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ON THE TWO-STEP MODULUS-BASED MATRIX SPLITTING ITERATION METHOD FOR HORIZONTAL LINEAR COMPLEMENTARITY PROBLEMS

Cuixia Li[1,†] and Shiliang Wu[1,2]

Abstract In this paper, based on the previous work by Zheng and Vong [Numer. Algor., 86 (2021) 1791-1810], we further discuss the two-step modulus-based matrix splitting (TMMS) iteration method for solving the horizontal linear complementarity problems. The new convergence conditions of the TMMS method are obtained, which are weaker than those of the aforementioned paper.

Keywords Horizontal linear complementarity problem, TMMS method, convergence.

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1. Introduction

The horizontal linear complementarity problems, denoted by HLCP(q, A, B), is to find a pair of real vectors r and $z \in \mathbb{R}^n$ such that

$$Az - Br + q = 0,\ z, r \geq 0 \text{ and } z^T r = 0, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are two given matrices, $q \in \mathbb{R}^n$ is a given vector.

At present, the HLCP(q, A, B) has many applications in several fields, including control theory, hydrodynamic lubrication, nonlinear networks [7,8,17,18]. Beyond that, the HLCP(q, A, B) can be regarded as a generalization form of the well-known linear complementarity problem (LCP) [4,16].

For solving the HLCP(q, A, B), many methods have been designed over the years, including the interior point method [21], the neural network [9], the projected splitting method [12], and so on. Modulus-based matrix splitting (MMS) methods for solving LCP are instead more recent and popular, see [1]. Based on the idea of the MMS method, by reformulating the HLCP(q, A, B) as an implicit fixed-point form, Mezzadri and Galligani in [13] have successfully extended the MMS method to the HLCP(q, A, B). Not only that, in [14], the numerical results confirm that the computational efficiency of the MMS method overmatches the interior point method and the projected splitting method. Weaker hypothesis and larger convergence area of the MMS method with H_+-matrices were given [22]. Recently, to accelerate the MMS method, by using the technique of the two-step splitting, Zheng and Vong in [23] presented a two-step modulus-based matrix splitting (TMMS) iteration method for solving the HLCP(q, A, B), gave the convergence

[†]The corresponding author.
[1]School of Mathematics, Yunnan Normal University, Kunming, Yunnan 650500, China
[2]Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Yunnan Normal University, Kunming 650500, China
Email: lixiatkynu@126.com(C. Li), wushiliang1999@126.com(S. Wu)

conditions for the TMMS method, including the case of accelerated overrelaxation splitting, confirmed the effectiveness of the TMMS method by some numerical experiments. In the meantime, in [23], they left over an interested problem that "*How to improve the convergence theorems is worth studying in the future*". See [5,11,15,19,20] for some recent works.

In this paper, to answer this equation, we further study the TMMS method for solving the HLCP(q, A, B) and establish some new sufficient conditions for the convergence of the TMMS method, which improve the convergence theorems in the previously published work in [23].

Next, we briefly summarize some notations, definitions and lemmas.

Let $A \in \mathbb{R}^{n\times n}$. Then $A = D_A - L_A - U_A = D_A - C_A$, where $D_A, -L_A, -U_A, -C_A$, respectively, denote the diagonal, the strictly lower triangular, the strictly upper triangular and non-diagonal matrices of matrix A. $A = M - N$ is called an H-splitting if $\langle M\rangle - |N|$ is a nonsingular M-matrix, where $\langle M\rangle$ denotes the comparison matrix of M with $\langle m\rangle_{ij} = |m_{ij}|$ for $i = j$ and $\langle m\rangle_{ij} = -|m_{ij}|$ for $i \neq j$; an H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$ with $|N| = (|n_{ij}|)$.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a nonsingular M-matrix if $A^{-1} \geq 0$ with $a_{ij} \leq 0$ for $i \neq j$; an H-matrix if $\langle A\rangle$ is a nonsingular M-matrix; an H_+-matrix if it is an H-matrix with $a_{ii} > 0$ for $i = 1, 2, \ldots n$, see [2]. $\rho(A)$ denotes the spectral radius of the matrix A.

