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THE EXISTENCE AND ULAM-HYERS STABILITY RESULTS FOR MULTI-POINT GENERALIZED CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract This paper addresses a class of multi-point boundary value problems involving generalized fractional derivatives with integral conditions. Specifically, we consider the following problem

$$\begin{aligned}
&{}^cD_{a^+}^{\alpha,h}u(t) + f(t,u(t)) = 0, \quad t \in [a,b],\\
&u^{(i)}(a) = 0, \ i = 0,1,2,...,n-2,\\
&u(b) = \sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} g(s)u(s)ds + \sum_{j=1}^{m-2} \lambda_j u(\eta_j).
\end{aligned}$$

We establish necessary and sufficient conditions for the existence and uniqueness of solutions to this problem. Our approach relies on the Banach fixed point theorem and Schaefer's fixed point theorem to prove the existence of solutions. Additionally, we introduce the concept of Ulam-Hyers stability for this class of boundary value problems and provide a stability analysis. To illustrate the applicability of our theoretical results, we present two concrete examples.

Keywords Boundary value problem, generalized Caputo fractional derivative and integral, Green's function, fixed point theorem, existence and uniqueness of the solution.

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1. Introduction

Fractional calculus, the branch of mathematics that deals with derivatives and integrals of non-integer order, has become an indispensable tool in modeling complex systems across various scientific and engineering disciplines. Unlike classical integer-order calculus, fractional calculus captures memory effects and long-range dependencies, making it particularly suitable for describing phenomena in viscoelasticity, fluid dynamics, control theory, and even financial markets [10,18,21]. Classical works by Oldham and Spanier [19], Miller and Ross [17], Podlubny [20], Kilbas et al. [12], and Samko et al. [22] laid the foundational theory, while more recent developments by Yoruk et al. [33], Hilfer [10] and Baleanu and Diethelm [6] have extended its applications to physics and engineering.

Among the many types of fractional derivatives, the generalized fractional derivative has gained prominence due to its flexibility in modeling a wide range of physical and biological processes. In particular, the Caputo derivative with respect to another function introduced by

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Almeida [4] and its extension to more general kernel functions [5] have expanded the analytical toolkit for modeling memory-dependent dynamics. These forms of fractional operators have been further utilized in electrochemistry [18], physics [10], and biological systems [1,3,30].

Boundary value problems (BVPs) involving fractional derivatives are of particular interest because they frequently arise in the study of real-world systems where the behavior of a function is constrained by conditions at multiple points within the domain. Multi-point boundary conditions, in particular, are often used to model systems where the state at one boundary is influenced by intermediate points, such as in heat conduction, beam deflection, or population dynamics [15,30,31]. When integral conditions are added to these problems, they further enrich the modeling capabilities, allowing for the incorporation of global information about the system's behavior [1,13,25,27,29,30].

In this context, Wang, Liang, and Wang [31] investigated a fractional differential equation of arbitrary order subject to multi-point integral boundary conditions. Using Green's function, the problem was reformulated as an integral equation and solved using fixed point theorems. Similarly, Wahash et al. [30] examined a boundary value problem involving a Caputo-type fractional derivative with respect to a function and integral boundary conditions. They employed the method of lower and upper solutions alongside fixed point theory to establish the existence of positive solutions.

Recent research has focused not only on existence theory but also on Ulam-Hyers stability and uniqueness results. For example, Mahmudov [14], Tatar and Torun [24], and Tunç [26, 28] provided sufficient conditions for the Ulam-Hyers or generalized Ulam-type stabilities of fractional differential equations. Lachouri and Ardjouni [13] extended these results to Hilfer-type fractional integro-differential equations with nonlocal boundary conditions. Further, Malghi et al. [15] analyzed hybrid generalized Caputo time-fractional systems, establishing both existence and stability criteria.

Various mathematical tools have been applied to fractional BVPs to establish the existence, uniqueness, and positivity of solutions, including fixed point theorems such as Schauder, Banach, and Krasnoselskii's [7,9,23], the method of upper and lower solutions [30], and generalized contraction principles [11]. Jleli and Samet [11] introduced a new generalization of the Banach contraction principle and applied it to fractional equations, while Agarwal et al. [2] and Zhou and Jiao [34] discussed existence and uniqueness results in Banach spaces for nonlocal and causal operator settings.

