as follows:

Theorem 1.1. *If V satisfies (V1)-(V2), then problem(1.1) has at least a ground state solution.*

Hereafter for the convenience of narration, we will use the following notations:

- $X := H^1(\mathbb{R}^N)$ is a Hilbert space in which an equivalent norm is defined as

$$\|u\| = [\int_{\mathbb{R}^N} (a|\nabla u|^2 + V(x)u^2)\mathrm{d}x]^{\frac{1}{2}}.$$

- $L^r(\mathbb{R}^N)(1 \leq r \leq \infty)$ denotes the Lebesgue space in which the norm is defined as follows

$$|u|_r = (\int_{\mathbb{R}^N} |u|^r \mathrm{d}x)^{1/r}.$$

- For any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, u_t is denoted as:

$$u_t = \begin{cases} 0, & t = 0, \\ \sqrt{t} u(\frac{x}{t}), & t > 0. \end{cases}$$

- $C, C_\varepsilon, C_1, C_2, ...$ denote positive constants which are possibly different in different lines.

2. Preliminaries

Problem(1.1) has a variational structure, i.e., the critical points of the functional

$$\begin{aligned} I(u) =& \frac{1}{2} \int_{\mathbb{R}^N} [a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x)^2 \\ &- \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \frac{1}{2q} \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \mathrm{d}x \end{aligned} \tag{2.1}$$

are weak solutions of equation (1.1).

Lemma 2.1. (Hardy-Littlewood-Sobolev inequality [14]) *Let* $0 < \alpha < N$, $p, q > 1$ *and* $1 \leq r < s < \infty$ *be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \ \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

1. For any $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, one has

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \right| \leq C(N, \alpha, p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}.$$

2. For any $f \in L^r(\mathbb{R}^N)$ one has

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq C(N, \alpha, r)\|f\|_{L^r(\mathbb{R}^N)}.$$

Lemma 2.2. (Pohozaev identity [1, 23, 24, 32]) *Suppose V(x) satisfies (V1)-(V2) and let* $u \in X$ *be a weak solution of Problem(1.1), then we have the following Pohozaev identity:*

$$\begin{aligned} 0 =& P_V(u) \\ =& \frac{a(N-2)}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{N}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x \\ &+ \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x)|u|^2 \mathrm{d}x + \frac{b(N-2)}{2}(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x)^2 \\ &- \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x - \frac{N+\alpha}{2q} \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \mathrm{d}x. \end{aligned} \tag{2.2}$$

In particular, if $V \equiv V_\infty$, we have:

$$\begin{aligned}0 =&P_\infty(u)\\ =&\frac{a(N-2)}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\frac{N}{2}\int_{\mathbb{R}^N}V_\infty|u|^2\mathrm{d}x+\frac{b(N-2)}{2}(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x)^2\\ &-\frac{N+\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\mathrm{d}x-\frac{N+\alpha}{2q}\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\mathrm{d}x.\end{aligned}\tag{2.3}$$

3. Ground state solution for the “limit problem” of equation (1.1)

In this section, we study the following limit problem associated to problem (1.1):

$$\begin{cases}-(a+b\int_{\mathbb{R}^N}|\nabla u|^2)\Delta u+V_\infty u=(I_\alpha*|u|^p)|u|^{p-2}u+(I_\alpha*|u|^q)|u|^{q-2}u, \text{ in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N).\end{cases}\tag{3.1}$$

The asscoiated energy functional is given by:

$$\begin{aligned}I_\infty(u)=&\frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla u|^2+V_\infty u^2]\mathrm{d}x+\frac{b}{4}\big(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\big)^2\\ &-\frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\mathrm{d}x-\frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\mathrm{d}x.\end{aligned}\tag{3.2}$$

Next, we prove the following results.

