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LEGENDRE SPECTRAL METHOD FOR THE NONLINEAR TIME FRACTIONAL MIXED SUB-DIFFUSION AND DIFFUSION-WAVE EQUATION WITH SMOOTH AND NON-SMOOTH SOLUTIONS*

Ying Lu[1,2,3], Wenping Chen[1,2,3,†], Xia Zhou[1,2,3], Lihua Jiang[1,2,3] and Xiaoli Li[1,2,3]

Abstract In this paper, we propose a Legendre spectral method for solving the nonlinear time-fractional mixed sub-diffusion and diffusion-wave equations with smooth and non-smooth solutions. By using the L1 scheme to discretise the Caputo fractional derivative in time and the Legendre spectral method in space, a fully discrete scheme is constructed. A prior estimate and the existence and uniqueness of numerical solution are derived. The stability and convergence of the fully discrete scheme are strictly proved, and the convergence order is proved to be $O(\tau^{\min\{3-\gamma,2-\alpha\}} + N^{1-s})$. Finally, numerical experiments are presented to verify the theoretical convergence results.

Keywords Mixed sub-diffusion and diffusion-wave equation, L1 scheme, Legendre spectral method, stability and convergence.

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1. Introduction

Fractional differential equations better describe some natural phenomena of physics and engineering, providing a more comprehensive explanation for materials with memory and genetic properties [20]. Due to the difficulty of finding the analytic solutions for fractional differential equations [25, 26], the numerical solution has received extensive attention. Here are some commonly used numerical methods: The finite difference method [12, 15], finite element method [5, 16], and spectral method [3, 9, 14], etc.

The time fractional diffusion and diffusion-wave equations replaced the time derivatives in the classical diffusion and wave equations by fractional derivatives of order $0 < \alpha < 1$, $1 < \gamma < 2$, respectively [18]. The fractional derivatives commonly used include the Caputo fractional deriva-

†The corresponding author.
1School of Mathematics & Computing Science, Guilin University of Electronic Technology, Guilin 541004, China
2Center for Applied Mathematics of Guangxi (GUET), Guilin 541004, China
3Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin 541004, China
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Email: 1977334666@qq.com(Y. Lu), wpchen@guet.edu.cn(W. Chen), xiazhou201612@guet.edu.cn(X. Zhou), anhuijlh@126.com(L. Jiang), 1371644917@qq.com(X. Li)

tive, the Riemann-Liouville fractional derivative, the Grünwald-Letnikov derivative. However, compared with classical diffusion and wave equations, time fractional mixed sub-diffusion and diffusion-wave equations are more consistent and accurate in depicting the anomalous diffusion process and modeling viscoelastic damping. In recent years, time fractional mixed sub-diffusion and diffusion-wave equations have been extensively studied [1,2,4,8,10,11,17,21,24].

In this paper, we use the Legendre spectral method to approximate the nonlinear time fractional mixed sub-diffusion and diffusion-wave equation:

$$\begin{cases} {}_0^C D_t^\alpha u + {}_0^C D_t^\gamma u - u_{xx} + g(u) = f(x,t), (x,t) \in (-1,1)\times(0,T], \\ u(-1,t) = 0, u(1,t) = 0, t \in [0,T], \\ u(x,0) = u_0(x), u_t(x,0) = \psi(x), x \in [-1,1], \end{cases} \tag{1.1}$$

where f is a given function, $g(u)$ is a nonlinear term satisfies $g'(u) > 0$ and $|g'(u)| < L$. Without loss of generality, let $g(0) = 0$. ${}_0^C D_t^\alpha u$ and ${}_0^C D_t^\gamma u$ denote the Caputo time fractional derivatives of u give by

$${}_0^C D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds,\ 0 < \alpha < 1,$$

and

$${}_0^C D_t^\gamma u = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{u''(s)}{(t-s)^{\gamma-1}} ds,\ 1 < \gamma < 2,$$

respectively.

There are several numerical methods to solve fractional diffusion and diffusion-wave equations. Chen et al. [8] used the finite difference/Legendre spectral method to approximate multi-term linear time fractional diffusion and diffusion-wave equations with variable coefficients, and obtained a convergence order is $O(\tau^2 + N^{1-m})$. Sun et al. [21] studied the time fractional mixed sub-diffusion and diffusion-wave equations using the standard finite difference method based on the L1 scheme, proposed a new analytical technique, and obtained the time convergence order of $O(\tau^{\min\{2-\alpha, 3-\gamma\}})$. Liu et al. [17] proposed a numerical solution for a novel linear time fractional mixed diffusion and diffusion-wave equation based on finite difference discretization in time and Legendre spectral approximation in space, and obtained the unconditional stability and convergence of the fully discrete spectral scheme. A meshless method for solving the nonlinear time fractional mixed diffusion and diffusion-wave equation is given in [4]. Cui [10] studied the 2-D multi-term time-fractional mixed diffusion and diffusion-wave equation using the compact finite difference method, achieving a convergence order of $O(\tau^2 + h_x^4 + h_y^4)$. By employing the L1 approximation coupled with the Crank-Nicolson scheme in temporal direction and EQ_1^{rot} nonconforming finite element in spatial direction, Fan et al. [11] presented the unconditional stable numerical approximation for the multi-term time-fractional mixed sub-diffusion and diffusion-wave equation on anisotropic meshes. Alikhanov et al. [2] introduced a second-order temporal convergence order linearized L2-type difference scheme to approximate the nonlinear time-fractional mixed sub-diffusion and diffusion-wave equation with delay. In [1] Alikhanov et al. provided a priori estimates of the exact solution and employed an L2-type formula to approximate the Caputo derivative for the time-fractional mixed sub-diffusion and diffusion-wave equation, and obtained the unconditional stability of the proposed difference scheme. In [24], Zhang et al. proposed two numerical schemes for multi-term time-fractional mixed sub-diffusion and diffusion-wave equation, a direct numerical scheme that uses quadratic Charles Hermite and Newton (H2N2) interpolation polynomials approximations in the temporal direction and finite element discretization in the spatial direction, and a new fast numerical scheme based on H2N2 interpolation and an efficient sum-of-exponentials approximation for the kernels.

In this paper, we construct a fully discrete scheme for the nonlinear mixed diffusion and diffusion wave equation with smooth and non-smooth solutions by using the L1 approximation for the temporal direction and the Legendre spectral method for the spatial direction. By a priori estimates, the well-posedness and convergence of the fully discrete scheme are analyzed, respectively.

The rest of this paper is organized as follows. In Section 2, the relevant symbols and lemmas are presented. The fully discrete spectral scheme is constructed in Section 3. Based on a priori estimate, the existence and uniqueness of the fully discrete scheme are given in section 4, and the stability and convergence of the fully discrete scheme are analyzed in Section 5. Some numerical examples are provided in Section 6 and some conclusions are made in the last section.

2. Preliminaries and notations

Some notations and lemmas are introduced in this section, which will be used in the context.

Let $H^m(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), 0 \le |\alpha| \le m\}$ represents Sobolev spaces equipped with norm $\|\cdot\|_m,\ m \ge 0$. In particular, $L^2(\Omega) = H^0(\Omega)$, $\|\cdot\|$ indicates the norm of $L^2(\Omega)$. $P_N^0(\Omega) = \{v \in P_N(\Omega) : v(\pm 1) = 0\}$, $H_0^1(\Omega) = \{v : v \in H^1(\Omega), v(\pm 1) = 0\}$.

Let $\pi_N^{1,0} : H_0^1(\Omega) \to P_N^0(\Omega)$ be the H_0^1-orthogonal projection operator, such that for all $u \in H_0^1(\Omega)$,

$$(\partial_x \pi_N^{1,0} u, \partial_x v) = (\partial_x u, \partial_x v), \forall v \in P_N^0(\Omega).$$

If the boundary conditions are non-homogeneous, the following transformation can be made

$$u(x,t) = \hat{u}(x,t) + \frac{x-1}{2}\hat{u}(-1,t) + \frac{-1-x}{2}\hat{u}(1,t),$$

then we can obtain homogeneous boundary conditions.

Lemma 2.1. [6] *For all $s \ge 1$, if $u \in H_0^1(\Omega) \cap H^s(\Omega)$, then*

$$\|u - \pi_N^{1,0} u\|_l \le C N^{l-s} \|u\|_s,\ l = 0,\ 1,$$

where C is a positive constant independent of N and u.

The Poincaré's inequality and the discrete Grönwall inequalities are given below.