Lemma 1.1. [6] *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix. Then $|A^{-1}| \leq \langle A\rangle^{-1}$.*

Lemma 1.2. [3] *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. Then there is a positive diagonal matrix D such that AD is a strictly diagonal dominant (s.d.d.) matrix.*

Lemma 1.3. [10] *Let $B \in \mathbb{R}^{n\times n}$ be an s.d.d. matrix. Then*

$$\|B^{-1}C\|_\infty \leq \max_{1\leq i\leq n} \frac{(|C|e)_i}{(\langle B\rangle e)_i},\ \textit{for}\ \forall\ C \in \mathbb{R}^{n\times n},$$

where $e = (1, 1, \ldots, 1)^T$.

2. The TMMS method

First, a brief review of the TMMS method in [23] is required.

If we take

$$z = \gamma^{-1}(|x| + x), r = \gamma^{-1}\Omega(|x| - x), A = M_A - N_A \text{ and } B = M_B - N_B$$

for the HLCP(q, A, B), then we obtain

$$(M_A + M_B\Omega)x = (N_A + N_B\Omega)x + (B\Omega - A)|x| - \gamma q, \tag{2.1}$$

where Ω is a positive diagonal matrix and $\gamma > 0$, see [13,23].

Further, let $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ and $B = M_{B_1} - N_{B_1} = M_{B_2} - N_{B_2}$, in [23], Zheng and Vong designed the following TMMS method for solving the HLCP(q, A, B).

The TMMS method [23]. Let matrix Ω be positive diagonal and $\gamma > 0$, and let $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ and $B = M_{B_1} - N_{B_1} = M_{B_2} - N_{B_2}$ be the two splittings of matrices A and B, respectively. Starting from an arbitrary initial vector $x^{(0)} \in \mathbb{R}^n$, compute $x^{(k+1)}$ by

$$\begin{cases} (M_{A_1} + M_{B_1}\Omega)x^{(k+\frac{1}{2})} = (N_{A_1} + N_{B_1}\Omega)x^{(k)} + (B\Omega - A)|x^{(k)}| - \gamma q, \\ (M_{A_2} + M_{B_2}\Omega)x^{(k+1)} = (N_{A_2} + N_{B_2}\Omega)x^{(k+\frac{1}{2})} + (B\Omega - A)|x^{(k+\frac{1}{2})}| - \gamma q. \end{cases} \tag{2.2}$$

Then set

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}), r^{(k+1)} = \frac{1}{\gamma}\Omega(|x^{(k+1)}| - x^{(k+1)}),$$

for $k = 0, 1, \ldots,$ until the iteration sequence $(z^{(k)}, r^{(k)})_{k=1}^{+\infty}$ converges.

Taking

$$\begin{cases} M_{A_1} = \frac{1}{\alpha}(D_A - \beta L_A), M_{A_2} = \frac{1}{\alpha}(D_A - \beta U_A), \\ M_{B_1} = \frac{1}{\alpha}(D_B - \beta L_B), M_{B_2} = \frac{1}{\alpha}(D_B - \beta U_B), \end{cases}$$

for the TMMS method, we can obtain the two-step modulus-based accelerated overrelaxation (TMAOR) method.

When $A, B \in \mathbb{R}^{n\times n}$ in (1.1) are two H_+-matrices, the following results are obtained, which are major results in [23].

Theorem 2.1. [23] *Let $A, B \in \mathbb{R}^{n\times n}$ be two H_+-matrices and $\Omega = (\omega_{jj}) \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix with*

$$|b_{ij}|\omega_{jj} \leq |a_{ij}| \ (i \neq j) \text{ and } sign(b_{ij}) = sign(a_{ij}), \ b_{ij} \neq 0, i, j = 1, 2, \ldots, n.$$