Several recent works have specifically generalized and nonlinear fractional differential equations with multi-point or integral conditions [1,3,32]. Alghanmi et al. [3] addressed coupled nonlinear systems with anti-periodic boundary conditions involving ρ fractional derivatives, and Yalçın et al. [32] focused on generalized Caputo-type equations with nonlinear structure. Abdo et al. [1] investigated positive solutions for boundary value problems involving generalized Caputo fractional derivatives, while Gouari et al. [8] studied models involving convergent series and singularities.

In 2018 Wang et al. [31] investigated the following boundary value problem

$$
\begin{aligned}
&D^{\alpha}x(t) + f(t, x(t)) = 0, \quad t \in [0,1],\\
&x^{(i)}(0) = 0, \quad i = 0, 1, 2, ..., n-2,\\
&x(1) = \sum_{j=1}^{m-2} \beta_j \int_0^{\eta_i} x(s)ds + \sum_{i=1}^{m-2} \gamma_i x(\eta_i).
\end{aligned}
$$

In this BVP, $n-1<\alpha\leq n,\ n\geqslant 3,\ n\in\mathbb{N}$. Moreover $m\geq 3$ is an integer, $0<\eta_1<\eta_2<...<\eta_{m-2}<1,\ \beta_i,\gamma_i>0$ and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous. This paper investigates a fractional differential equation of arbitrary order subject to multi-point integral boundary conditions. The problem is converted into an equivalent integral equation and subsequently solved. Green's function is derived from the obtained solution. Under specific assumptions on the function f, ensuring continuity and incorporating the properties of Green's function, a continuous operator is introduced in a normed Banach space. Utilizing Krasnoselkii's and Schauder's fixed point theorems, the existence and uniqueness of multiple solutions to the problem are established.

In 2020 Wahash et al. [30] investigated the following boundary value problem

$$\begin{aligned} &{}^cD^{r,h}_{0^+}=f(t,u(t)),\quad t\in[0,1],\\ &u(0)=\lambda\int_0^1 g(s)u(s)ds+d. \end{aligned}$$

In this BVP, $0<r\leq 1,\ \lambda\geq 0,\ d\in\mathbb{R}^+$, $g\in\mathcal{L}^1([0,1],\mathbb{R}^+)$ and $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous. This study explores a boundary value problem incorporating integral boundary conditions and a generalized Caputo-type fractional derivative defined with respect to an arbitrary function. The problem is reformulated as an equivalent integral equation. By employing the lower and upper solution method along with Banach and Schauder fixed point theorems, the existence and uniqueness of positive solutions are established.

Motivated by the aforementioned studies and the growing body of literature, in this paper we investigate a new class of multi-point boundary value problems with integral conditions, where the differential operator is a generalized fractional derivative. By constructing appropriate Green's functions and employing a combination of fixed point theory and comparison principles, we aim to establish sufficient conditions for the existence, uniqueness, and Ulam-Hyers stability of positive solutions. Specifically, we consider the following problem

$$\begin{aligned} &{}^cD^{\alpha,h}_{a^+}u(t)+f(t,u(t))=0,\quad t\in[a,b],\\ &u^{(i)}(a)=0,\ i=0,1,2,...,n-2,\\ &u(b)=\sum_{j=1}^{m-2}\beta_j\int_a^{\eta_j}g(s)u(s)ds+\sum_{j=1}^{m-2}\lambda_j u(\eta_j), \end{aligned}$$

where ${}^cD^{\alpha,h}_{a^+}$ is generalized Caputo fractional derivative, $n-1<\alpha\leq n,\ n\geqslant 3,\ n\in\mathbb{N}$ with

$$\begin{cases} n=\lfloor\alpha\rfloor+1, & \alpha\notin\mathbb{N},\\ n=\alpha, & \alpha\in\mathbb{N}, \end{cases}$$

and $h\in\mathcal{C}^n[a,b]$ is a continuously differentiable, increasing function with $h'(t)\neq 0$ for all $t\in[a,b]$. Moreover $a<\eta_1<\eta_2<...<\eta_{m-2}<b,\ \beta_j,\lambda_j>0, f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $g\in\mathcal{L}^1([a,b],\mathbb{R}^+)$. The presence of both multi-point and integral boundary conditions in the proposed problem introduces significant analytical challenges, as it requires balancing local differential constraints with global integral effects. These types of conditions frequently arise in modeling real-world phenomena such as heat conduction with memory, population dynamics with intermediate stages, or viscoelastic systems with hereditary properties.

