Journal of Applied Analysis and Computation
Volume 16, Number 2, April 2026, 555–578
Website:http://www.jaac-online.com
DOI:10.11948/20250151

OPTIMAL TIME-DECAY FOR EULER-FOURIER SYSTEM WITH DAMPING IN THE CRITICAL L^2 FRAMEWORK

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Abstract This paper is concerned with the large time behavior of solutions to the Euler-Fourier system with damping in $\mathbb{R}^d$ $(d \geq 1)$. In this study, a time-weighted energy argument has been developed within the L^2 framework to derive the optimal time-decay rates. A great part of our analysis relies on the study of a Lyapunov functional in the spirit of [18], which mainly depends on some elaborate use of non-classical Besov product estimates and interpolations. Furthermore, exhibiting a damped mode with faster time decay than the whole solution also plays a key role.

Keywords Euler-Fourier system with damping, critical regularity, optimal time-decay rate.

MSC(2010) 35Q31, 76N15.

1. Introduction

In this paper, we consider the following compressible Euler system with damping and heat conduction in $\mathbb{R}^+ \times \mathbb{R}^d$ $(d \geq 1)$:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = -\alpha \rho u, \\ (\rho \mathcal{E})_t + \mathrm{div}(\rho u \mathcal{E} + uP) = -\alpha \rho u^2 + k\Delta T, \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, u, T)(x, 0) = (\rho_0, u_0, T_0)(x) \to (\bar{\rho}, 0, \bar{T}), \qquad |x| \to \infty, \tag{1.2}$$

where $\bar{\rho} > 0$ and $\bar{T} > 0$ are some given constants. Here $\rho = \rho(x,t)$, $u = u(x,t) \in \mathbb{R}^d$, $T = T(x,t)$ and $P = P(x,t)$ denote the mass density, velocity, absolute temperature and pressure function of the fluid respectively. The total energy $\mathcal{E} = e + \frac{|u|^2}{2}$, where e is the internal energy. The parameter $\alpha > 0$ is the friction coefficient and $k > 0$ is the heat conductivity coefficient, respectively.

It is well known that all thermodynamics variables ρ, T, e, P as well as the entropy s can be denoted by functions of any two of them. The nonlinear coupling of ρ and T in the pressure function, which makes the overall estimation more complicated, is a challenging issue. Therefore, in this paper, we will consider ideal, polytropic fluids, so that the equations of state for the fluids are given by

$$P = R\rho T, \quad e = c_v T,$$

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where $R > 0$ and $c_v > 0$ are the universal gas constant and the specific heat at constant volume, respectively. Thus (1.1) can reduce to the following system:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P = -\alpha \rho u, \\ \rho T_t + \rho u \cdot \nabla T + \dfrac{1}{c_v} P \operatorname{div} u = \dfrac{k}{c_v} \Delta T, \\ (\rho, u, T)(x, 0) = (\rho_0, u_0, T_0)(x). \end{cases} \tag{1.3}$$

In this paper, we focus on the Cauchy problem (1.3) and investigate the asymptotic behavior of global solutions. Let us review some works related to the subject of this paper. In the one-dimensional space case, Nishida [22,23] firstly obtained the global existence of a smooth solution with small data and [14,15,22,23] for the large-time behavior of the solution, and the references therein. In multi-dimension space case, the global existence of the small smooth solutions was proved by different methods in [13, 25, 27, 33] and the large-time behavior of the solution was studied in [8, 13, 21, 26, 27] and the references therein. For the non-isentropic flow, the global existence of small smooth solutions to the Cauchy problem was proved in [16] in one dimension, and the large time behavior of these solutions can be seen in [17,19]. For multi-dimensions, Tan, Wu and Huang [28] proved the global existence of the small smooth solutions and showed that under the additional assumption that the initial data are bounded in the L^1 space. Subsequently, Wu and Miao [29] obtained the global existence and large time behavior of the solution when the initial data is close to its equilibrium in H^3-norm. In contrast to [28], this work removed additional L^1 assumptions for the initial value.

In this paper, our aim of this paper is to prove (more accurate) decay estimates in the critical regularity Besov spaces, which is close to the framework of weak solutions. Xu and Kawashima [31, 32] investigated partially dissipative hyperbolic system for balance laws, including Euler system with damping, and established the global existence and optimal decay estimates of classical solutions in critical Besov spaces under the Shizuta-Kawashima condition (see [24]). Recently, Crin-Barat and Danchin [2–4] made improvements to the results presented in [31, 32]. They proved the global existence and uniqueness of solutions on the $L^2 - L^p$ type hybrid homogeneous Besov spaces with different regularity exponents in low and high frequencies and derive the optimal time-decay rates of global solutions. In their study, a specific damped mode emerged, which is a key finding of their research and exhibits a faster time decay compared to the whole solution.

The rest of this paper is organized as follows. In Section 2, we state the main results and explain the strategies of this paper. In Section 3, we establish the low-frequency and high-frequency analysis for the Cauchy problem (2.1). In Section 4, we carry out the proofs of Theorems 2.2 on the optimal time-decay rates of the global solution. In the appendix, we present some notations of Besov spaces and recall related analysis tools used in this paper.

2. Main results and strategy of proof

2.1. Main results

Without loss of generality, set

$$R = c_v = \alpha = k = 1.$$

Define the perturbation

$$a := \rho - 1, \quad a_0 := \rho_0 - 1, \quad \theta := T - 1, \quad \theta_0 := T_0 - 1.$$

Then, the Cauchy problem (1.1)-(1.2) can be reformulated into

$$\begin{cases} \partial_t a + u \cdot \nabla a + (1+a)\mathrm{div} u = 0, \\ \partial_t u + \nabla a + u + \nabla\theta + u \cdot \nabla u + \dfrac{\theta - a}{a+1}\nabla a = 0, \\ \partial_t \theta + (1+\theta)\mathrm{div} u + u \cdot \nabla\theta - \dfrac{1}{a+1}\Delta\theta = 0, \\ (a, u, \theta)(x, 0) = (a_0, u_0, \theta_0)(x) \to (0,0,0), \quad |x| \to \infty. \end{cases} \tag{2.1}$$

It is clear that the second equation on u is a damping equation and the third equation on θ is a heat equation. In this paper, we focus on the Cauchy problem (2.1) and investigate the asymptotic behavior of global solution. First of all, let us take a look at the spectral analysis briefly, which was made by Chen, Tan and Wu [7]. It is not difficult to check that the linearized system (around the equilibrium) reads

$$\begin{cases} \partial_t a + \mathrm{div} u = 0, \\ \partial_t u + \nabla a + u + \nabla\theta = 0, \\ \partial_t \theta + \mathrm{div} u - \Delta\theta = 0 \end{cases}$$

with $a = \rho - 1$, $\theta = T - 1$. It follows from [7] that

$$\widehat{a}(t,\xi) \sim \begin{cases} C(e^{-c_+|\xi|^2 t}|\widehat{a}_0| + |\xi| e^{-\frac{1}{2}c_+|\xi|^2 t}|\widehat{u}_0| + e^{-\frac{1}{2}c_+|\xi|^2 t}|\widehat{\theta}_0|), & |\xi| \to 0, \\ C(e^{-\tilde{c}t}|\widehat{a}_0| + e^{-\tilde{c}t}|\widehat{u}_0| + |\xi|^{-1}e^{-\tilde{c}t}|\widehat{\theta}_0|), & |\xi| \to \infty, \end{cases}$$

$$\widehat{u}(t,\xi) \sim \begin{cases} C(|\xi| e^{-c_+|\xi|^2 t}|\widehat{a}_0| + e^{-\frac{1}{2}t}|\widehat{u}_0| + |\xi| e^{-\frac{1}{2}c_+|\xi|^2 t}|\widehat{\theta}_0|), & |\xi| \to 0, \\ C(e^{-\tilde{c}t}|\widehat{a}_0| + e^{-\tilde{c}t}|\widehat{u}_0| + |\xi|^{-1}e^{-\tilde{c}t}|\widehat{\theta}_0|), & |\xi| \to \infty, \end{cases}$$

and

$$\widehat{\theta}(t,\xi) \sim \begin{cases} C(e^{-\frac{1}{2}c_+|\xi|^2 t}|\widehat{a}_0| + |\xi| e^{-\frac{1}{2}c_+|\xi|^2 t}|\widehat{u}_0| + e^{-\frac{1}{2}c_+|\xi|^2 t}|\widehat{\theta}_0|), & |\xi| \to 0, \\ C(|\xi|^{-1}e^{-\tilde{c}t}|\widehat{a}_0| + |\xi|^{-1}e^{-\tilde{c}t}|\widehat{u}_0| + |\xi|^{-2}e^{-\tilde{c}t}|\widehat{\theta}_0|), & |\xi| \to \infty, \end{cases}$$

where $c_+, \tilde{c}$ are generic constants. One can observe that a is associated with a parabolic behavior at low frequencies and an exponentially damped behavior at high frequencies; while u is associated with exponentially damped behavior at both low and high frequencies; θ is associated with a parabolic behavior at both low and high frequencies. If the initial data $\|(a_0, \theta_0)\|_{L^1} + \|u_0\|_{L^{\frac{3}{2}}} < +\infty$, then one can conclude the optimal decay rates:

$$\|(a, \theta)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}}, \quad \|u\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}}.$$

However, there are few results to our knowledge on the global existence and large time behavior of solutions to (2.1) in spatially critical spaces. Then, we establish the global existence of the solution to the Cauchy problem (2.1) for the initial data close to the equilibrium state below. Recently, Gu and Liu [12] established the global existence of small strong solution to (2.1) in the L^2 critical hybrid Besov space. For convenience, we state it as follows (the reader is also referred to [12]).

Theorem 2.1. *For any $d \geq 1$, there exists a constant $\varepsilon_0 > 0$ such that if the initial data (a_0, u_0, θ_0) satisify $(a_0, u_0, \theta_0) \in \dot{B}^{\frac{d}{2}}_{2,1} \cap \dot{B}^{\frac{d}{2}+1}_{2,1}$, and*

$$\mathcal{X}_0 \triangleq \|(a_0, u_0, \theta_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(a_0, u_0, \theta_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \varepsilon_0, \tag{2.2}$$

then the Cauchy problem (2.1) admits a unique global strong solution (a, u, θ), which satisfies

$$\begin{cases} a^{\ell} \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1}) \cap \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}), \ a^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}) \cap \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}), \\ u^{\ell} \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1}) \cap \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1}), \ u^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}) \cap \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}), \\ \theta^{\ell} \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}}_{2,1}) \cap \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}), \ \theta^h \in \mathcal{C}_b(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+1}_{2,1}) \cap \widetilde{L}^2(\mathbb{R}^+; \dot{B}^{\frac{d}{2}+2}_{2,1}), \end{cases} \tag{2.3}$$

and

$$\begin{aligned} \mathcal{X} \triangleq & \|(a, u, \theta)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(a, u, \theta)\|^{h}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + (a, \theta)\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\ & + \|u\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(a, u)\|^{h}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|\theta\|^{h}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} \\ & \leq C\mathcal{X}_0, \qquad t > 0, \end{aligned} \tag{2.4}$$

for $C > 0$ a constant independent of time.

