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SHADOWING PROPERTIES AND CHAOTIC CHARACTERISTICS IN TIME-VARYING DISCRETE DYNAMICAL SYSTEMS*

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Abstract Let $f_{1,\infty}$ be a map sequence on a compact metric space (X,d). First, this paper defines NK-th iteration map sequence $f_{1,\infty}^{n[k]}$, where $f_{1,\infty}^{n[k]} = \{f^{nk}_{\frac{n(n-1)k}{2}+1}\}_{n=1}^{\infty}$ $(k \in \mathbb{N})$. It is proved that $f_{1,\infty}$ is $\mathcal{P}$-chaotic (or has pseudo-orbit shadowing property) if and only if $f_{1,\infty}^{n[k]}$ is also $\mathcal{P}$-chaotic (or has pseudo-orbit shadowing property). Where $\mathcal{P}$-chaos denotes one of the nine properties: Li-Yorke chaos, distributional chaotic, dense chaos, dense δ-chaos, generic chaos, generic δ-chaos, Li-Yorke sensitivity, sensitivity, spatio-temporal chaos. Then, the preservation of persistence, expansive, linking, topological stability, and six types of shadowing properties in time-varying discrete dynamical systems (T-VDDSs) under product operator or topological conjugacy are shown. Moreover, a homeomorphism that is expansive and has the eventual shadowing property necessarily implies topological stability.

Keywords Time-varying discrete dynamical systems, pseudo-orbits, shadowing properties, product systems.

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1. Introduction

The idea of time-varying discrete dynamical systems was initially presented by Kolyada and Snoha in [16]. Compared to autonomous discrete dynamical systems (ADDSs), T-VDDSs can more accurately depict the dynamic behavior of the system. And then it became an important component of research in topological dynamical systems. In recent years, the chaotic characteristics of T-VDDSs have increasingly garnered attention. Li and Yorke were the first to provide a strict mathematical definition of "chaos" in [19], which is known as Li-Yorke chaos. Canovas [5] analyzed a uniformly convergent sequence of continuous interval maps $f_{1,\infty}$ demonstrating that the chaotic behavior of sequences like $(f_n \circ \cdots \circ f_1)(x)$ remains invariant under iteration. Lasota first proposed the definition of general chaos in [24]. In [27], Snoha defined generic δ-chaos, dense chaos, and dense δ-chaos based on the concept of generic chaos, and studied the relationships between generic chaos and generic δ-chaos, generic δ-chaos and dense δ-chaos, as well as dense chaos and generic chaos. Building on the concept of Li-Yorke chaos, Schweizer and Smital further extended it by introducing the definition of distribution chaos in [26]. Blanchard et al.

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studied spatio-temporal chaos in [4]. Akin and Kolyada defined the concept of Li-Yorke sensitivity as a creative combination of Li-Yorke chaos and initial condition sensitivity in [1]. In [31], Wu and Zhu proved that some chaotic characteristics of uniformly convergent map sequences are preserved under iteration. Building on the work of the above, [2, 21, 33] investigated the chaotic characteristics of time-varying discrete systems.

Due to the different definitions of pseudo-orbits and tracking methods for shadowing property, many different shadowing definitions can be derived. In 1976, Conley [9] first provided definitions of classical pseudo-orbit and pseudo-orbit shadowing properties. In 2018, inspired by the definition of the classical pseudo-orbit shadowing property, Good and Meddaugh [14] introduced the concept of the eventual shadowing property. In 1999, Pilyugin [23] defined the limit shadowing property. In 2003, Sakai [25] defined the S-limit shadowing property. In 2014, Oprocha [20] defined the $M^{\alpha}(M_{\alpha})$ shadowing property, which is essentially a further extension of the $\bar{d}$ (or $\underline{d}$) shadowing property introduced by Dastjerdi [11]. In 2017, Garg and Das [13] defined the almost average shadowing property. In 2016, Wang et al. [30] defined a specific type of sequence asymptotic average shadowing property and discussed its connection with transitivity and chain transitivity. More research on shadowing property in T-VDDSs can be found in [15, 22, 34] and others.

In order to investigate whether the pseudo-orbit shadowing property of extended homeomorphisms implies certain stability results, Walters [29] proved that expandable homeomorphisms with pseudo-orbit shadowing property in compact metric spaces must have topological stability. In 1991, Choi et al. [6] studied chain recurrent sets in persistent dynamical systems and proved that a homeomorphism $f : X \to X$ that is both extensible and persistent implies topological stability. In 2013, Das et al. [10] obtained the same conclusion in the Hausdorff uniform space. In 2014, Thakkar and Das [28] verified that on a compact metric space, a time-varying map with both the shadowing property and the extensible property implies topological stability. In 2018, Chung and Lee [8] extended their research to the role of non-generating groups in compact metric spaces. In 2022, Yan and Zeng [32] defined topological stability in compact uniform spaces and proved that homeomorphisms $T : X \to X$ possesses the pseudo-orbit shadowing property and extensibility, indicating that they are topologically stable. For more relevant content, please refer to references [3, 7, 12, 17, 18, 35].

This paper defines the NK-th iterative system and studies some chaotic properties of this system. The preservation of six types of shadowing properties and related properties in product systems is investigated. Then, similar conclusions in topological conjugation cases are obtained. Finally, it is concluded that a homeomorphism with both the eventual shadowing property and expansivity implies topological stability. The main structure of this paper is divided into the following sections. In Section 2, the concept of the NK-th iteration system and some other fundamental concepts are introduced. In Section 3, the relevant results under the NK-th iteration system, along with their corresponding proofs are presented. In Section 4, the main results regarding the preservation of several types of shadowing properties, along with their corresponding proofs are presented. In Section 5, the preservation of several types of chaotic properties and their interrelationships are discussed. In Section 6, the differences between the NK-th iterate system and the K-th iterate system are presented.

2. Preliminaries

Assume that X is a compact metric space with a metric d. Let $f_{1,\infty} := \{f_n\}_{n=1}^{\infty}$ denote a sequence of continuous maps. The space $(X, d, f_{1,\infty})$ is called a time-varying discrete dynamical system (T-VDDSs), sometimes referred to as a non-autonomous discrete dynamical system. In this paper, it is denoted as $(X, f_{1,\infty})$ or simply X without confusing. If $f_1 = f_2 = f_3 = ... = f$, then the systems $(X, f_{1,\infty})$ are referred to as ADDSs. The set $Orb(x, f_{1,\infty}) = \{f_1^n(x) : n \in \mathbb{N}_0\}$ is defined to represent the orbit of any point $x \in X$. Where $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Furthermore, f_n^i is defined as $f_n^i = f_{n+i-1} \circ \cdots \circ f_n$ for $n \in \mathbb{N}_0$ and f_1^0 denotes the identity map. For any positive integer k, denote

$$\begin{aligned}
&f_1^k = f_k \circ \cdots \circ f_2 \circ f_1,\\
&f_{k+1}^{2k} = f_{3k} \circ \cdots \circ f_{k+2} \circ f_{k+1},\\
&\cdots,\\
&f_{\frac{i(i-1)k}{2}+1}^{ik} = f_{\frac{i(i+1)k}{2}} \circ \cdots \circ f_{\frac{i(i-1)k}{2}+2} \circ f_{\frac{i(i-1)k}{2}+1},\\
&\cdots.
\end{aligned}$$

In light of this, we define $f_{1,\infty}^{n[k]} = \{f_{\frac{n(n-1)k}{2}+1}^{nk}\}_{n=1}^{\infty}$, and $(X, f_{1,\infty}^{n[k]})$ is referred to as the NK-th iterate system. The NK-th iterate system $(X, f_{1,\infty}^{n[k]})$ is essentially an extension of the K-th iterate system $(X, f_{1,\infty}^{[k]})$ (see [31]). The current article considers the case that $f_{1,\infty}$ converges uniformly to a continuous map f.

Let $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ be two T-VDDSs. Define the sequence of continuous maps $f_{1,\infty} \times g_{1,\infty} = \{f_n \times g_n\}_{n=1}^{\infty}$. The space $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is referred to as the product system of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, where the metric ρ defined on the product system is expressed as

$$\rho((a, b), (c, d)) = \max\{d_1(a, c), d_2(b, d)\}$$

for any pairs $(a, b), (c, d) \in X \times Y$. In addition, for any $(a, b) \in X \times Y$ and $n \in \mathbb{N}_0$, $(f_1^n \times g_1^n)((a, b))$ is defined as

$$(f_1^n \times g_1^n)((a, b)) = (f_n \times g_n) \circ \cdots \circ (f_2 \times g_2) \circ (f_1 \times g_1)(a, b) = (f_1^n(a), g_1^n(b)).$$

Definition 2.1. Let $(X, d, f_{1,\infty})$ and $(Y, d, g_{1,\infty})$ be two T-VDDSs. For any $n \in \mathbb{N}_0$, if there exists a homeomorphism $\sigma : X \to Y$ such that $g_n \circ \sigma = \sigma \circ f_n$ for any $n \in \mathbb{N}_0$, then, $f_{1,\infty}$ and $g_{1,\infty}$ are said to be topologically conjugate to each other.

Definition 2.2. Assume that $(X, f_{1,\infty})$ is a T-VDDSs. For any $\delta > 0$ and any $p, q \in X$,

(1) if

$$\liminf_{n\to\infty} d\left(f_1^n(p), f_1^n(q)\right) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d\left(f_1^n(p), f_1^n(q)\right) \geq \delta,$$

then, the pair (p, q) is referred to as a Li-Yorke pair of modulus δ. Specifically, if

$$\liminf_{n\to\infty} d\left(f_1^n(p), f_1^n(q)\right) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d\left(f_1^n(p), f_1^n(q)\right) > 0,$$

then (p, q) is called a Li-Yorke pair;

(2) for $S \subset X$, if for any pair of distinct points $p, q \in S$, the pair (p, q) is a Li-Yorke pair, then S is called a Li-Yorke scramble set.

The system $(X, f_{1,\infty})$ (or the map sequence $f_{1,\infty}$) is defined as Li-Yorke chaotic if there exists an uncountable Li-Yorke chaotic set contained in X.

In this paper, $\mathrm{LY}(f_{1,\infty},\delta)$ represents the set of all Li-Yorke pairs of modulus δ, while $\mathrm{LY}(f_{1,\infty})$ denotes the set of all Li-Yorke pairs. It can be easily concluded that

$$\mathrm{LY}(f_{1,\infty}) = \bigcup_{\delta>0} \mathrm{LY}(f_{1,\infty},\delta).$$

Definition 2.3. Assume that $(X, f_{1,\infty})$ is a T-VDDSs.

(1) If there exists an $\varepsilon > 0$ such that for any $\delta > 0$ and any point $p \in X$, there exists a point $q \in X$ such that $d(p,q) < \delta$ and $d(f_1^n(p), f_1^n(q)) > \varepsilon$ for some $n \in \mathbb{N}_0$, then, $f_{1,\infty}$ is said to be sensitive;

(2) If for any $\delta > 0$ and any $p \in X$, there exists a point $q \in X$ such that $d(p,q) < \delta$ and the pair $(p,q) \in \mathrm{LY}(f_{1,\infty})$, then, $f_{1,\infty}$ is said to be spatio-temporally chaotic;

(3) If there exists an $\varepsilon > 0$ such that for any $\delta > 0$ and any point $p \in X$, there exists a point $q \in X$ satisfying $d(p,q) < \delta$ and the pair $(p,q) \in \mathrm{LY}(f_{1,\infty},\varepsilon)$, then, $f_{1,\infty}$ is said to be Li-Yorke sensitive.

A set $S \subseteq X$ is said to be of first category in X if S is a countable union of nowhere dense subsets of X.

Definition 2.4. Assume that $(X, f_{1,\infty})$ is a T-VDDSs. For $\delta > 0$, if

(1) if X^2-$\mathrm{LY}(f_{1,\infty})$ is a first category subset of X^2, then $f_{1,\infty}$ is generic chaos;

(2) if X^2-$\mathrm{LY}(f_{1,\infty},\delta)$ is a first category subset of X^2, then $f_{1,\infty}$ is generic δ-chaos;

(3) if $\overline{\mathrm{LY}(f_{1,\infty})} = X^2$, i.e., $\mathrm{LY}(f_{1,\infty})$ is dense in X^2, then $f_{1,\infty}$ is dense chaos;

(4) if $\overline{\mathrm{LY}(f_{1,\infty},\delta)} = X^2$, i.e., $\mathrm{LY}(f_{1,\infty},\delta)$ is dense in X^2, then $f_{1,\infty}$ is dense δ-chaos.

In this paper, the upper distributional function of $f_{1,\infty}$ is denoted by $F^*_{p,q}(t, f_{1,\infty})$, and the lower distributional function of $f_{1,\infty}$ is denoted by $F_{p,q}(t, f_{1,\infty})$, where

$$F^*_{p,q}(t, f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{[0,t)}(d(f_1^j(p), f_1^j(q))),$$

and

$$F_{p,q}(t, f_{1,\infty}) = \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{[0,t)}(d(f_1^j(p), f_1^j(q)))$$

for any $p, q \in X$, $\chi_{[0,t)}(\lambda)$ denotes the characteristic function, i.e., if $\lambda \le t$, then $\chi_{[0,t)}(\lambda) = 1$, and if $\lambda > t$, then $\chi_{[0,t)}(\lambda) = 0$.

Definition 2.5. Assume that $(X, f_{1,\infty})$ is a T-VDDSs. Let $S \subset X$ be an uncountable subset. If for any $p, q \in S$, $F^*_{p,q}(t, f_{1,\infty}) = 1$ for all $t > 0$ and $F_{p,q}(\varepsilon, f_{1,\infty}) = 0$ for some $\varepsilon > 0$, then, $(X, f_{1,\infty})$ is distributional chaotic.