The motivation for studying this class of fractional boundary value problems stems from both theoretical and practical considerations. Many natural and engineered systems exhibit memory effects, spatial heterogeneity, and delayed interactions that cannot be adequately described by classical integer-order models. Generalized fractional derivatives, particularly those defined with respect to another function, provide a flexible framework for incorporating these complex dynamics. Furthermore, the inclusion of multi-point and integral boundary conditions allows the modeling of systems where the state at a boundary is influenced by cumulative or distributed interactions within the domain. These features are especially relevant in fields such as viscoelasticity, biological systems with feedback control, and thermal systems with distributed memory. Despite their significance, such problems have not been thoroughly analyzed in the existing literature, particularly in terms of establishing the existence, uniqueness, and Ulam-Hyers stability of solutions. This motivates the present study, which aims to address these gaps by developing a rigorous analytical framework using Green's functions and fixed point theory.

Our primary objective is to establish existence, uniqueness, and Ulam-Hyers stability results for a new class of fractional boundary value problems involving a generalized Caputo fractional derivative. The considered problem incorporates both multi-point and Riemann-Stieltjes integral-type boundary conditions, which provide a more flexible modeling framework by capturing both discrete and cumulative influences within the domain.

To tackle the analytical complexity, we first transform the original fractional differential equation into an equivalent integral equation. This transformation allows us to construct the associated Green's function, which plays a pivotal role in the representation and analysis of solutions. We then employ two fundamental tools from nonlinear functional analysis: the Banach contraction principle, which is used to ensure uniqueness under contraction-type conditions, and Schaefer's fixed point theorem, which is effective in proving the existence of solutions under more general settings.

Beyond the qualitative properties of solutions, we also investigate their Ulam-Hyers stability, a concept particularly relevant for fractional-order systems due to their nonlocal memory effects. This form of stability ensures that small perturbations in the data or initial conditions do not cause significant deviations in the solution, thereby confirming the robustness and reliability of the model from both theoretical and applied perspectives.

To support the analytical results, we present two concrete examples, which not only validate the main results but also illustrate the practical implementation of the proposed methods. These examples highlight how the general theory can be applied to specific problems with varying structures and parameters.

The remainder of this paper is organized as follows: Section 2 presents preliminary concepts, including essential definitions, lemmas, and fundamental theorems related to generalized fractional derivatives and fixed point theory. Section 3 is devoted to the reformulation of the proposed boundary value problem as an integral equation and the construction of the corresponding Green's function. Section 4 establishes the main results concerning the existence, uniqueness, and Ulam-Hyers stability of solutions using appropriate fixed point theorems. Also, Section 5 contains illustrative examples that demonstrate the applicability of the theoretical findings.Finally, Section 6 summarizes the conclusions and discussion.

2. Preliminaries

This section introduces key definitions, lemmas, and fixed point theorems that are essential to the content of the paper and will be utilized in the third section.

Definition 2.1. [12] Let h be a continuously differentiable and increasing function on $[a, b]$ with $h'(t) \neq 0$ for all $t \in [a, b]$. For $\alpha > 0$ the left-sided h-Riemann-Liouville fractional integral of an integrable function $u : [a, b] \to \mathbb{R}$ is defined as

$$I_{a^+}^{\alpha,h} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t h'(s)(h(t) - h(s))^{\alpha-1} u(s) ds,$$

where $\Gamma(.)$ represents the Gamma function.

Definition 2.2. [5] Let $n - 1 < \alpha < n$ and let $u : [a, b] \to \mathbb{R}$ be a differentiable function on the closed interval $[a, b]$. Assume that $h \in \mathcal{C}^n[a, b]$ is continuously differentiable and increasing function satisfaying $h'(t) \neq 0$ for all $t \in [a, b]$. The left-sided h-Riemann-Liouville fractional derivative of order α for the function u is defined as

$$D_{a^+}^{\alpha,h} u(t) = \left[\frac{1}{h'(t)} \frac{d}{dt}\right]^n I_{a^+}^{n-\alpha,h} u(t),$$

which can be rewritten in integral form as

$$D_{a^+}^{\alpha,h} u(t) = \frac{1}{\Gamma(n - \alpha)} \left[\frac{1}{h'(t)} \frac{d}{dt}\right]^n \int_a^t h'(s)(h(t) - h(s))^{n-\alpha-1} u(s) ds.$$

Here n is determined as $n = \lfloor \alpha \rfloor + 1$ when α is not an integer, and $n = \alpha$ if α is an integer, where $\lfloor \alpha \rfloor$ denotes the greatest integer less than or equal to α.