Lemma 2.2. [7] *For any $u(x) \in C^1[-1,1]$, with $u(-1) = u(1) = 0$, we have*

$$\|u\| \le \frac{1}{\sqrt{2}} \|u_x\|.$$

Lemma 2.3. [19] *Assume that $\{k_n\}$ and $\{g_n\}$ are nonnegative sequence, and the sequence $\{\phi_n\}$ satisfies*

$$\begin{aligned} &\phi_0 \le g_0, \\ &\phi_n \le g_{n-1} + \sum_{l=0}^{n-1} k_l \phi_l,\ n \ge 1, \end{aligned}$$

where $g_0 \geq 0$. *Then the sequence* $\{\phi_n\}$ *satisfies*

$$\phi_n \leq g_{n-1} \exp\left(\sum_{l=0}^{n-1} k_l\right), \ n \geq 1.$$

Lemma 2.4. [13] *Let* k, B *and* a_μ, b_μ, c_μ, γ_μ, *for integer* $\mu \geq 0$, *be nonnegative numbers such that*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq k\sum_{\mu=0}^{n} \gamma_\mu a_\mu + k\sum_{\mu=0}^{n} c_\mu + B, \ n \geq 0.$$

Suppose that $k\gamma_\mu < 1$, *for all* μ, *set* $\delta_\mu = (1 - k\gamma_\mu)^{-1}$, *then*

$$a_n + k\sum_{\mu=0}^{n} b_\mu \leq \exp(k\sum_{\mu=0}^{n} \delta_\mu\gamma_\mu)(k\sum_{\mu=0}^{n} c_\mu + B).$$

3. The construction of fully discrete scheme

Let M be a positive integer, $t_k = k\tau$, $k = 0, 1, ..., M$, where $\tau = \frac{T}{M}$ is the time step size. To simplify the notation, assume $u^n = u(x, t_n)$. For the grid function $w = \{w^n | 0 \leq n \leq M\}$, we define

$$w^{n-\frac{1}{2}} = \frac{1}{2}(w^n + w^{n-1}), \ \delta_t w^{n-\frac{1}{2}} = \frac{1}{\tau}(w^n - w^{n-1}).$$

In order to perform a numerical approximation of the Caputo fractional derivatives of $\alpha(0 < \alpha < 1)$ and $\gamma(1 < \gamma < 2)$, we use the $L1$ approximation as in [22].

Suppose $u(t) \in C^2[t_0, t_n]$ and $0 < \alpha < 1$. Then

$$\begin{aligned}{}_0^C D_t^\alpha u(t_n) &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(a_0 u^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u^k - a_{n-1}u^0\right) + O(\tau^{2-\alpha}) \\ &= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n} a_{n-k}\delta_t u^{k-\frac{1}{2}} + O(\tau^{2-\alpha}),\end{aligned}$$

where $a_l = (l+1)^{1-\alpha} - l^{1-\alpha}$, $a_0 = 1$, $a_l > a_{l+1}(l \geq 0)$. For any $0 < \alpha < 1$, we have $C_1 l^{1-\alpha} \leq (a_l)^{-1} \leq C_2 l^{1-\alpha}$, where C_1 and C_2 are positive constants.

Suppose $u(t) \in C^3[t_0, t_n]$ and $1 < \gamma < 2$. Then

$$\begin{aligned}&\frac{1}{2}\left({}_0^C D_t^\gamma u(t_n) + {}_0^C D_t^\gamma u(t_{n-1})\right) \\ =&\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}\left(b_0\delta_t u^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\delta_t u^{k-\frac{1}{2}} - b_{n-1}u'(t_0)\right) + O(\tau^{3-\gamma}),\end{aligned}$$

where $b_l = (l+1)^{2-\gamma} - l^{2-\gamma}$, $l \geq 0$, $b_l \geq (2-\gamma)M^{1-\gamma}$. For any $1 < \gamma < 2$, we have

$$0 < b_k < b_{k-1} < \cdots < b_1 < b_0 = 1.$$

The following lemma plays a vital role in the theoretical analysis.

Lemma 3.1. [21] *For $\psi,\ V_1, V_2, ..., V_M \in P_N^0(\Omega)$, it holds that*

$$\sum_{n=1}^{m}(\frac{a_0}{2}V^n+\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-1-k}}{2}V^k,V^n)\geq-\frac{a_0}{2}\sum_{n=1}^{m}\|V^n\|^2,\ 1\leq m\leq M,$$

and

$$\begin{aligned}&\sum_{n=1}^{m}(b_0V^n-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})V^k-b_{n-1}\psi,V^n)\\ \geq&\frac{1}{2}(\sum_{k=1}^{m}b_{m-k}\|V^k\|^2-\sum_{n=1}^{m}b_{n-1}\|\psi\|^2),\ 1\leq m\leq M.\end{aligned}$$

Specifically, using $\frac{G(u^n)-G(u^{n-1})}{u^n-u^{n-1}}$ to approximate $\frac{g(u^n)+g(u^{n-1})}{2}$, where $G(u)=\int_0^u g(s)ds$. The Legendre spectral method is used to discrete the space direction, the fully discrete scheme of problem (1.1) is: Find $u_N^n \in P_N^0$, such that

$$\begin{aligned}&\frac{1}{\mu}\left(b_0\delta_t u_N^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\delta_t u_N^{k-\frac{1}{2}}-b_{n-1}\psi,v\right)\\ &+\frac{1}{\lambda}\left(\frac{a_0}{2}\delta_t u_N^{n-\frac{1}{2}}+\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}\delta_t u_N^{k-\frac{1}{2}},v\right)\\ &+(u_{N_x}^{n-\frac{1}{2}},v_x)+\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}},v\right)\\ =&(f^{n-\frac{1}{2}},v),\ \forall v\in P_N^0,\\ u_N^0 =&\pi_N^{1,0}u^0,\end{aligned}\tag{3.1}$$

where $\lambda=\tau^{\alpha-1}\Gamma(2-\alpha)$ and $\mu=\tau^{\gamma-1}\Gamma(3-\gamma)$.

4. Existence and uniqueness of the fully discrete scheme

In order to analyze the existence and uniqueness of the numerical solutions, we first introduce the following Brouwer fixed point theorem and a priori estimate of the numerical solution.

Lemma 4.1. [23] *Let X be a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and let P be a continuous mapping from X into itself such that*

$$(P(\xi),\xi)>0,\ for\ \|\xi\|=k>0.$$

Then there exists $\xi\in X,\ \|\xi\|\leq k$, such that $P(\xi)=0$.

Lemma 4.2. *Suppose that $\{u_N^j\}_{j=0}^{n-1}$ be the solution of problem (3.1), if $\tau\leq\tau_0,\ \tau_0=\left(\frac{T^{1-\gamma}\Gamma(2-\alpha)}{2\Gamma(2-\gamma)}\right)^{\frac{1}{1-\alpha}}$, then $u_{N_x}^n$ satisfies*

$$\|u_{N_x}^n\|^2\leq A,\quad 1\leq n\leq M,$$

where $A>0$ is independent of M.

Proof. Taking $v=\delta_t u_N^{n-\frac{1}{2}}$ and summing for n from 1 to m in (3.1), then

$$\begin{aligned}
&\frac{1}{\mu}\sum_{n=1}^{m}\left(b_0\delta_t u_N^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\delta_t u_N^{k-\frac{1}{2}}-b_{n-1}\psi,\delta_t u_N^{n-\frac{1}{2}}\right)\\
&+\frac{1}{\lambda}\sum_{n=1}^{m}\left(\frac{a_0}{2}\delta_t u_N^{n-\frac{1}{2}}+\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}\delta_t u_N^{k-\frac{1}{2}},\delta_t u_N^{n-\frac{1}{2}}\right)+\sum_{n=1}^{m}(u_{N_x}^{n-\frac{1}{2}},\delta_t u_{N_x}^{n-\frac{1}{2}})\\
=&-\sum_{n=1}^{m}\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}},\delta_t u_N^{n-\frac{1}{2}}\right)+\sum_{n=1}^{m}(f^{n-\frac{1}{2}},\delta_t u_N^{n-\frac{1}{2}}).
\end{aligned}\tag{4.1}$$

For the first and second terms on the left of (4.1), according to Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{\mu}\sum_{n=1}^{m}\left(b_0\delta_t u_N^{n-\frac{1}{2}}-\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\delta_t u_N^{k-\frac{1}{2}}-b_{n-1}\psi,\delta_t u_N^{n-\frac{1}{2}}\right)\\
\geq&\frac{1}{2\mu}\left(\sum_{k=1}^{m}b_{m-k}\|\delta_t u_N^{k-\frac{1}{2}}\|^2-\sum_{n=1}^{m}b_{n-1}\|\psi\|^2\right),
\end{aligned}\tag{4.2}$$

and

$$\frac{1}{\lambda}\sum_{n=1}^{m}\left(\frac{a_0}{2}\delta_t u_N^{n-\frac{1}{2}}+\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}\delta_t u_N^{k-\frac{1}{2}},\delta_t u_N^{n-\frac{1}{2}}\right)\geq-\frac{1}{2\lambda}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2,\tag{4.3}$$

respectively.