Definition 2.6. Let $\{a_i\}_{i=0}^{+\infty}$ be a sequence of points in X. For any $\delta > 0$, the sequence $\{a_i\}_{i=0}^{+\infty}$ is called

(1) a δ pseudo-orbit of $f_{1,\infty}$ if $d(f_{i+1}(a_i), a_{i+1}) < \delta$ for any $i \in \mathbb{N}_0$;

(2) an asymptotic pseudo-orbit of $f_{1,\infty}$ if

$$\lim_{i\to\infty} d(f_{i+1}(a_i), a_{i+1}) = 0;$$

(3) an asymptotic δ pseudo-orbit if $\{a_i\}_{i=0}^{+\infty}$ satisfies both conditions of being a δ pseudo-orbit and an asymptotic pseudo-orbit;

(4) an almost δ average pseudo-orbit of $f_{1,\infty}$ if

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(f_{i+1}(a_i),a_{i+1})<\delta;$$

(5) a $\{n_i\}_{i=0}^{+\infty}$ asymptotic average pseudo-orbit of $f_{1,\infty}$ if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d(f^{n_{i+1}}(a_i),a_{i+1})=0$$

for given a sequence $\{n_i\}_{i=0}^{+\infty}$ of non-negative integers;

(6) a δ ergodic pseudo-orbit of $f_{1,\infty}$ if

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n:d(f_{i+1}(a_i),a_{i+1})<\delta\}|=1.$$

Definition 2.7. Assume that $f_{1,\infty}$ is a sequence of continuous maps on X.

(1) If for any $\varepsilon>0$, there exist a $\delta>0$ and a point $p\in X$ such that p ε-shadows any δ pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, i.e., $d(f_1^i(p),a_i)\le\varepsilon$ for any $i\in\mathbb{N}_0$, then, the system $(X,f_{1,\infty})$ is said to have the pseudo-orbit shadowing property (PSP);

(2) If for any $\varepsilon>0$, there exist a $\delta>0$ and a point $p\in X$ such that p eventually ε-shadows any δ pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, i.e., there is an $N\in\mathbb{N}_0$ such that $d(f_1^i(p),a_i)\le\varepsilon$ for every $i\ge N$, then, the system $(X,f_{1,\infty})$ is said to have the eventual shadowing property (ESP);

(3) If there exists a point $p\in X$ such that p asymptotically shadows any asymptotic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, i.e.,

$$\lim_{i\to\infty}d(f_1^i(p),a_i)=0,$$

then, the system $(X,f_{1,\infty})$ is said to have the limit shadowing property (LSP);

(4) If for any $\varepsilon>0$, one can find a $\delta>0$ such that

(i) there exists a point $p\in X$ such that p ε-shadows any δ pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, and

(ii) there exists a point $q\in X$ such that q asymptotically ε-shadows (i.e., ε-shadows and asymptotically shadows simultaneously) any asymptotic δ pseudo-orbit $\{b_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, then, the system $(X,f_{1,\infty})$ is said to have the s-limit shadowing property (s-LSP);

(5) If for every $\varepsilon>0$, there exist a $\delta>0$ and a point $p\in X$ such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ satisfies

$$\limsup_{n\to\infty}\frac{1}{n}|\{0\le i<n:d(f_1^i(p),a_i)<\varepsilon\}|>\alpha,$$

where $\alpha\in[0,1)$, then, the system $(X,f_{1,\infty})$ is said to have the $\mathcal{M}^\alpha$ shadowing property ($\mathcal{M}^\alpha$-SP);

(6) If for every $\varepsilon>0$, there exist a $\delta>0$ and a point $p\in X$ such that any δ ergodic pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ satisfies

$$\liminf_{n\to\infty}\frac{1}{n}|\{0\le i<n:d(f_1^i(p),a_i)<\varepsilon\}|>\alpha,$$

where $\alpha\in[0,1)$, then, the system $(X,f_{1,\infty})$ is said to have the $\mathcal{M}_\alpha$ shadowing property ($\mathcal{M}_\alpha$-SP);

(7) If for every $\varepsilon > 0$, there exist a $\delta > 0$ and a point $p \in X$ such that p ε-shadows in average any almost δ average pseudo-orbit $\{x_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, i.e.,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_1^i(p), a_i) < \varepsilon,$$

then, the system $(X, f_{1,\infty})$ is said to have the almost average shadowing property (ALASP);

(8) If there exists a point $p \in X$ such that p sequence asymptotic shadows in average any $\{n_i\}_{i=0}^{+\infty}$ asymptotic average pseudo-orbit $\{a_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f_1^{n_0+n_1+\ldots+n_i}(p), a_i) = 0,$$

then, the system $(X, f_{1,\infty})$ is said to have the sequence asymptotic average shadowing property ($\{n_i\}_{i=0}^{+\infty}$-AASP).

Definition 2.8. Assume that $f_{1,\infty}$ is a sequence of homeomorphisms on X.

(1) For any $p, q \in X$, if there exists an $\varepsilon > 0$ such that $d(f_1^i(p), f_1^i(q)) \le \varepsilon$ for all $i \in \mathbb{N}_0$ implies $p = q$, then $f_{1,\infty}$ is said to be expansive, and ε is called an expansivity constant;

(2) Let

$$d_0(f_{1,\infty}, \varphi_{1,\infty}) = \sup\{d(f_1^i(p), \varphi_1^i(p)) : p \in X\}.$$

$f_{1,\infty}$ is called α-persistent (or β-persistent) if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $d_0(f_{1,\infty}, \varphi_{1,\infty}) < \delta$, then there exists a $q \in X$ satisfying

$$d(f_1^i(q), \varphi_1^i(p)) < \varepsilon \quad (\text{or } d(f_1^i(p), \varphi_1^i(q)) < \varepsilon)$$

for all $i \in \mathbb{N}_0$. If $f_{1,\infty}$ is both α-persistent and β-persistent, then it is called persistent;

(3) $f_{1,\infty}$ is said to be topologically stable if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $g_{1,\infty} : X \to X$ is any homeomorphism satisfying

$$d(f_{1,\infty}, g_{1,\infty}) = \sup_{p\in X} d(f_1^i(p), g_1^i(p))$$

for any $i \in \mathbb{N}_0$, and $d(f_{1,\infty}, g_{1,\infty}) \le \delta$, then, there exists a continuous map $\sigma : X \to X$ such that

$$d(f_1^i(\sigma(p)), g_1^i(p)) < \varepsilon \quad \text{and} \quad d(\sigma(p), \mathrm{Id}_X) < \varepsilon$$

for any $p \in X$ and any $i \in \mathbb{N}$.

Definition 2.9. Assume $f_{1,\infty}$ is a sequence of continuous maps on X. Let $\varepsilon > 0$. For any points $p, q \in X$, if there exist an integer $m \ge 1$ and a point $z \in X$ such that for any $0 \le j \le m$,

$$f_1^m(z) = q \quad \text{and} \quad d(f_1^j(p), f_1^j(z)) \le \varepsilon,$$

then, p is said to be ε-linked to q by $f_{1,\infty}$.

For any $\varepsilon > 0$, if p is ε-linked to q by $f_{1,\infty}$, then, p is said to be linked to q by $f_{1,\infty}$. If for any $p \in A \subset X$, there exist some $q \in A$ such that p is linked to q by $f_{1,\infty}$, then, the set A is said to be linked by $f_{1,\infty}$.

3. The chaotic characteristics in NK-th iterate system

In this section, under the assumption that $f_{1,\infty}$ converges uniformly to a continuous map f, several lemmas are used to demonstrate that $f_{1,\infty}$ is $\mathcal{P}$-chaotic if and only if $f_{1,\infty}^{n[k]}$ exhibits $\mathcal{P}$-chaos as well. In this context, $\mathcal{P}$-chaos denotes one of the nine properties: Li-Yorke chaos, distributional chaos, sensitivity, dense chaos, dense δ-chaos, generic chaos, generic δ-chaos, Li-Yorke sensitivity, and spatio-temporal chaos.

Lemma 3.1. *Assume $f_{1,\infty}$ converges uniformly to f. It follows that:*

(1) The sequence $\{f^{nk}_{\frac{n(n-1)k}{2}+1}\}_{n=1}^{\infty}$ converges uniformly to f^k for every $k \in \mathbb{N}_0$.

(2) Moreover, the sequence $\{f^{m_n k}_{\frac{m_n(m_n-1)k}{2}+1}\}_{n=1}^{\infty}$ also converges uniformly to f^k, where $\{m_n\}_{n=1}^{\infty}$ is any strictly increasing sequence of positive integers.

Proof. Since $f_{1,\infty}$ converges uniformly to f, for any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}_0$ such that for any $p \in X$ and $n \geq n_0$, it holds that $d(f_n(p), f(p)) < \varepsilon$.

Mathematical induction is used here to prove this conclusion. First, prove the case for $k = 1$. Since f is uniformly continuous, combining with Lemma 2.1 from [31] and the inequality

$$\begin{aligned}
&d(f^{n}_{\frac{n(n-1)}{2}+1}(p), f^n(p))\\
=&d(f_{\frac{n(n+1)}{2}}(f^{n-1}_{\frac{n(n-1)}{2}+1}(p)), f(f^{n-1}(p)))\\
\leq&d(f_{\frac{n(n+1)}{2}}(f^{n-1}_{\frac{n(n-1)}{2}+1}(p)), f(f^{n-1}_{\frac{n(n-1)}{2}+1}(p))) + d(f(f^{n-1}_{\frac{n(n-1)}{2}+1}(p)), f(f^{n-1}(p))),
\end{aligned}$$

the sequence $(f^{n}_{\frac{n(n-1)}{2}+1})_{n=1}^{\infty}$ converges uniformly to f.

Assume that the case for $k = s$ has been established. Next, to prove the case $k = s + 1$. Since

$$\begin{aligned}
&d(f^{n(s+1)}_{\frac{n(n-1)(s+1)}{2}+1}(p), f^{n(s+1)}(p))\\
=&d(f^{ns}_{\frac{n(n+1)(s+1)}{2}+n+1}(f^{n}_{\frac{n(n-1)(s+1)}{2}+1}(p)), f^{ns}(f^n(p)))\\
\leq&d(f^{ns}_{\frac{n(n+1)(s+1)}{2}+n+1}(f^{n}_{\frac{n(n-1)(s+1)}{2}+1}(p)), f^{ns}(f^{n}_{\frac{n(n-1)(s+1)}{2}+1}(p)))\\
&+ d(f^{ns}(f^{n}_{\frac{n(n-1)(s+1)}{2}+1}(p)), f^{ns}(f^n(p)))
\end{aligned}$$

for any $p \in X$, applying Lemma 2.1 in [31], the sequence $(f^{n(s+1)}_{\frac{n(n-1)(s+1)}{2}+1})_{n=1}^{\infty}$ converges uniformly to f^{s+1}. Therefore, the conclusion is correct. □

Corollary 3.1. *Assume $f_{1,\infty}$ converges uniformly to f. For any $\varepsilon > 0$ and $k \in \mathbb{N}$, there exist a $\xi(\varepsilon) > 0$ and an integer $N(k) \in \mathbb{N}_0$ such that for any $p, q \in X$ with $d(p, q) < \xi(\varepsilon)$ and any $n \geq N(k)$,*

$$d(f^{nk}_{\frac{n(n-1)k}{2}+1}(p), f^{nk}_{\frac{n(n-1)k}{2}+1}(q)) < \frac{\varepsilon}{2}.$$

Proof. By the triangle inequality,

$$\begin{aligned}
&d(f^{nk}_{\frac{n(n-1)k}{2}+1}(p), f^{nk}_{\frac{n(n-1)k}{2}+1}(q))\\
\leq&d(f^{nk}_{\frac{n(n-1)k}{2}+1}(p), f^k(p)) + d(f^k(p), f^k(q)) + d(f^k(q), f^{nk}_{\frac{n(n-1)k}{2}+1}(q)).
\end{aligned}$$

Then, by Lemma 3.1, the conclusion is proved. □

Lemma 3.2. *Assume $f_{1,\infty}$ converges uniformly to f. For any $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists a $\xi : 0 < \xi < \varepsilon$, satisfying*

$$LY(f_{1,\infty}, \varepsilon) \subset LY(f_{1,\infty}^{n[k]}, \xi) \subset LY(f_{1,\infty}, \xi).$$

Proof. For any fixed $\varepsilon > 0$, according to Corollary 3.1, there exist a ξ with $0 < \xi < \varepsilon$ and $N_0 = \max\{N(1), \dots, N(k)\} \in \mathbb{N}_0$ such that for any $p, q \in X$ with $d(p, q) < \xi$, and all $n \geq N_0$,

$$d((f^{ni}_{\frac{n(n-1)i}{2}+1})(p), (f^{ni}_{\frac{n(n-1)i}{2}+1})(q)) < \varepsilon$$

for every $1 \leq i \leq k$. In LY $(f_{1,\infty}, \varepsilon)$, for any pair of points (p, q), according to the definition, one can find an increasing sequence $\{m_l\}_{l=1}^{\infty}$ such that

$$\lim_{l\to\infty} d(f_1^{m_l}(p), f_1^{m_l}(q)) = \limsup_{n\to\infty} d(f_1^n(p), f_1^n(q)) \geq \varepsilon.$$