Definition 2.3. [5] Let $n - 1 < \alpha < n$ and let $h \in \mathcal{C}^n[a, b]$ be a continuously differentiable, increasing function with $h'(t) \neq 0$ for all $t \in [a, b]$. Suppose $u \in \mathcal{C}^{n-1}[a, b]$. The fractional derivative of order α order in the sense of the h-Caputo definition is given by

$${}^c D_{a^+}^{\alpha,h} u(t) = D_{a^+}^{\alpha,h} \left[u(t) - \sum_{j=0}^{n-1} \frac{u_h^{[j]}(a)}{j!} (h(t) - h(a))^j \right],$$

where the function $u_h^{[j]}(t)$ is defined as $u_h^{[j]}(t) = \left[\frac{1}{h'(t)} \frac{d}{dt}\right]^j u(t)$. For non-integer values of α, where $n = \lfloor \alpha \rfloor + 1$, the h-Caputo fractional derivative is expressed as

$${}^c D_{a^+}^{\alpha,h} u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t h'(s)(h(t) - h(a))^{n-\alpha-1} u_h^{[n]}(s) ds.$$

When α is a natural number, the fractional derivative reduces to

$${}^c D_{a^+}^{\alpha,h} u(t) = u_h^{[n]}(t).$$

Lemma 2.1. [5] *Let $\alpha > 0$. The function $u : [a, b] \to \mathbb{R}$ satisfies the following fundamental properties:*

1. *For any $u \in \mathcal{C}[a, b]$, the h-Riemann-Liouville fractional integral and the h-Caputo fractional derivative are inverse operations, meaning*

$${}^c D_{a^+}^{\alpha,h} I_{a^+}^{\alpha,h} u(t) = u(t).$$

2. *If u is sufficiently smooth, specifically $u \in \mathcal{C}^{n-1}[a,b]$, then the application of the h-Riemann-Liouville fractional integral to the h-Caputo fractional derivative satisfies the identity*

$$I_{a^+}^{\alpha,h}{}^cD_{a^+}^{\alpha,h}u(t) = u(t) - \sum_{j=0}^{n-1} c_j(h(t)-h(a))^j \tag{2.1}$$

where the constants c_j are defined as $c_j = \frac{u^{[j]}(a)}{j!}$.

Theorem 2.1. (Banach fixed point theorem) [7] *Consider a Banach space U with K as its closed subset. If the mapping A is a contraction on K, there exists a unique element $u \in K$ satisfying $A(u) = u$.*

Theorem 2.2. (Schaefer's fixed point theorem) [23] *Let U be a Banach space and consider a continuous and compact operator $A : U \to U$. If the set $\{u \in U : u = \lambda Au, \text{ for some } \lambda \in (0,1)\}$ is bounded, then A possesses at least one fixed point in U.*

3. Main results

In this section, in order to establish sufficient conditions for the existence and uniqueness of the solutions to the boundary value problem, firstly, the problem is expressed as an integral equation and Green's functions are created. Then we will aim to prove the existence theorems.

Lemma 3.1. *Let $n-1 < \alpha \leq n$ and suppose $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider the following multi-point h-Caputo fractional boundary value problem*

$${}^cD_{a^+}^{\alpha,h}u(t) + f(t,u(t)) = 0, \quad t \in [a,b], \tag{3.1}$$

$$u^{(i)}(a) = 0, i = 0,1,2,...,n-2, \tag{3.2}$$

$$u(b) = \sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} g(s)u(s)ds + \sum_{j=1}^{m-2} \lambda_j u(\eta_j). \tag{3.3}$$