For Besov spaces, the readers can refer to Definitions 5.1-5.2 in the appendix. Next, we study the optimal time-decay rates of global solutions to system (2.1).

Theorem 2.2. *For any $d \geq 1$, let (a, u, θ) be the corresponding global solution to (2.1) constructed in Theorem 2.1. If the low-frequency part of the initial data (a_0, u_0, θ_0) additionally fulfills*

$$(a_0, u_0, \theta_0)^{\ell} \in \dot{B}^{-\sigma_1}_{2,\infty} \quad for \quad \sigma_1 \in (-\frac{d}{2}, \frac{d}{2}], \tag{2.5}$$

then it holds for any $t \in \mathbb{R}^+$ that

$$\|(a, u, \theta)(t)\|_{\dot{B}^{\sigma}_{2,1}} \leq C\delta_0(1+t)^{-\frac{1}{2}(\sigma+\sigma_1)}, \quad \sigma \in (-\sigma_1, \frac{d}{2}], \tag{2.6}$$

where $C > 0$ is a generic constant independent of time and

$$\delta_0 \triangleq \|(a_0, u_0, \theta_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}} + \|(a_0, u_0, \theta_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}. \tag{2.7}$$

Furthermore, if $d \geq 2$, then the following time-decay estimate holds:

$$\|u\|_{\dot{B}^{\sigma}_{2,1}} \leq C\delta_0(1+t)^{-\frac{1}{2}(1+\sigma+\sigma_1)}, \quad \sigma \in (-\sigma_1, \frac{d}{2} - 1]. \tag{2.8}$$

Remark 2.1. Due to the embedding $L^1 \hookrightarrow \dot{B}^{-\frac{d}{2}}_{2,\infty}$, the above assumption (2.5) with $\sigma_1 = \frac{d}{2}$ covers the classical L^1 condition as first presented by Matsumura and Nishda in [20]. Moreover, it should be noted that the smallness of the norm $\|(a_0, u_0, \theta_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,1}}$ is no longer needed. The crucial part of proof of Theorem 2.2 lies in the evolution of negative Besov norms at low frequencies (see Lemma 4.1 for details).

Remark 2.2. More precisely, one can obtain the classical decay rates. For instance, if taking $\sigma = \sigma_1 = \frac{3}{2}$, then the solution decays in the spatially critical space L^∞ at the rate of $\mathcal{O}(t^{-\frac{3}{2}})$. If taking $\sigma = 0,\ \sigma_1 = \frac{3}{2}$, then the solution decays in L^2 at the rate of $\mathcal{O}(t^{-\frac{3}{4}})$ and the damping term u in the velocity equation is at the enhanced rate of $\mathcal{O}(t^{-\frac{5}{4}})$, which were shown by Chen, Tan and Wu [7] in the framework of Sobolev spaces with higher regularity.

2.2. Strategy of proofs

In the proof of Theorem 2.2, we adapt the work by Xin-Xu [30] about optimal time-decay rates for the compressible Navier-Stokes equations without additional smallness assumptions. Motivated by the interesting works [5, 18], we establish a new time-weighted functional $\mathcal{X}_M(t)$, defined in (2.4) with the time weighted $(1+\tau)^M$ for $M > 1 + \frac{1}{2}(\frac{d}{2} + \sigma_1)$. Actually, we derive the following time-weighted inequality,

$$
\begin{aligned}
&\|(1+\tau)^M (a,u,\theta)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(1+\tau)^M (a,\theta)\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\
&+ \|(1+\tau)^M u\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(1+\tau)^M (a,u,\theta)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\
&+ \|(1+\tau)^M (a,u)\|^{h}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(1+\tau)^M \theta\|^{h}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} \\
\lesssim &\left(\|(a,u,\theta)^{\ell}\|_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|(a,u,\theta)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \right) (1+t)^{M-\frac{1}{2}(\frac{d}{2}+\sigma_1)}.
\end{aligned}
$$

This leads to the desired decay estimate (2.6) with the aid of interpolation tricks. Here, the crucial part of the decay proof lies in the nonlinear evolution of the $\dot{B}^{-\sigma_1}_{2,\infty}$-norm in low frequencies, which is a weaker assumption than the usual L^1 assumption. Then, the faster time decay rates in (2.8) are obtained by capturing the damping effects in the equation $(2.1)_2$. In addition, compared with the isentropic case, the non-isentropic case includes a temperature equation, which gives rise to some more complex nonlinear terms.

Notation. Throughout the paper, we denote by C harmless positive constants that may change from one line to the other. We sometimes write $A \lesssim B$ instead of $A \le CB$. Likewise, $A \sim B$ means that $C_1 B \le A \le C_2 B$ with absolute constants C_1, C_2. For any Banach space X and $f, g \in X$, we agree that $\|(f,g)\|_X \triangleq \|f\|_X + \|g\|_X$. For all $T > 0$ and $\varrho \in [1,\infty]$, we denote by $L^\varrho_T(X) \triangleq L^\varrho([0,T];X)$ the set of measurable function $f : [0,T] \to X$ such that $t \mapsto \|f(t)\|_X$ is in $L^\varrho(0,T)$.

3. Low-frequency and high-frequency analysis

We are going to establish a Lyapunov-type inequality for energy norms by using a pure energy argument. For clarity, the proof is divided into several steps.

3.1. Low-frequency analysis

In this subsection, we first establish the a priori estimates of solutions to the Cauchy problem (2.1) in the low-frequency region $\{\xi \in \mathbb{R}^3 \mid |\xi| \leq \frac{8}{3}\}$. Note that (2.3) implies

$$\begin{aligned} &\sup_{(x,t)\in\mathbb{R}^d\times(0,T)} |a(x,t)| \leq \frac{1}{2} \Rightarrow \frac{1}{2} \leq \varrho = 1+a \leq \frac{3}{2}, \\ &\sup_{(x,t)\in\mathbb{R}^d\times(0,T)} |\theta(x,t)| \leq \frac{1}{2} \Rightarrow \frac{1}{2} \leq T = 1+\theta \leq \frac{3}{2}, \quad \text{if } \mathcal{X}_0 \ll 1. \end{aligned} \tag{3.1}$$

The property (3.1) will be used to handle the nonlinear terms $\frac{\theta-a}{a+1}$ and $\frac{1}{1+a}$ in (2.1) by the continuity of composition estimates in Proposition 5.1. For any $j \in \mathbb{Z}$, applying the operator $\dot{\Delta}_j$ to (2.1), we get

$$\begin{cases} \partial_t \dot{\Delta}_j a + \dot{\Delta}_j \mathrm{div} u + \dot{\Delta}_j \mathrm{div}(au) = 0, \\ \partial_t \dot{\Delta}_j u + \dot{\Delta}_j \nabla a + \dot{\Delta}_j u + \dot{\Delta}_j \nabla\theta + \dot{\Delta}_j (u\cdot\nabla u) + \dot{\Delta}_j (\frac{\theta-a}{a+1}\nabla a) = 0, \\ \partial_t \dot{\Delta}_j \theta - \dot{\Delta}_j \Delta\theta + \dot{\Delta}_j \mathrm{div} u + \dot{\Delta}_j \mathrm{div}(u\theta) + \dot{\Delta}_j (\frac{a}{a+1}\Delta\theta) = 0. \end{cases} \tag{3.2}$$

First, we derive a low-frequency Lyapunov type inequality of (3.2).

Lemma 3.1. *Let* (a, u, θ) *be any strong solution to the Cauchy problem* (2.1)*. Then, it holds for any* $j \leq 0$ *that*

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_{1,j}(t) + \mathcal{D}_{1,j}(t) \\ \lesssim &\|\dot{\Delta}_j(au)\|_{L^2} \|(\dot{\Delta}_j \nabla a, \dot{\Delta}_j u)\|_{L^2} \\ &+ \|(\dot{\Delta}_j(u\cdot\nabla u), \dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a))\|_{L^2} \|(\dot{\Delta}_j u, \dot{\Delta}_j \nabla a)\|_{L^2} \\ &+ \|(\dot{\Delta}_j(u\theta), \dot{\Delta}_j(\frac{a}{a+1}\nabla\theta))\|_{L^2} \|\dot{\Delta}_j \nabla\theta\|_{L^2} + \|\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\|_{L^2} \|\dot{\Delta}_j \theta\|_{L^2}, \end{aligned} \tag{3.3}$$

where $\mathcal{E}_{1,j}(t)$ *and* $\mathcal{D}_{1,j}(t)$ *are defined by*

$$\begin{cases} \mathcal{E}_{1,j}(t) \triangleq \frac{1}{2}\|(\dot{\Delta}_j a, \dot{\Delta}_j u, \dot{\Delta}_j \theta)\|_{L^2}^2 + \eta_1 \int_{\mathbb{R}^d} \dot{\Delta}_j \nabla a \dot{\Delta}_j u dx, \\ \mathcal{D}_{1,j}(t) \triangleq \|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j \nabla\theta\|_{L^2}^2 + \eta_1 \|\dot{\Delta}_j \nabla a\|_{L^2}^2 - \eta_1 \|\dot{\Delta}_j divu\|_{L^2}^2 \\ \qquad\qquad + \eta_1 \int_{\mathbb{R}^d} \dot{\Delta}_j u \dot{\Delta}_j \nabla a dx + \eta_1 \int_{\mathbb{R}^d} \dot{\Delta}_j \nabla\theta \dot{\Delta}_j \nabla a dx \end{cases} \tag{3.4}$$

with constant $\eta_1 \in (0,1)$ *to be determined later.*

Proof. Taking the L^2 inner product of $(3.2)_1$, $(3.2)_2$ and $(3.2)_3$ with $\dot{\Delta}_j a$, $\dot{\Delta}_j u$ and $\dot{\Delta}_j \theta$, respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j a\|_{L^2}^2 = -\int_{\mathbb{R}^d} \dot{\Delta}_j \mathrm{div} u \dot{\Delta}_j a dx - \int_{\mathbb{R}^d} \dot{\Delta}_j \mathrm{div}(au) \dot{\Delta}_j a dx, \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j u\|_{L^2}^2 = -\int_{\mathbb{R}^d} \dot{\Delta}_j \nabla a \dot{\Delta}_j u dx - \int_{\mathbb{R}^d} \dot{\Delta}_j \nabla\theta \dot{\Delta}_j u dx$$

$$
-\int_{\mathbb{R}^d}\dot{\Delta}_j(u\cdot\nabla u)\dot{\Delta}_j u dx-\int_{\mathbb{R}^d}\dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a)\dot{\Delta}_j u dx, \tag{3.6}
$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\theta\|_{L^2}^2+\|\dot{\Delta}_j\nabla\theta\|_{L^2}^2=&-\int_{\mathbb{R}^d}\dot{\Delta}_j\mathrm{div}(u\theta)\dot{\Delta}_j\theta dx-\int_{\mathbb{R}^d}\dot{\Delta}_j\mathrm{div}u\dot{\Delta}_j\theta dx\\
&-\int_{\mathbb{R}^d}\dot{\Delta}_j(\frac{a}{a+1}\Delta\theta)\dot{\Delta}_j\theta dx.
\end{aligned} \tag{3.7}
$$