Set

$$\begin{aligned}
M_0 &= \{\frac{n(n+1)k}{2} : n \in \mathbb{N}\} \cap \{m_l : l \in \mathbb{N}_0\},\\
M_1 &= \{\frac{n(n+1)k}{2} + 1 : n \in \mathbb{N}\} \cap \{m_l : l \in \mathbb{N}_0\},\\
&\vdots\\
M_{(n+1)k-1} &= \{\frac{n(n+1)k}{2} + (n+1)k - 1 : n \in \mathbb{N}\} \cap \{m_l : l \in \mathbb{N}_0\}.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \cup_{i=0}^{(n+1)k-1} M_i = \{m_l : l \in \mathbb{N}_0\},$$

it follows that there must exist at least one $i_0 \in \{0, 1, \dots, (n+1)k - 1\}$ such that M_{i_0} is countably infinite. Let $M_{i_0} = \{\frac{n(n+1)k}{2} + i_0 : n \in \mathbb{N}\} \cap \{m_l : l \in \mathbb{N}_0\} = \{m_l^{(i_0)}\}_{l=1}^{\infty}$, where $i_0 \in \{0, 1, \dots, (n+1)k - 1\}$. Then, there exist some $l_0 \in \mathbb{N}_0$ such that

$$d(f_1^{m_l^{(i_0)}}(p), f_1^{m_l^{(i_0)}}(q)) = d(f^{i_0}_{m_l^{(i_0)}-i_0+1}(f_1^{m_l^{(i_0)}-i_0}(p)), f^{i_0}_{m_l^{(i_0)}-i_0+1}(f_1^{m_l^{(i_0)}-i_0}(q))) \geq \frac{\varepsilon}{2}$$

for any $l \geq l_0$. Together with the selection of ξ, for any $l \geq \max\{l_0, N_0 + i_0\}$, it holds that

$$d(f_1^{m_l^{(i_0)}-i_0}(p), f_1^{m_l^{(i_0)}-i_0}(q)) \geq \xi.$$

Since $\frac{n(n+1)k}{2}\Big|(m_l^{(i_0)} - i_0)$, it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} d(f^{nk}_{\frac{n(n-1)k}{2}+1} \circ \cdots \circ f^{2k}_{k+1} \circ f_1^k(p), f^{nk}_{\frac{n(n-1)k}{2}+1} \circ \cdots \circ f^{2k}_{k+1} \circ f_1^k(q))\\
\geq &\limsup_{l\to\infty} d(f_1^{m_l^{(i_0)}-i_0}(p), f_1^{m_l^{(i_0)}-i_0}(q))\\
\geq &\xi.
\end{aligned}$$

Similarly, since

$$\liminf_{n\to\infty} d(f_1^n(p), f_1^n(q)) = 0,$$

it can be shown that

$$\liminf_{n\to\infty} d(f^{nk}_{\frac{n(n-1)k}{2}+1}\circ\cdots\circ f^{2k}_{k+1}\circ f^{k}_{1}(p), f^{nk}_{\frac{n(n-1)k}{2}+1}\circ\cdots\circ f^{2k}_{k+1}\circ f^{k}_{1}(q))=0.$$

This implies that

$$\mathrm{LY}(f_{1,\infty},\varepsilon)\subset \mathrm{LY}(f^{n[k]}_{1,\infty},\xi).$$

Since

$$\mathrm{LY}(f^{n[k]}_{1,\infty},\xi)\subset \mathrm{LY}(f_{1,\infty},\xi),$$

then, $\mathrm{LY}(f_{1,\infty},\varepsilon)\subset \mathrm{LY}(f^{n[k]}_{1,\infty},\xi)\subset \mathrm{LY}(f_{1,\infty},\xi)$. □

Remark 3.1. Since $\mathrm{LY}(f_{1,\infty})=\bigcup_{\delta>0}\mathrm{LY}(f_{1,\infty},\delta)$, using Lemma 3.2, it follows that $\mathrm{LY}(f_{1,\infty})=\mathrm{LY}(f^{n[k]}_{1,\infty})$.

Lemma 3.3. *If $f_{1,\infty}$ is sensitive and converges uniformly to f, then $f^{n[k]}_{1,\infty}$ is also sensitive, where $k\in\mathbb{N}$.*

Proof. According to Corollary 3.1, there exist a $\xi>0$ and an $N_0\geq k$ such that for any $q,p\in X$ with $d(q,p)<\xi$ and all $n\geq N_0$,

$$d(f^{ni}_{\frac{n(n-1)i}{2}+1}(q), f^{ni}_{\frac{n(n-1)i}{2}+1}(p))<\delta$$

for each $0\leq i\leq k$. For any fixed $k>0$, Assume that $f_{1,\infty}$ is sensitive. Then, according to the definition, there exists a $\delta>0$ such that for any $\varepsilon>0$ and any point $x\in X$, there exists a point $y\in X$ with $d(x,y)<\varepsilon$, and y satisfies

$$d(f_1^{n_{(x,\varepsilon)}}(x), f_1^{n_{(x,\varepsilon)}}(y))>\delta$$

for some $n_{(x,\varepsilon)}\in\mathbb{N}_0$. Since f_1^i is uniformly continuous for $1\leq i\leq 2N_0$, there exists an $\varepsilon^*>0$ such that for any pair $a,b\in X$ with $d(a,b)\leq\varepsilon^*$, $d(f_1^i(a),f_1^i(b))<\delta$ holds for all $1\leq i\leq 2N_0$. Thus, for any $\varepsilon<\varepsilon^*$, with $n_{(x,\varepsilon)}>2N_0\geq 2k$, there exist a $l\in\mathbb{N}$ and an $i_0\in\{0,1,\ldots,(l+1)k-1\}$ such that $n_{(x,\varepsilon)}-i_0=\frac{l(l+1)k}{2}$. In light of the fact that

$$d(f_1^{n_{(x,\varepsilon)}}(x), f_1^{n_{(x,\varepsilon)}}(y))=d(f^{i_0}_{n_{(x,\varepsilon)}-i_0+1}(f_1^{n_{(x,\varepsilon)}-i_0}(x)), f^{i_0}_{n_{(x,\varepsilon)}-i_0+1}(f_1^{n_{(x,\varepsilon)}-i_0}(y)))>\delta$$

for any $\varepsilon<\varepsilon^*$, $n_{(x,\varepsilon)}-i_0+1\geq N_0$, by appropriately choosing ξ, it can be obtained that

$$d(f_1^{n(x,\varepsilon)-i_0}(x), f_1^{n(x,\varepsilon)-i_0}(y))\geq\xi$$

for any $\varepsilon<\varepsilon^*$.

This means that, as long as $\frac{n(n+1)k}{2}\Big|(n_{(x,\varepsilon)}-i_0)$, it can be concluded that $f^{n[k]}_{1,\infty}$ is sensitive. □

Theorem 3.1. *Assume $f_{1,\infty}$ converges uniformly to f. Then, $f_{1,\infty}$ is $\mathcal{P}$-chaotic if and only if $f^{n[k]}_{1,\infty}$ is also $\mathcal{P}$-chaotic, where $k\in\mathbb{N}$.*

Proof. Applying Lemmas 3.2 and 3.3, one can easily prove this conclusion. □

Before proving the preservation of distributional chaos in $f_{1,\infty}^{n[k]}$, we first provide the following basic definitions. Put

$$\xi_n(t, f_{1,\infty}, p, q) := \sum_{j=1}^{n} \chi_{[0,t)}(d(f_1^j(p), f_1^j(q))),$$

$$\hat{\xi}_n(t, f_{1,\infty}, p, q) := \sum_{j=1}^{n} \chi_{[t,+\infty)}(d(f_1^j(p), f_1^j(q))),$$

$$\xi_n(t, f_{1,\infty}^{n[k]}, p, q) := \sum_{j=1}^{n} \chi_{[0,t)}(d(f_1^{\frac{j(j+1)}{2}k}(p), f_1^{\frac{j(j+1)}{2}k}(q))).$$

Clearly, $\xi_n(t, f_{1,\infty}, p, q) + \hat{\xi}_n(t, f_{1,\infty}, p, q) = n$. Meanwhile, the upper distributional function and lower distributional function of $f_{1,\infty}$ can be denoted as

$$F_{p,q}^*(t, f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n}\xi_n(t, f_{1,\infty}, p, q)$$

and

$$F_{p,q}(t, f_{1,\infty}) = \liminf_{n\to\infty} \frac{1}{n}\xi_n(t, f_{1,\infty}, p, q).$$

Lemma 3.4. *Assume $f_{1,\infty}$ converges uniformly to f. Then, for any $k \in \mathbb{N}$ and any pair $p, q \in X$,*

(1) if $F_{p,q}^*(\zeta, f_{1,\infty}^{n[k]}) = 1$ for all $\zeta > 0$, then $F_{p,q}^*(\varepsilon, f_{1,\infty}) = 1$ for all $\varepsilon > 0$;

(2) if $F_{p,q}(\zeta, f_{1,\infty}^{n[k]}) = 0$ for some $\zeta > 0$, then there exists an $\varepsilon > 0$ such that $F_{p,q}(\varepsilon, f_{1,\infty}) = 0$.

Proof. (1) Assume that $F_{p,q}^*(\xi, f_{1,\infty}^{n[k]}) = 1$ for all $\zeta > 0$. By Corollary 3.1, there exist an $\varepsilon > 0$ and an $N \in \mathbb{N}_0$ such that for any $1 \le i \le k$, any pair $p, q \in X$ with $d(p, q) < \zeta$, and any $n \ge N$, the inequality

$$d(f_{\frac{n(n-1)i}{2}+1}^{ni}(p), f_{\frac{n(n-1)i}{2}+1}^{ni}(q)) < \varepsilon$$

is satisfied. Considering $s \ge N$, if

$$d(f_1^{\frac{s(s-1)k}{2}}(p), f_1^{\frac{s(s-1)k}{2}}(q)) < \zeta,$$

then for any $1 \le i \le k$,

$$d(f_{\frac{s(s-1)i}{2}+1}^{si}(f_1^{\frac{s(s-1)k}{2}}(p)), f_{\frac{s(s-1)i}{2}+1}^{si}(f_1^{\frac{s(s-1)k}{2}}(q))) = d(f_1^{si}(p), f_1^{si}(q)) < \varepsilon.$$

Furthermore, it can be deduced that

$$(\sum_{s=1}^{j} \chi_{[0,\zeta)}(d(f_1^{\frac{s(s+1)}{2}k}(p), f_1^{\frac{s(s+1)}{2}k}(q))) - N)k \le \sum_{s=1}^{jk} \chi_{[0,\varepsilon)}(d(f_1^s(p), f_1^s(q))).$$

So,

$$(\xi_j(\zeta, f_{1,\infty}^{n[k]}, p, q) - N)k \le \xi_{jk}(\varepsilon, f_{1,\infty}, p, q). \tag{1-1}$$

Thus, $F_{p,q}^*(\varepsilon, f_{1,\infty}) \ge F_{x,y}^*(\zeta, f_{1,\infty}^{n[k]}) = 1$. Therefore $F_{p,q}^*(\varepsilon, f_{1,\infty}) = 1$.

(2) Assume that $F_{p,q}(\zeta, f_{1,\infty}^{n[k]}) = 0$ for some $\zeta > 0$. By Corollary 3.1, there exist an $\varepsilon > 0$ and an $N_1 \in \mathbb{N}_0$ such that for any $1 \leq i \leq k$, any $a, b \in X$ with $d(a,b) < \zeta$, and for all $n \geq N_1$, the inequality

$$d(f_{\frac{i(i-1)k}{2}+1}^{ik}(a), f_{\frac{i(i-1)k}{2}+1}^{ik}(b)) < \varepsilon$$

is satisfied. This implies that for any $j \geq N_1$,

$$d(f_1^{jk}(p), f_1^{jk}(q)) \geq \varepsilon = d(f_{\frac{i(i-1)}{2}k+1}^{jk}(f_1^{\frac{j(j-1)}{2}k}(p)), f_{\frac{i(i-1)}{2}k+1}^{jk}(f_1^{\frac{j(j-1)}{2}k}(q)) \geq \varepsilon,$$

and for any $1 \leq i \leq k$, $d(f_1^{\frac{j(j-1)}{2}k}(p), f_1^{\frac{j(j-1)}{2}k}(q)) \geq \zeta$ holds.

Then

$$(\sum_{s=1}^{j} \chi_{[0,\zeta)}(d(f_1^{\frac{s(s+1)}{2}k}(p), f_1^{\frac{s(s+1)}{2}k}(q))) - N_1)k \leq \sum_{s=1}^{jk} \chi_{[0,\varepsilon)}(d(f_1^s(p), f_1^s(q))).$$

So,

$$(\hat{\xi}_j(\zeta, f_{1,\infty}^{n[k]}, p, q) - N_1)k \leq \hat{\xi}_{jk}(\varepsilon, f_{1,\infty}, p, q). \tag{1-2}$$

Combining with

$$\limsup_{n\to\infty} \frac{1}{n}\hat{\xi}_n(\zeta, f_{1,\infty}^{n[k]}, p, q) = 1 - \liminf_{n\to\infty} \frac{1}{nk}\xi_{nk}(\zeta, f_{1,\infty}^{n[k]}, p, q) = 1,$$

since

$$\limsup_{n\to\infty} \frac{1}{jk}\hat{\xi}_{jk}(\varsigma, f_{1,\infty}, p, q) \geq \limsup_{n\to\infty} \frac{1}{j}\hat{\xi}_j(\zeta, f_{1,\infty}^{n[k]}, p, q) = 1,$$

then

$$\limsup_{n\to\infty} \frac{1}{jk}\hat{\xi}_{jk}(\varsigma, f_{1,\infty}, p, q) = 1.$$

One has that

$$F_{p,q}(\varsigma, f_{1,\infty}) = 1 - \limsup_{n\to\infty} \frac{1}{n}\hat{\xi}_n(\varsigma, f_{1,\infty}, p, q) = 0.$$

□

Lemma 3.5. *Assume $f_{1,\infty}$ converges uniformly to f. Then, for any $k \in \mathbb{N}$ and any pair $p, q \in X$,*

(1) if $F_{p,q}^*(\varepsilon, f_{1,\infty}) = 1$ for all $\varepsilon > 0$, then $F_{p,q}^*(\zeta, f_{1,\infty}^{n[k]}) = 1$ for all $\zeta > 0$;

(2) if $F_{p,q}(\varepsilon, f_{1,\infty}) = 0$ for some $\varepsilon > 0$, then there exists a $\zeta > 0$ such that $F_{p,q}(\zeta, f_{1,\infty}^{n[k]}) = 0$.