The function $u \in \mathcal{C}^{n-1}[a,b]$ is a solution of the boundary value problem (3.1)-(3.3) if and only if u satisfies the following integral equation

$$u(t) = \int_a^b h'(s)G(t,s)f(s,u(s))ds \tag{3.4}$$

where $G(t,s)$ represents the Green's function, given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} G_1(t,s), & s \leq \eta_1, \\ G_\gamma(t,s), & \eta_{\gamma-1} \leq s \leq \eta_\gamma, \ 2 \leq \gamma \leq m-2, \\ G_{m-1}(t,s), & s \geq \eta_{m-2}, \end{cases} \tag{3.5}$$

where the function components are defined as follows:

$$G_1(t,s)=\begin{cases}\dfrac{(h(t)-h(a))^{n-1}}{\Delta}\left\{-\sum_{j=1}^{m-2}\beta_j\Theta_j(s)-\sum_{j=1}^{m-2}\lambda_j(h(\eta_j)-h(s))^{\alpha-1}+(h(b)-h(s))^{\alpha-1}\right\}\\ -(h(t)-h(s))^{\alpha-1},\ a\leq s\leq t\leq \eta_1,\\ \dfrac{(h(t)-h(a))^{n-1}}{\Delta}\left\{-\sum_{j=1}^{m-2}\beta_j\Theta_j(s)-\sum_{j=1}^{m-2}\lambda_j(h(\eta_j)-h(s))^{\alpha-1}+(h(b)-h(s))^{\alpha-1}\right\},\\ a\leq t\leq s\leq \eta_1,\end{cases}$$

$$G_\gamma(t,s)=\begin{cases}\dfrac{(h(t)-h(a))^{n-1}}{\Delta}\left\{-\sum_{j=\gamma}^{m-2}[\beta_j\Theta_j(s)+\lambda_j(h(\eta_j)-h(s))^{\alpha-1}]+(h(b)-h(s))^{\alpha-1}\right\}\\ -(h(t)-h(s))^{\alpha-1},\ \eta_{\gamma-1}\leq s\leq t\leq \eta_\gamma,\\ \dfrac{(h(t)-h(a))^{n-1}}{\Delta}\left\{-\sum_{j=\gamma}^{m-2}[\beta_j\Theta_j(s)+\lambda_j(h(\eta_j)-h(s))^{\alpha-1}]+(h(b)-h(s))^{\alpha-1}\right\},\\ \eta_{\gamma-1}\leq t\leq s\leq \eta_\gamma,\end{cases}$$

and

$$G_{m-1}(t,s)=\begin{cases}\dfrac{(h(t)-h(a))^{n-1}}{\Delta}(h(b)-h(s))^{\alpha-1}-(h(t)-h(s))^{\alpha-1}\ ,\ \eta_{m-2}\leq s\leq t\leq b,\\ \dfrac{(h(t)-h(a))^{n-1}}{\Delta}(h(b)-h(s))^{\alpha-1}\ ,\ \eta_{m-2}\leq t\leq s\leq b.\end{cases}$$

Here,

$$\Delta:=(h(b)-h(a))^{n-1}-\nabla\neq 0 \tag{3.6}$$

where

$$\nabla:=\sum_{j=1}^{m-2}\beta_j\int_a^{\eta_j}g(s)(h(s)-h(a))^{n-1}ds+\sum_{j=1}^{m-2}\lambda_j(h(\eta_j)-h(a))^{n-1}. \tag{3.7}$$

Additionally, the function $\Theta_j(s)$ is given by

$$\Theta_j(s):=\int_s^{\eta_j}g(\tau)(h(\tau)-h(s))^{\alpha-1}d\tau. \tag{3.8}$$

Proof. Initially, let us assume that $u\in\mathcal{C}^{n-1}[a,b]$ is a solution to the boundary value problem defined by equations (3.1)-(3.3). Define $r(t)\equiv f(t,u(t))$ and by integrating the differential equation (3.1) from a to t, we obtain:

$$I_{a^+}^{\alpha,h}\ {}^cD_{a^+}^{\alpha,h}u(t)=-I_{a^+}^{\alpha,h}r(t)$$

by applying identity (2.1), we can express the following

$$u(t) = c_0 + c_1(h(t) - h(a)) + c_2(h(t) - h(a))^2 + ... + c_{n-1}(h(t) - h(a))^{n-1} - I_{a^+}^{\alpha,h} r(t).$$

Given the boundary conditions $u^{(i)}(a) = 0$, it follows that $c_i = 0$ for $i = 0, 1, 2, ..., n-2$. Consequently, the function $u(t)$ can be expressed as

$$u(t) = c_{n-1}(h(t) - h(a))^{n-1} - \frac{1}{\Gamma(\alpha)} \int_a^t h'(s)(h(t) - h(s))^{\alpha-1} r(s) ds.$$

By evaluating at $t = b$, we obtain

$$u(b) = c_{n-1}(h(b) - h(a))^{n-1} - \frac{1}{\Gamma(\alpha)} \int_a^b h'(s)(h(b) - h(s))^{\alpha-1} r(s) ds.$$