By performing an integration by parts in the third term on the right-hand side of (3.7), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\dot{\Delta}_j(\frac{a}{a+1}\Delta\theta)\dot{\Delta}_j\theta dx\\
=&\int_{\mathbb{R}^d}\dot{\Delta}_j\mathrm{div}\big(\frac{a}{a+1}\nabla\theta\big)\dot{\Delta}_j\theta dx-\int_{\mathbb{R}^d}\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\dot{\Delta}_j\theta dx\\
=&-\int_{\mathbb{R}^d}\dot{\Delta}_j\big(\frac{a}{a+1}\nabla\theta\big)\dot{\Delta}_j\nabla\theta dx-\int_{\mathbb{R}^d}\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\dot{\Delta}_j\theta dx.
\end{aligned}
$$

By applying Cauchy-Schwarz's inequality, the combination of (3.5)-(3.7) leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^2}^2+\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\nabla\theta\|_{L^2}^2\\
\leq&\|\dot{\Delta}_j(au)\|_{L^2}\|\dot{\Delta}_j\nabla a\|_{L^2}+\|\dot{\Delta}_j(u\cdot\nabla u)\|_{L^2}\|\dot{\Delta}_j u\|_{L^2}\\
&+\|\dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a)\|_{L^2}\|\dot{\Delta}_j u\|_{L^2}+\|\left(\dot{\Delta}_j(u\theta),\dot{\Delta}_j\big(\frac{a}{a+1}\nabla\theta\big)\right)\|_{L^2}\|\dot{\Delta}_j\nabla\theta\|_{L^2}\\
&+\|\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\|_{L^2}\|\dot{\Delta}_j\theta\|_{L^2}.
\end{aligned} \tag{3.8}
$$

In order to obtain the dissipation of a, we make use of $(3.2)_1$ and $(3.2)_2$ to have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}\dot{\Delta}_j\nabla a\dot{\Delta}_j u dx+\|\dot{\Delta}_j\nabla a\|_{L^2}^2-\|\dot{\Delta}_j\mathrm{div}u\|_{L^2}^2\\
&+\int_{\mathbb{R}^d}\dot{\Delta}_j u\dot{\Delta}_j\nabla a dx+\int_{\mathbb{R}^d}\dot{\Delta}_j\nabla\theta\dot{\Delta}_j\nabla a dx\\
\leq&\|\dot{\Delta}_j\mathrm{div}(au)\|_{L^2}\|\dot{\Delta}_j\mathrm{div}u\|_{L^2}\\
&+\left(\|\dot{\Delta}_j(u\cdot\nabla u)\|_{L^2}+\|\dot{\Delta}_j\big(\frac{\theta-a}{a+1}\nabla a\big)\|_{L^2}\right)\|\dot{\Delta}_j\nabla a\|_{L^2}.
\end{aligned} \tag{3.9}
$$

According to (3.8)-(3.9) and the Bernstein inequality, (3.3) follows. □

3.2. High-frequency analysis

In this subsection, we establish solutions to the Cauchy problem (2.1) in the high-frequency region $\{\xi\in\mathbb{R}^d \mid |\xi|\geq\frac{3}{8}\}$. To this end, we show a high-frequency Lyapunov type inequality of (3.2).

Lemma 3.2. *Let (a,u,θ) be any strong solution to the Cauchy problem (2.1). Then, it holds for any $j\geq -1$ that*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{2,j}(t)+\mathcal{D}_{2,j}(t)\\
\leq&\frac{1}{2}\Big\|\frac{\partial}{\partial t}\Big(\frac{1+\theta}{(1+a)^2}\Big)\Big\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}^2+\Big\|\nabla\Big(\frac{1+\theta}{1+a}\Big)\Big\|_{L^\infty}\|\dot{\Delta}_j u\|_{L^2}\|\dot{\Delta}_j a\|_{L^2}\\
&+\frac{1}{2}\Big\|\mathrm{div}\Big(\frac{1+\theta}{(1+a)^2}\Big)\Big\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}^2+\frac{1}{2}\|\mathrm{div}u\|_{L^\infty}\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j(u\theta)\|_{L^2}\|\dot{\Delta}_j\nabla\theta\|_{L^2}\\
&+\|\nabla(\frac{a}{1+a})\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^2}\|\dot{\Delta}_j\theta\|_{L^2}+\|\frac{a}{1+a}\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^2}^2\\
&+\|\tilde{R}_j^1\|_{L^2}\|\frac{1+\theta}{(1+a)^2}\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}+\|\tilde{R}_j^2\|_{L^2}\|\dot{\Delta}_j u\|_{L^2}+\|\tilde{R}_j^3\|_{L^2}\|\dot{\Delta}_j\theta\|_{L^2}\\
&+\eta_2 2^{-2j}\|\dot{\Delta}_j\mathrm{div}(au)\|_{L^2}\|\dot{\Delta}_j\mathrm{div}u\|_{L^2}+\eta_2 2^{-2j}\|\dot{\Delta}_j(u\cdot\nabla u)\|_{L^2}\|\dot{\Delta}_j\nabla a\|_{L^2}\\
&+\eta_2 2^{-2j}\|\dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a)\|_{L^2}\|\dot{\Delta}_j\nabla a\|_{L^2},
\end{aligned}\tag{3.10}$$

where $\mathcal{E}_{2,j}(t)$ and $\mathcal{D}_{2,j}(t)$ are defined by

$$\begin{cases}
\mathcal{E}_{2,j}(t)\triangleq\frac{1}{2}\left(\int_{\mathbb{R}^d}\frac{1+\theta}{(1+a)^2}(\dot{\Delta}_j a)^2+(\dot{\Delta}_j u)^2+(\dot{\Delta}_j\theta)^2dx\right)\\
\qquad\qquad+\eta_2 2^{-2j}\int_{\mathbb{R}^d}\dot{\Delta}_j\nabla a\dot{\Delta}_j u dx,\\
\mathcal{D}_{2,j}(t)\triangleq\|\dot{\Delta}_j u\|_{L^2}^2+\|\dot{\Delta}_j\nabla\theta\|_{L^2}^2+\eta_2 2^{-2j}\|\dot{\Delta}_j\nabla a\|_{L^2}^2-\eta_2 2^{-2j}\|\dot{\Delta}_j\mathrm{div}u\|_{L^2}^2\\
\qquad\qquad+\eta_2 2^{-2j}\int_{\mathbb{R}^d}\dot{\Delta}_j u\dot{\Delta}_j\nabla a dx+\eta_2 2^{-2j}\int_{\mathbb{R}^d}\dot{\Delta}_j\nabla\theta\dot{\Delta}_j\nabla a dx
\end{cases}\tag{3.11}$$

with constant $\eta_2\in(0,1)$ to be determined later.

Proof. The Cauchy problem (3.2) can be reformulated as

$$\begin{cases}
\partial_t\dot{\Delta}_j a+u\dot{\Delta}_j\nabla a+(1+a)\dot{\Delta}_j\mathrm{div}u=\tilde{R}_j^1,\\
\partial_t\dot{\Delta}_j u+\dot{\Delta}_j u+\dot{\Delta}_j\nabla\theta+u\dot{\Delta}_j\nabla u+\dfrac{1+\theta}{1+a}\dot{\Delta}_j\nabla a=\tilde{R}_j^2,\\
\partial_t\dot{\Delta}_j\theta+\dot{\Delta}_j\mathrm{div}u-\dot{\Delta}_j\Delta\theta+\dot{\Delta}_j\mathrm{div}(u\theta)-\dfrac{a}{a+1}\dot{\Delta}_j\Delta\theta=\tilde{R}_j^3,
\end{cases}\tag{3.12}$$

where

$$\begin{aligned}
\tilde{R}_j^1&=-[\dot{\Delta}_j,(1+a)]\mathrm{div}u-[\dot{\Delta}_j,u]\nabla a,\\
\tilde{R}_j^2&=-[\dot{\Delta}_j,u]\nabla u-[\dot{\Delta}_j,\frac{1+\theta}{1+a}]\nabla a,\\
\tilde{R}_j^3&=[\dot{\Delta}_j,\frac{a}{a+1}]\Delta\theta.
\end{aligned}\tag{3.13}$$

By similar arguments to Lemma 3.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1+\theta}{(1+a)^2}(\dot{\Delta}_j a)^2dx+\int_{\mathbb{R}^d}\frac{1+\theta}{1+a}\dot{\Delta}_j\mathrm{div}u\dot{\Delta}_j a dx+\int_{\mathbb{R}^d}\frac{1+\theta}{(1+a)^2}u\dot{\Delta}_j\nabla a\dot{\Delta}_j a dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\left(\frac{1+\theta}{(1+a)^2}\right)(\dot{\Delta}_j a)^2dx + \int_{\mathbb{R}^d}\tilde{R}_j^1\frac{1+\theta}{(1+a)^2}\dot{\Delta}_j adx, \tag{3.14}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j u\|_{L^2}^2 + \int_{\mathbb{R}^d}u\dot{\Delta}_j\nabla u\dot{\Delta}_j udx\\ &+\int_{\mathbb{R}^d}\dot{\Delta}_j\nabla\theta\dot{\Delta}_j udx + \int_{\mathbb{R}^d}\frac{1+\theta}{1+a}\dot{\Delta}_j\nabla a\dot{\Delta}_j udx = \int_{\mathbb{R}^d}\tilde{R}_j^2\dot{\Delta}_j udx,\end{aligned} \tag{3.15}$$

as well as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\theta\|_{L^2}^2 + \|\dot{\Delta}_j\nabla\theta\|_{L^2}^2 + \int_{\mathbb{R}^d}\dot{\Delta}_j\mathrm{div}u\dot{\Delta}_j\theta dx + \int_{\mathbb{R}^d}\dot{\Delta}_j\mathrm{div}(u\theta)\dot{\Delta}_j\theta dx\\ &-\int_{\mathbb{R}^d}\frac{a}{1+a}\dot{\Delta}_j\Delta\theta\dot{\Delta}_j\theta dx = \int_{\mathbb{R}^d}\tilde{R}_j^3\dot{\Delta}_j\theta dx.\end{aligned} \tag{3.16}$$