Lemma 3.1. *I_∞ is not bounded from below.*

Proof. For all $u \in X\backslash\{0\}$ and $t > 0$, we have:

$$\begin{aligned}I_\infty(u_t)&=I_\infty\big(\sqrt{t}u(t^{-1}x)\big)\\ &=\frac{at^{N-1}}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\frac{t^{N+1}}{2}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x+\frac{bt^{2N-2}}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\ &\quad-\frac{t^{p+N+\alpha}}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\mathrm{d}x-\frac{t^{q+N+\alpha}}{2q}\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\mathrm{d}x\\ &\to-\infty\end{aligned}$$

as $t \to \infty$, since $\alpha > N - 2$, then we can get the conclusion. □

Next we define $\mathcal{M}_\infty = \{u \in X\backslash\{0\} : G_\infty(u) = 0\}$, where

$$\begin{aligned}G_\infty(u)=&\frac{1}{2}\langle I'_\infty(u),u\rangle+P_\infty(u)\\ =&\frac{a(N-1)}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\frac{N+1}{2}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x+\frac{b(N-1)}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\ &-\frac{p+N+\alpha}{2p}\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\mathrm{d}x-\frac{q+N+\alpha}{2q}\int_{\mathbb{R}^N}(I_\alpha*|u|^q)|u|^q\mathrm{d}x\\ =&\left.\frac{\mathrm{d}I_\infty(u_t)}{\mathrm{d}t}\right|_{t=1}.\end{aligned}\tag{3.3}$$

Remark 3.2. For $t > 0$, we set

$$\begin{aligned}\gamma(t) =& I_\infty(u_t)\\ =& \frac{at^{N-1}}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{t^{N+1}}{2}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x + \frac{bt^{2N-2}}{4}\Big(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\Big)^2\\ &- \frac{t^{p+N+\alpha}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\mathrm{d}x - \frac{t^{q+N+\alpha}}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\mathrm{d}x.\end{aligned} \tag{3.4}$$

Lemma 3.3. *Let $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ be positive constants and $u \in X\backslash\{0\}$. Then the function*

$$\eta(t) = c_1t^{N-1} + c_2t^{N+1} + c_3t^{2N-2} - c_4t^{p+N+\alpha} - c_5t^{q+N+\alpha} \text{ for } t \geq 0$$

has a unique positive critical point which corresponds to its maximum.

Proof. By elementary calculation we can obtain the result. □

Lemma 3.4. *For any $u \in X\backslash\{0\}$, there exists a unique $t_0 > 0$ such that $u_{t_0} \in \mathcal{M}_\infty$. Moreover, $I_\infty(u_{t_0}) = \max\limits_{t>0} I_\infty(u_t)$.*

Proof. Consider the function $\eta(t)$ defined above. By Lemma 3.3, $\eta(t)$ has only one critical point $t_0 > 0$ corresponding to its maximum. Then $\eta(t_0) = \max\limits_{t>0}\eta(t)$ and $\eta'(t_0) = 0$. Since $I_\infty(u_t)$ has the form of $\eta(t)$, then it follows that $G_\infty(u_{t_0}) = t_0\eta'(t_0) = 0$. This implies $u_{t_0} \in \mathcal{M}_\infty$ and $I_\infty(u_{t_0}) = \max\limits_{t>0} I_\infty(u_t)$. □

Lemma 3.5. *The functional I_∞ possesses the mountain-pass geometry, that is*

1. *there exist $\rho, \delta > 0$ such that $I_\infty \geq \delta$ for all $\|u\| = \rho$;*
2. *there exists $e \in H^1(\mathbb{R}^N)$ such that $\|e\| > \rho$ and $I_\infty(e) < 0$.*

Proof. (1) By Lemma 2.1(1), one can get

$$I_\infty(u) \geq \frac{1}{2}\|u\|^2 - C_1\|u\|^{2p} - C_2\|u\|^{2q}.$$

Thus there exists $\rho, \delta > 0$ such that $I_\infty \geq \delta$ for all $\|u\| = \rho > 0$ small enough.
(2) For any $u \in X\backslash\{0\}$, by the definition of $I_\infty(u_t)$, one can have $I_\infty(u_t) < 0$ for $t > 0$ large. Note that

$$\|u_t\|^2 = at^{N-1}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + t^{N+1}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x.$$

So we can take $e = u_{t_0}$ with $t_0 > 0$ large, then we get $\|e\| > \rho$ and $I_\infty(e) < 0$. □

Hence we define the mountain-pass level of I_∞:

$$c_\infty = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\infty(\gamma(t)) > 0,$$

where: $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$.