Proof. (1) Since $F_{p,q}^*(\varepsilon, f_{1,\infty}) = 1$ for all $\varepsilon > 0$, and

$$\xi_n(t, f_{1,\infty}, p, q) + \hat{\xi}_n(t, f_{1,\infty}, p, q) = n,$$

then

$$\limsup_{n\to\infty} \frac{1}{nk}\xi_{nk}(\varepsilon, f_{1,\infty}, p, q) = 1 - \limsup_{n\to\infty} \frac{1}{nk}\hat{\xi}_{nk}(\varepsilon, f_{1,\infty}, p, q).$$

This implies that

$$\limsup_{n\to\infty} \frac{1}{nk}\hat{\xi}_{nk}(\varepsilon, f_{1,\infty}, p, q) = 0.$$

Combining with equation (1-2), one can obtain that

$$\limsup_{n\to\infty} \frac{1}{nk}\hat{\xi}_{nk}(\varepsilon, f_{1,\infty}, p, q) \geq \limsup_{n\to\infty} \frac{1}{n}\hat{\xi}_n(\varsigma, f_{1,\infty}^{n[k]}, p, q).$$

So

$$\limsup_{n\to\infty} \frac{1}{n}\hat{\xi}_n(\varsigma, f_{1,\infty}^{n[k]}, p, q) = 0.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n}\xi_n(\varsigma, f_{1,\infty}^{n[k]}, p, q) = 1 - \limsup_{n\to\infty} \frac{1}{n}\hat{\xi}_n(\varsigma, f_{1,\infty}^{n[k]}, p, q) = 1.$$

This indicates that $F^*_{p,q}(\varsigma, f_{1,\infty}^{n[k]}) = 1$ for all $\zeta > 0$.

(2) Since $F_{p,q}(\varepsilon, f_{1,\infty}) = 0$ for some $\varepsilon > 0$, then,

$$\liminf_{n\to\infty} \frac{1}{nk}\xi_{nk}(\varepsilon, f_{1,\infty}, p, q) = 0.$$

Combining with equation (1-1), one can get that

$$\liminf_{n\to\infty} \frac{1}{nk}\xi_{nk}(\varepsilon, f_{1,\infty}, p, q) \geq \liminf_{n\to\infty} \frac{1}{n}\xi_n(\zeta, f_{1,\infty}^{n[k]}, p, q).$$

So

$$\liminf_{n\to\infty} \frac{1}{n}\xi_n(\zeta, f_{1,\infty}^{n[k]}, p, q) = 0.$$

Thus, $F_{p,q}(\zeta, f_{1,\infty}^{n[k]}) = 0$. □

Theorem 3.2. *Assume $f_{1,\infty}$ converges uniformly to f. Then, $f_{1,\infty}$ is distributional chaotic if and only if $f_{1,\infty}^{n[k]}$ is also distributional chaotic.*

Proof. According to Lemma 3.4 and Lemma 3.5, one can easily get this conclusion. □

4. The retentivity of shadowing properties

This section proves that if a map sequence $f_{1,\infty}$ possesses the PSP, then the NK-th iterate system also possesses this property. And that if a time-varying discrete system $f_{1,\infty}$ possesses six different types of shadowing property, then both the product system and the conjugate system have the corresponding shadowing property as well. Moreover, this section investigates the preservation of persistence, expansive, linking, and topological stability within product systems and conjugate systems.

Theorem 4.1. *Assume that $f_{1,\infty}$ has the PSP, then $f_{1,\infty}^{n[k]}$ also possesses the PSP.*

Proof. Assume $f_{1,\infty}$ has the PSP. For any $\varepsilon > 0$, there exist a $\delta > 0$ and a $p \in X$ such that the point p ε-shadows any δ pseudo-orbit of $f_{1,\infty}$. Let $\{m_i\}_{i=0}^{+\infty}$ be a δ pseudo-orbit of $f_{1,\infty}^{n[k]}$. Based on the definition, it follows that

$$d(f^{(i+1)k}_{\frac{i(i+1)k}{2}+1}(m_i), m_{i+1}) < \delta$$

for $i \in \mathbb{N}_0$. For $0 < j < k$, let $n_{\frac{i(i+1)k}{2}+(i+1)j} = f^{(i+1)j}_{\frac{i(i+1)k}{2}+1}(m_i)$, then

$$f_{\frac{i(i+1)k}{2}+(i+1)j+1}(f^{(i+1)j}_{\frac{i(i+1)k}{2}+1}(m_i)) = f^{(i+1)j+1}_{\frac{i(i+1)k}{2}+1}(m_i) = n_{\frac{i(i+1)k}{2}+(i+1)j+1}.$$

So

$$\begin{aligned}&d(f^{(i+1)k}_{\frac{i(i+1)k}{2}+1}(m_i), m_{i+1})\\ &= d(f_{\frac{i(i+1)k}{2}+(i+1)j+1}(f^{(i+1)j}_{\frac{i(i+1)k}{2}+1}(m_i)), f^{(i+1)j+1}_{\frac{i(i+1)k}{2}+1}(m_i))\\ &= d(f_{\frac{i(i+1)k}{2}+(i+1)j+1}(n_{\frac{i(i+1)k}{2}+(i+1)j}), n_{\frac{i(i+1)k}{2}+(i+1)j+1})\\ &= 0\\ &< \delta.\end{aligned}$$

This means that $\{n_i\}_{i=0}^{+\infty}$ is a δ pseudo-orbit of $f_{1,\infty}$. Since $f_{1,\infty}$ possesses the PSP, a point $p \in X$ can be found such that p ε-shadows the δ pseudo-orbit $\{n_i\}_{i=0}^{+\infty}$. Then $d(f_1^i(p), n_i) \le \varepsilon$ for any $i \in \mathbb{N}_0$. One can take the value of the index i being $\frac{i(i+1)k}{2}$, let $n_{\frac{i(i+1)k}{2}} = m_i$. So

$$d(f_1^i(p), n_i) = d(f_1^{\frac{i(i+1)k}{2}}(p), n_{\frac{i(i+1)k}{2}}) = d(f_1^{\frac{i(i+1)k}{2}}(p), m_i) \le \varepsilon.$$

This indicates that the point p ε-shadows the δ pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ of $f^{n[k]}_{1,\infty}$. Therefore, it can be confirmed that $f^{n[k]}_{1,\infty}$ possesses the PSP. □

Theorem 4.2. *Let $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ be two T-VDDSs. $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is the product T-VDDSs, where $\rho = max\{d_1, d_2\}$. Then*

(1) $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have ESP if and only if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has ESP;

(2) $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have LSP if and only if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has LSP;

(3) $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have s-LSP if and only if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has s-LSP;

(4) $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have $\mathcal{M}^\alpha$-SP (or$\mathcal{M}_\alpha$-SP) if and only if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has $\mathcal{M}^\alpha$-SP (or$\mathcal{M}_\alpha$-SP);

(5) $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have ALASP if and only if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ hs ALASP;

(6) $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have $\{n_i\}_{i=0}^{+\infty}$-AASP if and only if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has $\{n_i\}_{i=0}^{+\infty}$-AASP.

Proof. (1) (Necessity) Assume that $f_{1,\infty}$ and $g_{1,\infty}$ have the ESP. For any $\varepsilon_1 > 0$, then there exist a $\delta_1 > 0$ and a $p \in X$ such that the point p eventual ε_1-shadows any δ_1 pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, i.e., there exist a $p \in X$ and an $N_1 \in \mathbb{N}_0$ such that for all $i \ge N_1$, $d_1(f_1^i(p), m_i) \le \varepsilon_1$. Similarly, for any $\varepsilon_2 > 0$, then there exist a $\delta_2 > 0$ and a $q \in Y$ such that the point q eventual ε_2-shadows any δ_2 pseudo-orbit $\{n_i\}_{i=0}^{+\infty}$ of $g_{1,\infty}$, i.e., there exist a $q \in Y$ and an $N_2 \in \mathbb{N}_0$ such that $d_2(g_1^i(q), n_i) \le \varepsilon_2$ for all $i \ge N_2$.

Since

$$d_1(f_1^{i+1}(m_i), m_{i+1}) \le \delta_1 \quad \text{and} \quad d_2(g_1^{i+1}(n_i), n_{i+1}) \le \delta_2,$$

choose $\delta = \max\{\delta_1, \delta_2\}$, then,

$$\begin{aligned}&\rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1}))\\ &= \rho((f_1^{i+1}(m_i), g_1^{i+1}(n_i)), (m_{i+1}, n_{i+1}))\end{aligned}$$

$$\begin{aligned}&= \max\{d_1(f_1^{i+1}(m_i), m_{i+1}), d_2(g_1^{i+1}(n_i), n_{i+1})\}\\ &\leq \delta.\end{aligned}$$

Thus, it follows that $\{(m_i, n_i)\}_{i=0}^{+\infty}$ is a δ pseudo-orbit for the product system $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$.

Since $f_{1,\infty}$ and $g_{1,\infty}$ possesses the ESP, then there exist a $p \in X$, a $q \in Y$ and $N_1, N_2 \in \mathbb{N}_0$ such that for all $i \geq N_1$, $d_1(f_1^i(p), m_i) \leq \varepsilon_1$, and for all $i \geq N_2$, $d_2(g_1^i(q), n_i) \leq \varepsilon_2$, choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, $N = \max\{N_1, N_2\}$, then

$$\begin{aligned}&\rho((f_1^i \times g_1^i)(p, q), (m_i, n_i))\\ &= \rho((f_1^i(p), g_1^i(q)), (m_i, n_i))\\ &= \max\{d_1(f_1^i(p), m_i), d_2(g_1^i(q), n_i)\}\\ &\leq \varepsilon\end{aligned}$$

for every $i \geq N$. This implies that the point $(p, q) \in X \times Y$ can eventually ε-shadow the δ pseudo-orbit $\{(m_i, n_i)\}_{i=0}^{+\infty}$. Consequently, it follows that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ possesses the ESP.

(Sufficiency) Assume that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has the ESP. Let $\varepsilon > 0$, then there exists a $\delta > 0$ such that any δ pseudo-orbit $\{(m_i, n_i)\}_{i=0}^{+\infty}$ of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ can be eventually ε-shadowed, i.e., there exist a $(p, q) \in X \times Y$ and an $N \in \mathbb{N}_0$ such that for all $i \geq N$, $\rho((f_1^i \times g_1^i)(p, q), (m_i, n_i)) \leq \varepsilon$. Since $\{(m_i, n_i)\}_{i=0}^{+\infty}$ is a δ pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$, then

$$\begin{aligned}&\rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1}))\\ &= \max\{d_1(f_1^{i+1}(m_i), m_{i+1}), d_2(g_1^{i+1}(n_i), n_{i+1})\}\\ &\leq \delta.\end{aligned}$$

Thus,

$$d_1(f_1^{i+1}(m_i), m_{i+1}) \leq \delta \quad \text{and} \quad d_2(g_1^{i+1}(n_i), n_{i+1}) \leq \delta.$$

Therefore, $\{m_i\}_{i=0}^{+\infty}$ and $\{n_i\}_{i=0}^{+\infty}$ are δ pseudo-orbits of $f_{1,\infty}$ and $g_{1,\infty}$, respectively. Since $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has the ESP, then there exist a $(p, q) \in X \times Y$ and an $N \in \mathbb{N}_0$ such that $\rho((f_1^i \times g_1^i)(p, q), (m_i, n_i)) \leq \varepsilon$ for every $i \geq N$, i.e.

$$\rho((f_1^i \times g_1^i)(p, q), (m_i, n_i)) = \max\{d_1(f_1^i(p), m_i), d_2(g_1^i(q), n_i)\} \leq \varepsilon$$

for every $i \geq N$. Thus,

$$d_1(f_1^i(p), m_i) \leq \varepsilon \quad \text{and} \quad d_2(g_1^i(q), n_i) \leq \varepsilon.$$

This implies that the point $p \in X$ can eventually ε-shadows the δ pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$. Consequently, it follows that $(X, d_1, f_{1,\infty})$ possesses the ESP. Similarly, it can also be proven that $(Y, d_2, g_{1,\infty})$ possesses the ESP.

(2) The method in (1) can also be easily used to prove this.

(3) (Necessity) Since $f_{1,\infty}$ possesses the s-LSP, then there exist an asymptotic δ_1 pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ and a δ_1 pseudo-orbit $\{c_i\}_{i=0}^{+\infty}$. Thus, $\{m_i\}_{i=0}^{+\infty}$ is both a δ_1 pseudo-orbit and an asymptotic pseudo-orbit of $(X, d_1, f_{1,\infty})$. So

$$d_1(f_{i+1}(m_i), m_{i+1}) < \delta_1 \quad \text{and} \quad \lim_{i\to\infty} d_1(f_{i+1}(m_i), m_{i+1}) = 0$$

for any $i \in \mathbb{N}_0$. Similarly, since $g_{1,\infty}$ possesses the s-LSP, then there exist an asymptotic δ_2 pseudo-orbit $\{n_i\}_{i=0}^{+\infty}$ and a δ_2 pseudo-orbit $\{d_i\}_{i=0}^{+\infty}$. Thus, $\{n_i\}_{i=0}^{+\infty}$ is both a δ_2 pseudo-orbit and an asymptotic pseudo-orbit of $(Y, d_2, g_{1,\infty})$. So

$$d_2(g_{i+1}(n_i), n_{i+1}) < \delta_2 \quad \text{and} \quad \lim_{i\to\infty} d_2(g_{i+1}(n_i), n_{i+1}) = 0$$

for any $i \in \mathbb{N}_0$. Choose $\delta = \max\{\delta_1, \delta_2\}$. Since

$$\rho((f_1^i \times g_1^i)(m_{i+1}, n_{i+1}), (m_i, n_i)) = \max\{d_1(f_1^i(m_{i+1}), m_i), d_2(g_1^i(n_{i+1}), n_i)\},$$

then,

$$\lim_{i\to\infty} \rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) = 0$$

and

$$\rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) < \delta$$

for any $i \in \mathbb{N}_0$. Thus, $\{(m_i, n_i)\}_{i=0}^{+\infty}$ is an asymptotic pseudo-orbit and a δ pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$. Therefore, $\{(m_i, n_i)\}_{i=0}^{+\infty}$ is an asymptotic δ pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$. It is easy to obtain that $\{(c_i, d_i)\}_{i=0}^{+\infty}$ is a δ pseudo-orbit of $f_{1,\infty} \times g_{1,\infty}$.