The combination of (3.14)-(3.16) leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^d}\frac{1+\theta}{(1+a)^2}(\dot{\Delta}_j a)^2 + (\dot{\Delta}_j u)^2 + (\dot{\Delta}_j\theta)^2dx\right) + \|\dot{\Delta}_j u\|_{L^2}^2 + \|\dot{\Delta}_j\nabla\theta\|_{L^2}^2\\ \le&\frac{1}{2}\Big\|\frac{\partial}{\partial t}\Big(\frac{1+\theta}{(1+a)^2}\Big)\Big\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}^2 + \Big\|\nabla\Big(\frac{1+\theta}{1+a}\Big)\Big\|_{L^\infty}\|\dot{\Delta}_j u\|_{L^2}\|\dot{\Delta}_j a\|_{L^2}\\ &+\frac{1}{2}\Big\|\mathrm{div}\Big(\frac{1+\theta}{(1+a)^2}u\Big)\Big\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}^2 + \frac{1}{2}\|\mathrm{div}u\|_{L^\infty}\|\dot{\Delta}_j u\|_{L^2}^2\\ &+\|\dot{\Delta}_j(u\theta)\|_{L^2}\|\dot{\Delta}_j\nabla\theta\|_{L^2} + \|\nabla(\frac{a}{1+a})\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^2}\|\dot{\Delta}_j\theta\|_{L^2}\\ &+\|\frac{a}{1+a}\|_{L^\infty}\|\dot{\Delta}_j\nabla\theta\|_{L^2}^2 + \|\tilde{R}_j^1\|_{L^2}\|\frac{1+\theta}{(1+a)^2}\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^2}\\ &+\|\tilde{R}_j^2\|_{L^2}\|\dot{\Delta}_j u\|_{L^2} + \|\tilde{R}_j^3\|_{L^2}\|\dot{\Delta}_j\theta\|_{L^2}.\end{aligned} \tag{3.17}$$

By virtue of (3.9) and (3.17), the Bernstein inequality and the fact $2^{-j}\le 2$, (3.10) holds. □

4. The proof of Theorem 2.2

In this section, we establish the decay estimates of the global solution given by Theorem 2.2. Our method is partially inspired by the time-weighted Lyapunov energy method in [18] which contains the following two key steps:
(1) The evolution of negative Besov norm under particular regularity $-\sigma_1$;
(2) Time-weighted estimates and asymptotic behaviors.

4.1. The regularity evolution of negative Besov norms

In this subsection, we turn to bound the evolution of negative Besov norm, which is the main ingredient in deriving the Lyapunov-type inequality for energy norms.

Lemma 4.1. *If* (a,u,θ) *is the global solution to the Cauchy problem* (2.1) *given by Theorem* 2.1*, then for all* $t>0$ *the following inequality holds:*

$$\begin{aligned}\mathcal{X}_{L,\sigma_1}(t) \triangleq& \|(a,u,\theta)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|a\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})} + \|u\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|\theta\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})}\\ \le& C\delta_0.\end{aligned} \tag{4.1}$$

Here $C>0$ *is a uniform constant, and* δ_0 *is defined by* (2.7).

Proof. Recall that $\mathcal{E}_{1,j}(t)$ and $\mathcal{D}_{1,j}(t)$ given by (3.4) satisfy the Lyapunov type inequality (3.3). One can show for any $j \leq 0$ that

$$(\frac{1}{2}-\frac{\eta_1}{2})\|(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^2}^2 \leq \mathcal{E}_{1,j}(t) \leq (\frac{1}{2}+\frac{\eta_1}{2})\|(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^2}^2 \tag{4.2}$$

and

$$\begin{aligned}\mathcal{D}_{1,j}(t) \geq C\eta_1 2^{2j}\|\dot{\Delta}_j a\|_{L^2}^2 + (1-2^{2j}\eta_1 - C\eta_1)\|\dot{\Delta}_j u\|_{L^2}^2\\ + 2^{2j}(1-C\eta_1)\|\dot{\Delta}_j\theta\|_{L^2}^2,\end{aligned} \tag{4.3}$$

where $C>1$ denotes a sufficiently large constant independent of time. Choosing a sufficiently small constant $\eta_1 \in (0,1)$, we deduce by (4.2)-(4.3) for any $j \leq 0$ that

$$\mathcal{E}_{1,j}(t) \sim \|(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^2}^2 \tag{4.4}$$

and

$$\mathcal{D}_{1,j}(t) \gtrsim 2^{2j}\|(\dot{\Delta}_j a,\dot{\Delta}_j\theta)\|_{L^2}^2 + \|\dot{\Delta}_j u\|_{L^2}^2. \tag{4.5}$$

By first integrating (3.3) over $[0,t]$, then combining the inequalities in (4.4)-(4.5), and finally taking the square root of both sides of the resulting inequality, we obtain

$$\begin{aligned}&\|(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^\infty_t(L^2)} + 2^j\|(\dot{\Delta}_j a,\dot{\Delta}_j\theta)\|_{L^2_t(L^2)} + \|\dot{\Delta}_j u\|_{L^2_t(L^2)}\\ \lesssim&\|(\dot{\Delta}_j a_0,\dot{\Delta}_j u_0,\dot{\Delta}_j\theta_0)\|_{L^2} + \Big\|\Big(\dot{\Delta}_j(au),\dot{\Delta}_j(u\cdot\nabla u),\dot{\Delta}_j(u\theta)\Big)\Big\|_{L^2_t(L^2)}\\ &+\Big\|\Big(\dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a),\dot{\Delta}_j\big(\frac{a}{a+1}\nabla\theta\big)\Big)\Big\|_{L^2_t(L^2)} + \|\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\|_{L^1_t(L^2)}.\end{aligned} \tag{4.6}$$

Multiplying (4.6) by $2^{-\sigma_1 j}$ and taking the supremum on both $[0,t]$ and $j \leq 0$, we get

$$\begin{aligned}&\|(a,u,\theta)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|(a,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})} + \|u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1}_{2,\infty})}\\ \lesssim&\|(a_0,u_0,\theta_0)\|^\ell_{\dot{B}^{-\sigma_1}_{2,\infty}} + \|(au,u\cdot\nabla u,u\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1}_{2,\infty})}\\ &+\|(\frac{\theta-a}{a+1}\nabla a,\frac{a}{a+1}\nabla\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|\nabla(\frac{a}{a+1})\nabla\theta\|^\ell_{\widetilde{L}^1_t(\dot{B}^{-\sigma_1}_{2,\infty})}.\end{aligned} \tag{4.7}$$

In what follows, we focus on the nonlinear terms on the right-hand side of (4.7). According to Proposition 5.1 and Proposition 5.2 with $s_1=\frac{d}{2}, s_2=-\sigma_1$, one can get

$$\begin{aligned}&\|(au,u\cdot\nabla u,u\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1}_{2,\infty})}\\ \lesssim&\|(a,u,\theta)\|_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})}\|(u,\nabla u)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\ \lesssim&\,(\mathcal{X}_{L,\sigma_1}+\mathcal{X}(t))\,\mathcal{X}(t),\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\|(\frac{\theta-a}{a+1}\nabla a,\frac{a}{a+1}\nabla\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1}_{2,\infty})}\\ \lesssim&\|(a,\theta)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|a\|_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})} + \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|\theta\|_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})}\\ \lesssim&\,(\mathcal{X}_{L,\sigma_1}+\mathcal{X}(t))\,\mathcal{X}(t),\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\|\nabla(\frac{a}{a+1})\nabla\theta\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{-\sigma_1}_{2,\infty})} &\lesssim \|\frac{a}{a+1}\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|\theta\|_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})}\\ &\lesssim \|a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|\theta\|_{\widetilde{L}^2_t(\dot{B}^{-\sigma_1+1}_{2,\infty})}\\ &\lesssim \mathcal{X}(t)\left(\mathcal{X}_{L,\sigma_1}+\mathcal{X}(t)\right).\end{aligned} \tag{4.10}$$

Substituting the estimates (4.8)-(4.10) into (4.7), we have

$$\mathcal{X}_{L,\sigma_1} \lesssim \|(a_0,u_0,\theta_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}} + \mathcal{X}(t)\left(\mathcal{X}_{L,\sigma_1}+\mathcal{X}(t)\right).$$

Making use of (2.4), $\mathcal{X}(t) \lesssim \mathcal{X}(0) \ll 1$ and $\|(a_0,u_0,\theta_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}} + \mathcal{X}(0) \sim \delta_0$, we prove (4.1). □

4.2. Time-weighted Lyapunov approach

Next, we introduce a new time-weighted energy functional

$$\begin{aligned}\mathcal{X}_M(t) \triangleq &\|(1+\tau)^M(a,u,\theta)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(1+\tau)^M(a,\theta)\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\ &+ \|(1+\tau)^M u\|^{\ell}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(1+\tau)^M(a,u,\theta)\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\ &+ \|(1+\tau)^M(a,u)\|^{h}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(1+\tau)^M\theta\|^{h}_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})},\end{aligned} \tag{4.11}$$

where $M>0$ is chosen sufficiently large. Consequently, we have the following time-weighted Lyapunov estimates.

Proposition 4.1. *Let (a,u,θ) be the global solution to the Cauchy problem (2.1) given by Theorem 2.1. Under the assumption of Theorem 2.2, it holds that*

$$\mathcal{X}_M \lesssim \delta_0(1+t)^{M-\frac{1}{2}(\frac{d}{2}+\sigma_1)} \tag{4.12}$$

for $M > 1+\frac{1}{2}(\frac{d}{2}+\sigma_1)$ and $t>0$, where $\delta_0 \triangleq \|(a_0,u_0,\theta_0)\|^{\ell}_{\dot{B}^{-\sigma_1}_{2,\infty}} + \|(a_0,u_0,\theta_0)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$.

Proof. The proof is separated into several steps.