Let

$$m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u),$$

then for any $u \in \mathcal{M}_\infty$, we have

$$I_\infty(u) = I_\infty(u) - \frac{1}{2(N-1)}G_\infty(u) \geq \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x \geq 0.$$

Hence m_∞ is well defined. In addition, by the similar argument as Chapter 4 [32], we have the following characterization

$$c_\infty = \inf_{u\in X\setminus\{0\}} \max_{t>0} I_\infty(u_t) = m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u).$$

Theorem 3.6. *Problem (3.1) has a ground state solution.*

Proof. Let $\{u_n\} \subset \mathcal{M}_\infty$ be a minimizing sequence of I_∞ in $\mathcal{M}_\infty$, i.e., $I_\infty(u_n) \to m_\infty$ as $n \to \infty$. Similar to the proof of lemma in [10] and [8], we know that $\{u_n\} \subset \mathcal{M}_\infty$ is also a (PS) sequence of I_∞, i.e $I_\infty(u_n) \to m_\infty$ and $I'_\infty(u_n) \to 0$ as $n \to \infty$. It follows from $I'_\infty(u_n) \to 0$ and $G_\infty(u_n) = 0$ that

$$\begin{aligned}
&m_\infty + o(1)\|u_n\| \\
\geq\ & I_\infty(u_n) \\
=\ & I_\infty(u_n) - \frac{1}{2(N-1)} G_\infty(u_n) \\
=\ & \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x + \frac{N-3}{4(N-1)}\int_{\mathbb{R}^N} V_\infty u_n^2 \mathrm{d}x + \frac{p+\alpha+2-N}{4(N-1)p}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x \\
&+ \frac{q+\alpha+2-N}{4(N-1)q}\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \mathrm{d}x \\
\geq\ & \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x.
\end{aligned}$$

Therefore, $\{|\nabla u_n|_2^2\}$ is bounded. Next we prove that $\{|u_n|_2^2\}$ is also bounded. By Lemma 2.1(1) and the definition of $G_\infty(u_n)$ and Sobolev inequality, we have

$$\begin{aligned}
&\frac{N+1}{2}\int_{\mathbb{R}^N} V_\infty u_n^2 \mathrm{d}x \\
=\ & \frac{p+N+\alpha}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \frac{q+N+\alpha}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \mathrm{d}x \\
&- \frac{a(N-1)}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x - \frac{b(N-1)}{2}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x\right)^2 \\
\leq\ & \frac{p+N+\alpha}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \frac{q+N+\alpha}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \mathrm{d}x \\
\leq\ & C(|u_n|_{\frac{2N}{N-2}}^{2p} + |u_n|_2^{2q}) \\
\leq\ & c(|\nabla u_n|_2^{2p} + |\nabla u_n|_2^{2q}),
\end{aligned}$$

which implies $\{|u_n|_2^2\}$ is bounded. Then let $\delta = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^p \mathrm{d}x$. We claim $\delta > 0$. On the contrary, by Lions' concentration compactness principle in [23], we have $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x \to 0$ and $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \mathrm{d}x \to 0$. Together with $P_\infty(u_n) = 0$, we can deduce $u_n \to 0$ in X. This is a contradiction with the fact that $c_\infty > 0$. Hence $\delta > 0$ and there exists $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^p \mathrm{d}x \geq \frac{\delta}{2} > 0$. We set $v_n(x) = u_n(x + y_n)$, then $\|u_n\| = \|v_n\|$, $\int_{B_1(0)} |v_n|^p \mathrm{d}x > \frac{\delta}{2}$ and $I_\infty(v_n) \to m_\infty$, $G_\infty(v_n) = 0$. Therefore there exists $v_0 \in X\setminus\{0\}$ such that

$$\begin{cases} v_n \rightharpoonup v_0 \text{ in } X, \\ v_n \to v_0 \text{ in } L^s_{\text{loc}}(\mathbb{R}^N), \ \forall\ s \in [1, 2^*), \\ v_n \to v_0 \text{ a.e. on } \mathbb{R}^N. \end{cases}$$