Let $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ and $B = M_{B_1} - N_{B_1} = M_{B_2} - N_{B_2}$ be two H-compatible splittings of A and B, respectively. Let D be a positive diagonal matrix such that $\langle A\rangle D$ is an s.d.d. matrix. Then, the iteration sequence $(z^{(k)}, r^{(k)})_{k=1}^{+\infty}$ generated by the TMMS method converges to the unique solution (z^*, r^*) of the HLCP(q, A, B) from any initial vector $x^{(0)} \in \mathbb{R}^n$ provided that

$$(D_A - D^{-1}\langle A\rangle D)e < \Omega D_B e. \tag{2.3}$$

Theorem 2.2. [23] *Let $A, B \in \mathbb{R}^{n\times n}$ be two H_+-matrices. Assume that the positive diagonal matrix Ω satisfies $\Omega \geq D_A D_B^{-1}$. Further, for $i, j = 1, 2, \ldots, n$, let $|b_{ij}|\omega_{jj} \leq |a_{ij}|$ $(i \neq j)$ and $sign(b_{ij}) = sign(a_{ij})(b_{ij} \neq 0)$. Then, the iteration sequence $(z^{(k)}, r^{(k)})_{k=1}^{+\infty}$ generated by the TMAOR method converges to the unique solution (z^*, r^*) of the HLCP(q, A, B) from any initial vector $x^{(0)} \in \mathbb{R}^n$ provided that*

$$0 < \beta \leq \alpha < \frac{1}{\rho[(D_A + D_B\Omega)^{-1}(D_B\Omega + |C_A|)]}. \tag{2.4}$$

To improve the convergence conditions of Theorem 2.1, we give the following result, see Theorem 2.3.

Theorem 2.3. *Let $A, B \in \mathbb{R}^{n\times n}$ be two H_+-matrices and $\Omega = (\omega_{jj}) \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix with*

$$|b_{ij}|\omega_{jj} \leq |a_{ij}| \ (i \neq j) \text{ and } sign(b_{ij}) = sign(a_{ij}), \ b_{ij} \neq 0, i, j = 1, 2, \ldots, n.$$

Let $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ be two H-splittings of A, $B = M_{B_1} - N_{B_1} = M_{B_2} - N_{B_2}$ be two H-compatible splittings of B, and $M_{A_1} + \Omega M_{B_1}$ and $M_{A_2} + \Omega M_{B_2}$ be two H_+-matrices. Then, the iteration sequence $(z^{(k)}, r^{(k)})_{k=1}^{+\infty}$ generated by the TMMS method converges to the unique solution (z^*, r^*) of the HLCP(q, A, B) from any initial vector $x^{(0)} \in \mathbb{R}^n$, provided that

$$[D_A - \frac{1}{2}D^{-1}(\min\{\langle A\rangle + \langle M_{A_1}\rangle - |N_{A_1}|, \langle A\rangle + \langle M_{A_2}\rangle - |N_{A_2}|\})D]e < D_B\Omega e \tag{2.5}$$

where D is a positive diagonal matrix such that $(\langle M_{A_1}\rangle - |N_{A_1}|)D$ and $(\langle M_{A_2}\rangle - |N_{A_2}|)D$ are two s.d.d. matrices.

Proof. Based on Eq. (2.1), together with the TMMS method, we obtain

$$\begin{cases}(M_{A_1}+M_{B_1}\Omega)x^*=(N_{A_1}+N_{B_1}\Omega)x^*+(B\Omega-A)|x^*|-\gamma q,\\(M_{A_2}+M_{B_2}\Omega)x^*=(N_{A_2}+N_{B_2}\Omega)x^*+(B\Omega-A)|x^*|-\gamma q,\end{cases}\tag{2.6}$$

where $x^*=\frac{\gamma}{2}(z^*-\Omega^{-1}r^*)$ with (z^*,r^*) being a solution of the HLCP(q,A,B).