For the third term on the left side of equation (4.1), using the Hölder's and Young's inequalities, we get

$$\sum_{n=1}^{m}(u_{N_x}^{n-\frac{1}{2}},\delta_t u_{N_x}^{n-\frac{1}{2}})=\frac{1}{2\tau}(\|u_{N_x}^m\|^2-\|u_{N_x}^0\|^2).\tag{4.4}$$

The first term on the right side of equation (4.1) is estimated as follows, it holds that

$$\begin{aligned}
\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}}=&\frac{1}{u_N^n-u_N^{n-1}}\int_{u_N^{n-1}}^{u_N^n}g(s)ds\\
=&\int_0^1 g(u_N^{n-1}+\theta(u_N^n-u_N^{n-1}))d\theta.
\end{aligned}\tag{4.5}$$

By mean value theorem, $g(0)=0$ and (4.5), yields

$$-\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}},\delta_t u_N^{n-\frac{1}{2}}\right)=-\left(\int_0^1 g'(\xi)\left(u_N^{n-1}+\theta(u_N^n-u_N^{n-1})\right)d\theta,\delta_t u_N^{n-\frac{1}{2}}\right).\tag{4.6}$$

According to $g'>0$, the Hölder's and Young's inequalities, we can infer that

$$\begin{aligned}
&-\left(\int_0^1 g'(\xi)\left(\theta(u_N^n-u_N^{n-1})\right)d\theta,\delta_t u_N^{n-\frac{1}{2}}\right)\leq 0,\\
&-\left(\int_0^1 g'(\xi)u_N^{n-1}d\theta,\delta_t u_N^{n-\frac{1}{2}}\right)\leq c\frac{\mu}{b_{m-1}}\|u_N^{n-1}\|^2+\frac{b_{m-1}}{8\mu}\|\delta_t u_N^{n-\frac{1}{2}}\|^2.
\end{aligned}$$

Substituting above two inequalities into (4.6), we get

$$\sum_{n=1}^{m}\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}},\delta_t u_N^{n-\frac{1}{2}}\right)\le c\frac{\mu}{b_{m-1}}\sum_{n=1}^{m}\|u_N^{n-1}\|^2+\frac{b_{m-1}}{8\mu}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2. \tag{4.7}$$

To the second term on the right of (4.1), using the Hölder's and Young's inequalities, we obtain

$$\sum_{n=1}^{m}\left(f^{n-\frac{1}{2}},\delta_t u_N^{n-\frac{1}{2}}\right)\le \frac{2\mu}{b_{m-1}}\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2+\frac{b_{m-1}}{8\mu}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2. \tag{4.8}$$

Taking (4.2)-(4.4) and(4.7)-(4.8) into (4.1), we conclude that

$$\begin{aligned}
&\frac{1}{2\mu}\left(\sum_{k=1}^{m}b_{m-k}\|\delta_t u_N^{k-\frac{1}{2}}\|^2-\sum_{n=1}^{m}b_{n-1}\|\psi\|^2\right)-\frac{1}{2\lambda}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2\\
&+\frac{1}{2\tau}(\|u_{N_x}^m\|^2-\|u_{N_x}^0\|^2)\\
\le&c\frac{\mu}{b_{m-1}}\sum_{n=1}^{m}\|u_N^{n-1}\|^2+\frac{b_{m-1}}{8\mu}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2+\frac{2\mu}{b_{m-1}}\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2\\
&+\frac{b_{m-1}}{8\mu}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2,
\end{aligned} \tag{4.9}$$

the inequality (4.9) can be rewritten as

$$\begin{aligned}
&\frac{b_{m-1}}{4\mu}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2-\frac{1}{2\lambda}\sum_{n=1}^{m}\|\delta_t u_N^{n-\frac{1}{2}}\|^2+\frac{1}{2\tau}\|u_{N_x}^m\|^2\\
\le&\frac{1}{2\mu}\sum_{n=1}^{m}b_{n-1}\|\psi\|^2+\frac{1}{2\tau}\|u_{N_x}^0\|^2+c\frac{\mu}{b_{m-1}}\sum_{n=1}^{m}\|u_N^{n-1}\|^2+\frac{2\mu}{b_{m-1}}\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2.
\end{aligned} \tag{4.10}$$

Using Poincaré's inequality, we derive that

$$\|u_{N_x}^m\|^2\le\frac{\tau m}{\mu}\|\psi\|^2+\|u_{N_x}^0\|^2+\frac{c\mu\tau}{b_{m-1}}\sum_{n=1}^{m}\|u_{N_x}^{n-1}\|^2+\frac{4\tau\mu}{b_{m-1}}\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2.$$

By Lemma 2.3, we can obtain

$$\|u_{N_x}^m\|^2\le\left(\frac{T}{\mu}\|\psi\|^2+\frac{4\mu T}{b_{m-1}}\sum_{n=1}^{m}\|f^{n-\frac{1}{2}}\|^2\right)\exp\left(1+\frac{c\mu T}{b_{m-1}}\right)\triangleq A,$$

where A is a positive constant.

The proof is complete. □

Next, we analyze the existence of the numerical solution.

Theorem 4.1. *Suppose that* $\{u_N^j\}_{j=0}^{n-1}$ *are given, there exists the solution* u_N^n *satisfying* (3.1).

Proof. Define a mapping $Z: P_N^0(\Omega) \to P_N^0(\Omega)$, for all $w \in P_N^0(\Omega)$, such that:

$$\begin{aligned}(Z(w),v) =&\frac{b_0}{\mu}(\frac{w}{\tau},v) - \frac{1}{\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\delta_t u_N^{k-\frac{1}{2}},v) - \frac{b_{n-1}}{\mu}(\psi,v) + \frac{1}{2\lambda}(\frac{w}{\tau},v)\\ &+\frac{1}{\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}(\delta_t u_N^{k-\frac{1}{2}},v) + \frac{1}{2}(w_x,v_x) + (u_{N_x}^{n-1},v_x)\\ &+\left(\frac{G(w+u_N^{n-1})-G(u_N^{n-1})}{w},v\right) - (f^{n-\frac{1}{2}},v),\ \forall v \in P_N^0.\end{aligned} \tag{4.11}$$

Taking $v = w$, then

$$\begin{aligned}(Z(w),w) =&\frac{1}{\mu\tau}\|w\|^2 + \frac{1}{2\lambda\tau}\|w\|^2 + \frac{1}{2}\|w_x\|^2 + (u_{N_x}^{n-1},w_x)\\ &+\frac{1}{\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}(\delta_t u_N^{k-\frac{1}{2}},w) - \frac{1}{\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\delta_t u_N^{k-\frac{1}{2}},w)\\ &-\frac{b_{n-1}}{\mu}(\psi,w) + \left(\frac{G(w+u_N^{n-1})-G(u_N^{n-1})}{w},w\right) - (f^{n-\frac{1}{2}},w).\end{aligned} \tag{4.12}$$

Using the Hölder's and Young's inequalities, we obtain

$$(u_{N_x}^{n-1},w_x) \geq -\frac{1}{2}\|u_{N_x}^{n-1}\|^2 - \frac{1}{2}\|w_x\|^2, \tag{4.13}$$

$$\frac{1}{\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}(\delta_t u_N^{k-\frac{1}{2}},w) \geq -\frac{2}{\lambda\tau}\|u_N^{n-1}\|^2 - \frac{2}{\lambda\tau}\|u_N^0\|^2 - \frac{1}{4\lambda\tau}\|w\|^2, \tag{4.14}$$

$$-\frac{1}{\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\delta_t u_N^{k-\frac{1}{2}},w) \geq -\frac{2\lambda}{\mu^2\tau}\|u_N^{n-1}\|^2 - \frac{2\lambda}{\mu^2\tau}\|u_N^0\|^2 - \frac{1}{4\lambda\tau}\|w\|^2, \tag{4.15}$$

and

$$-\frac{b_{n-1}}{\mu}(\psi,w) \geq -\frac{\tau}{\mu}\|\psi\|^2 - \frac{1}{4\mu\tau}\|w\|^2. \tag{4.16}$$

By (4.5) and mean value theorem, it holds

$$\left(\frac{G(w+u_N^{n-1})-G(u_N^{n-1})}{w},w\right) = \left(\int_0^1 g'(\xi)\left(u_N^{n-1}+\theta w\right)d\theta,w\right). \tag{4.17}$$