It follows from $I'_\infty(u_n) \to 0$ that

$$\begin{aligned} &a\int_{\mathbb{R}^N} a\nabla v_0\nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} V_\infty v_0\varphi \mathrm{d}x + b(\int_{\mathbb{R}^N}|\nabla v_0|^2\mathrm{d}x)\int_{\mathbb{R}^N} a\nabla v_0\nabla\varphi \mathrm{d}x \\ &= \int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\Gamma(\frac{N+\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}}\Big(\frac{|v_0(x)|^p|v_0(y)|^{p-1}}{|x-y|^{N-\alpha}} + \frac{|v_0(x)|^q|v_0(y)|^{q-1}}{|x-y|^{N-\alpha}}\Big)\varphi(y)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{3.5}$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then v_0 is a critical point for I_∞. Therefore $\langle I'_\infty(v_0), v_0\rangle$ and $P_\infty(v_0) = 0$, which implies that $v_0 \in \mathcal{M}_\infty$.

Denote $v_n^1 = v_n - v_0$, then $v_n^1 \rightharpoonup 0$ in X as $n \to \infty$. By lemma 2.4 in [23], we have

$$\int_{\mathbb{R}^N}|v_n^1|^2\mathrm{d}x = \int_{\mathbb{R}^N}|v_n|^2\mathrm{d}x - \int_{\mathbb{R}^N}|v_0|^2\mathrm{d}x + o(1), \tag{3.6}$$

$$\int_{\mathbb{R}^N}|\nabla v_n^1|^2\mathrm{d}x = \int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla v_0|^2\mathrm{d}x + o(1), \tag{3.7}$$

$$\int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^p)|v_n^1|^p\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * |v_n|^p)|v_n|^p\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * |v_0|^p)|v_0|^p\mathrm{d}x + o(1), \tag{3.8}$$

$$\int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^q)|v_n^1|^q\mathrm{d}x = \int_{\mathbb{R}^N}(I_\alpha * |v_n|^q)|v_n|^q\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * |v_0|^q)|v_0|^q\mathrm{d}x + o(1). \tag{3.9}$$

Together with $\langle I'_\infty(v_0), v_0\rangle = 0$ and $\langle I'_\infty(v_n), v_n\rangle = o(1)$, we have

$$\begin{aligned} &a\int_{\mathbb{R}^N}|\nabla v_n^1|^2\mathrm{d}x + \int_{\mathbb{R}^N} V_\infty|v_n^1|^2\mathrm{d}x + b\big(\int_{\mathbb{R}^N}|\nabla v_n^1|^2\mathrm{d}x\big)^2 \\ &-\int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^p)|v_n^1|^p\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^q)|v_n^1|^q\mathrm{d}x = o(1). \end{aligned} \tag{3.10}$$

Consequently,

$$\begin{aligned} I_\infty(v_n) \geq\ & \frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla v_n^1|^2 + V_\infty|v_n^1|^2]\mathrm{d}x + \frac{b}{4}\big(\int_{\mathbb{R}^N}|\nabla v_n^1|^2\mathrm{d}x\big)^2 \\ &- \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^p)|v_n^1|^p\mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^q)|v_n^1|^q\mathrm{d}x + I_\infty(v_0) + o(1) \\ =\ & \frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla v_n^1|^2 + V_\infty|v_n^1|^2]\mathrm{d}x + \frac{b}{4}\big(\int_{\mathbb{R}^N}|\nabla v_n^1|^2\mathrm{d}x\big)^2 - \frac{1}{2p}\int_{\mathbb{R}^N}[a|\nabla v_n^1|^2 + V_\infty|v_n^1|^2]\mathrm{d}x \\ &- \frac{b}{2p}\big(\int_{\mathbb{R}^N}|\nabla v_n^1|^2\mathrm{d}x\big)^2 + \big(\frac{1}{2p} - \frac{1}{2q}\big)\int_{\mathbb{R}^N}(I_\alpha * |v_n^1|^q)|v_n^1|^q\mathrm{d}x + I_\infty(v_0) + o(1) \\ \geq\ & I_\infty(v_0) + o(1). \end{aligned} \tag{3.11}$$