- **Step1. Low-frequency estimates.**

Let us begin with the Lyapunov type inequality (3.3) in the low-frequency regime. Multiplying (3.3) by $(1+t)^{2M}$ and using the fact

$$(1+t)^{2M}\frac{d}{dt}\mathcal{E}_{1,j}(t) = \frac{d}{dt}\left((1+t)^{2M}\mathcal{E}_{1,j}(t)\right) - 2M(1+t)^{2M-1}\mathcal{E}_{1,j}(t),$$

we obtain

$$\begin{aligned}&\frac{d}{dt}\left((1+t)^{2M}\mathcal{E}_{1,j}(t)\right) + (1+t)^{2M}\mathcal{D}_{1,j}(t)\\ \lesssim&(1+t)^{2M-1}\mathcal{E}_{1,j}(t)\\ &+ (1+t)^{2M}\|\left(\dot{\Delta}_j(au),\dot{\Delta}_j(u\cdot\nabla u),\dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a)\right)\|_{L^2}\|(\dot{\Delta}_j u,\dot{\Delta}_j\nabla a)\|_{L^2}\\ &+ (1+t)^{2M}\|\left(\dot{\Delta}_j(u\theta),\dot{\Delta}_j(\frac{a}{a+1}\nabla\theta)\right)\|_{L^2}\|\dot{\Delta}_j\nabla\theta\|_{L^2}\\ &+ (1+t)^{2M}\|\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\|_{L^2}\|\dot{\Delta}_j\theta\|_{L^2}\end{aligned} \tag{4.13}$$

for $j \le 0$, which together with (4.4) and (4.5) implies that

$$
\begin{aligned}
&\|(1+t)^M(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^2}+2^j\|(1+\tau)^M(\dot{\Delta}_j a,\dot{\Delta}_j\theta)\|_{L^2_tL^2}+\|(1+\tau)^M\dot{\Delta}_j u\|_{L^2_tL^2}\\
\lesssim&\|(\dot{\Delta}_j a_0,\dot{\Delta}_j u_0,\dot{\Delta}_j\theta_0)\|_{L^2}+\|(1+\tau)^{M-\frac12}(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|_{L^2_tL^2}\\
&+\left\|(1+\tau)^M\left(\dot{\Delta}_j(au),\dot{\Delta}_j(u\cdot\nabla u),\dot{\Delta}_j(u\theta),\dot{\Delta}_j(\frac{\theta-a}{a+1}\nabla a),\dot{\Delta}_j\big(\frac{a}{a+1}\nabla\theta\big)\right)\right\|_{L^2_t(L^2)}\\
&+\|(1+\tau)^M\dot{\Delta}_j\big(\nabla(\frac{a}{a+1})\nabla\theta\big)\|_{L^1_t(L^2)}.
\end{aligned}
\tag{4.14}
$$

Then, we multiply (4.14) by $2^{\frac{d}{2}j}$, take the supremum on $[0,t]$, and then sum over $j\le 0$ to get

$$
\begin{aligned}
&\|(1+\tau)^M(a,u,\theta)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac d2}_{2,1})}+\|(1+\tau)^M(a,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2+1}_{2,1})}+\|(1+\tau)^M u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2}_{2,1})}\\
\lesssim&\|(a_0,u_0,\theta_0)\|^\ell_{\dot{B}^{\frac d2}_{2,1}}+\|(1+\tau)^{M-\frac12}(a,u,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2}_{2,1})}\\
&+\|(1+\tau)^M(au,u\cdot\nabla u,u\theta,\frac{\theta-a}{a+1}\nabla a,\frac{a}{a+1}\nabla\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2}_{2,1})}\\
&+\|(1+\tau)^M\nabla(\frac{a}{a+1})\nabla\theta\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac d2}_{2,1})}.
\end{aligned}
\tag{4.15}
$$

It is worth emphasized that the second term on the right-hand side of (4.15) plays a key role in the derivation of decay rates. To bound it, it follows from Lemma 5.2 and Young's inequality to deduce

$$
\begin{aligned}
&\|(1+\tau)^{M-\frac12}(a,u,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac d2}_{2,1})}\\
\le&C\|(1+\tau)^{M-1}(a,u,\theta)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac d2}_{2,1})}+\frac14\|(1+\tau)^M(a,u,\theta)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac d2}_{2,1})},
\end{aligned}
\tag{4.16}
$$

where the first term can be handled as follows:

$$
\begin{aligned}
&\|(1+\tau)^{M-1}(a,u,\theta)\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac d2}_{2,1})}\\
\lesssim&\int_0^t(1+\tau)^{M-1}\|(a,u,\theta)^\ell\|_{\dot{B}^{\frac d2}_{2,1}}\,d\tau+\int_0^t(1+\tau)^{M-1}\|(a,u,\theta)^h\|_{\dot{B}^{\frac d2}_{2,1}}\,d\tau.
\end{aligned}
\tag{4.17}
$$

To control the first term on the right-hand side of (4.17), we deduce from Lemma 5.2 with $s_1=-\sigma_1, s_2=\frac d2+1, p=2$ and $\eta_0=\frac{1}{\sigma_1+1+\frac d2}$ that

$$
\begin{aligned}
&\int_0^t(1+\tau)^{M-1}\|(a,u,\theta)^\ell\|_{\dot{B}^{\frac d2}_{2,1}}\,d\tau\\
\lesssim&\int_0^t(1+\tau)^{M-1}\|(a,u,\theta)^\ell\|^{\eta_0}_{\dot{B}^{-\sigma_1}_{2,\infty}}\|(a,u,\theta)^\ell\|^{1-\eta_0}_{\dot{B}^{\frac d2+1}_{2,\infty}}d\tau\\
\lesssim&\|(a,u,\theta)^\ell\|^{\eta_0}_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})}\int_0^t\|(1+\tau)^M(a,u,\theta)^\ell\|^{1-\eta_0}_{\dot{B}^{\frac d2+1}_{2,1}}(1+\tau)^{M\eta_0-1}d\tau\\
\lesssim&\left(\|(a,u,\theta)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})}\right)^{\eta_0}
\end{aligned}
$$

$$\times\left(\|(1+\tau)^M(a,u,\theta)^\ell\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{1-\eta_0}\|(1+\tau)^{M\eta_0-1}\|_{L_t^{\frac{2}{1+\eta_0}}}. \tag{4.18}$$

On the other hand, by taking advantage of (5.5) and the dissipation properties of (a,u,θ) for high frequencies, it is easy to verify that

$$\begin{aligned}
&\int_0^t(1+t)^{M-1}\|(a,u)^h\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,d\tau\\
&\lesssim\left(\|(a,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right)^{\eta_0}\int_0^t\left(\|(1+\tau)^M(a,u)\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)^{1-\eta_0}(1+\tau)^{M\eta_0-1}d\tau\\
&\lesssim\left(\|(1+\tau)^M(a,u)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{1-\eta_0}\\
&\quad\times\left(\|(a,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{\eta_0}\|(1+\tau)^{M\eta_0-1}\|_{L_t^{\frac{2}{1+\eta_0}}},
\end{aligned}\tag{4.19}$$

and

$$\begin{aligned}
&\int_0^t(1+t)^{M-1}\|\theta^h\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\,d\tau\\
&\lesssim\left(\|\theta\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right)^{\eta_0}\int_0^t\left(\|(1+\tau)^M\theta\|^h_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)^{1-\eta_0}(1+\tau)^{M\eta_0-1}d\tau\\
&\lesssim\left(\|(1+\tau)^M\theta\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})}\right)^{1-\eta_0}\left(\|\theta\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{\eta_0}\|(1+\tau)^{M\eta_0-1}\|_{L_t^{\frac{2}{1+\eta_0}}}.
\end{aligned}\tag{4.20}$$

By combining (4.16)-(4.20), Young's inequality and the fact that

$$\|(1+\tau)^{M\eta_0-1}\|_{L_t^{\frac{2}{1+\eta_0}}}\lesssim\left((1+t)^{M-\frac12(\frac{d}{2}+\sigma_1)}\right)^{\eta_0},$$

we deduce

$$\begin{aligned}
&\|(1+\tau)^{M-\frac12}(a,u,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\
&\le\left(\|(a,u,\theta)^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})}+\|(a,u,\theta)^h\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)(1+t)^{M-\frac12(\frac{d}{2}+\sigma_1)}\\
&\quad+\frac14\left(\|(1+\tau)^M(a,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|(1+\tau)^Mu\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\right)\\
&\quad+\frac14\left(\|(1+\tau)^M(a,u)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|(1+\tau)^M\theta\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})}\right).
\end{aligned}\tag{4.21}$$

According to Proposition 5.1 and Proposition 5.2, the nonlinearities on the right-hand side of (4.15) can be estimated by

$$\|(1+\tau)^M(au,u\cdot\nabla u,u\theta,\frac{\theta-a}{a+1}\nabla a,\frac{a}{a+1}\nabla\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}$$

$$
\begin{aligned}
&\lesssim \|(a,u,\theta)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|(1+\tau)^M(u,\nabla u)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\
&\quad + \|(a,\theta)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|(1+\tau)^M a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})}\|(1+\tau)^M\theta\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
&\lesssim \mathcal{X}(t)\mathcal{X}_M(t)
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
\|(1+\tau)^M\nabla(\frac{a}{a+1})\nabla\theta\|^\ell_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} &\lesssim \|a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|(1+\tau)^M\theta\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
&\lesssim \mathcal{X}(t)\mathcal{X}_M(t).
\end{aligned}
\tag{4.23}
$$

Substituting (4.21)-(4.23) into (4.15), we get

$$
\begin{aligned}
&\|(1+\tau)^M(a,u,\theta)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|(1+\tau)^M(a,\theta)\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(1+\tau)^M u\|^\ell_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}\\
\lesssim &\|(a_0,u_0,\theta_0)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}}\\
&+\left(\|(a,u,\theta)^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)(1+t)^{M-\frac{1}{2}(\frac{d}{2}+\sigma_1)} + \mathcal{X}(t)\mathcal{X}_M(t).
\end{aligned}
\tag{4.24}
$$

- **Step2. High-frequency estimates.**

We recall that the Lyapunov type inequality (3.10) holds for $\mathcal{E}_{2,j}$ and $\mathcal{D}_{2,j}$ given by (3.11). It is easy to verify for any $j \geq -1$ that

$$
(\frac{1}{2}-\eta_2)\|(\dot{\Delta}_j a,\dot{\Delta}_j u)\|^2_{L^2} + \frac{1}{2}\|\dot{\Delta}_j\theta\|^2_{L^2} \leq \mathcal{E}_{2,j}(t) \leq (\frac{1}{2}+\eta_2)\|(\dot{\Delta}_j a,\dot{\Delta}_j u)\|^2_{L^2} + \frac{1}{2}\|\dot{\Delta}_j\theta\|^2_{L^2}, \tag{4.25}
$$

and

$$
\mathcal{D}_{2,j}(t) \gtrsim \frac{\eta_2}{2}\|\dot{\Delta}_j a\|^2_{L^2} + (1-\frac{3}{2}\eta_2)\|\dot{\Delta}_j u\|^2_{L^2} + (1-\frac{1}{2}\eta_2)2^{2j}\|\dot{\Delta}_j\theta\|^2_{L^2}, \tag{4.26}
$$

where in the first inequality one has used the key fact from (3.1) that

$$
\int_{\mathbb{R}^d}\frac{1+\theta}{(1+a)^2}(\dot{\Delta}_j a)^2dx \leq \|\frac{1+\theta}{(1+a)^2}\|_{L^\infty}\|\dot{\Delta}_j a\|^2_{L^2} \sim \|\dot{\Delta}_j a\|^2_{L^2}.
$$

By (4.25)-(4.26), for any $j \geq -1$, one can choose a sufficiently small constant $\eta_2 \in (0,1)$ so that we have

$$
\mathcal{E}_{2,j}(t) \sim \|(\dot{\Delta}_j a,\dot{\Delta}_j u,\dot{\Delta}_j\theta)\|^2_{L^2} \tag{4.27}
$$

and

$$
\mathcal{D}_{2,j}(t) \gtrsim \|(\dot{\Delta}_j a,\dot{\Delta}_j u)\|^2_{L^2} + 2^{2j}\|\dot{\Delta}_j\theta\|^2_{L^2}. \tag{4.28}
$$