Therefore, we have

$$\begin{aligned}&\left(\int_0^1 g'(\xi)\theta w d\theta,w\right) \geq 0,\\ &\left(\int_0^1 g'(\xi)u_N^{n-1}d\theta,w\right) \leq c\mu\tau\|u_N^{n-1}\|^2 + \frac{1}{4\mu\tau}\|w\|^2.\end{aligned}$$

Substituting the above inequalities into (4.17), we deduce that

$$\left(\frac{G(w+u_N^{n-1})-G(u_N^{n-1})}{w},w\right) \geq -c\mu\tau\|u_N^{n-1}\|^2 - \frac{1}{4\mu\tau}\|w\|^2. \tag{4.18}$$

Similarly, the following inequality can be obtained

$$-(f^{n-\frac{1}{2}}, w) \geq -\mu\tau \|f^{n-\frac{1}{2}}\|^2 - \frac{1}{4\mu\tau}\|w\|^2. \tag{4.19}$$

Taking (4.13)-(4.16) and (4.18)-(4.19) into (4.12), then

$$\begin{aligned}(Z(w), w) \geq & \frac{1}{4\mu\tau}\|w\|^2 - \frac{1}{2}\|u_{N_x}^{n-1}\|^2 - \frac{2}{\lambda\tau}\|u_N^{n-1}\|^2 - \frac{2}{\lambda\tau}\|u_N^0\|^2 - \frac{2\lambda}{\mu^2\tau}\|u_{N_x}^{n-1}\|^2 \\ & - \frac{2\lambda}{\mu^2\tau}\|u_{N_x}^0\|^2 - \frac{\tau}{\mu}\|\psi\|^2 - c\mu\tau\|u_N^{n-1}\|^2 - \mu\tau\|f^{n-\frac{1}{2}}\|^2.\end{aligned} \tag{4.20}$$

By the Poincaré's inequality and Lemma 4.2, we deduce that

$$\begin{aligned}(Z(w), w) \geq & \frac{1}{4\mu\tau}\|w\|^2 - \frac{2\mu^2 + 2\lambda^2}{\mu^2\lambda\tau}\|u_N^0\|^2 - \frac{\mu^2\lambda\tau + 4\mu^2 + 4\lambda^2 + c\mu^3\lambda\tau^2}{2\mu^2\lambda\tau}A \\ & - \frac{\tau}{\mu}\|\psi\|^2 - \mu\tau\|f^{n-\frac{1}{2}}\|^2.\end{aligned}$$

Denote

$$\begin{aligned}K \geq & \Big(\frac{4(2\mu^2 + 2\lambda^2)}{\mu\lambda}\|u_N^0\|^2 + \frac{2(\mu^2\lambda\tau + 4\mu^2 + 4\lambda^2 + c\mu^3\lambda\tau^2)}{\mu\lambda}A + 4\tau^2\|\psi\|^2 \\ & + 4\mu^2\tau^2\|f^{n-\frac{1}{2}}\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Then $(Z(w), w) \geq 0$, if $\|w\| = K$. According to Lemma 4.1, there is $w^{n-1} \in P_N^0$, $\|w^{n-1}\| > 0$ satisfying $Z(w^{n-1}) = 0$. Let $u_N^n = w^{n-1} + u_N^{n-1}$, then u_N^n exists. □

We give the uniqueness analysis as follows.

Theorem 4.2. *The solution u_N^n of problem (3.1) is unique.*

Proof. Suppose $\{u_N^n\}$ and $\{\tilde{u}_N^n\}$ are the solutions of problem (3.1) with the same initial value $u_N^0 = \tilde{u}_N^0$. Denote $\eta_N^n = u_N^n - \tilde{u}_N^n$, we have

$$\begin{aligned}& \frac{1}{\mu}\left(b_0\delta_t\eta_N^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\delta_t\eta_N^{k-\frac{1}{2}}, v\right) \\ & + \frac{1}{\lambda}\left(\frac{a_0}{2}\delta_t\eta_N^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\frac{a_{n-k} + a_{n-k-1}}{2}\delta_t\eta_N^{k-\frac{1}{2}}, v\right) \\ & + (\eta_{N_x}^{n-\frac{1}{2}}, v_x) + \left(\frac{G(u_N^n) - G(u_N^{n-1})}{u_N^n - u_N^{n-1}} - \frac{G(\tilde{u}_N^n) - G(\tilde{u}_N^{n-1})}{\tilde{u}_N^n - \tilde{u}_N^{n-1}}, v\right) = 0.\end{aligned}$$

Taking $v = \delta_t\eta_N^{n-\frac{1}{2}}$ in above equation, we get

$$\begin{aligned}& \frac{1}{\mu}\|\delta_t\eta_N^{n-\frac{1}{2}}\|^2 - \frac{1}{\mu}\sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})(\delta_t\eta_N^{k-\frac{1}{2}}, \delta_t\eta_N^{n-\frac{1}{2}}) + \frac{1}{2\lambda}\|\delta_t\eta_N^{n-\frac{1}{2}}\|^2 \\ & + \frac{1}{\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k} + a_{n-k-1}}{2}(\delta_t\eta_N^{k-\frac{1}{2}}, \delta_t\eta_N^{n-\frac{1}{2}}) + \frac{1}{2\tau}\|\eta_{N_x}^n\|^2 - \frac{1}{2\tau}\|\eta_{N_x}^{n-1}\|^2 \\ & + \left(\frac{G(u_N^n) - G(u_N^{n-1})}{u_N^n - u_N^{n-1}} - \frac{G(\tilde{u}_N^n) - G(\tilde{u}_N^{n-1})}{\tilde{u}_N^n - \tilde{u}_N^{n-1}}, \delta_t\eta_N^{n-\frac{1}{2}}\right) = 0.\end{aligned} \tag{4.21}$$

Using the Hölder's and Young's inequalities, then we obtain

$$\frac{1}{\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\delta_t\eta_N^{k-\frac{1}{2}},\delta_t\eta_N^{n-\frac{1}{2}}) \le \frac{1}{\mu\tau^2}\|\eta_N^{n-1}\|^2+\frac{1}{2\mu}\|\delta_t\eta_N^{n-\frac{1}{2}}\|^2+\frac{1}{\mu\tau^2}\|\eta_N^0\|^2, \tag{4.22}$$

and

$$-\frac{1}{\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}(\delta_t\eta_N^{k-\frac{1}{2}},\delta_t\eta_N^{n-\frac{1}{2}}) \le \frac{1}{\tau^2\lambda}\|\eta_N^{n-1}\|^2+\frac{1}{2\lambda}\|\delta_t\eta_N^{n-\frac{1}{2}}\|^2+\frac{1}{\tau^2\lambda}\|\eta_N^0\|^2. \tag{4.23}$$

By (4.5) and mean value theorem, it holds

$$\begin{aligned}&\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}}-\frac{G(\tilde{u}_N^n)-G(\tilde{u}_N^{n-1})}{\tilde{u}_N^n-\tilde{u}_N^{n-1}},\delta_t\eta_N^{n-\frac{1}{2}}\right)\\ =&\left(\int_0^1 g'(\xi)(\eta_N^{n-1}+\theta(\eta_N^n-\eta_N^{n-1}))d\theta,\delta_t\eta_N^{n-\frac{1}{2}}\right).\end{aligned}$$

Then we can obtain

$$-(\int_0^1 g'(\xi)(\theta(\eta_N^n-\eta_N^{n-1}))d\theta,\delta_t\eta_N^{n-\frac{1}{2}}) \le 0,$$

and

$$-(\int_0^1 g'(\xi)\eta_N^{n-1}d\theta,\delta_t\eta_N^{n-\frac{1}{2}}) \le c\mu\|\eta_N^{n-1}\|^2+\frac{1}{2\mu}\|\delta_t\eta_N^{n-\frac{1}{2}}\|^2.$$

Further we can infer that

$$\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}}-\frac{G(\tilde{u}_N^n)-G(\tilde{u}_N^{n-1})}{\tilde{u}_N^n-\tilde{u}_N^{n-1}},\delta_t\eta_N^{n-\frac{1}{2}}\right) \le c\mu\|\eta_N^{n-1}\|^2+\frac{1}{2\mu}\|\delta_t\eta_N^{n-\frac{1}{2}}\|^2. \tag{4.24}$$

Substituting (4.22)-(4.24) into (4.21), then

$$\|\eta_{N_x}^n\|^2 \le (\frac{2}{\mu\tau}+\frac{2}{\tau\lambda}+c\tau\mu)\|\eta_N^{n-1}\|^2+(\frac{2}{\mu\tau}+\frac{2}{\tau\lambda})\|\eta_N^0\|^2+\|\eta_{N_x}^{n-1}\|^2. \tag{4.25}$$