Hence $I_\infty(v_0) \leq m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u)$. Noting that $v_0 \in \mathcal{M}_\infty$, thus $I_\infty(v_0) \geq m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u)$ which implies $I_\infty(v_0) = m_\infty = \inf_{u\in\mathcal{M}_\infty} I_\infty(u)$. Then v_0 is a ground state solution of Eq. (3.1). The proof is thus complete. □

4. Ground state solution for problem (1.1)

In this section, we prove the main theorem.

Proposition 4.1. (See [9]) *Let $(X, \|\cdot\|)$ be a Banach space and $T \subset \mathbb{R}^+$ be an interval. We* *consider a family of C^1 functions on X of the form:*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \forall \lambda \in T,$$

with $B(u) \geq 0, \forall\ u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$. We assume that there are 2 points $v_1, v_2 \in X$ such that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \forall \lambda \in T,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$, then for almost every $\lambda \in T$, there is a bounded $(PS)_{c_\lambda}$ sequence in X.

Set T=$[\delta, 1]$, where δ is a positive constant. We investigate a family of functionals on X with the following form:

$$\begin{aligned} I_{V,\lambda}(u) =& \frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\bigg)^2 \\ & - \frac{\lambda}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p\mathrm{d}x - \frac{\lambda}{2q}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^q\big)|u|^q\mathrm{d}x,\ \forall \lambda \in [\delta, 1]. \end{aligned}$$

Then let $I_{V,\lambda}(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^N}[a|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{b}{4}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\bigg)^2 \to +\infty,\ \text{as } \|u\| \to +\infty$$

and

$$B(u) = \frac{1}{2p}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^p\big)|u|^p\mathrm{d}x + \frac{1}{2q}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^q\big)|u|^q\mathrm{d}x \geq 0.$$

Lemma 4.2. *Assume (V2) holds, then we have*

1. *there exists a $v \in X\backslash\{0\}$ such that $I_{V,\lambda}(v) \leq 0$ for all $\lambda \in [\delta, 1]$,*
2. *$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{V,\lambda}(\gamma(t)) > \max\{I_{V,\lambda}(0), I_{V,\lambda}(v)\}$ for all $\lambda \in [\delta, 1]$, where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = v\}.$$

Proof. (1) Fix $u \in X\backslash\{0\}$, then for $\forall \lambda \in [\delta, 1]$ and $t > 0$, we have

$$\begin{aligned} I_{V,\lambda}(u_t) &\leq I_{V_\infty,\delta}(u_t) \\ &= \frac{at^{N-1}}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{t^{N+1}}{2}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x \\ &\quad + \frac{bt^{2N-2}}{4}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\bigg)^2 - \frac{\delta t^{p+N+\alpha}}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\mathrm{d}x \\ &\quad - \frac{\delta t^{q+N+\alpha}}{2q}\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\mathrm{d}x \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$. Taking $v = u_t$ with t large, we have $I_{V,\lambda}(u_t) \leq I_{V_\infty,\delta}(u_t) < 0$.
(2) By Lemma 2.1(1), we have $I_{V,\lambda}(u) \geq \frac{1}{2}\|u\|^2 - C_3\|u\|^{2p} - C_4\|u\|^{2q}$. Since $p > 1$, we see $I_{V,\lambda}(u)$ has a strictly local minimum at 0, i.e., there exists $r > 0$ such that

$$b = \inf_{\|u\|=r} I_{V,\lambda}(u) > 0 = I_{V,\lambda}(0) \geq I_{V,\lambda}(v)$$

and hence taking $u_t = v$, we get $c_\lambda > \max\{I_{V,\lambda}(0), I_{V,\lambda}(u_t)\} = 0$. □