By subtracting (2.6) from (2.2), we have

$$\begin{cases}(M_{A_1}+M_{B_1}\Omega)(x^{(k+\frac{1}{2})}-x^*)=(N_{A_1}+N_{B_1}\Omega)(x^{(k)}-x^*)\\\qquad\qquad+(B\Omega-A)(|x^{(k)}|-|x^*|),\\(M_{A_2}+M_{B_2}\Omega)(x^{(k+1)}-x^*)=(N_{A_2}+N_{B_2}\Omega)(x^{(k+\frac{1}{2})}-x^*)\\\qquad\qquad+(B\Omega-A)(|x^{(k+\frac{1}{2})}|-|x^*|).\end{cases}\tag{2.7}$$

By Lemma 1.1, making use of the absolute value for the first equality of (2.7) results in

$$\begin{aligned}|x^{(k+\frac{1}{2})}-x^*|=&|(M_{A_1}+M_{B_1}\Omega)^{-1}((N_{A_1}+N_{B_1}\Omega)(x^{(k)}-x^*)\\&+(B\Omega-A)(|x^{(k)}|-|x^*|))|\\\le&|(M_{A_1}+M_{B_1}\Omega)^{-1}(N_{A_1}+N_{B_1}\Omega)(x^{(k)}-x^*)|+|(M_{A_1}+M_{B_1}\Omega)^{-1}|\\&\times|(B\Omega-A)|\cdot||x^{(k)}|-|x^*||\\\le&|(M_{A_1}+M_{B_1}\Omega)^{-1}(N_{A_1}+N_{B_1}\Omega)||x^{(k)}-x^*|\\&+|(M_{A_1}+M_{B_1}\Omega)^{-1}||B\Omega-A||x^{(k)}-x^*|\\\le&\langle M_{A_1}+M_{B_1}\Omega\rangle^{-1}[|N_{A_1}+N_{B_1}\Omega|+|B\Omega-A|]|x^{(k)}-x^*|\\=&K_1|x^{(k)}-x^*|,\end{aligned}$$

where

$$K_1=\bar{M}_1^{-1}\bar{N}_1,\bar{M}_1=\langle M_{A_1}+M_{B_1}\Omega\rangle,\bar{N}_1=|N_{A_1}+N_{B_1}\Omega|+|B\Omega-A|.$$

Next, we consider two cases: $D_B\Omega e\ge D_Ae$ and

$$[D_A-\frac{1}{2}D^{-1}(\langle A\rangle+\langle M_{A_1}\rangle-|N_{A_1}|)D]e<D_B\Omega e.$$

Case I. Obviously, $D_B\Omega e\ge D_Ae$ is equal to $D_B\Omega\ge D_A$. In this case, we have

$$|B\Omega-A|=\langle B\rangle\Omega-\langle A\rangle.$$

Since $A=M_{A_1}-N_{A_1}$ is an H-splitting of A, matrix $\langle M_{A_1}\rangle-|N_{A_1}|$ is a nonsingular M-matrix. By Lemma 1.2, then the existence of such a matrix D satisfies

$$(\langle M_{A_1}\rangle-|N_{A_1}|)De>0.$$

Noting that

$$\langle M_{A_1}\rangle-|N_{A_1}|\le\langle A\rangle,\ \langle M_{B_1}\Omega\rangle-|N_{B_1}\Omega|=\langle B\rangle\Omega.$$

Then, we have

$$\begin{aligned}(\bar{M}_1-\bar{N}_1)De =&(\langle M_{A_1}+M_{B_1}\Omega\rangle-|N_{A_1}+N_{B_1}\Omega|-|B\Omega-A|)De\\ \geq&(\langle M_{A_1}\rangle+\langle M_{B_1}\rangle\Omega-|N_{A_1}|-|N_{B_1}|\Omega+\langle A\rangle-\langle B\rangle\Omega)De\\ \geq&2(\langle M_{A_1}\rangle-|N_{A_1}|)De\\ >&0.\end{aligned}$$

Thus, based on Lemma 1.3, we have

$$\|D^{-1}K_1D\|_\infty=\|(\bar{M}_1D)^{-1}\bar{N}_1D\|_\infty\leq\max_{1\leq i\leq n}\frac{((\bar{N}_1D)De)_1}{(\bar{M}_1De)_i}<1.$$