By Poincaré's inequality, we deduce that

$$\|\eta_{N_x}^n\|^2 \le (1+\frac{1}{\mu\tau}+\frac{1}{\tau\lambda}+c\tau\mu)\|\eta_{N_x}^{n-1}\|^2+(\frac{1}{\mu\tau}+\frac{1}{\tau\lambda})\|\eta_{N_x}^0\|^2. \tag{4.26}$$

Using Lemma 2.3, we have

$$\|\eta_{N_x}^n\|^2 \le 0\cdot\exp(1+\frac{2}{\mu\tau}+\frac{2}{\tau\lambda}+c\tau\mu)=0.$$

Applying Poincaré's inequality again, we have $\eta_N^n=0$, then $u_N^n=\tilde{u}_N^n$. Then the solution is unique. The proof is complete. □

5. Stability and convergence of the fully discrete scheme

Suppose $\{\hat{u}_N^n\}_{n=0}^{M-1} \in P_N^0(\Omega)$, for any $v \in P_N^0(\Omega)$,

$$\begin{aligned}
&\frac{1}{\mu}\left(b_0\delta_t\hat{u}_N^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\delta_t\hat{u}_N^{k-\frac{1}{2}} - b_{n-1}\hat{\psi}, v\right)\\
&+\frac{1}{\lambda}\left(\frac{a_0}{2}\delta_t\hat{u}_N^{n-\frac{1}{2}} + \sum_{k=1}^{n-1}\frac{a_{n-k}+a_{n-k-1}}{2}\delta_t\hat{u}_N^{k-\frac{1}{2}}, v\right)\\
&+\left(\hat{u}_{N_x}^{n-\frac{1}{2}}, v_x\right) + \left(\frac{G(\hat{u}_N^n)-G(\hat{u}_N^{n-1})}{\hat{u}_N^n-\hat{u}_N^{n-1}}, v\right) = (\hat{f}^{n-\frac{1}{2}}, v),\\
&\hat{u}_N^0 = \pi_N^{1,0}\hat{u}_0.
\end{aligned} \tag{5.1}$$

Following the same lines as in the proof of Theorem 4.2, we can obtain the stability of the fully discrete scheme.

Theorem 5.1. *Suppose $\{u_N^n\}_{k=0}^{M-1}$ is solution of (3.1), $\{\hat{u}_N^n\}_{k=0}^{M-1}$ is solution of (5.1), it holds*

$$\|u_{N_x}^n - \hat{u}_{N_x}^n\|^2 \le \exp\left(1+\frac{4}{\mu\tau}+\frac{4}{\tau\lambda}+c\tau\mu\right)\left(\frac{2T}{\mu}\|\psi-\hat{\psi}\|^2 + 2T\mu\|f^{n-\frac{1}{2}}-\hat{f}^{n-\frac{1}{2}}\|^2\right).$$

Next, the convergence analysis of the fully discrete scheme (3.1) is given.

Theorem 5.2. *Let u be exact solution of (1.1), u_N^n is solution of (3.1). Assume that $u \in L^\infty(0,T;H^s(\Omega))$, ${}_0^C D_t^\alpha u \in L^\infty(0,T;H^s(\Omega))$, ${}_0^C D_t^\gamma u \in L^\infty(0,T;H^s(\Omega))$, $\psi \in H^s(\Omega)$, where $s \ge 1$. Thus there exists a positive constant C independent of τ and N, such that*

$$\|u^n - u_N^n\|^2 \le C(\tau^{\min\{6-2\gamma,4-2\alpha\}} + N^{2-2s}).$$

Proof. Let $u^n - u_N^n = (u^n - \pi_N^{1,0}u^n) + (\pi_N^{1,0}u^n - u_N^n) = \hat{e}_N^n + \tilde{e}_N^n$. By the original equation (1.1) and the fully discrete scheme (3.1), we have the following error equation

$$\begin{aligned}
&\frac{\tau}{\mu}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2 + \frac{\tau}{2\lambda}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2 + \frac{1}{2}\|\tilde{e}_{N_x}^n\|^2 + \tau\left(\frac{g(u^n)+g(u^{n-1})}{2}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}\right)\\
&-\tau\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}\right) = \frac{1}{2}\|\tilde{e}_{N_x}^{n-1}\|^2 + \sum_{i=1}^{5} r_i,
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
r_1 &= \tau(D_t^\gamma\pi_N^{1,0}u^{n-\frac{1}{2}}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}) - \tau({}_0^C D_t^\gamma u^{n-\frac{1}{2}}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}),\\
r_2 &= \tau(D_t^\alpha\pi_N^{1,0}u^{n-\frac{1}{2}}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}) - \tau({}_0^C D_t^\alpha u^{n-\frac{1}{2}}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}),\\
r_3 &= \frac{\tau}{\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})(\delta_t\tilde{e}_N^{k-\frac{1}{2}}, \delta_t\tilde{e}_N^{n-\frac{1}{2}}),\\
r_4 &= \frac{\tau b_{n-1}}{\mu}(\pi_N^{1,0}\psi - \psi, \delta_t\tilde{e}_N^{n-\frac{1}{2}}),
\end{aligned}$$

and

$$r_5=-\frac{\tau}{\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}-a_{n-k-1}}{2}(\delta_t\tilde{e}_N^{k-\frac{1}{2}},\delta_t\tilde{e}_N^{n-\frac{1}{2}}).$$

For nonlinear terms, by (4.5) and mean value theorem, the following inequality holds

$$\begin{aligned}&\tau\left(\frac{g(u^n)+g(u^{n-1})}{2},\delta_t\tilde{e}_N^{n-\frac{1}{2}}\right)-\tau\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}},\delta_t\tilde{e}_N^{n-\frac{1}{2}}\right)\\ \leq&\frac{c}{2}(\hat{e}_N^n+\tilde{u}_N^n,\delta_t\tilde{e}_N^{n-\frac{1}{2}})+\frac{c}{2}(\hat{e}_N^{n-1}+\tilde{u}_N^{n-1},\delta_t\tilde{e}_N^{n-\frac{1}{2}}).\end{aligned}\tag{5.3}$$

Applying the Hölder's and Young's inequalities, we derive that

$$\begin{aligned}&\frac{c}{2}(\hat{e}_N^n+\tilde{u}_N^n,\delta_t\tilde{e}_N^{n-\frac{1}{2}})+\frac{c}{2}(\hat{e}_N^{n-1}+\tilde{u}_N^{n-1},\delta_t\tilde{e}_N^{n-\frac{1}{2}})\\ \leq&c\tau\|\hat{e}_N^n\|^2+c\tau\|\tilde{e}_N^n\|^2+c\tau\|\hat{e}_N^{n-1}\|^2+c\tau\|\tilde{e}_N^{n-1}\|^2+\frac{3\tau b_{n-1}}{8\mu}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2.\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}&\tau\left(\frac{g(u^n)+g(u^{n-1})}{2},\delta_t\tilde{e}_N^{n-\frac{1}{2}}\right)-\tau\left(\frac{G(u_N^n)-G(u_N^{n-1})}{u_N^n-u_N^{n-1}},\delta_t\tilde{e}_N^{n-\frac{1}{2}}\right)\\ \leq&c\tau\|\tilde{e}_N^n\|^2+c\tau\|\tilde{e}_N^{n-1}\|^2+cN^{-2s}\|u\|^2_{L^\infty(0,T;H^s)}+\frac{3\tau b_{n-1}}{8\mu}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2.\end{aligned}\tag{5.4}$$

Now, we estimate $r_i(i=1,2,...,5)$.

Due to Hölder's and Young's inequalities, then

$$r_1\leq\frac{2\tau\mu}{b_{n-1}}\|D_t^\gamma\pi_N^{1,0}u^{n-\frac{1}{2}}-{}_0^CD_t^\gamma u^{n-\frac{1}{2}}\|^2+\frac{\tau b_{n-1}}{8\mu}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2,\tag{5.5}$$

$$r_2\leq\frac{3\tau\lambda}{2}\|D_t^\alpha\pi_N^{1,0}u^{n-\frac{1}{2}}-{}_0^CD_t^\alpha u^{n-\frac{1}{2}}\|^2+\frac{\tau}{6\lambda}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2,\tag{5.6}$$

$$r_3\leq\frac{\tau}{2\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\|\delta_t\tilde{e}_N^{k-\frac{1}{2}}\|^2+\frac{\tau}{2\mu}(b_0-b_{n-1})\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2,\tag{5.7}$$

$$r_4\leq\frac{3\tau b_{n-1}^2\lambda}{2\mu^2}\|\pi_N^{1,0}\psi-\psi\|^2+\frac{\tau}{6\lambda}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2,\tag{5.8}$$

and

$$r_5\leq\frac{3\tau}{2\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}-a_{n-k-1}}{2}\|\delta_t\tilde{e}_N^{k-\frac{1}{2}}\|^2+\frac{\tau}{6\lambda}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2,\tag{5.9}$$

respectively.