For any $q \geq -1$, multiplying the Lyapunov inequality (3.10) by $(1+t)^{2M}$ gives

$$
\begin{aligned}
&\frac{d}{dt}\left((1+t)^{2M}\mathcal{E}_{2,j}(t)\right) + (1+t)^{2M}\mathcal{D}_{2,j}(t)\\
\lesssim &(1+t)^{2M-1}\mathcal{E}_{2,j}(t) + (1+t)^{2M}J_1 + (1+t)^{2M}J_2\\
&+(1+t)^{2M}J_3 + 2^{-2j}(1+t)^{2M}J_4,
\end{aligned}
\tag{4.29}
$$

where

$$
\begin{aligned}
J_1 \triangleq & \Big\|\frac{\partial}{\partial t}\Big(\frac{1+\theta}{(1+a)^2}\Big)\Big\|_{L^\infty}\|\dot\Delta_j a\|_{L^2}^2+\Big\|\nabla\Big(\frac{1+\theta}{1+a}\Big)\Big\|_{L^\infty}\|\dot\Delta_j u\|_{L^2}\|\dot\Delta_j a\|_{L^2}\\
&+\Big\|\mathrm{div}\Big(\frac{1+\theta}{(1+a)^2}u\Big)\Big\|_{L^\infty}\|\dot\Delta_j a\|_{L^2}^2+\|\mathrm{div}u\|_{L^\infty}\|\dot\Delta_j u\|_{L^2}^2,\\
J_2 \triangleq & \|\Big(\dot\Delta_j(u\theta),\|\frac{a}{1+a}\|_{L^\infty}\|\dot\Delta_j\nabla\theta\|_{L^2},\|\nabla(\frac{a}{1+a})\|_{L^\infty}\|\dot\Delta_j\theta\|_{L^2}\Big)\|_{L^2}\|\dot\Delta_j\nabla\theta\|_{L^2},\\
J_3 \triangleq & \|\tilde R_j^1\|_{L^2}\|\frac{1+\theta}{(1+a)^2}\|_{L^\infty}\|\dot\Delta_j a\|_{L^2}+\|\tilde R_j^2\|_{L^2}\|\dot\Delta_j u\|_{L^2}+\|\tilde R_j^3\|_{L^2}\|\dot\Delta_j\theta\|_{L^2},\\
J_4 \triangleq & \|\dot\Delta_j\mathrm{div}(au)\|_{L^2}\|\dot\Delta_j\mathrm{div}u\|_{L^2}+\|\Big(\dot\Delta_j(u\cdot\nabla u),\dot\Delta_j\big(\frac{\theta-a}{a+1}\nabla a\big)\Big)\|_{L^2}\|\dot\Delta_j\nabla a\|_{L^2}.
\end{aligned}
$$

By similar arguments as in (4.13)-(4.14) and combining (4.27)-(4.28), the inequality (4.29) implies

$$
\begin{aligned}
&(1+t)^M\|(\dot\Delta_j a,\dot\Delta_j u,\dot\Delta_j\theta)\|_{L^2}+\|(1+\tau)^M(\dot\Delta_j a,\dot\Delta_j u)\|_{L^2_tL^2}+2^j\|(1+\tau)^M\dot\Delta_j\theta\|_{L^2_tL^2}\\
\lesssim&\|(\dot\Delta_j a_0,\dot\Delta_j u_0,\dot\Delta_j\theta_0)\|_{L^2}+\|(1+\tau)^{M-\frac12}(\dot\Delta_j a,\dot\Delta_j u,\dot\Delta_j\theta)\|_{L^2_tL^2}\\
&+\Big(\int_0^t(1+\tau)^{2M}J_1\,d\tau\Big)^{1/2}+\Big(\int_0^t(1+\tau)^{2M}J_2\,d\tau\Big)^{1/2}\\
&+\Big(\int_0^t(1+\tau)^{2M}J_3\,d\tau\Big)^{1/2}+2^{-j}\Big(\int_0^t(1+\tau)^{2M}J_4\,d\tau\Big)^{1/2}.
\end{aligned}
\tag{4.30}
$$

Furthermore, it is straightforward to obtain

$$
\begin{aligned}
&\|(1+\tau)^M(a,u,\theta)\|^h_{\widetilde L^\infty_t(\dot B^{\frac d2+1}_{2,1})}+\|(1+\tau)^M(a,u)\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}+\|(1+\tau)^M\theta\|^h_{\widetilde L^2_t(\dot B^{\frac d2+2}_{2,1})}\\
\le&\|(a_0,u_0,\theta_0)\|^h_{\dot B^{\frac d2+1}_{2,1}}+\|(1+\tau)^{M-\frac12}(a,u,\theta)\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}+\sum_{i=i}^4 I_i,
\end{aligned}
\tag{4.31}
$$

where

$$
\begin{aligned}
I_1\triangleq&\Big\|\frac{\partial}{\partial t}\Big(\frac{1+\theta}{(1+a)^2}\Big)\Big\|_{L^2_t(L^\infty)}\|(1+\tau)^Ma\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}\\
&+\Big\|\nabla\Big(\frac{1+\theta}{1+a}\Big)\Big\|_{L^\infty_t(L^\infty)}\|(1+\tau)^Mu\|_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}\\
&+\Big\|\mathrm{div}\Big(\frac{1+\theta}{(1+a)^2}u\Big)\Big\|_{L^2_t(L^\infty)}\|(1+\tau)^Ma\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}\\
&+\|\mathrm{div}u\|_{L^2_t(L^\infty)}\|(1+\tau)^Mu\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})},\\
I_2\triangleq&\|(1+\tau)^Mu\theta\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}+\|\nabla(\frac{a}{1+a})\|_{L^2_t(L^\infty)}\|(1+\tau)^M\nabla\theta\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})}\\
&+\|\frac{a}{1+a}\|_{L^\infty_t(L^\infty)}\|(1+\tau)^M\nabla\theta\|^h_{\widetilde L^2_t(\dot B^{\frac d2+1}_{2,1})},\\
I_3\triangleq&\sum_{j\ge-1}2^{(\frac d2+1)j}\Big(\|(1+\tau)^M\tilde R_j^1\|_{L^1_t(L^2)}+\|(1+\tau)^M\tilde R_j^2\|_{L^1_t(L^2)}+\|(1+\tau)^M\tilde R_j^3\|_{L^1_t(L^2)}\Big),
\end{aligned}
$$

$$I_4 \triangleq \|(1+\tau)^M\Big(\mathrm{div}(au), (u\cdot\nabla u), (\frac{\theta-a}{a+1}\nabla a)\Big)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

By employing a similar procedure leading to (4.16)-(4.21), one can get

$$\begin{aligned}
&\|(1+\tau)^{M-\frac{1}{2}}(a,u,\theta)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
\le& C\int_0^t \|(1+\tau)^{M-1}(a,u,\theta)\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}}d\tau + \frac{1}{4}\|(1+\tau)^M(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
\le& C\|(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}(1+t)^{M-\frac{1}{2}(\frac{3}{2}+\sigma_1)} + \frac{1}{4}\|(1+\tau)^M(a,u)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
&+\frac{1}{4}\|(1+\tau)^M\theta\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} + \frac{1}{4}\|(1+\tau)^M(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}.
\end{aligned} \tag{4.32}$$

The right-hand side of (4.31) can be controlled below. As in (2.1), one can show

$$\big\|\frac{\partial}{\partial t}\big(\frac{1+\theta}{(1+a)^2}\big)\big\|_{L^\infty} \lesssim (1+\|(a,\theta)\|_{L^\infty})\|(\partial_t a, \partial_t\theta)\|_{L^\infty},$$

where

$$\begin{aligned}
&\partial_t a = -u\cdot\nabla a - (1+a)\mathrm{div}u,\\
&\partial_t\theta = -(1+\theta)\mathrm{div}u - u\cdot\nabla\theta + \frac{1}{a+1}\Delta\theta.
\end{aligned}$$

Furthermore, straightforward computations lead to

$$\begin{aligned}
&\|\partial_t a\|_{L^\infty} \lesssim \|u\|_{L^\infty}\|\nabla a\|_{L^\infty} + \big(1+\|a\|_{L^\infty}\big)\|\mathrm{div}u\|_{L^\infty},\\
&\|\partial_t \theta\|_{L^\infty} \lesssim \big(1+\|\theta\|_{L^\infty}\big)\|\mathrm{div}u\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla\theta\|_{L^\infty} + \big(1+\|a\|_{L^\infty}\big)\|\Delta\theta\|_{L^\infty}.
\end{aligned}$$

With the above preparation, due to (3.1) we have the following estimates:

$$\begin{cases}
\quad\big\|\frac{\partial}{\partial t}\big(\frac{1+\theta}{(1+a)^2}\big)\big\|_{L^2_t(L^\infty)}\\
\lesssim \|u\|_{L^\infty_t(L^\infty)}\|\nabla a\|_{L^2_t(L^\infty)} + \Big(1+\|a\|_{L^\infty_t(L^\infty)}+\|\theta\|_{L^\infty_t(L^\infty)}\Big)\|\mathrm{div}u\|_{L^2_t(L^\infty)}\\
\quad +\|u\|_{L^\infty_t(L^\infty)}\|\nabla\theta\|_{L^2_t(L^\infty)} + \Big(1+\|a\|_{L^\infty_t(L^\infty)}\Big)\|\Delta\theta\|_{L^2_t(L^\infty)},\\
\Big\|\nabla\Big(\frac{1+\theta}{1+a}\Big)\Big\|_{L^\infty_t(L^\infty)} \lesssim \|(\nabla a,\nabla\theta)\|_{L^\infty_t(L^\infty)},\\
\quad\Big\|\mathrm{div}\Big(\frac{1+\theta}{(1+a)^2}u\Big)\Big\|_{L^2_t(L^\infty)}\\
\lesssim \Big\|\nabla\Big(\frac{1+\theta}{(1+a)^2}\Big)\Big\|_{L^\infty_t(L^\infty)}\|u\|_{L^2_t(L^\infty)} + \Big\|\frac{1+\theta}{(1+a)^2}\Big\|_{L^\infty_t(L^\infty)}\|\mathrm{div}u\|_{L^2_t(L^\infty)}\\
\lesssim \|(\nabla a,\nabla\theta)\|_{L^\infty_t(L^\infty)}\|u\|_{L^2_t(L^\infty)} + \|\mathrm{div}u\|_{L^2_t(L^\infty)}.
\end{cases} \tag{4.33}$$

Thence it follows by $\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d_x) \hookrightarrow L^\infty(\mathbb{R}^d_x)$ and (5.5) that

$$I_1 \lesssim \big(\mathcal{X}(t)+\mathcal{X}^2(t)\big)\,\mathcal{X}_M(t). \tag{4.34}$$