Let $f_{1,\infty}$ possesses the s-LSP, for any $\varepsilon_1 > 0$, one can find a $\delta_1 > 0$ such that

(i) there exists a point $p \in X$ such that p can ε_1-shadows any δ_1 pseudo-orbit $\{c_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$,

and

(ii) there exists a point $r \in X$ such that r can asymptotically ε_1-shadows any asymptotic δ_1 pseudo-orbit $\{m_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$, i.e.,

$$\lim_{i\to\infty} d_1(f_1^i(r), m_i) = 0$$

and $d_1(f_1^i(r), m_i) \leq \varepsilon_1$ for every $i \geq N$.

Similarly, let $g_{1,\infty}$ possesses the s-LSP, for any $\varepsilon_1 > 0$, one can find a $\delta_2 > 0$ such that

(i) there exists a point $q \in Y$ such that q can ε_2-shadows any δ_2 pseudo-orbit $\{d_i\}_{i=0}^{+\infty} \subset Y$ of $g_{1,\infty}$,

and

(ii) there exists a point $s \in Y$ such that s can asymptotically ε_2-shadows any asymptotic δ_2 pseudo-orbit $\{n_i\}_{i=0}^{+\infty} \subset Y$ of $g_{1,\infty}$, i.e., $\lim_{i\to\infty} d_2(g_1^i(s), n_i) = 0$ and $d_2(g_1^i(s), n_i) \leq \varepsilon_2$ for every $i \geq N$.

Choose $\delta = \max\{\delta_1, \delta_2\}$, $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Since

$$\rho((f_1^i \times g_1^i)(p, q), (m_i, c_i)) = \max\{d_1(f_1^i(p), m_i), d_2(g_1^i(q), c_i)\} < \varepsilon,$$

then,

(i) for every $\varepsilon > 0$, there exist a $\delta > 0$ and a point $(p, q) \in X \times Y$ such that (p, q) can ε-shadows any δ pseudo-orbit $\{(c_i, d_i)\}_{i=0}^{+\infty}$ of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$.

(ii) Since

$$\lim_{i\to\infty} \rho((f_1^i \times g_1^i)(r, s), (m_i, n_i)) = \lim_{i\to\infty} \max\{d_1(f_1^i(r), m_i), d_2(g_1^i(s), n_i)\} = 0$$

and

$$\rho((f_1^i \times g_1^i)(r, s), (m_i, n_i)) = \max\{d_1(f_1^i(r), m_i), d_2(g_1^i(s), n_i)\} < \varepsilon,$$

then, one can obtain that there is a point $(r, s) \in X \times Y$ such that (r, s) can asymptotically ε-shadows any asymptotic δ pseudo-orbit $\{m_i, n_i\}_{i=0}^{+\infty}$ of $f_{1,\infty} \times g_{1,\infty}$.

Combining (i) and (ii), it can be concluded that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ possesses the s-LSP.

(4) Let $\delta_1 > 0$ and $\delta_2 > 0$, for a δ_1 ergodic-pseudo orbit $\{m_i\}_{i=0}^{+\infty}$ of $(X, d_1, f_{1,\infty})$ and a δ_2 ergodic-pseudo orbit $\{n_i\}_{i=0}^{+\infty}$ of $(Y, d_2, g_{1,\infty})$, since

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d_1(f_{i+1}(m_i), m_{i+1}) < \delta_1\}| = 1$$

and

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d_2(g_{i+1}(n_i), n_{i+1}) < \delta_2\}| = 1,$$

choose $\delta = \max\{\delta_1, \delta_2\}$, then,

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : \rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) < \delta\}| \\ =&\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : \max\{d_1(f_{i+1}(m_i), m_{i+1}), d_2(g_{i+1}(n_i), n_{i+1})\} < \delta\}| \\ =&1.\end{aligned}$$

Thus, $\{(m_i, n_i)\}_{i=0}^{+\infty}$ is a δ ergodic pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$.

Let $\{(m_i, n_i)\}_{i=0}^{+\infty}$ be a δ ergodic pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$. That is, there exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : \rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) < \delta\}| = 1.$$

So,

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d_1(f_{i+1}(m_i), m_{i+1}) < \delta\}| = 1$$

and

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d_2(g_{i+1}(n_i), n_{i+1}) < \delta\}| = 1.$$

Thus, $\{m_i\}_{i=0}^{+\infty}$ is a δ ergodic pseudo-orbit of $(X, d_1, f_{1,\infty})$ and $\{n_i\}_{i=0}^{+\infty}$ is a δ ergodic pseudo-orbit of $(Y, d_2, g_{1,\infty})$.

(Necessity) Since $f_{1,\infty}$ and $g_{1,\infty}$ both possesses the $\mathcal{M}^\alpha$-SP. For every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exist a $\alpha_1 > 0$ and a $\alpha_2 > 0$ such that for every δ_1 ergodic pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ and every δ_2 ergodic pseudo-orbit $\{n_i\}_{i=0}^{+\infty}$ of $g_{1,\infty}$, there exist a point $p \in X$ and a point $q \in Y$ such that

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d_1(f_1^i(p), m_i) < \varepsilon_1\}| > \alpha_1$$

and

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : d_2(g_1^i(q), n_i) < \varepsilon_2\}| > \alpha_2.$$

Choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, $\alpha = \min\{\alpha_1, \alpha_2\}$. One can get that

$$\limsup_{n\to\infty} \frac{1}{n}|\{0 \le i < n : \rho((f_1^i \times g_1^i)(p, q), (m_i, n_i)) < \varepsilon\}| > \alpha.$$

Thus, for every $\varepsilon > 0$ there is a $\alpha > 0$ such that for every δ ergodic pseudo-orbit $\{(m_i, n_i)\}_{i=0}^{+\infty}$ of $f_{1,\infty} \times g_{1,\infty}$, there is a $(p, q) \in X \times Y$ such that

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \le i < n : \rho(f_1^i \times g_1^i(p, q), (m_i, n_i)) < \varepsilon\}| > \alpha.$$

Therefore, $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ possesses the $\mathcal{M}^{\alpha}$-SP.

(Sufficiency) Let $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ possesses the $\mathcal{M}^{\alpha}$-SP. For every $\varepsilon > 0$, there exists a $\alpha > 0$ such that for every δ ergodic pseudo-orbit $\{m_i \times n_i\}_{i=0}^{+\infty}$ of $f_{1,\infty} \times g_{1,\infty}$, there is a $(p, q) \in X \times Y$ satisfying that

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \le i < n : \rho((f_1^i \times g_1^i)(p, q), (m_i, n_i)) < \varepsilon\}| > \alpha.$$

Then,

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \le i < n : d_1(f_1^i(p), m_i) < \varepsilon\}| > \alpha$$

and

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \le i < n : d_2(g_1^i(q), n_i) < \varepsilon\}| > \alpha.$$

Thus, for every $\varepsilon > 0$, there exists a $\alpha > 0$ such that for every δ ergodic pseudo-orbit $\{m_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$, there is a $p \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \le i < n : d_1(f_1^i(p), m_i) < \varepsilon\}| > \alpha.$$

Therefore, $f_{1,\infty}$ possesses the $\mathcal{M}^{\alpha}$-SP. Similarly, $g_{1,\infty}$ also possesses the $\mathcal{M}^{\alpha}$-SP.

Using similar methods, one can also draw conclusions about the $\mathcal{M}_{\alpha}$-SP.

(5) Let $\delta_1 > 0$ and $\delta_2 > 0$, then for an almost δ_1 average pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ of $(X, d_1, f_{1,\infty})$ and an almost δ_2 average pseudo-orbit $\{n_i\}_{i=0}^{+\infty}$ of $(Y, d_2, g_{1,\infty})$, since

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_1(f_{i+1}(m_i), m_{i+1}) < \delta_1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_2(f_{i+1}(n_i), n_{i+1}) < \delta_2,$$

choose $\delta = \max\{\delta_1, \delta_2\}$, then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) < \delta.$$

Thus, $\{(m_i, n_i)\}_{i=0}^{+\infty}$ is an almost δ average pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$.

By the definition of almost δ average pseudo-orbit,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) < \delta.$$

Since

$$\rho((f_1^{i+1} \times g_1^{i+1})(m_i, n_i), (m_{i+1}, n_{i+1})) = \max\{d_1(f_{i+1}(m_i), m_{i+1}), d_2(g_{i+1}(n_i), n_{i+1})\},$$

then,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_1(f_{i+1}(m_i),m_{i+1})<\delta \quad\text{and}\quad \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_2(f_{i+1}(n_i),n_{i+1})<\delta.$$

So, $\{m_i\}_{i=0}^{+\infty}$ is an almost δ average pseudo-orbit of $(X,d_1,f_{1,\infty})$ and $\{n_i\}_{i=0}^{+\infty}$ is an almost δ average pseudo-orbit of $(Y,d_2,g_{1,\infty})$.

(Necessity) Sine $f_{1,\infty}$ and $g_{1,\infty}$ possesses the ALASP. For any $\varepsilon_1>0$ and $\varepsilon_2>0$, there exist a $\delta_1>0$ and a $\delta_2>0$ such that for every almost δ_1 average pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$ and every almost δ_2 average pseudo orbit $\{n_i\}_{i=0}^{+\infty}$ of $g_{1,\infty}$, there exist a $p\in X$ and a $q\in Y$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_1(f_1^i(p),m_i)<\varepsilon_1 \quad\text{and}\quad \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_2(g_1^i(q),n_i)<\varepsilon_2.$$

Choose $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$. Then,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\rho((f_1^i\times g_1^i)(p,q),(m_i,n_i))\\ =&\lim_{i\to\infty}\sup\frac{1}{n}\sum_{i=0}^{n-1}\max\{d_1(f_1^i(p),m_i),d_2(g_1^i(q),n_i)\}\\ <&\varepsilon.\end{aligned}$$

Thus, for every $\varepsilon>0$ there is a $\delta>0$ such that for every almost δ average pseudo-orbit $\{m_i,n_i\}_{i=0}^{+\infty}\subset X\times Y$ of $f_{1,\infty}\times g_{1,\infty}$, there is a $(p,q)\in X\times Y$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\rho((f_1^i\times g_1^i)(p,q),(m_i,n_i))<\varepsilon.$$

Thus, $(X\times Y,\rho,f_{1,\infty}\times g_{1,\infty})$ possesses the ALASP.

(Sufficiency) Let $f_{1,\infty}\times g_{1,\infty}$ possesses the ALASP, that is for every $\varepsilon>0$, there is a $\delta>0$ such that for every almost δ average pseudo-orbit $\{(m_i,n_i)\}_{i=0}^{+\infty}\subset X\times Y$ of $f_{1,\infty}\times g_{1,\infty}$, there is a $(p,q)\in X\times Y$ satisfying that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\rho((f_1^i\times g_1^i)(p,q),(m_i,n_i))<\varepsilon.$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_1(f_1^i(p),m_i)<\varepsilon \quad\text{and}\quad \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_2(y_1^i(q),n_i)<\varepsilon.$$

Then, for every $\varepsilon>0$, there exist a $\delta>0$ and a point $p\in X$ such that p ε-shadows in average any almost δ average pseudo-orbit $\{m_i\}_{i=0}^{+\infty}\subset X$ of $f_{1,\infty}$, i.e.,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}d_1(f_1^i(p),m_i)<\varepsilon.$$

Thus, $f_{1,\infty}$ possesses the ALASP. Similarly, it can be inferred that $g_{1,\infty}$ also possesses the ALASP.

(6) The proof is similar to (5). □

Theorem 4.3. *Let* $(X, d_1, f_{1,\infty})$ *and* $(Y, d_2, g_{1,\infty})$ *are two T-VDDSs. If* $f_{1,\infty}$ *is topologically conjugate to* $g_{1,\infty}$*, then*

(1) $(X, d_1, f_{1,\infty})$ *has ESP if and only if* $(Y, d_2, g_{1,\infty})$ *also has ESP;*

(2) $(X, d_1, f_{1,\infty})$ *has LSP if and only if* $(Y, d_2, g_{1,\infty})$ *also has LSP;*

(3) $(X, d_1, f_{1,\infty})$ *has s-LSP if and only if* $(Y, d_2, g_{1,\infty})$ *also has s-LSP;*

(4) $(X, d_1, f_{1,\infty})$ *has* $\mathcal{M}^\alpha$*-SP(or* $\mathcal{M}_\alpha$*-SP) if and only if* $(Y, d_2, g_{1,\infty})$ *also have* $\mathcal{M}^\alpha$*-SP (or* $\mathcal{M}_\alpha$*-SP);*

(5) $(X, d_1, f_{1,\infty})$ *has ALASP if and only if* $(Y, d_2, g_{1,\infty})$ *also has ALASP;*

(6) $(X, d_1, f_{1,\infty})$ *has* $\{n_i\}_{i=0}^{+\infty}$*-AASP if and only if* $(Y, d_2, g_{1,\infty})$ *also has* $\{n_i\}_{i=0}^{+\infty}$*-AASP.*

Proof. (1) (Necessity) Assume $(X, d_1, f_{1,\infty})$ possesses the ESP. For every $\varepsilon_1 > 0$, there exist a $\delta_1 > 0$ and a $p \in X$ such that p can eventually ε_1-shadows any δ_1 pseudo-orbit $\{m_i\}_{i=0}^{+\infty}$ of $(X, d_1, f_{1,\infty})$. That is, there exist a $p \in X$ and an $N \in \mathbb{N}_0$ such that $d_1(f_1^i(p), m_i) \le \varepsilon_1$ for every $i \ge N$. Since $f_{1,\infty}$ is topologically conjugate to $g_{1,\infty}$, there exists a homeomorphism $\sigma : X \to Y$ such that $\sigma \circ f_n = g_n \circ \sigma$ for all n. By the uniform continuity of σ, for every $\delta_2 > 0$ and $x_1, y_1 \in X$, there is a $\delta_1 > 0$ such that $d_1(x_1, y_1) < \delta_1$ implies $d_2(\sigma(x_1), \sigma(y_1)) < \delta_2$.