Lemma 4.3. (See [9]) *Under the assumptions of Proposition 4.1, the map $\lambda \mapsto c_\lambda$ is non-increasing and left continuous.*

By Theorem 3.6, we conclude that for all $\lambda \in [\delta, 1]$, the "limit problem" of the following type:

$$\begin{cases} -(a + b\int_{\mathbb{R}^N} |\nabla u|^2)\Delta u + V_\infty u = \lambda[\big(I_\alpha * |u|^p\big)|u|^{p-2}u + \big(I_\alpha * |u|^q\big)|u|^{q-2}u], \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{4.1}$$

has a ground state solution $u_\lambda \in H^1(\mathbb{R}^N)$, that is, for $\forall \lambda \in [\delta, 1]$, there exists $u_\lambda \in \mathcal{M}_\lambda = \{u \in X\backslash\{0\} : G_{\infty,\lambda}(u) = 0\}$ such that $I'_{V_\infty,\lambda}(u_\lambda) = 0$ and $I_{V_\infty,\lambda}(u_\lambda) = m_\lambda = \inf\limits_{u\in\mathcal{M}_\lambda} I_{V_\infty,\lambda}(u)$. Here

$$\begin{aligned} I_{V_\infty,\lambda}(u) = &\frac{1}{2}\int_{\mathbb{R}^N} [a|\nabla u|^2 + V_\infty u^2]\mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x\Big)^2 \\ &- \frac{\lambda}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\mathrm{d}x - \frac{\lambda}{2q}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q\mathrm{d}x, \\ G_{\infty,\lambda}(u) = &\frac{a(N-1)}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{N+1}{2}\int_{\mathbb{R}^N} V_\infty u^2\mathrm{d}x + \frac{b(N-1)}{2}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x\Big)^2 \\ &- \lambda\big[\frac{p+N+\alpha}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p\mathrm{d}x + \frac{q+N+\alpha}{2q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q\mathrm{d}x\big]. \end{aligned}$$

Lemma 4.4. *Suppose that (V1)-(V2) hold and $V(x) \not\equiv V_\infty$, then $c_\lambda < m_\lambda$ for $\forall\lambda \in [\delta, 1]$.*

Proof. Let u_λ be the minimizer of m_λ. By Lemma 4.2, there exists $\widetilde{t} \in (0, t_0)$ such that

$$\begin{aligned} c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{V,\lambda}(\gamma(t)) &\le \max_{0<t<t_0} I_{V,\lambda}\big(\sqrt{t}u_\lambda(\frac{x}{t})\big) = I_{V,\lambda}\big(\sqrt{\widetilde{t}}u_\lambda(\frac{x}{\widetilde{t}})\big) \\ &< I_{V_\infty,\lambda}\big(\sqrt{\widetilde{t}}u_\lambda(\frac{x}{\widetilde{t}})\big) \le \max_{t>0} I_{V_\infty,\lambda}\big(\sqrt{t}u_\lambda(\frac{x}{t})\big) = I_{V_\infty,\lambda}(u_\lambda) = m_\lambda. \end{aligned}$$

□

Next we provide the following global compactness lemma.

Lemma 4.5. (See [3, 8, 34]) *Suppose that (V1)-(V2) hold. For $c > 0$ and $\lambda \in [\delta, 1]$, let $\{u_n\} \subset X$ be a bounded $(PS)_c$ sequence for $I_{V,\lambda}$. Then there exist a $u_0 \in X$ and $A \in \mathbb{R}$ such that $J'_{V,\lambda}(u_0) = 0$, where*

$$\begin{aligned} J_{V,\lambda}(u) = &\frac{a + bA^2}{2}\int_{\mathbb{R}^N} a|\nabla u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\mathrm{d}x \\ &- \frac{\lambda}{2p}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\mathrm{d}x - \frac{\lambda}{2q}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q\mathrm{d}x. \end{aligned}$$