Taking (5.4)-(5.9) into (5.2), we conclude that

$$\begin{aligned}&\frac{\tau}{2\mu}\|\delta_t\tilde{e}_N^{n-\frac{1}{2}}\|^2+\frac{1}{2}\|\tilde{e}_{N_x}^n\|^2\\ \leq&\frac{1}{2}\|\tilde{e}_{N_x}^{n-1}\|^2+c\tau(\|\tilde{e}_N^n\|^2+\|\tilde{e}_N^{n-1}\|^2)+\frac{\tau}{2\mu}\sum_{k=1}^{n-1}(b_{n-k-1}-b_{n-k})\|\delta_t\tilde{e}_N^{k-\frac{1}{2}}\|^2\\ &+\frac{3\tau}{2\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k}-a_{n-k-1}}{2}\|\delta_t\tilde{e}_N^{k-\frac{1}{2}}\|^2+F,\end{aligned}\tag{5.10}$$

where

$$F = cN^{-2s}\|u\|^2_{L^\infty(0,T;H^s)} + \frac{2\tau\mu}{b_{n-1}}\|D_t^\gamma \pi_N^{1,0} u^{n-\frac{1}{2}} - {}_0^C D_t^\gamma u^{n-\frac{1}{2}}\|^2 + \frac{3\tau\lambda}{2}\|D_t^\alpha \pi_N^{1,0} u^{n-\frac{1}{2}} - {}_0^C D_t^\alpha u^{n-\frac{1}{2}}\|^2 + \frac{3\tau b_{n-1}^2\lambda}{2\mu^2}\|\pi_N^{1,0}\psi - \psi\|^2.$$

By means of $\sum_{k=1}^{n-1}(b_{n-k-1} - b_{n-k})\|\delta_t \tilde{e}_N^{k-\frac{1}{2}}\|^2 = \sum_{k=0}^{n-2} b_k\|\delta_t \tilde{e}_N^{n-1-k-\frac{1}{2}}\|^2 - \sum_{k=0}^{n-1} b_k\|\delta_t \tilde{e}_N^{n-k-\frac{1}{2}}\|^2$, yields

$$\begin{aligned} &\frac{\tau}{2\mu}\sum_{k=0}^{n-1} b_k\|\delta_t \tilde{e}_N^{n-k-\frac{1}{2}}\|^2 + \frac{1}{2}\|\tilde{e}_{N_x}^n\|^2 \\ \le &\frac{1}{2}\|\tilde{e}_{N_x}^{n-1}\|^2 + c\tau(\|\tilde{e}_N^n\|^2 + \|\tilde{e}_N^{n-1}\|^2) + \frac{\tau}{2\mu}\sum_{k=0}^{n-2} b_k\|\delta_t \tilde{e}_N^{n-1-k-\frac{1}{2}}\|^2 \\ &+ \frac{3\tau}{2\lambda}\sum_{k=1}^{n-1}\frac{a_{n-k} - a_{n-k-1}}{2}\|\delta_t \tilde{e}_N^{k-\frac{1}{2}}\|^2 + F. \end{aligned} \tag{5.11}$$

By (5.11), we get

$$\frac{1}{2}\|\tilde{e}_{N_x}^n\|^2 \le \frac{1}{2}\|\tilde{e}_{N_x}^{n-1}\|^2 + c\tau\|\tilde{e}_N^n\|^2 + c\tau\|\tilde{e}_N^{n-1}\|^2 + F + \tau^{-1}\sum_{k=0}^{n-1} l_k\|\tilde{e}_N^k\|^2, \tag{5.12}$$

where $l_0 = \frac{3}{\lambda}$, $l_1 = \frac{6}{\lambda} + \frac{1}{\mu}$, $l_i = \frac{6}{\lambda} + \frac{2}{\mu}(i = 2, ..., n-2)$, $l_{n-1} = \frac{3}{\lambda} + \frac{1}{\mu}$.

By the Poincaré's inequality, we deduce that

$$\|\tilde{e}_{N_x}^n\|^2 \le \sum_{k=0}^{n} y_k\|\tilde{e}_{N_x}^k\|^2 + 2F,$$

where $y_0 = 2\tau^{-1}l_0$, $y_1 = 2\tau^{-1}l_1$, $y_i = 2\tau^{-1}l_i(i = 2, ..., n-2)$, $y_{n-1} = 1 + cT + 2\tau^{-1}l_{n-1}$, $y_n = cT$.

According to Lemma 2.4, we obtain

$$\|\tilde{e}_{N_x}^n\|^2 \le \exp(\sum_{k=0}^{n}\delta_\mu y_k)\cdot 2F. \tag{5.13}$$

Applying Lemma 2.1, we derive that

$$\begin{aligned} &\|D_t^\alpha \pi_N^{1,0} u^{n-\frac{1}{2}} - {}_0^C D_t^\alpha u^{n-\frac{1}{2}}\| \\ =&\|(D_t^\alpha u^{n-\frac{1}{2}} - {}_0^C D_t^\alpha u^{n-\frac{1}{2}}) + (\pi_N^{1,0}\, {}_0^C D_t^\alpha u^{n-\frac{1}{2}} - {}_0^C D_t^\alpha u^{n-\frac{1}{2}}) \\ &+ [\pi_N^{1,0}(D_t^\alpha u^{n-\frac{1}{2}} - {}_0^C D_t^\alpha u^{n-\frac{1}{2}}) - (D_t^\alpha u^{n-\frac{1}{2}} - {}_0^C D_t^\alpha u^{n-\frac{1}{2}})]\| \\ \le& c\tau^{2-\alpha} + cN^{-s}\|{}_0^C D_t^\alpha u^{n-\frac{1}{2}}\|_{L^\infty(0,T;H^s)}. \end{aligned} \tag{5.14}$$

Similarly, we have

$$\|D_t^\gamma \pi_N^{1,0} u^{n-\frac{1}{2}} - {}_0^C D_t^\gamma u^{n-\frac{1}{2}}\| \le c\tau^{3-\gamma} + cN^{-s}\|{}_0^C D_t^\gamma u^{n-\frac{1}{2}}\|_{L^\infty(0,T;H^s)}, \tag{5.15}$$

and

$$\|\pi_N^{1,0}\psi - \psi\| \leq cN^{-s}\|\psi\|_{H^s}. \tag{5.16}$$

Combining (5.14)-(5.16), the inequality (5.13) can be rewritten as

$$\begin{aligned}\|\tilde{e}_{N_x}^n\|^2 \leq& \exp(\sum_{k=0}^{n}\delta_\mu y_k)\{cN^{-2s}\|u\|^2_{L^\infty(0,T;H^s)} + c\tau^{6-2\gamma} + c\tau^{4-2\alpha} \\ &+ cN^{-2s}\|{}_0^CD_t^\gamma u^{n-\frac{1}{2}}\|^2_{L^\infty(0,T;H^s)} + cN^{-2s}\|{}_0^CD_t^\alpha u^{n-\frac{1}{2}}\|^2_{L^\infty(0,T;H^s)} \\ &+ cN^{-2s}\|\psi\|^2_{H^s}\}.\end{aligned} \tag{5.17}$$

By Lemma 2.1, we have

$$\|\hat{e}_{N_x}^n\|^2 \leq cN^{2-2s}\|u\|^2_{L^\infty(0,T;H^s)}. \tag{5.18}$$

Using the triangular inequality $\|u_x^n - u_{N_x}^n\| \leq \|\hat{e}_{N_x}^n\| + \|\tilde{e}_{N_x}^n\|$, we get

$$\begin{aligned}&\|u_x^n - u_{N_x}^n\|^2 \\ \leq& \exp(\sum_{k=0}^{n}\delta_\mu y_k)\{cN^{-2s}\|u\|^2_{L^\infty(0,T;H^s)} + c\tau^{6-2\gamma} + c\tau^{4-2\alpha} \\ &+ cN^{-2s}\|{}_0^CD_t^\gamma u^{n-\frac{1}{2}}\|^2_{L^\infty(0,T;H^s)} + cN^{-2s}\|{}_0^CD_t^\alpha u^{n-\frac{1}{2}}\|^2_{L^\infty(0,T;H^s)} \\ &+ cN^{-2s}\|\psi\|^2_{H^s}\} + cN^{2-2s}\|u\|^2_{L^\infty(0,T;H^s)} \\ =& C(\tau^{\min\{6-2\gamma,4-2\alpha\}} + cN^{2-2s}).\end{aligned} \tag{5.19}$$

According to Poincaré's inequality, it holds that

$$\|u^n - u_N^n\|^2 \leq C(\tau^{\min\{6-2\gamma,4-2\alpha\}} + cN^{2-2s}).$$

The proof is complete. □

6. Numerical experiments

In this section, we give two numerical examples with smooth and non-smooth solutions to verify the theoretical results. $Error_2 := \max\|u^n - u_N^n\|$ and $Error_\infty := \max\|u^n - u_N^n\|_\infty$ denotes the max L^2-norm error and the max L^∞-norm error between the exact solution and numerical solution, respectively. The rate of convergence is $Rate_2 := \log_2 \frac{Error_2(M)}{Error_2(2M)}$.