Let $\sigma(m_i) = n_i$. The following prove that $\{n_i\}_{i=0}^{+\infty}$ is a δ_2 pseudo-orbit of $(Y, d_2, g_{1,\infty})$.

Since $d_1(f_1^{i+1}(m_i), m_{i+1}) < \delta_1$ if and only if $d_2(\sigma(f_1^{i+1}(m_i)), \sigma(m_{i+1})) < \delta_2$ if and only if $d_2(g_1^{i+1}(\sigma(m_i)), n_{i+1}) < \delta_2$ if and only if $d_2(g_1^{i+1}(n_i), n_{i+1}) < \delta_2$, then, $\{n_i\}_{i=0}^{+\infty}$ is a δ_2 pseudo-orbit of $(Y, d_2, g_{1,\infty})$. By the uniform continuity of σ, for any $\varepsilon_2 > 0$, there is a $\varepsilon_1 > 0$ such that $d_1(x_1, y_1) < \varepsilon_1$ implies $d_2(\sigma(x_1), \sigma(y_1)) < \varepsilon_2$. Since $d_1(f_1^i(p), m_i) < \varepsilon_1$ if and only if $d_2(\sigma(f_1^i(p)), \sigma(m_i)) < \varepsilon_2$ if and only if $d_2(g_1^i(\sigma(p)), n_i) < \varepsilon_2$ for every $i \ge N$, then, for $\varepsilon_2 > 0$, there exists a $\delta_2 > 0$ such that every δ_2 pseudo-orbit $\{n_i\}_{i=0}^{+\infty}$ of $(Y, d_2, g_{1,\infty})$ can be eventually ε_2-shadowed by some points of $(Y, d_2, g_{1,\infty})$. That is, there exist a $\sigma(p) \in Y$ and an $N \in \mathbb{N}_0$ such that $d_2(g_1^i(\sigma(p)), n_i) \le \varepsilon_2$ for every $i \ge N$. Thus, $(Y, d_2, g_{1,\infty})$ possesses the ESP.

(Sufficiency) This is obvious.

(2) The proof is similar to (1).

(3) (Necessity) Since $(X, d_1, f_{1,\infty})$ possesses the s-LSP, then for every $\varepsilon_1 > 0$, one can find a $\delta_1 > 0$ such that

(i) there exists a $p \in X$ such that p can ε_1-shadows any δ_1 pseudo-orbit $\{m_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$,

and

(ii) there exists a $q \in X$ such that q can asymptotically ε_1-shadows any asymptotic δ_1 pseudo-orbit $\{n_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$.

Since $f_{1,\infty}$ is topologically conjugate to $g_{1,\infty}$, there exists a homeomorphism $\sigma : X \to Y$ such that $\sigma \circ f_n = g_n \circ \sigma$ for all n. By the uniform continuity of σ, for every $\delta_2 > 0$ and any $x_1, y_1 \in X$, there is a $\delta_1 > 0$ such that $d_1(x_1, y_1) < \delta_1$ implies $d_2(\sigma(x_1), \sigma(y_1)) < \delta_2$. Let $\sigma(m_i) = c_i$ and $\sigma(n_i) = d_i$. One can infer from the proof of (1) that $\{c_i\}_{i=0}^{+\infty}$ is a δ_2 pseudo-orbit of $(Y, d_2, g_{1,\infty})$. The following proves that $\{d_i\}_{i=0}^{+\infty}$ is an asymptotic δ_2 pseudo-orbit of $(Y, d_2, g_{1,\infty})$.

Since $d_1(f_1^{i+1}(n_i), n_{i+1}) < \delta_1$, then $d_2(\sigma(f_1^{i+1}(n_i)), \sigma(n_{i+1})) < \delta_2$. That is,

$$d_2(\sigma(f_1^{i+1}(n_i)), \sigma(n_{i+1})) = d_2(g_1^{i+1}(\sigma(n_i)), d_{i+1}) = d_2(g_1^{i+1}(d_i), d_{i+1}) < \delta_2.$$

And because

$$\begin{aligned}\lim_{i\to\infty} d_1(f_{i+1}(n_i), n_{i+1}) &= \lim_{i\to\infty} d_2(\sigma(f_{i+1}(n_i)), \sigma(n_{i+1}))\\ &= \lim_{i\to\infty} d_2(g_{i+1}(\sigma(n_i)), d_{i+1})\end{aligned}$$

$$
\begin{aligned}
&= \lim_{i\to\infty} d_2(g_{i+1}(d_i), d_{i+1}) \\
&=0,
\end{aligned}
$$

thus, $\{d_i\}_{i=0}^{+\infty}$ is an asymptotic δ_2 pseudo-orbit of $(Y, d_2, g_{1,\infty})$. It can be obtained that $\{d_i\}_{i=0}^{+\infty}$ is an asymptotic δ_2 pseudo-orbit of $(Y, d_2, g_{1,\infty})$. In fact, let $(X, d_1, f_{1,\infty})$ have s-limit shadowing. For every $\varepsilon_1 > 0$, one can find a $\delta_1 > 0$ such that

(i) there exists a point $p \in X$ such that p can ε_1-shadows any δ_1 pseudo-orbit $\{m_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$, i.e., there exist a $p \in X$ and an $N \in \mathbb{N}_0$ such that $d_1(f_1^i(p), m_i) \leq \varepsilon_1$ for every $i \geq N$, and

(ii) there exists a point $q \in X$ such that q can asymptotically ε_1-shadows any asymptotic δ_1 pseudo-orbit $\{n_i\}_{i=0}^{+\infty} \subset X$ of $f_{1,\infty}$, i.e., there exist a $q \in X$ and an $N \in \mathbb{N}_0$ such that

$$\lim_{i\to\infty} d_1(f_1^i(q), n_i) = 0 \quad \text{and} \quad d_1(f_1^i(q), n_i) \leq \varepsilon_1$$

for every $i \geq N$.

By the uniform continuity of σ, for any $\varepsilon_2 > 0$, there is a $\varepsilon_1 > 0$ such that $d_1(x_1, y_1) < \varepsilon_1$ implies $d_2(\sigma(x_1), \sigma(y_1)) < \varepsilon_2$. Since $d_1(f_1^i(p), m_i) < \varepsilon_1$, then

$$d_2(\sigma(f_1^i(p)), \sigma(m_i)) = d_2(g_1^i(\sigma(p)), c_i) < \varepsilon_2.$$

For $\varepsilon_2 > 0$, there exist a $\delta_2 > 0$ and a $\sigma(p) \in Y$ such that $\sigma(p)$ can ε_2-shadows any δ_2 pseudo-orbit $\{c_i\}_{i=0}^{+\infty}$ of $(Y, d_2, g_{1,\infty})$, i.e., there is an $N \in \mathbb{N}_0$ such that $d_2(g_1^i(\sigma(p)), c_i) \leq \varepsilon_2$ for every $i \geq N$. Since $d_1(f_1^i(q), n_i) < \varepsilon_1$, then

$$d_2(\sigma(f_1^i(q)), \sigma(n_i)) = d_2(g_1^i(\sigma(q)), d_i) < \varepsilon_2.$$

And because

$$\lim_{i\to\infty} d_1(f_{i+1}(q), n_i) = \lim_{i\to\infty} d_2(\sigma(f_{i+1}(q)), \sigma(n_i)) = \lim_{i\to\infty} d_2(g_{i+1}(\sigma(q)), d_i) = 0,$$

similarly, for $\varepsilon_2 > 0$, there exist a $\delta_2 > 0$ and a $\sigma(q) \in Y$ such that $\sigma(q)$ can asymptotically ε_2-shadows any asymptotic δ_2 pseudo-orbit $\{d_i\}_{i=0}^{+\infty} \subset Y$ of $g_{1,\infty}$.

Thus, $(Y, d_2, g_{1,\infty})$ possesses the s-LSP.

(Sufficiency) It is obvious.

The proofs of (4)-(6) are similar. □

5. The retentivity of some other chaotic characteristics

Theorem 5.1. *Let $f_{1,\infty}$ and $g_{1,\infty}$ be two sequences of homeomorphisms on compact metric spaces $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, respectively. Then*

(1) $f_{1,\infty}$ and $g_{1,\infty}$ both are expansive if and only if $f_{1,\infty} \times g_{1,\infty}$ is expansive;

(2) $f_{1,\infty}$ and $g_{1,\infty}$ both are persistent if and only if $f_{1,\infty} \times g_{1,\infty}$ is persistent;

(3) $f_{1,\infty}$ and $g_{1,\infty}$ both are topologically stable if and only if $f_{1,\infty} \times g_{1,\infty}$ is topologically stable;

(4) $f_{1,\infty}$ and $g_{1,\infty}$ both are linked if and only if $f_{1,\infty} \times g_{1,\infty}$ is linked.

Proof. (1) (Necessity) Let $f_{1,\infty}$ and $g_{1,\infty}$ are expansive. For any $p, q \in X$, there exists an $\varepsilon_1 > 0$ such that $d_1(f_1^i(p), f_1^i(q)) \leq \varepsilon_1$ for all $i \in \mathbb{N}_0$ implies $p = q$. Similarly, there exists an

$\varepsilon_2 > 0$ such that $d_2(g_1^i(r), g_1^i(s)) \leq \varepsilon_2$ for all $i \in \mathbb{N}_0$ implies $r = s$. Choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Then,

$$\begin{aligned}\rho((f_1^i \times g_1^i)(p,r), (f_1^i \times g_1^i)(q,s)) &= \rho((f_1^i(p), g_1^i(r)), (f_1^i(q), g_1^i(s)))\\ &= \max\{d_1(f_1^i(p), f_1^i(q)), d_2(g_1^i(r), g_1^i(s))\}\\ &\leq \varepsilon.\end{aligned}$$

Since

$$d_1(f_1^i(p), f_1^i(q)) \leq \varepsilon_1 \quad \text{and} \quad d_2(g_1^i(r), g_1^i(s)) \leq \varepsilon_2$$

implies

$$\rho((f_1^i \times g_1^i)(p,r), (f_1^i \times g_1^i)(q,s)) \leq \varepsilon,$$

the above equations imply $p = q$ and $r = s$, that is $(p,r) = (q,s)$. Therefore, there exists an $\varepsilon > 0$ such that, $\rho((f_1^i \times g_1^i)(p,r), (f_1^i \times g_1^i)(q,s)) \leq \varepsilon$ implies $(p,r) = (q,s)$. Hence, $f_{1,\infty} \times g_{1,\infty}$ is expansive.

(Sufficiency) Let $f_{1,\infty} \times g_{1,\infty}$ is expansive. There exists an $\varepsilon > 0$ such that $\rho((f_1^i \times g_1^i)(p,r), (f_1^i \times g_1^i)(q,s)) \leq \varepsilon$ for all $i \in \mathbb{N}_0$ implies $(p,r) = (q,s)$. Since

$$\rho((f_1^i \times g_1^i)(p,r), (f_1^i \times g_1^i)(q,s)) = \max\{d_1(f_1^i(p), f_1^i(q)), d_2(g_1^i(r), g_1^i(s))\} \leq \varepsilon,$$

then,

$$d_1(f_1^i(p), f_1^i(q)) \leq \varepsilon \quad \text{and} \quad d_2(g_1^i(r), g_1^i(s)) \leq \varepsilon$$

imply $p = q$ and $r = s$. Hence, there exists an $\varepsilon > 0$ such that $d_1(f_1^i(p), f_1^i(q)) \leq \varepsilon$ implies $p = q$ and $d_2(g_1^i(r), g_1^i(s)) \leq \varepsilon$ implies $r = s$. Therefore, $f_{1,\infty}$ and $g_{1,\infty}$ are expansive.

(2) (Necessity) Let $f_{1,\infty}$ be α-persistent. For any a $\varepsilon_1 > 0$, there exists a $\delta_1 > 0$ such that, if $d_0(f_{1,\infty}, \varphi_{1,\infty}) = \sup\{d_1(f_1^i(p), \varphi_1^i(p)) : p \in X\} < \delta_1$, then there is an $r \in X$ satisfying $d_1(f_1^i(r), \varphi_1^i(p)) < \varepsilon_1$ for all $i \in \mathbb{N}_0$. Similarly, let $g_{1,\infty}$ be α-persistent. For any $\varepsilon_2 > 0$, there exists a $\delta_2 > 0$ such that if $d_0(g_{1,\infty}, \phi_{1,\infty}) = \sup\{d_2(g_1^i(q), \phi_1^i(q)) : q \in Y\} < \delta_2$, then there is a $s \in Y$ satisfying $d_2(g_1^i(s), \phi_1^i(q)) < \varepsilon_2$ for all $i \in \mathbb{N}_0$. Let $d_0(f_{1,\infty} \times g_{1,\infty}, \varphi_{1,\infty} \times \phi_{1,\infty}) = \sup\{\rho((f_1^i \times g_1^i)(p,q), (\varphi_1^i \times \phi_1^i)(p,q)) : (p,q) \in X \times Y\}$ for all $i \in \mathbb{N}_0$. Choose $\delta = \max\{\delta_1, \delta_2\}$ and $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$. Since

$$d_0(f_{1,\infty}, \varphi_{1,\infty}) = \sup\{d_1(f_1^i(p), \varphi_1^i(p)) : p \in X\} < \delta_1$$

and

$$d_0(g_{1,\infty}, \phi_{1,\infty}) = \sup\{d_2(g_1^i(q), \phi_1^i(q)) : q \in Y\} < \delta_2,$$

then,

$$\begin{aligned}&d_0(f_{1,\infty} \times g_{1,\infty}, \varphi_{1,\infty} \times \phi_{1,\infty})\\ &= \sup\{\rho((f_1^i \times g_1^i)(p,q), (\varphi_1^i \times \phi_1^i)(p,q)) : (p,q) \in X \times Y\}\\ &= \sup\{\rho((f_1^i(p), g_1^i(q)), (\varphi_1^i(p), \phi_1^i(q))) : p \in X, q \in Y\}\\ &= \sup\{\max\{d_1(f_1^i(p), \varphi_1^i(p)), d_2(g_1^i(q), \phi_1^i(q))\} : p \in X, q \in Y\}\\ &< \delta.\end{aligned}$$

So, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_0(f_{1,\infty} \times g_{1,\infty}, \varphi_{1,\infty} \times \phi_{1,\infty}) < \delta$ and for $(p,q) \in X \times Y$. By definition, there exists an $r \in X$ satisfying $d_1(f_1^i(r), \varphi_1^i(p)) < \varepsilon_1$ for all

$i \in \mathbb{N}_0$, and a $s \in Y$ satisfying $d_2(g_1^i(s), \phi_1^i(q)) < \varepsilon_2$ for all $i \in \mathbb{N}_0$. Therefore, for $(r,s) \in X \times Y$, one can get that

$$\begin{aligned}\rho((f_1^i \times g_1^i)(r,s), (\varphi_1^i \times \phi_1^i)(p,q)) &= \rho((f_1^i(r), g_1^i(s)), (\varphi_1^i(p), \phi_1^i(q))) \\ &= \max\{d_1(f_1^i(r), \varphi_1^i(p)), d_2(g_1^i(s), \phi_1^i(q))\} \\ &< \varepsilon.\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d_0(f_{1,\infty} \times g_{1,\infty}, \varphi_{1,\infty} \times \phi_{1,\infty}) < \delta$ and $(p,q) \in X \times Y$, then one can find a $(r,s) \in X \times Y$ satisfying

$$\rho((f_1^i \times g_1^i)(r,s), (\varphi_1^i \times \phi_1^i)(p,q)) < \varepsilon$$

for all $i \in \mathbb{N}_0$. Therefore, $f_{1,\infty} \times g_{1,\infty}$ is α-persistent.