Case II. Clearly,

$$|B\Omega-A|=|A|-|B|\Omega\geq 0.$$

Similar to Case I, we have

$$\begin{aligned}(\bar{M}_1-\bar{N}_1)De &= (\langle M_{A_1}+M_{B_1}\Omega\rangle-|N_{A_1}+N_{B_1}\Omega|-|A|+|B|\Omega)De\\ &\geq (\langle M_{A_1}\rangle+\langle M_{B_1}\rangle\Omega-|N_{A_1}|-|N_{B_1}|\Omega-|A|+|B|\Omega)De\\ &\geq (\langle M_{A_1}\rangle-|N_{A_1}|-|A|+\langle M_{B_1}\rangle\Omega-|N_{B_1}|\Omega+|B|\Omega)De\\ &\geq (\langle M_{A_1}\rangle-|N_{A_1}|-|A|+\langle B\rangle\Omega+|B|\Omega)De\\ &= (\langle M_{A_1}\rangle-|N_{A_1}|-|A|+2D_B\Omega)De\\ &= (\langle M_{A_1}\rangle-|N_{A_1}|-D_A-|C_A|+2\Omega D_B)De\\ &= (\langle M_{A_1}\rangle-|N_{A_1}|+D_A-|C_A|+2\Omega D_B-2D_A)De\\ &= (\langle M_{A_1}\rangle-|N_{A_1}|+\langle A\rangle+2\Omega D_B-2D_A)De\\ &> 0.\end{aligned}$$

It is to check that

$$\|D^{-1}K_1D\|_\infty<1.$$

Summarizing Case I and Case II, we know that if

$$[D_A-\frac{1}{2}D^{-1}(\langle A\rangle+\langle M_{A_1}\rangle-|N_{A_1}|\})D]e<D_B\Omega e, \tag{2.8}$$

then $\|D^{-1}K_1D\|_\infty<1$.

Similarly, making use of the second equality of (2.7), we have

$$|x^{(k+1)}-x^*|\leq K_2|x^{(k+\frac{1}{2})}-x^*|,$$

where

$$K_2=\bar{M}_2^{-1}\bar{N}_2, \bar{M}_2=\langle M_{A_2}+\Omega M_{B_2}\rangle, \bar{N}_2=|N_{A_2}+\Omega N_{B_2}|+|A-\Omega B|.$$

Therefore,

$$|x^{(k+1)}-x^*|\leq K_2K_1|x^{(k)}-x^*|.$$

With the same discussion, if

$$[D_A-\frac{1}{2}D^{-1}(\langle A\rangle+\langle M_{A_2}\rangle-|N_{A_2}|\})D]e<D_B\Omega e, \tag{2.9}$$

then $\|D^{-1}K_2D\|_\infty < 1$, too.

Combining (2.8) with (2.9), obviously, under the condition (2.5), the next inequality holds:

$$\begin{aligned}\rho(K_2K_1) =&\rho(D^{-1}K_2K_1D)\\ =&\rho(D^{-1}K_2DD^{-1}K_1D)\\ =&\|(D^{-1}K_2DD^{-1}K_1D\|_\infty\\ \leq&\|D^{-1}K_2D\|_\infty\|D^{-1}K_1D\|_\infty\\ <&1.\end{aligned}$$

This implies that the TMMS method is convergent. □

Comparing Theorem 2.1 with Theorem 2.3, the latter does not require that $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ must be two H-compatible splittings of A. From the view of this point, Theorem 2.3 may be weaker than Theorem 2.1. In addition, when $A = M_{A_1} - N_{A_1} = M_{A_2} - N_{A_2}$ becomes two H-compatible splittings of A in Theorem 2.3, Theorem 2.3 reduces to Theorem 2.1. In a way, Theorem 2.3 improves the convergence conditions of Theorem 2.1.

By investigating Theorem 2.2, it is obtained under the assumption that one parameter must be greater than the other. Here, we will show how we can avoid this assumption and, consequently, improve the convergence area. That is to say, we will obtain another convergence area that allows more freedom in the choice of the two parameters and, in that way, improves the original one. Specifically, see Theorem 2.4.