Moreover, either

1. *$u_n \to u_0$ strongly in $H^1(\mathbb{R}^N)$, or*
2. *there exists a finite (possibly empty) set $\{u_n\}_{n=1,\ldots,k} \subset X$ of nontrivial solutions of*

$$-(a + bA^2)\Delta u + V_\infty u = \lambda[(I_\alpha * |u|^p)|u|^{p-2}u + (I_\alpha * |u|^q)|u|^{q-2}u],$$

and $y_n^i \subset \mathbb{R}^N,\ i = 1, 2, 3, ..., k(k \in N^+)$, *such that*

$$|y_n^i| \to \infty,\ |y_n^i - y_n^j| \to \infty (i \neq i),\ \text{as } n \ \to \infty,$$
$$c + \frac{bA^4}{4} = J_{V,\lambda}(u_0) + \sum_{i=1}^{k} J_{V_\infty,\lambda}(u_i),$$
$$\left\| u_n - u_0 - \sum_{i=1}^{k} (\cdot - y_n^i) u_i \right\| \to 0,$$
$$A^2 = |\nabla u_0|_2^2 + \sum_{i=1}^{k} |\nabla u_i|_2^2,$$

where

$$\begin{aligned} J_{V_\infty,\lambda}(u) =& \frac{a + bA^2}{2} \int_{\mathbb{R}^N} a|\nabla u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty u^2 \mathrm{d}x \\ & - \frac{\lambda}{2p} \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p \mathrm{d}x - \frac{\lambda}{2q} \int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q \mathrm{d}x. \end{aligned}$$

Lemma 4.6. (See [3, 8, 34]) *Suppose that (V1)-(V2) hold. For* $\lambda \in [\delta, 1]$, *let* $\{u_n\} \subset X$ *be a bounded* $(PS)_{c_\lambda}$ *sequence for* $I_{V,\lambda}$. *Then there exists a nontrivial* $u_\lambda \in X$ *such that* $u_n \to u_\lambda$ *strongly in* X.

Now, we can prove the main theorem.

Proof of Theorem 1.1. In the view of Proposition 4.1 and Lemma 4.2, we see for a.e.$\lambda \in [\delta, 1]$ there exists a bounded sequence $\{u_n\} \subset X$ such that $I_{V,\lambda}(u_n) \to c_\lambda,\ I'_{V,\lambda}(u_n) \to 0$. By Lemma 4.6, $I_{V,\lambda}$ has a nontrivial critical point $u_\lambda \in X$ and $I_{V,\lambda}(u_\lambda) = c_\lambda$ for a.e. $\lambda \in [\delta, 1]$. Choosing an arbitrary sequence $\{\lambda_n\} \subset [\delta, 1]$ with $\lambda_n \to 1^-$, then we get a sequence $\{u_{\lambda_n}\} \subset X$ such that $I'_{V,\lambda_n}(u_{\lambda_n}) = 0$ and $I_{V,\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}$. Next we show that $\{u_{\lambda_n}\}$ is bounded in X. By (V1) and Hardy inequality, using the argument in Theorem 3.6, we can deduce both $|\nabla u_{\lambda_n}|_2$ and $|u_{\lambda_n}|_2$ are bounded.Thus $\{u_{\lambda_n}\}$ is bounded in X.

On the other hand, since $\lambda_n \to 1^-$, by Lemma 4.3, we have

$$\begin{aligned} &\lim_{n\to\infty} I(u_{\lambda_n}) \\ =& \lim_{n\to\infty} I_{V,1}(u_{\lambda_n}) \\ =& \lim_{n\to\infty} \left[I_{V,\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1)\left(\frac{1}{2p} \int_{\mathbb{R}^N} \big(I_\alpha * |u_{\lambda_n}|^p\big)|u_{\lambda_n}|^p \mathrm{d}x + \frac{1}{2q} \int_{\mathbb{R}^N} \big(I_\alpha * |u_{\lambda_n}|^q\big)|u_{\lambda_n}|^q \mathrm{d}x \right) \right] \\ =& \lim_{n\to\infty} c_{\lambda_n} \\ =& c_1 \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \big\langle I'(u_{\lambda_n}), \varphi \big\rangle \\ =& \lim_{n\to\infty} \big\langle I'_{V,1}(u_{\lambda_n}), \varphi \big\rangle \\ =& \lim_{n\to\infty} \left[\big\langle I'_{V,\lambda_n}(u_{\lambda_n}), \varphi \big\rangle + (\lambda_n - 1)\left(\int_{\mathbb{R}^N} \big(I_\alpha * |u_{\lambda_n}|^p\big)|u_{\lambda_n}|^p \mathrm{d}x + \int_{\mathbb{R}^N} \big(I_\alpha * |u_{\lambda_n}|^q\big)|u_{\lambda_n}|^q \mathrm{d}x \right) \right] \\ =& 0. \end{aligned}$$