(Sufficiency) Let $f_{1,\infty} \times g_{1,\infty}$ be α-persistent. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d_0(f_{1,\infty} \times g_{1,\infty}, \varphi_{1,\infty} \times \phi_{1,\infty}) < \delta$ and $(p,q) \in X \times Y$, then there is a $(r,s) \in X \times Y$ satisfying

$$\rho((f_1^i \times g_1^i)(r,s), (\varphi_1^i \times \phi_1^i)(p,q)) < \varepsilon$$

for all $i \in \mathbb{N}_0$. Since

$$\begin{aligned}&d_0(f_{1,\infty} \times g_{1,\infty}, \varphi_{1,\infty} \times \phi_{1,\infty}) \\ &= \sup\{\rho((f_1^i \times g_1^i)(p,q), (\varphi_1^i \times \phi_1^i)(p,q)) : (p,q) \in X \times Y\} \\ &= \sup\{\max\{d_1(f_1^i(p), \varphi_1^i(p)), d_2(g_1^i(q), \phi_1^i(q))\} : p \in X, q \in Y\}\} \\ &< \delta,\end{aligned}$$

then

$$d_0(f_{1,\infty}, \varphi_{1,\infty}) = \sup\{d_1(f_1^i(p), \varphi_1^i(p)) : p \in X\} < \delta$$

and

$$d_0(g_{1,\infty}, \phi_{1,\infty}) = \sup\{d_2(g_1^i(q), \phi_1^i(q)) : q \in Y\} < \delta.$$

Since there exists a $(r,s) \in X \times Y$ satisfying

$$\rho((f_1^i \times g_1^i)(r,s), (\varphi_1^i \times \phi_1^i)(p,q)) < \varepsilon,$$

then

$$d_1(f_1^i(r), \varphi_1^i(p)) < \varepsilon \quad \text{and} \quad d_2(g_1^i(s), \phi_1^i(q)) < \varepsilon.$$

Hence, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d_0(f_{1,\infty}, \varphi_{1,\infty}) < \delta$ and $p \in X$, then there is an $r \in X$ satisfying $d_1(f_1^i(r), \varphi_1^i(p)) < \varepsilon$ for all $i \in \mathbb{N}_0$. Therefore, $f_{1,\infty}$ is α-persistent, and $g_{1,\infty}$ is α-persistent too.

Similarly, it is not difficult to show that $f_{1,\infty}$ and $g_{1,\infty}$ are β-persistent if and only if $f_{1,\infty} \times g_{1,\infty}$ is β-persistent. Hence, $f_{1,\infty}$ and $g_{1,\infty}$ are persistent if and only if $f_{1,\infty} \times g_{1,\infty}$ is persistent.

(3) (Necessity) Let $f_{1,\infty}$ be topologically stable on $(X, d_1, f_{1,\infty})$. For every $\varepsilon_1 > 0$, there exists a $\delta_1 > 0$ such that if $\psi_{1,\infty} : X \to X$ is a homeomorphism map sequence with

$$d_1(f_{1,\infty}, \psi_{1,\infty}) = \sup_{p \in X} d_1(f_1^i(p), \psi_1^i(p)) \leq \delta_1 \quad (i \in \mathbb{N}_0),$$

then there is a continuous map $\mu : X \to X$ with $d(f_1^i(\mu(p)), \psi_1^i(p)) < \varepsilon_1$ and $d(\mu(p), \mathrm{Id}_X) \leq \varepsilon_1$ for any $p \in X$ and $i \in \mathbb{N}_0$.

Let $g_{1,\infty}$ also be topologically stable on $(Y, d_2, g_{1,\infty})$. For every $\varepsilon_2 > 0$, there exists a $\delta_2 > 0$ such that if $\omega_{1,\infty} : Y \to Y$ is a homeomorphism map sequence with

$$d_2(g_{1,\infty}, \omega_{1,\infty}) = \sup_{q \in Y} d_2(g_1^i(q), \omega_1^i(q)) \leq \delta_2 \quad (i \in \mathbb{N}_0),$$

then there is a continuous map $\nu : Y \to Y$ with $d(g_1^i(\nu(q)), \omega_1^i(q)) < \varepsilon_2$ and $d_2(\nu(q), \mathrm{Id}_Y) \leq \varepsilon_2$ for any $q \in Y$ and $i \in \mathbb{N}_0$.

Let $f_{1,\infty} \times g_{1,\infty} : X \times Y \to X \times Y$ be a homeomorphism of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$. Choose $\delta = \max\{\delta_1, \delta_2\}$, and let $\psi_{1,\infty} \times \omega_{1,\infty} : X \times Y \to X \times Y$ be any homeomorphism of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ satisfying

$$\begin{aligned} \rho(f_{1,\infty} \times g_{1,\infty}, \psi_{1,\infty} \times \omega_{1,\infty}) &= \sup_{(p,q) \in X \times Y} \rho((f_1^i \times g_1^i)(p,q), (\psi_1^i \times \omega_1^i)(p,q)) \\ &= \sup_{(p,q) \in X \times Y} \rho((f_1^i(p), g_1^i(q)), (\psi_1^i(p), \omega_1^i(q))) \\ &= \sup_{p \in X, q \in Y} \max\left\{d_1(f_1^i(p), \psi_1^i(p)), d_2(g_1^i(q), \omega_1^i(q))\right\} \\ &< \delta \end{aligned}$$

for $i \in \mathbb{N}_0$. Choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, and let $\mu \times \nu : X \to X$ be a continuous map which satisfies

$$\begin{aligned} &\rho((f_1^i \times g_1^i)(\mu(p), \nu(q)), (\psi_1^i \times \omega_1^i)(p,q)) \\ &= \rho((f_1^i(\mu(p)), g_1^i(\nu(q))), (\psi_1^i(p), \omega_1^i(q))) \\ &= \max\left\{d_1(f_1^i(\mu(p)), \psi_1^i(p)), d_2(g_1^i(\nu(q)), \omega_1^i(q))\right\} \\ &< \varepsilon \end{aligned}$$

and

$$\begin{aligned} \rho((\mu \times \nu)(p,q), \mathrm{Id}_{X \times Y}) &= \rho(\mu(p) \times \nu(q), \mathrm{Id}_X \times \mathrm{Id}_Y) \\ &= \max\{d_1(\mu(p), \mathrm{Id}_X), d_2(\nu(q), \mathrm{Id}_Y)\} \\ &\leq \varepsilon \end{aligned}$$

for any $(p,q) \in X \times Y$ and $i \in \mathbb{N}_0$. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\psi_{1,\infty} \times \omega_{1,\infty} : X \times Y \to X \times Y$ is a homeomorphism map sequence with

$$\rho(f_{1,\infty} \times g_{1,\infty}, \psi_{1,\infty} \times \omega_{1,\infty}) = \sup_{(p,q) \in X \times Y} \rho((f_1^i \times g_1^i)(p,q), (\psi_1^i \times \omega_1^i(p,q)) < \delta,$$

then, there is a continuous map $\mu \times \nu : X \times Y \to X \times Y$ satisfying

$$\rho((f_1^i \times g_1^i)(\mu(p), \nu(q)), (\psi_1^i \times \omega_1^i)(p,q)) < \varepsilon$$

and

$$\rho((\mu \times \nu)(p,q), \mathrm{Id}_{X \times Y}) < \varepsilon$$

for any $(p,q) \in X \times Y$ and $i \in \mathbb{N}_0$. Therefore, $f_{1,\infty} \times g_{1,\infty}$ is topologically stable.

(Sufficiency) Let $f_{1,\infty} \times g_{1,\infty}$ be topologically stable. Thus, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\psi_{1,\infty} \times \omega_{1,\infty} : X \times Y \to X \times Y$ is a homeomorphism map sequence with

$$\rho(f_{1,\infty} \times g_{1,\infty}, \psi_{1,\infty} \times \omega_{1,\infty}) = \sup_{(p,q) \in X \times Y} \rho((f_1^i \times g_1^i)(p,q), (\psi_1^i \times \omega_1^i)(p,q)) < \delta \quad (i \in \mathbb{N}_0),$$

then there is a continuous product map $\mu \times \nu : X \times Y \to X \times Y$ satisfying

$$\rho((f_1^i \times g_1^i)(\mu(p), \nu(q)), (\psi_1^i \times \omega_1^i)(p, q)) < \varepsilon$$

and

$$\rho((\mu \times \nu)(p, q), \mathrm{Id}_{X\times Y}) < \varepsilon$$

for any $(p, q) \in X \times Y$ and $i \in \mathbb{N}_0$.

Let $f_{1,\infty}$ be a homeomorphism map sequence of $(X, d_1, f_{1,\infty})$, $\psi_{1,\infty} : X \to X$ be a homeomorphism map sequence of $(X, d_1, f_{1,\infty})$, $g_{1,\infty}$ be a homeomorphism map sequence of $(Y, d_2, g_{1,\infty})$, and $\omega_{1,\infty} : Y \to Y$ be any homeomorphism map sequence of $(Y, d_2, g_{1,\infty})$. Then, $\psi_{1,\infty} \times \omega_{1,\infty} : X \times Y \to X \times Y$ is a homeomorphism map sequence of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$, and since

$$\begin{aligned}
&\rho(f_{1,\infty} \times g_{1,\infty}, \psi_{1,\infty} \times \omega_{1,\infty}) \\
=&\sup_{p\in X, q\in Y} \max\left\{d_1(f_1^i(p), \psi_1^i(p)), d_2(g_1^i(q), \omega_1^i(q))\right\} \\
<&\delta
\end{aligned}$$

for $i \in \mathbb{N}_0$, then,

$$d_1(f_{1,\infty}, \psi_{1,\infty}) = \sup_{p\in X} d_1(f_1^i(p), \psi_1^i(p)) \leq \delta(i \in \mathbb{N}_0)$$

and

$$d_2(g_{1,\infty}, \omega_{1,\infty}) = \sup_{q\in Y} d_2(g_1^i(q), \omega_1^i(q)) \leq \delta \quad (i \in \mathbb{N}_0).$$

Since $\mu \times \nu : X \times Y \to X \times Y$ is a continuous map with

$$\rho((f_1^i \times g_1^i)(\mu(p), \nu(q)), (\psi_1^i \times \omega_1^i)(p, q)) < \varepsilon,$$

where

$$\begin{aligned}
&\rho((f_1^i \times g_1^i)(\mu(p), \nu(q)), (\psi_1^i \times \omega_1^i)(p, q)) \\
=&\max\left\{d_1(f_1^i(\mu(p)), \psi_1^i(p)), d_2(g_1^i(\nu(q)), \omega_1^i(q))\right\} \\
<&\varepsilon.
\end{aligned}$$

Then,

$$d(f_1^i(\mu(p)), \psi_1^i(p)) < \varepsilon \quad \text{and} \quad d(g_1^i(\nu(q)), \omega_1^i(q)) < \varepsilon.$$

Since

$$\rho((\mu \times \nu)(p, q), \mathrm{Id}_{X\times Y}) = \max\{d_1(\mu(p), \mathrm{Id}_X), d_2(\nu(q), \mathrm{Id}_Y)\} \leq \varepsilon,$$

then

$$d_1(\mu(p), \mathrm{Id}_X) \leq \varepsilon \quad \text{and} \quad d_2(\nu(q), \mathrm{Id}_Y) \leq \varepsilon,$$

where $p \in X$, $q \in Y$, and $n \in \mathbb{N}$. Hence, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\psi_{1,\infty} : X \to X$ is any homeomorphism with

$$d_1(f_{1,\infty}, \psi_{1,\infty}) = \sup_{p\in X} d_1(f_1^i(p), \psi_1^i(p)) \leq \delta \quad (i \in \mathbb{N}_0),$$

then there is a continuous map $\mu : X \to X$ satisfying $d(f_1^i(\mu(p)), \psi_1^i(p)) < \varepsilon$ and $d_1(\mu(p), \mathrm{Id}_X) \leq \varepsilon$ for any $p \in X$ and $n \in \mathbb{N}$. So, the homeomorphism map sequence $f_{1,\infty}$ is topologically stable on $(X, d_1, f_{1,\infty})$. Similarly, the homeomorphism map sequence $g_{1,\infty}$ is topologically stable too.

(4) In this proof, the set should have invariance.