Theorem 2.4. *Let $A, B \in \mathbb{R}^{n\times n}$ be two H_+-matrices. Assume that the positive diagonal matrix Ω satisfies $\Omega \geq D_AD_B^{-1}$ and $|C_B|\Omega \leq |C_A|$. Then, the iteration sequence $(z^{(k)}, r^{(k)})_{k=1}^{+\infty}$ generated by the TMAOR method converges to the unique solution (z^*, r^*) of the HLCP(q, A, B) from any initial vector $x^{(0)} \in \mathbb{R}^n$ provided that*

$$0 < \max\{\alpha, 2\beta - \alpha\}\rho[(D_A + D_B\Omega)^{-1}(D_B\Omega + |C_A|)] < \min\{1, \alpha\}. \tag{2.10}$$

Proof. First, by simple passages, we find

$$1 + \alpha - |1 - \alpha| = 2\min\{1, \alpha\}$$

and

$$\begin{aligned}&\beta|L_B\Omega + L_A| + |(\alpha - \beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A)| + \alpha|C_A - C_B\Omega| + 2\alpha D_B\Omega\\ \leq&2\max\{\alpha, 2\beta - \alpha\}(|C_A| + D_B\Omega).\end{aligned} \tag{2.11}$$

The inequality (2.11) is obtained by using this fact $|C_B|\Omega \leq |C_A|$, which implies that

$$|C_B\Omega + C_A| + |C_A - C_B\Omega| = 2|C_A|.$$

This is because $|a + b| + |a - b| = 2|a|$ for $|a| \geq |b|$, $a, b \in \mathbb{R}$.

Second, let us consider the error of the TMAOR method

$$\begin{aligned}x^{(k+\frac{1}{2})} - x_* =&(D_B\Omega + D_A - \beta(L_B\Omega + L_A))^{-1}((1 - \alpha)(D_B\Omega + D_A)\\ &+ (\alpha - \beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A))(x^{(k)} - x_*)\\ &+ \alpha(B\Omega - A)(|x^{(k)}| - |x_*|).\end{aligned}$$

Based on Lemma 1.1, we have

$$|(D_B\Omega + D_A - \beta(L_B\Omega + L_A))^{-1}| \leq (D_B\Omega + D_A - \beta|L_B\Omega + L_A|)^{-1}.$$

Further,

$$|x^{(k+\frac{1}{2})} - x_*| \leq \mathcal{W}_1(\alpha,\beta)|x^{(k)} - x_*|, \tag{2.12}$$

where

$$\mathcal{W}_1(\alpha,\beta) = \mathcal{M}_1^{-1}\mathcal{N}_1$$

with

$$\mathcal{M}_1 = D_B\Omega + D_A - \beta|L_B\Omega + L_A|$$

and

$$\mathcal{N}_1 = |(1-\alpha)(D_B\Omega + D_A) + (\alpha-\beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A)| + \alpha|B\Omega - A|.$$

Under the condition (2.10), we have

$$\begin{aligned}
\mathcal{M}_1 - \mathcal{N}_1 =& D_B\Omega + D_A - \beta|L_B\Omega + L_A| - |(1-\alpha)(D_B\Omega + D_A) \\
&+ (\alpha-\beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A)| - \alpha|B\Omega - A| \\
=& D_B\Omega + D_A - \beta|L_B\Omega + L_A| - |1-\alpha|(D_B\Omega + D_A) \\
&- |(\alpha-\beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A)| \\
&- \alpha|(D_B\Omega - D_A) + C_A - C_B\Omega| \\
=& D_B\Omega + D_A - \beta|L_B\Omega + L_A| - |1-\alpha|(D_B\Omega + D_A) \\
&- |(\alpha-\beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A)| \\
&- \alpha(D_B\Omega - D_A) - \alpha|C_A - C_B\Omega| \\
=& 1 + \alpha - |1-\alpha|)(D_B\Omega + D_A) - \beta|L_B\Omega + L_A| \\
&- |(\alpha-\beta)(L_B\Omega + L_A) + \alpha(U_B\Omega + U_A)| \\
&- \alpha|C_A - C_B\Omega| - 2\alpha D_B\Omega \\
\geq& (\min\{1,\alpha\}(D_B\Omega + D_A) - \max\{\alpha, 2\beta - \alpha\}(|C_A| + D_B\Omega).
\end{aligned}$$