According to Proposition 5.1 and Proposition 5.2, one can get

$$
\begin{aligned}
I_2 \lesssim & \|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|(1+\tau)^M\theta\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\
& + \big(\|a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\big)\|(1+\tau)^M\theta\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} \\
\lesssim & \mathcal{X}(t)\mathcal{X}_M(t).
\end{aligned}
\tag{4.35}
$$

It follows from the commutator estimate in Proposition 5.3 that

$$
\begin{aligned}
I_3 \lesssim & \|(1+\tau)^M a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(1+\tau)^M a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|\theta\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} \\
& + \|(1+\tau)^M u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|u\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \Big\|\frac{\theta-a}{1+a}\Big\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|(1+\tau)^M a\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\
\lesssim & \mathcal{X}(t)\mathcal{X}_M(t).
\end{aligned}
\tag{4.36}
$$

Due to Proposition 5.2, we get

$$
\begin{aligned}
I_4 \lesssim & \|(a,u,\theta)\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\|(1+\tau)^M(a,u)\|_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \\
\lesssim & \mathcal{X}(t)\mathcal{X}_M(t).
\end{aligned}
\tag{4.37}
$$

Hence, in view of (4.32), (4.34)-(4.37), it holds that

$$
\begin{aligned}
& \|(1+\tau)^M(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(1+\tau)^M(a,u)\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|(1+\tau)^M\theta\|^h_{\widetilde{L}^2_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} \\
\lesssim & \|(a_0,u_0,\theta_0)\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}(1+t)^{M-\frac{1}{2}(\frac{3}{2}+\sigma_1)} \\
& + \mathcal{X}_M(t)(\mathcal{X}(t)+\mathcal{X}^2(t)).
\end{aligned}
\tag{4.38}
$$

- **Step 3. The gain of time-weighted estimates.**

By combining (4.24) and (4.38), we conclude that

$$
\begin{aligned}
\mathcal{X}_M(t) \lesssim & \|(a_0,u_0,\theta_0)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(a_0,u_0,\theta_0)\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \\
& + \left(\|(a,u,\theta)^\ell\|_{\widetilde{L}^\infty_t(\dot{B}^{-\sigma_1}_{2,\infty})} + \|(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)(1+t)^{M-\frac{1}{2}(\frac{d}{2}+\sigma_1)} \\
& + (\mathcal{X}(t)+\mathcal{X}^2(t))\mathcal{X}_M(t).
\end{aligned}
$$

Together with the global existence result (Theorem 2.1) that implies $\mathcal{X}(t) \lesssim \varepsilon_0 \ll 1$, Lemma 4.1 and

$$
\|(a,u,\theta)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} \lesssim \|(a_0,u_0,\theta_0)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(a_0,u_0,\theta_0)\|^h_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \lesssim \varepsilon_0 \lesssim \delta_0,
$$

we end up with (4.12). Therefore, the proof of Proposition 4.1 is finished. □

4.3. The optimal decay

This subsection is devoted to the proof of Theorem 2.2. It follows from Proposition 4.1 that

$$\mathcal{X}_M \lesssim \delta_0(1+t)^{M-\frac{1}{2}(\frac{d}{2}+\sigma_1)} \tag{4.39}$$

for all $t>0$ and any suitably large M. Therefore, after dividing (4.39) by $(1+t)^M$, it is easy to get

$$\begin{aligned}\|(a,u,\theta)(t)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} &\lesssim \|(a,u,\theta)(t)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(a,u,\theta)(t)\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\\ &\lesssim \delta_0(1+t)^{-\frac{1}{2}(\frac{d}{2}+\sigma_1)}, \quad t\geq 1.\end{aligned} \tag{4.40}$$

Then it follows from (4.1), (4.40), and the interpolation inequality (5.6) that

$$\begin{aligned}\|(a,u,\theta)^{\ell}(t)\|_{\dot{B}^{\sigma}_{2,1}} &\lesssim \|(a,u,\theta)^{\ell}(t)\|^{\frac{\frac{d}{2}-\sigma}{\frac{d}{2}+\sigma_1}}_{\dot{B}^{-\sigma_1}_{2,\infty}} \|(a,u,\theta)^{\ell}(t)\|^{\frac{\sigma+\sigma_1}{\frac{d}{2}+\sigma_1}}_{\dot{B}^{\frac{d}{2}}_{2,1}}\\ &\lesssim \delta_0(1+t)^{-\frac{1}{2}(\sigma+\sigma_1)}, \quad \sigma\in(-\sigma_1,\frac{d}{2}).\end{aligned} \tag{4.41}$$

Regarding the corresponding high-frequency norm, one has

$$\|(a,u,\theta)^{h}(t)\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \|(a,u,\theta)(t)\|^{h}_{\dot{B}^{\frac{3}{2}}_{2,1}} \lesssim \delta_0(1+t)^{-\frac{1}{2}(\frac{d}{2}+\sigma_1)}. \tag{4.42}$$

By (4.40)-(4.42), the optimal time-decay estimates in (2.6) hold.

4.4. Improved time decay rates of u

First, we present the proof of (2.8). Recall $(2.1)_2$ can be rewritten as

$$\partial_t u + u = -\nabla a - \nabla\theta - u\cdot\nabla u - \frac{\theta-a}{a+1}\nabla a. \tag{4.43}$$

Performing a routine procedure yields

$$\begin{aligned}\|u\|_{\dot{B}^{\sigma}_{2,1}} &\lesssim e^{-t}\|u_0\|_{\dot{B}^{\sigma}_{2,1}}\\ &\quad + \int_0^t e^{-(t-\tau)}\big(\|(a,\theta)\|_{\dot{B}^{\sigma+1}_{2,1}} + \|u\cdot\nabla u\|_{\dot{B}^{\sigma}_{2,1}} + \|\frac{\theta-a}{a+1}\nabla a\|_{\dot{B}^{\sigma}_{2,1}}\big)d\tau.\end{aligned} \tag{4.44}$$

According to (2.7) and (5.5), one arrives at

$$\|u_0\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \|u_0^{\ell}\|_{\dot{B}^{-\sigma_1}_{2,\infty}} + \|u_0\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \mathcal{X}_0, \quad \sigma\in(-\sigma_1,\frac{d}{2}+1].$$

If $\sigma\in(-\sigma_1,\frac{d}{2}-1]$, it follows from (2.6) that

$$\|(a,\theta)\|_{\dot{B}^{\sigma+1}_{2,1}} \lesssim \delta_0(1+t)^{-\frac{1}{2}(\sigma+\sigma_1+1)}. \tag{4.45}$$

Then, we deduce from (2.6), (5.7) and Proposition 5.1 that

$$\|u\cdot\nabla u\|_{\dot{B}^{\sigma}_{2,1}} \lesssim \|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|u\|_{\dot{B}^{\sigma+1}_{2,1}} \lesssim \delta_0^2(1+t)^{-\frac{1}{2}(\frac{d}{2}+\sigma+2\sigma_1+1)}, \tag{4.46}$$

$$\|\frac{\theta-a}{a+1}\nabla a\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\|(a,\theta)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|a\|_{\dot{B}^{\sigma+1}_{2,1}}\lesssim\delta_0^2(1+t)^{-\frac{1}{2}(\frac{d}{2}+\sigma+2\sigma_1+1)}. \tag{4.47}$$

Together with (4.45)-(4.47), we conclude that

$$\begin{aligned}\|u\|_{\dot{B}^{\sigma}_{2,1}}&\lesssim e^{-t}\|u_0\|_{\dot{B}^{\sigma}_{2,1}}+\delta_0\int_0^t e^{-(t-\tau)}(1+t)^{-\frac{1}{2}(\sigma+\sigma_1+1)}d\tau\\&\quad+\delta_0^2\int_0^t e^{-(t-\tau)}(1+t)^{-\frac{1}{2}(\frac{d}{2}+\sigma+2\sigma_1+1)}d\tau\\&\lesssim\delta_0(1+t)^{-\frac{1}{2}(\sigma+\sigma_1+1)}\end{aligned} \tag{4.48}$$

with $\sigma\in(-\sigma_1,\frac{d}{2}-1]$. Therefore, the following decay rates holds:

$$\|u\|_{\dot{B}^{\sigma}_{2,1}}\lesssim\|u\|^{\ell}_{\dot{B}^{\sigma}_{2,1}}+\|u\|^{h}_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\lesssim\delta_0(1+t)^{-\frac{1}{2}(1+\sigma+\sigma_1)}. \tag{4.49}$$

Hence, (2.8) is followed by (4.44)-(4.49) directly. The proof of Theorem 2.2 is complete.

5. Appendix

5.1. Littlewood-Paley decomposition and Besov space

Let us briefly review the definition of Besov spaces based on the Littlewood-Paley decomposition. The interested reader is referred to Chapter 2 and Chapter 3 of [1] for more details. Firstly, let's introduce the homogeneous Littlewood-Paley decomposition. For that purpose, we fix some smooth radial non increasing function χ with $\operatorname{Supp}\chi\subset B\left(0,\frac{4}{3}\right)$ and $\chi\equiv1$ on $B\left(0,\frac{3}{4}\right)$, then set $\varphi(\xi)=\chi(\xi/2)-\chi(\xi)$ so that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\cdot)=1,\quad\operatorname{Supp}\varphi\subset\left\{\xi\in\mathbb{R}^d:\frac{3}{4}\le|\xi|\le\frac{8}{3}\right\}.$$

For any $j\in\mathbb{Z}$, define the homogeneous dyadic blocks $\dot{\Delta}_j$ by

$$\dot{\Delta}_j f\triangleq\varphi(2^{-j}D)f=\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f)=2^{jd}h(2^j\cdot)*f\quad\text{with}\quad h\triangleq\mathcal{F}^{-1}\varphi,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse. The following Littlewood-Paley decomposition of f:

$$f=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j f, \tag{5.1}$$

holds true modulo polynomials for any tempered distribution f. In order to have equality in the sense of tempered distributions, we consider only elements of the set $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions f such that

$$\lim_{j\to-\infty}\|\dot{S}_j f\|_{L^\infty}=0, \tag{5.2}$$

where $\dot{S}_j f$ stands for the low frequency cut-off defined by $\dot{S}_j f\triangleq\chi(2^{-j}D)f$. Indeed, if (5.2) is fulfilled, then (5.1) holds in $\mathcal{S}'(\mathbb{R}^d)$. For convenience, we denote by $\mathcal{S}'_h(\mathbb{R}^d)$ the subspace of tempered distributions satisfying (5.2).

Based on those dyadic blocks, Besov spaces are defined as follows.

Definition 5.1. For $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, the homogeneous Besov spaces $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} \triangleq \left\{ f \in \mathcal{S}'_h : \|f\|_{\dot{B}^s_{p,r}} < +\infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}_{j\in\mathbb{Z}}\|_{l^r(\mathbb{Z})}. \tag{5.3}$$

The mixed space-time Besov spaces are also used, which was introduced by J. Y. Chemin and N. Lerner [6].