That is $\{u_{\lambda_n}\}$ is a bounded $(PS)_{c_1}$ sequence for I. Again by Lemma 4.6, there exists $u_0 \in X$ such that $I(u_0) = c_1,\ I'(u_0) = 0$, which means u_0 is a nontrivial solution of Problem(1.1).

Finally, we show the existence of ground state solutions. Set $m = \inf\limits_{S} I(u)$, where $S = \{u \in X\backslash\{0\} : I'(u) = 0\}$. Now we claim $0 < m < \infty$. Since $u_0 \in S$, we see $m \leq c_1 < \infty$. For any $u \in S$, we have

$$\begin{aligned}
0 &= \langle I'(u), u\rangle \\
&= a\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x + b\Big(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\Big)^2 \\
&\quad - \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p \mathrm{d}x - \int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q \mathrm{d}x \\
&\geq \|u\|^2 - C_5\|u\|^{2p} - C_6\|u\|^{2q}.
\end{aligned}$$

This implies $\|u\| \geq \delta$ for some $\delta > 0$. On the other hand, by the Pohozaev identity, i.e., $P_V(u) = 0$. Then by (V1) and Hardy's inequality, we get:

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{4(N-1)}[\langle I'(u), u\rangle + 2P_V(u)] \\
&= \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{N-3}{4(N-1)}\int_{\mathbb{R}^N} V(x)|u|^2 \mathrm{d}x - \frac{1}{4(N-1)}\int_{\mathbb{R}^N} (\nabla V(x), x)u^2 \mathrm{d}x \\
&\quad + \frac{p+\alpha+2-N}{4(N-1)p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \mathrm{d}x + \frac{q+\alpha+2-N}{4(N-1)q}\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q \mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{4(N-1)}\int_{\mathbb{R}^N} (\nabla V(x), x)u^2 \mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{A}{8(N-1)}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{A}{16}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \mathrm{d}x \\
&\geq \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{A}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \\
&= \frac{a-A}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x.
\end{aligned}$$

This implies $m \geq 0$. In the following, let us rule out $m = 0$. If $m = 0$, then there exists minimizing sequence $\{u_n\} \subset S$ such that $I(u_n) \to 0$, which implies $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x = 0$. Since $\langle I'(u_n), u_n\rangle = 0$, we can infer $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx = 0$. Therefore $\lim\limits_{n\to\infty} \|u_n\|^2 = 0$, which contradicts to $\|u_n\| > \delta$. This proves our claim.

Then let $\{u_n\} \subset S$ be a minimizing sequence such that $I'(u_n) = 0$ and $I(u_n) \to m$. By similar argument as above, we can conclude that $\{u_n\}$ is bounded. Arguing by Lemma 4.6, there exists a $u \in X$ such that $u_n \to u$ strongly in X. Consequently, $I'(u) = 0,\ I(u) = m$. This means u is a ground state solution for Problem (1.1). The proof is completed. □

Data availability. No data was collected or produced for this study.

Conflicts of interest. The authors declare that they have no conflicts of interest.

Acknowledgements. This work was supported by Scientific research start-up fund (Grant No. F701108M03), National Natural Science Foundation of China (Grant Nos. 11771198, 11901276) and Science and Technology project of Education Department of Jiangxi Province (Grant No. GJJ2403004).

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Received May 2025; Accepted October 2025; Available online November 2025.