This implies that $\mathcal{M}_1 - \mathcal{N}_1$ a nonsingular M-matrix. Therefore, from Lemma 1.2, there exists a positive diagonal matrix $\bar{D}$ such that $(\mathcal{M}_1 - \mathcal{N}_1)\bar{D}$ is a strictly diagonal dominant matrix. By using Lemma 1.3, we obtain

$$\|\bar{D}^{-1}\mathcal{W}_1(\alpha,\beta)\bar{D}\|_\infty = \|(\mathcal{M}_1\bar{D})^{-1}\mathcal{N}_1\bar{D}\|_\infty \leq \max \frac{(\mathcal{N}_1\bar{D})e_i}{(\mathcal{M}_1\bar{D})e_i} < 1.$$

Similarly, by the second equality of (2.7), we have

$$|x^{(k+1)} - x_*| \leq \mathcal{W}_2(\alpha,\beta)|x^{(k+\frac{1}{2})} - x_*|, \tag{2.13}$$

where

$$\begin{aligned}
\mathcal{W}_2(\alpha,\beta) =& (D_B\Omega + D_A - \beta|U_B\Omega + U_A|)^{-1}(|(1-\alpha)(D_B\Omega + D_A) \\
&+ (\alpha-\beta)(U_B\Omega + U_A) + \alpha(L_B\Omega + L_A)| + \alpha|B\Omega - A|).
\end{aligned}$$

By the same discussion as above, under the condition (2.10), we still have

$$\|\bar{D}^{-1}\mathcal{W}_2(\alpha,\beta)\bar{D}\|_\infty < 1.$$

Combining (2.12) with (2.13), we have

$$|x^{(k+1)} - x_*| \le \mathcal{W}_2(\alpha,\beta)\mathcal{W}_1(\alpha,\beta)|x^{(k)} - x_*|.$$

The rest of proof is similar to the proof of Theorem 2.3, which is omitted. □

Remark 2.1. The convergence conditions for the TMAOR method in Theorem 2.4 is without one parameter being greater than the other. This implies that Theorem 2.4 provides the relaxation convergence area, which chooses two parameters α and β freely in a way. Beyond that, we use "$|C_B|\Omega \le |C_A|$" instead of "Further, for $i,j = 1,2,\ldots,n$, let $|b_{ij}|\omega_{jj} \le |a_{ij}|(i \ne j)$ and $sign(b_{ij}) = sign(a_{ij})(b_{ij} \ne 0)$" in Theorem 2.2. Obviously, it relaxes the convergence conditions of Theorem 2.2 as well. Compared with Theorem 2.2, Theorem 2.4 indeed improve the convergence area for the TMAOR method.

Corollary 2.1. *Under the conditions of Theorem 2.4, the following results are valid,*

(1) the TMSOR method is convergent for

$$\rho[(D_A + D_B\Omega)^{-1}(D_B\Omega + |C_A|)] < \min\{\frac{1}{\alpha}, 1\};$$

(2) the TMGS method is convergent for $\rho[(D_A + D_B\Omega)^{-1}(D_B\Omega + |C_A|)] < 1$*;*

(3) the TMJ method is convergent for $\rho[(D_A + D_B\Omega)^{-1}(D_B\Omega + |C_A|)] < 1$*.*

Remark 2.2. By investigating the results in Theorem 2.4 and Corollary 2.1, actually, these convergence conditions can guarantee the convergence of the corresponding the modulus-based AOR (MAOR), successive overrelaxation (MSOR), Gauss-Seidel (MGS) and Jacobi (MJ) method in [13] for solving the HLCP(q, A, B). In theory, our convergence theorems remedies the gap for the relaxation versions of the MMS method in [13].

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