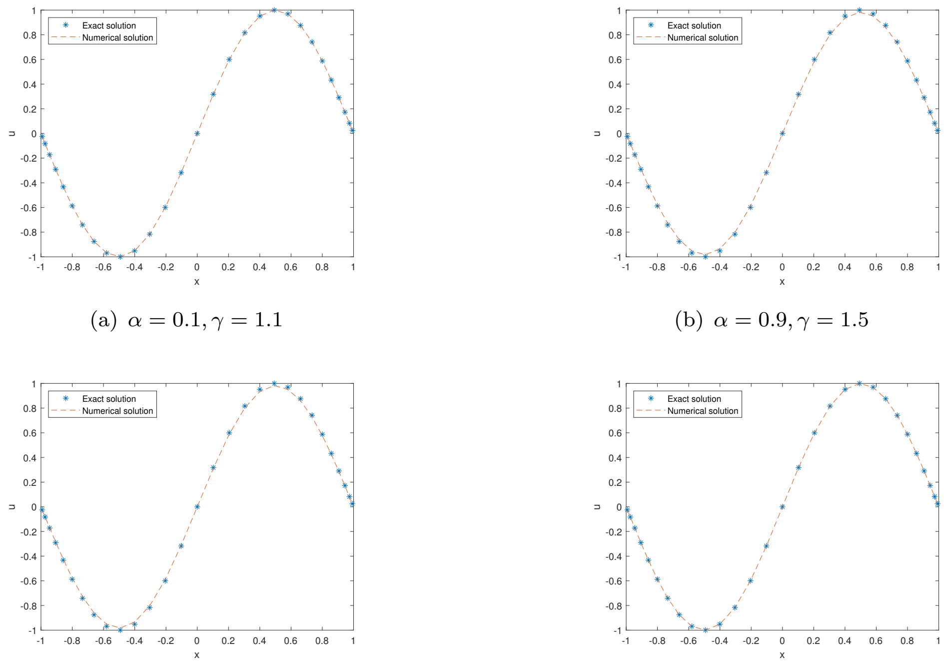
Example 6.1. Consider the problem (1.1) with $u_t(x,0) = \psi(x)$, $g(u) = u^3 - u$. The exact solution is

$$u(x,t) = t^3\sin(\pi x),$$

the source term is

$$\begin{aligned}f(x,t) =& \frac{6}{\Gamma(3-\alpha)}t^{3-\alpha}\sin(\pi x) + \frac{6}{\Gamma(4-\gamma)}t^{3-\gamma}\sin(\pi x) + \pi^2 t^3\sin(\pi x) \\ &+ t^9\sin^3(\pi x) - t^3\sin(\pi x).\end{aligned}$$

Figure 1 shows the comparison between the numerical solution and the exact solution, where $N = 30,\ M = 1000,\ T = 1,\ 0 < \alpha < 1,\ 1 < \gamma < 2$. Certainly, the numerical solutions we have acquired closely align with the exact solution. What's more, it demonstrates that the fully discrete scheme (3.1) is numerical stability.



(a) $\alpha = 0.1, \gamma = 1.1$ (b) $\alpha = 0.9, \gamma = 1.5$

(c) $\alpha = 0.5, \gamma = 1.5$ (d) $\alpha = 0.9, \gamma = 1.1$

Figure 1. Exact solutions and numerical solutions for $N = 30,\ M = 1000$ in Example 6.1.

Now, we validate temporal accuracy by selecting a sufficiently large value N in order to reduce the effect of spatial error. Taking $N = 500$ and $T = 1$, Tables 1 and 2 show the maximum L^2 and L^∞ error and accuracy. We can observe that the temporal accuracy is minimum value of $2-\alpha$ and $3-\gamma$, which is consistent with the theoretical results, as the values of α and γ change.

Table 1. L^2 error, convergence order with various values of M in Example 6.1.

M	$\alpha = 0.9, \gamma = 1.1$		$\alpha = 0.5, \gamma = 1.5$		$\alpha = 0.1, \gamma = 1.1$	
	Error	Order	Error	Order	Error	Order
5	2.2347e-04	–	2.5941e-02	–	9.1529e-03	–
10	9.9430e-05	1.1683	9.0526e-03	1.5188	2.5159e-03	1.8631
20	4.4966e-05	1.1449	3.1577e-03	1.5194	6.6238e-04	1.9254
40	2.0602e-05	1.1260	1.1057e-03	1.5139	1.7273e-04	1.9391
80	9.5128e-06	1.1149	3.8828e-04	1.5098	4.4982e-05	1.9411
$\min\{2-\alpha, 3-\gamma\}$		1.1		1.5		1.9

In terms of time direction, we compare the scheme of this paper with that of [4]. Taking $N = 500$ and $T = 1$, it can be seen from the Table 3 that the convergence orders of the two schemes are similar, but the error in this paper is smaller.

Next, the spatial accuracy is verified, and τ is taken small enough to avoid the influence of temporal error. Here we take $\tau = 10^{-3}$ and $T = 1$, Figure 2 shows the L^2 errors with respect

Table 2. L^∞ error, convergence order with various values of M in Example 6.1.

M	$\alpha=0.9,\gamma=1.1$		$\alpha=0.5,\gamma=1.5$		$\alpha=0.1,\gamma=1.1$	
	Error	Order	Error	Order	Error	Order
5	2.2347e-04	–	1.4453e-04	–	5.2283e-05	–
10	9.9430e-05	1.1683	5.0305e-05	1.5226	1.4379e-05	1.8624
20	4.4966e-05	1.1449	1.7502e-05	1.5232	3.7819e-06	1.9268
40	2.0602e-05	1.1260	6.1160e-06	1.5168	9.8495e-07	1.9410
80	9.5128e-06	1.1149	2.1446e-06	1.5119	2.5615e-07	1.9431
$\min\{2-\alpha,3-\gamma\}$		1.1		1.5		1.9

Table 3. Comparison of the L^∞ error, convergence order with various values of M in Example 6.1.

			Scheme in [4]		Scheme in this paper	
α	γ	M	Error	Order	Error	Order
0.3	1.7	5	4.7296e-02	–	2.0529e-04	–
		10	1.9832e-02	1.25	8.2230e-05	1.32
		20	7.8848e-03	1.33	3.3671e-05	1.29
		40	3.1419e-03	1.32	1.3659e-05	1.30
		80	1.2796e-03	1.30	5.5264e-06	1.31
0.5	1.6	5	4.4138e-02	–	1.7968e-04	–
		10	1.7805e-02	1.31	6.5987e-05	1.45
		20	6.6204e-03	1.42	2.4206e-05	1.45
		40	2.4329e-03	1.44	8.9718e-06	1.43
		80	9.1216e-04	1.41	3.3496e-06	1.42
0.7	1.5	5	4.8158e-02	–	2.3147e-04	–
		10	1.9711e-02	1.28	9.1610e-05	1.34
		20	7.3573e-03	1.42	3.6755e-05	1.32
		40	2.6980e-03	1.44	1.4753e-05	1.32
		80	1.0035e-03	1.42	5.9127e-06	1.32

to N in semi-log scale in four cases. We can observe the error decay exponentially in maximum L^2-norm, which verified the correctness of the theoretical results.