(Necessity) Let a set $A \subset X$ be linked by $f_{1,\infty}$. That is, for any $\varepsilon_1 > 0$ and any point $p \in A$, p is ε_1-linked to a point $q \in A$ by $f_{1,\infty}$, i.e., there exist an integer $m_1 \geq 1$ and a point r such that $f_1^{m_1}(r) = q$ and $d(f_1^i(p), f_1^i(r)) \leq \varepsilon_1$ for $i = 0, \ldots, m_1$. Similarly, a set $B \subset Y$ is linked by $g_{1,\infty}$. That is, for any $\varepsilon_2 > 0$ and any point $u \in B$, u is ε_2-linked to a point $v \in B$ by $g_{1,\infty}$, i.e., there exist an integer $m_2 \geq 1$ and a point s such that $g_1^{m_2}(s) = v$ and $d(g_1^i(u), g_1^i(s)) \leq \varepsilon_2$ for $i = 0, \ldots, m_2$. Choose $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and $m = \min\{m_1, m_2\}$. Assume that $m_1 < m_2$. For every $\varepsilon > 0$, there exists an integer $m \geq 1$, for any point $(p, u) \in A \times B$, there is a point (r, s) such that

$$f_1^m(r) \times g_1^m(s) = (f_1^m(r), g_1^m(s)) = (f_1^{m_1}(r), g_1^{m_1}(s)) = (q, h)$$

and $g_{m_1+1}^{m_2}(h) = v$. Since

$$\rho((f_1^i \times g_1^i)(p, u), (f_1^i \times g_1^i)(r, s)) = \max\{d_1(f_1^i(p), f_1^i(r)), d_2(g_1^i(u), g_1^i(s))\} < \varepsilon$$

for $i = 0, \ldots, m$, then the point $(p, u) \in A \times B$ is ε-linked to a point $(q, h) \in A \times B$ by $f_{1,\infty} \times g_{1,\infty}$. Therefore, the set $A \times B \subset X \times Y$ is linked by $f_{1,\infty} \times g_{1,\infty}$.

(Sufficiency) Let a set $A \times B \subset X \times Y$ is linked by $f_{1,\infty} \times g_{1,\infty}$. That is, for any $\varepsilon > 0$, any point $(p, u) \in A \times B$, (p, u) is ε-linked to a point $(q, v) \in A \times B$ by $f_{1,\infty} \times g_{1,\infty}$, i.e., there exist an integer $m \geq 1$ and a point (r, s) such that $(f_1^m \times g_1^m)(r, s) = (q, v)$ and

$$\rho((f_1^i \times g_1^i)(p, u), (f_1^i \times g_1^i)(r, s)) < \varepsilon$$

for $i = 0, \ldots, m$. Since $(f_1^m \times g_1^m)(r, s) = (f_1^m(r), g_1^m(s)) = (q, v)$, then $f_1^m(r) = q$ and $g_1^m(s) = v$. Since

$$\rho((f_1^i \times g_1^i)(p, u), (f_1^i \times g_1^i)(r, s)) = \max\{d_1(f_1^i(p), f_1^i(r)), d_2(g_1^i(u), g_1^i(s))\} < \varepsilon,$$

then,

$$d_1(f_1^i(p), f_1^i(r)) < \varepsilon \quad \text{and} \quad d_2(g_1^i(u), g_1^i(s)) < \varepsilon.$$

Therefore, for every $\varepsilon > 0$, the any point $p \in A$ is ε-linked to a point $q \in A$ by $f_{1,\infty}$. Thus, the set $A \subset X$ is linked by $f_{1,\infty}$.

Similarly, the set $B \subset Y$ is linked by $g_{1,\infty}$. □

The following will explore the relationships between these chaotic properties. Many scholars have investigated whether an expansive homeomorphism possessing the PSP implies topological stability, and affirmative answers have been obtained under non-autonomous conditions or other scenarios. Now, this paper will further investigate whether an expansive homeomorphism possessing the ESP implies topological stability in T-VDDSs.

Lemma 5.1. *Let $f_{1,\infty} : X \to X$ be an expansive homeomorphism. Given $\lambda > 0$, there exists an $N^* > 0$ such that $d(f_1^n(p), f_1^n(q)) \leq \varepsilon(f)$ for $n < N^*$ implies $d(p, q) < \lambda$, where $\varepsilon(f)$ is the expansivity constant of $f_{1,\infty} : X \to X$.*

Proof. This proof is derived from Theorem 4.1 in [29], and is not elaborated here. □

Lemma 5.2. *Let $f_{1,\infty} : X \to X$ be an expansive homeomorphism that possesses the ESP. Let $\varepsilon(f)$ is the expansivity constant of $f_{1,\infty} : X \to X$, for all $\varepsilon < \frac{\varepsilon(f)}{2}$, there exists a unique point $p \in X$ that eventual ε-shadows a given δ pseudo-orbit $\{p_i\}_{i=0}^{+\infty}$.*

Proof. Hypothesis p is not the unique point that eventual ε-shadows $\{p_i\}_{i=0}^{+\infty}$. There exist $p, q \in X$ and $N_1, N_2 \in \mathbb{N}_0$ such that for all $i \geq N_1$, $d(f_1^i(p), p_i) \leq \varepsilon$ and for all $i \geq N_2$, $d(f_1^i(q), p_i) \leq \varepsilon$. Choose $N = \max(N_1, N_2)$. Thus,

$$d(f_1^i(p), f_1^i(q)) \leq d(f_1^i(p), p_i) + d(p_i, f_1^i(q)) \leq 2\varepsilon \leq \varepsilon(f)$$

for $i \geq N$. By the expansiveness of $f_{1,\infty} : X \to X$, it follows that $p = q$. Thus, this means that p is the unique point that eventually ε-shadows the δ pseudo-orbit $\{p_i\}_{i=0}^{+\infty}$. ☐

Theorem 5.2. *If an expansive homeomorphism $f_{1,\infty} : X \to X$ with expansivity constant $\varepsilon(f)$ has the ESP, this means that $f_{1,\infty}$ is topologically stable.*

Proof. Let $\varepsilon < \frac{\varepsilon(f)}{3}$ and $\delta(\varepsilon) > 0$. Let $g_{1,\infty} : X \to X$ be a homeomorphism map sequence with $d(f_{1,\infty}, g_{1,\infty}) < \delta$. For all $i \in \mathbb{N}_0$ and $p \in X$, since

$$d(f_{i+1}(g_1^i(p)), g_1^{i+1}(p)) = d(f_{i+1}(g_1^i(p)), g_{i+1}(g_1^i(p))) \leq \delta,$$

this implies that $\{g_1^i(p)\}_{i=0}^{+\infty}$ is a δ pseudo-orbit for $f_{1,\infty}$. By Lemma 5.2, there exists a unique point $a = h(p) \in X$ such that $h(p)$ eventual ε-shadows the δ pseudo-orbit $\{g_1^i(p)\}_{i=0}^{+\infty}$. Define a map $h : X \to X$ such that $d(f_1^i(h(p)), g_1^i(p)) < \varepsilon$ for all $n > N$, where $N \in \mathbb{N}_0$ and $p \in X$. Clearly, if $i = 0$, then $d(h, Id) < \varepsilon$. Thus, this is sufficient to show that $h(p) \in X$ eventual ε-shadows the δ pseudo-orbit $\{g_1^i(p)\}_{i=0}^{+\infty}$.

Next, it will be proved that h is a continuous map. For any $n \leq N^*$, Noting that $g_{1,\infty}$ is uniformly equicontinuous on X, there exists an $\eta > 0$ such that $d(p, q) < \eta$ implies $d(g_1^i(p), g_1^i(q)) < \frac{\varepsilon(f)}{3}$ for $0 \leq i \leq N^*$. So

$$\begin{aligned} d(f_1^i(h(p)), f_1^i(h(q))) &\leq d(f_1^i(h(p)), g_1^i(p)) + d(g_1^i(p), g_1^i(q)) + d(g_1^i(q), f_1^i(h(q))) \\ &\leq \varepsilon + \frac{\varepsilon(f)}{3} + \varepsilon \\ &< \varepsilon(f). \end{aligned}$$

By combining Lemma 5.1, $d(h(p), h(q)) < \lambda$ is obtained. Consequently, $d(p, q) < \eta$ leads to $d(h(p), h(q)) < \lambda$. This implies that h is a continuous map. ☐

6. Some supplements

This section briefly explains the connection and differences between the NK-th iterate system and the K-th iterate system. As can be seen from Lemma 3.1 in this paper and Lemma 2.1 in [31], under the assumption that $f_{1,\infty}$ converges uniformly to a continuous map f, both sequences $\{f^{nk}_{\frac{n(n-1)k}{2}+1}\}_{n=1}^{\infty}$ and $\{f_n^k\}_{n=1}^{\infty}$ converge uniformly to f^k. However, this does not preclude the existence of substantial differences between $(X, f_{1,\infty}^{[k]})$ and $(X, f_{1,\infty}^{n[k]})$.

First, there is a fundamental difference in the iteration step lengths between the two systems. Unlike the K-th iterate system, which has a constant iteration step length of k, the NK-th iterate system features an iteration step length that increases progressively in the form of $k, 2k, 3k, \ldots$.

Next, it will be illustrated with an example that the behaviors of these two systems can also differ in certain cases.

Example 6.1. Let $X = [0, 1]$. Define two maps $\phi(x)$ and $\psi(x)$ as follows.

$$\phi(x) = 1 - x, x \in [0, 1],$$

$$\psi(x)=\begin{cases}4x, & x\in[0,\frac{1}{4}),\\ -4x+2, & x\in[\frac{1}{4},\frac{1}{2}),\\ 4x-2, & x\in[\frac{1}{2},\frac{3}{4}),\\ -4x+4, & x\in[\frac{3}{4},1].\end{cases}$$

A sequence of maps is defined as

$$f_n(x)=\begin{cases}\phi(x), \text{for } n \text{ being odd number}, x\in[0,1],\\ \psi(x), \text{for } n \text{ being even number}, x\in[0,1].\end{cases}$$

The following investigate the characteristics of the K-th iterate system and the NK-th iterate system generated by the given sequence. Assume $k=2$, and define the map sequence corresponding to the K-th iterate system as $\{g_1=f_2\circ f_1,\, g_2=f_4\circ f_3,\, g_3=f_6\circ f_5,\ldots\}$, and the sequence of maps corresponding to the NK-th iterate system as $\{h_1=f_2\circ f_1,\, h_2=f_6\circ f_5\circ f_4\circ f_3,\, h_3=f_{12}\circ f_{11}\circ f_{10}\circ f_9\circ f_8\circ f_7,\ldots\}$. Subsequently, the function graphs of these maps are plotted using MATLAB, and the results are shown in Figure 1 and Figure 2.

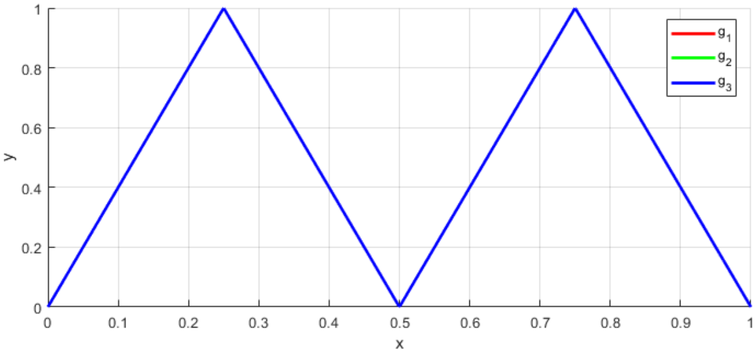

Figure 1. The function graphs of g_1, g_2 and g_3 in $f_{1,\infty}^{[2]}$.

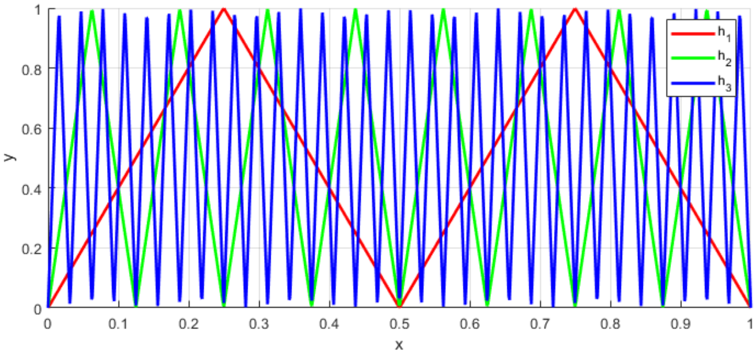

Figure 2. The function graphs of h_1, h_2 and h_3 in $f_{1,\infty}^{n[2]}$.

From Figure 1, it can be observed that only the graph of g_3 is visible, for the reason that $g_1 = g_2 = g_3$, which leads to the overlapping of the graphs. This also indicates that the K-th iterate system has become an ADDS. In contrast, Figure 2 shows that the sequence of maps in the NK-th iterate system behaves quite differently, and it remains non-autonomous in this case. Due to its exponentially increasing iteration step, the NK-th iterate system can converge more rapidly to the target attractor in chaotic systems. In addition, its distinctive step-size property offers unique advantages in block-wise processing. For example, in the multi-timescale analysis of high-dimensional systems such as climate models, it is often necessary to dynamically adjust the time step in order to reduce computational cost.

7. Conclusions

This paper studies some chaotic characteristics of the NK-th iteration system and investigates the preservation of various types of shadowing and chaotic properties. The results show that $\mathcal{P}$-chaos in $f_{1,\infty}$ is preserved in the NK-th iteration system $f_{1,\infty}^{n[k]}$. In T-VDDSs, six types of shadowing properties, as well as expansiveness, persistence, connectivity, and topological stability, are shown to be preserved under the product operator or topological conjugation. However, whether expansiveness, persistence, connectivity, and topological stability are still preserved under topological conjugation alone remains an open question. Furthermore, it is proved that expansive homeomorphisms with the eventual shadowing property are topologically stable. Whether homeomorphisms with other types of shadowing properties imply topological stability remains a topic for further research.

Statements and declarations

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