Definition 5.2. For $T > 0$, $s \in \mathbb{R}$, $1 \le r, \varrho \le \infty$, the homogeneous Chemin-Lerner space $\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})$ is defined by

$$\widetilde{L}^\varrho_T(\dot{B}^s_{p,r}) \triangleq \left\{ f \in L^\theta(0,T;\mathcal{S}'_h) : \|f\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} < +\infty \right\},$$

where

$$\|f\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \triangleq \|\{2^{js}\|\dot{\Delta}_j f\|_{L^\varrho_T(L^p)}\}_{j\in\mathbb{Z}}\|_{l^r(\mathbb{Z})}. \tag{5.4}$$

For notational simplicity, index T will be omitted if $T = +\infty$. We also use the following functional space:

$$\mathcal{C}_b(\mathbb{R}_+;\dot{B}^s_{p,r}) \triangleq \left\{ f \in \mathcal{C}(\mathbb{R}_+;\dot{B}^s_{p,r}) \mid \|f\|_{\widetilde{L}^\infty(\mathbb{R}_+;\dot{B}^s_{p,r})} < +\infty \right\}.$$

The above norm (5.4) may be linked with those of the standard spaces $L^\theta_T(\dot{B}^s_{p,r})$ by means of Minkowski's inequality.

Remark 5.1. It holds that

$$\|f\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \le \|f\|_{L^\varrho_T(\dot{B}^s_{p,r})} \ \text{ if } \ r \ge \varrho; \qquad \|f\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \ge \|f\|_{L^\varrho_T(\dot{B}^s_{p,r})} \ \text{ if } \ r \le \varrho.$$

Restricting the norms in (5.3) and (5.4) to the low frequency part and high frequency part of distributions will be fundamental to our approach. For example [18], we fix some integer j_0 (the value of which will follow from the proofs of our main results) and put*

$$\|f\|^\ell_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}_{j\le j_0}\|_{l^r} \text{ and } \|f\|^h_{\dot{B}^s_{p,r}} \triangleq \|\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}_{j\ge j_0-1}\|_{l^r},$$

$$\|f\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \triangleq \|\{2^{js}\|\dot{\Delta}_j f\|_{L^\varrho_T(L^p)}\}_{j\le j_0}\|_{l^r} \text{ and } \|f\|^h_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \triangleq \|\{2^{js}\|\dot{\Delta}_j f\|_{L^\varrho_T(L^p)}\}_{j\ge j_0-1}\|_{l^r}.$$

Define

$$f^\ell := \sum_{j\le -1} \dot{\Delta}_j f, \qquad f^h := f - f^\ell = \sum_{j\ge 0} \dot{\Delta}_j f.$$

It is easy to check for any $s' > 0$ that

$$\begin{cases} \|f^\ell\|_{\dot{B}^s_{p,r}} \lesssim \|f\|^\ell_{\dot{B}^s_{p,r}} \lesssim \|f\|^\ell_{\dot{B}^{s-s'}_{p,r}}, \\ \|f^h\|_{\dot{B}^s_{p,1}} \lesssim \|f\|^h_{\dot{B}^s_{p,r}} \lesssim \|f\|^h_{\dot{B}^{s+s'}_{p,r}}, \\ \|f^\ell\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \lesssim \|f\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \lesssim \|f\|^\ell_{\widetilde{L}^\varrho_T(\dot{B}^{s-s'}_{p,r})}, \\ \|f^h\|_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \lesssim \|f\|^h_{\widetilde{L}^\varrho_T(\dot{B}^s_{p,r})} \lesssim \|f\|^h_{\widetilde{L}^\varrho_T(\dot{B}^{s+s'}_{p,r})}. \end{cases} \tag{5.5}$$

We also need hybrid Besov spaces for which regularity assumptions are different in low frequencies and high frequencies [9]. We are going to recall the definition and properties.

*Note that for technical reasons, we need to a small overlap between low and high frequencies.

Definition 5.3. Let $s, t \in \mathbb{R}$. We define

$$\|f\|_{\dot{B}^{s,t}_{2,1}} = \sum_{j \le j_0} 2^{js}\|\dot{\Delta}_j f\|_{L^2} + \sum_{j > j_0} 2^{jt}\|\dot{\Delta}_j f\|_{L^2}.$$

Let $m = -[\frac{d}{2} + 1 - s]$, we then define

$$\dot{B}^{s,t}_{2,1}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\dot{B}^{s,t}_{2,1}} < \infty\}, \quad \text{if} \quad m < 0,$$
$$\dot{B}^{s,t}_{2,1}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_m : \|f\|_{\dot{B}^{s,t}_{2,1}} < \infty\}, \quad \text{if} \quad m \ge 0.$$

Remark 5.2. We have the following properties.

- $\dot{B}^{s,s}_{2,1} = \dot{B}^{s}_{2,1}$,
- if $s \le t$ then $\dot{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} \cap \dot{B}^{t}_{2,1}$. Otherwise, $\dot{B}^{s,t}_{2,1} = \dot{B}^{s}_{2,1} + \dot{B}^{t}_{2,1}$,
- if $s_1 \le s_2$ and $t_1 \ge t_2$, then $\dot{B}^{s_1,t_1}_{2,1} \hookrightarrow \dot{B}^{s_2,t_2}_{2,1}$.

We recall some basic properties of Besov spaces and product estimates which will be used repeatedly in this paper. The first lemma is the so-called Bernstein inequalities.

Lemma 5.1. *Let $k \in \mathbb{N}$, $1 \le a \le b \le \infty$, C is a constant and f is an any function in L^p, then we have if $\operatorname{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^d : |\xi| \le R\lambda\}$ for some $R > 0$*

$$\|D^k f\|_{L^b} \le C^{1+k}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}.$$

More generally, If $\operatorname{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^d : R_1\lambda \le |\xi| \le R_2\lambda\}$ for some $0 < R_1 < R_2$, we have

$$C^{-k-1}\lambda^k\|u\|_{L^a} \le \|D^k u\|_{L^a} \le C^{k+1}\lambda^k\|u\|_{L^a}.$$

Due to the Bernstein inequalities, the Besov spaces have many properties:

Lemma 5.2. *Let $1 \le p, r, r_1, r_2 \le \infty$.*

- *Completeness: $\dot{B}^s_{p,r}$ is a Banach space whenever $s < \frac{d}{p}$ or $s \le \frac{d}{p}$ and $r = 1$.*
- *Embedding: For any $s \in \mathbb{R}, 1 \le p_1 \le p_2 \le \infty$, and $1 \le r_1 \le r_2 \le \infty$, it holds that*

$$\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}.$$

  Moreover, For any $1 \le p \le q \le \infty$, we have the continuous embedding

$$\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty} \hookrightarrow \dot{B}^\sigma_{q,\infty} \text{ for } \sigma = -d(\frac{1}{p} - \frac{1}{q}) < 0.$$

- *Interpolation: The following inequalities are satisfied for $1 \le p,\ r_1,\ r_2,\ r \le \infty$, $s_1 < s_2$ and $\theta \in (0,1)$:*

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \lesssim \|f\|^\theta_{\dot{B}^{s_1}_{p,r_1}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,r_2}} \quad \text{with} \quad \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}.$$

  Particularly, we have the following optimal interpolation formula

$$\|f\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{\theta(1-\theta)(s_2-s_1)}\|f\|^\theta_{\dot{B}^{s_1}_{p,\infty}}\|f\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}. \tag{5.6}$$

System (2.1) also involves multivariate compositions of functions (through $\frac{\theta-a}{a+1}$, $\frac{1}{a+1}$ that are bounded thanks to the following result:

Proposition 5.1. ([34]) *Let $m \in \mathbb{N}$ and $s > 0$. Let G be a function in $\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^d)$ such that $G(0,\cdots,0)=0$. Then for every real valued functions $f_1,\cdots,f_m \in \dot{B}^s_{p,r} \cap L^\infty$, the function $G(f_1,\cdots,f_m)$ belons to $\dot{B}^s_{p,r}\cap L^\infty$ and we have*

$$\|G(f_1,\cdots,f_m)\|_{\dot{B}^s_{p,r}} \le C\|(f_1,\cdots,f_m)\|_{\dot{B}^s_{p,r}}$$

with C depending only on $\|f_i\|_{L^\infty}$ $(i=1,\cdots,m)$, G'_{f_i} (and higher derivatives), s, p and d.

In the case $s > -\min(\frac{d}{p},\frac{d}{p*})$, then $f_1,\cdots,f_m \in \dot{B}^s_{p,r}\cap \dot{B}^{\frac{d}{p}}_{p,1}$ implies that $G(f_1,\cdots,f_m) \in \dot{B}^s_{p,r}\cap\dot{B}^{\frac{d}{p}}_{p,1}$ and we have

$$\|G(f_1,\cdots,f_m)\|_{\dot{B}^s_{p,r}} \le C\left(1+\|f_1\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+\cdots+\|f_m\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\right)\|(f_1,\cdots,f_m)\|_{\dot{B}^s_{p,r}}.$$

The following product estimates in Besov spaces play a fundamental role in our analysis of the nonlinear terms.

Proposition 5.2. *The following statements holds:*

- *Let $s>0$ and $1\le p,r\le\infty$. Then $\dot{B}^s_{p,r}\cap L^\infty$ is an algebra and*
$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}. \tag{5.7}$$
- *Let the real numbers s_1, s_2 and p satisfy $2\le p\le\infty$, $s_1\le\frac{d}{p}$, $s_2\le\frac{d}{p}$ and $s_1+s_2>0$. Then we have*
$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}. \tag{5.8}$$
- *Assume that s_1, s_2 and p satisfy $2\le p\le\infty$, $s_1\le\frac{d}{p}$, $s_2<\frac{d}{p}$ and $s_1+s_2\ge0$. Then it holds that*
$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,\infty}}. \tag{5.9}$$

Finally, the following commutator estimates will be useful to control the nonlinearities in high frequencies.

Proposition 5.3. *Let $1<p<\infty$, $1\le\varrho\le\infty$ and $s\in(-\frac{d}{p}-1,\frac{d}{p}]$. Then there exists a generic constant $C>0$ depending only on the dimension d and the regular index s*

$$\begin{cases} \|[\dot{\Delta}_j,f]g\|_{L^p} \le Cc_j 2^{-j(s+1)}\|f\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}}\|g\|_{\dot{B}^s_{p,1}}, \\ \|[\dot{\Delta}_j,f]g\|_{L^\varrho_t(L^p)} \le Cc_j 2^{-j(s+1)}\|f\|_{\widetilde{L}^{\varrho_1}_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|g\|_{\widetilde{L}^{\varrho_2}_t(\dot{B}^s_{p,1})}, \end{cases} \tag{5.10}$$

with $\frac{1}{\varrho}=\frac{1}{\varrho_1}+\frac{1}{\varrho_2}$ and the commutator $[A,B]\triangleq AB-BA$.

Conflicts of interest and data availability statement. The authors do not have any possible conflict of interest. Also, there is no associated data in this manuscript.

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Received May 2025; Accepted September 2025; Available online September 2025.