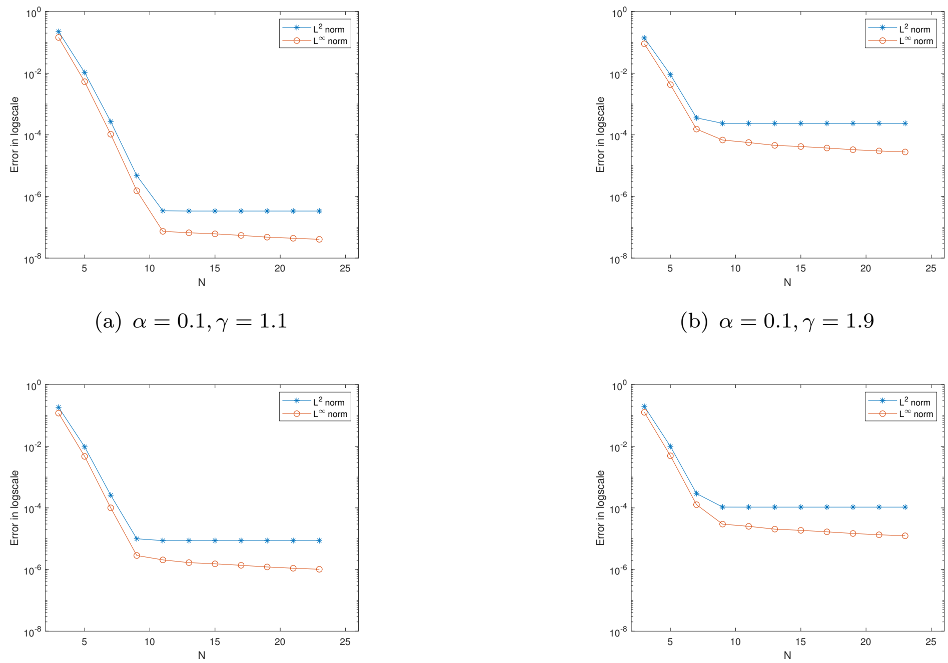
Example 6.2. Consider the problem (1.1) with $u_t(x,0)=\psi(x)$, $g(u)=u^2$. the exact solution is

$$u(x,t)=t^{2+\gamma}(1-x^2)x^{\frac{16}{3}},$$

the source term is

$$\begin{aligned} f(x,t)=&\frac{\Gamma(3+\gamma)}{2}t^2(1-x^2)x^{\frac{16}{3}}+\frac{\Gamma(3+\gamma)}{\Gamma(3-\alpha+\gamma)}t^{2-\alpha+\gamma}(1-x^2)x^{\frac{16}{3}} \\ &+t^{2+\gamma}(\frac{418}{9}x^{\frac{16}{3}}-\frac{208}{9}x^{\frac{10}{3}})+t^{4+2\gamma}(1-x^2)^2x^{\frac{32}{3}}. \end{aligned}$$

Figure 3 shows the exact solution on the left side and the numerical solution on the right side, where $N=30,\ M=1000,\ T=1,\ \alpha=0.9,\ \gamma=1.1$. The figures for these two solutions appear to be nearly identical.



(a) $\alpha = 0.1, \gamma = 1.1$ (b) $\alpha = 0.1, \gamma = 1.9$

(c) $\alpha = 0.5, \gamma = 1.5$ (d) $\alpha = 0.9, \gamma = 1.1$

Figure 2. L^∞ and L^2 errors versus N in Example 6.1.

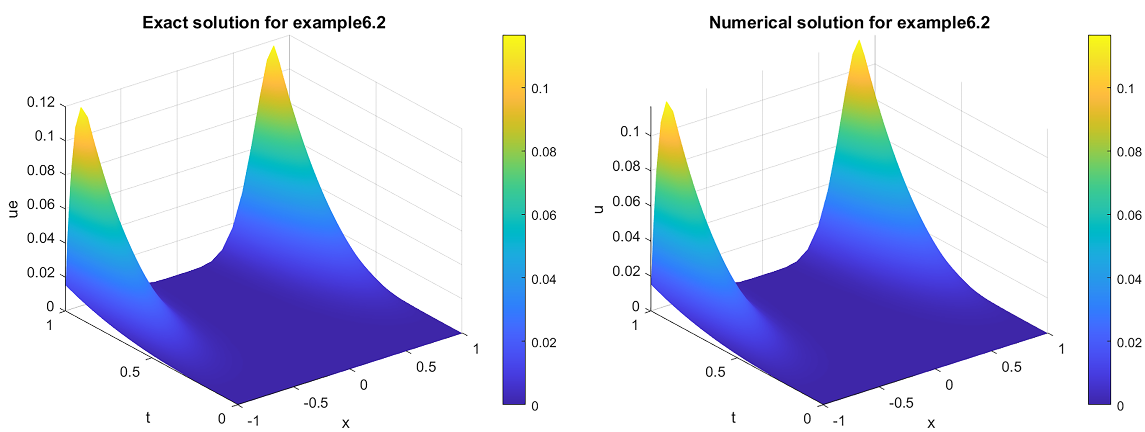


Figure 3. Exact solutions and numerical solutions for $N = 30,\ M = 1000$ in Example 6.2.

In order to reduce the effect of spatial error, we take N large enough. Here, take $T = 1$ and $N = 60$. Table 4 and Table 5 show temporal error and temporal accuracy with the maximum L^2-norm and the maximum L^∞-norm. We can see that the according orders are close to $\min\{2-\alpha, 3-\gamma\}$, aligning well with the agreement of our theoretical analysis.

Figures 4-5 show the L^2 and L^∞ errors with respect to N in log-log scale. Since its solution belongs to H^5, but not belongs to H^6. In the figure we can see the convergence rates between N^{-4} and N^{-5}. Spatial convergence order achieves spectral accuracy. This shows that the experimental results are in agreement with the theoretical results.

Table 4. L^2 error, convergence order with various values of M in Example 6.2.

M	$\alpha = 0.9, \gamma = 1.1$		$\alpha = 0.5, \gamma = 1.5$		$\alpha = 0.1, \gamma = 1.1$	
	Error	Order	Error	Order	Error	Order
8	2.9363e-04	–	2.3655e-04	–	2.8758e-05	–
16	1.3633e-04	1.1069	8.6020e-05	1.4594	7.8730e-06	1.8690
32	6.3370e-05	1.1053	3.0956e-05	1.4745	2.1397e-06	1.8795
64	2.9480e-05	1.1040	1.1065e-05	1.4842	5.7054e-07	1.9070
128	1.3722e-05	1.1033	3.9354e-06	1.4914	1.4245e-07	2.0018
$\min\{2-\alpha, 3-\gamma\}$		1.1		1.5		1.9

Table 5. L^∞ error, convergence order with various values of M in Example 6.2.

M	$\alpha = 0.9, \gamma = 1.1$		$\alpha = 0.5, \gamma = 1.5$		$\alpha = 0.1, \gamma = 1.1$	
	Error	Order	Error	Order	Error	Order
8	5.3559e-05	–	4.2581e-05	–	5.0384e-06	–
16	2.4926e-05	1.1035	1.5475e-05	1.4603	1.3809e-06	1.8674
32	1.1600e-05	1.1035	5.5684e-06	1.4746	3.7604e-07	1.8766
64	5.4007e-06	1.1030	1.9909e-06	1.4838	1.0093e-07	1.8976
128	2.5153e-06	1.1024	7.0876e-07	1.4901	2.6002e-08	1.9566
$\min\{2-\alpha, 3-\gamma\}$		1.1		1.5		1.9

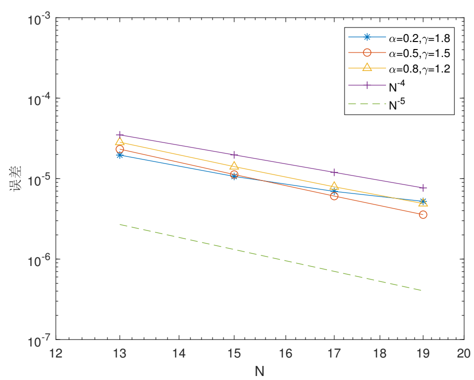


Figure 4. L^2 spatial convergence orders with various values of α and γ in Example 6.2.

7. Conclusion

In this paper, we have developed a Legendre spectral method for solving the nonlinear time fractional mixed sub-diffusion and diffusion-wave equation. The Caputo fractional derivative is discretized using the $L1$ scheme. The existence and uniqueness of the numerical solution, as well as the stability and convergence of the scheme, have been rigorously proven. The rate of convergence is $O(\tau^{\min\{3-\gamma,2-\alpha\}} + N^{1-s})$. Two numerical examples are presented, one with a smooth solution and the other with a non-smooth solution. It can be found that the spatial convergence order reaches spectral precision in examples. The numerical results are consistent with the theoretical analysis. We can consider that using this method to solve the 2D/3D nonlinear diffusion and diffusion-wave equations.

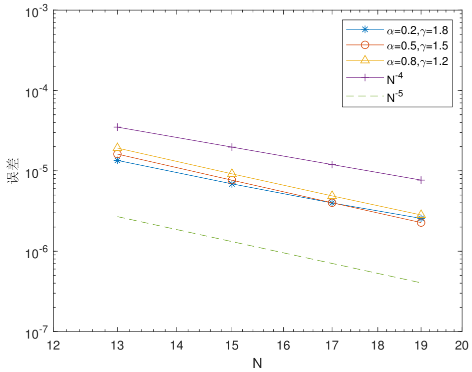


Figure 5. L^∞ spatial convergence orders with various values of α and γ in Example 6.2.

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