Incorporating the boundary condition (3.3) we have

$$\begin{aligned} &c_{n-1}(h(b) - h(a))^{n-1} - \frac{1}{\Gamma(\alpha)} \int_a^b h'(s)(h(b) - h(s))^{\alpha-1} r(s) ds \\ =&\sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} g(s) u(s) ds + \sum_{j=1}^{m-2} \lambda_j u(\eta_j). \end{aligned}$$

Substituting $u(s)$ and $u(\eta_j)$ into this equation, we obtain

$$\begin{aligned} &c_{n-1}(h(b) - h(a))^{n-1} - \frac{1}{\Gamma(\alpha)} \int_a^b h'(s)(h(b) - h(s))^{\alpha-1} r(s) ds \\ =&\sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} g(s) c_{n-1}(h(s) - h(a))^{n-1} ds - \sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} g(s) I_{a^+}^{\alpha,h} r(s) ds \\ &+ \sum_{j=1}^{m-2} \lambda_j c_{n-1}(h(\eta_j) - h(a))^{n-1} - \sum_{j=1}^{m-2} \lambda_j I_{a^+}^{\alpha,h} r(\eta_j). \end{aligned}$$

Rearranging terms and applying Fubini's theorem, we arrive at the final expression

$$\begin{aligned} c_{n-1} =&\frac{1}{\Gamma(\alpha)\Delta} \left\{ - \sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} \Theta_j(s) h'(s) r(s) ds - \sum_{j=1}^{m-2} \lambda_j \int_a^{\eta_j} h'(s)(h(\eta_j) - h(s))^{\alpha-1} r(s) ds \right. \\ &\left. + \int_a^b h'(s)(h(b) - h(s))^{\alpha-1} r(s) ds \right\}. \end{aligned}$$

Finally, substituting this into the equation for $u(t)$, we obtain

$$\begin{aligned} u(t) =&\frac{1}{\Gamma(\alpha)\Delta} \left\{ - \sum_{j=1}^{m-2} \beta_j \int_a^{\eta_j} h'(s) \Theta_j(s)(h(t) - h(a))^{n-1} r(s) ds \right. \\ &- \sum_{j=1}^{m-2} \lambda_j \int_a^{\eta_j} h'(s)(h(\eta_j) - h(s))^{\alpha-1}(h(t) - h(a))^{n-1} r(s) ds \end{aligned}$$

$$+ \int_a^b h'(s)(h(b)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}r(s)ds\Big\}$$
$$- \frac{1}{\Gamma(\alpha)}\int_a^t h'(s)(h(t)-h(s))^{\alpha-1}r(s)ds.$$

To express Green's functions $G_1(t,s),\ G_\gamma(t,s),\ G_{m-1}(t,s)$, $u(t)$ is written as

$$\begin{aligned}
u(t) =& \frac{1}{\Gamma(\alpha)\Delta}\Bigg\{-\sum_{j=1}^{m-2}\beta_j\int_a^{\eta_1} h'(s)\Theta_j(s)(h(t)-h(a))^{n-1}r(s)ds \\
&- \sum_{j=2}^{m-2}\beta_j\int_{\eta_1}^{\eta_2} h'(s)\Theta_j(s)(h(t)-h(a))^{n-1}r(s)ds \\
&- \sum_{j=3}^{m-2}\beta_j\int_{\eta_2}^{\eta_3} h'(s)\Theta_j(s)(h(t)-h(a))^{n-1}r(s)ds \\
&- \dots - \sum_{j=m-2}^{m-2}\beta_j\int_{\eta_{m-3}}^{\eta_{m-2}} h'(s)\Theta_j(s)(h(t)-h(a))^{n-1}r(s)ds \\
&- \sum_{j=1}^{m-2}\lambda_j\int_a^{\eta_1} h'(s)(h(\eta_j)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}r(s)ds \\
&- \sum_{j=2}^{m-2}\lambda_j\int_{\eta_1}^{\eta_2} h'(s)(h(\eta_j)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}r(s)ds \\
&- \dots - \sum_{j=m-2}^{m-2}\lambda_j\int_{\eta_{m-3}}^{\eta_{m-2}} h'(s)(h(\eta_j)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}r(s)ds \\
&+ \int_a^b h'(s)(h(b)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}r(s)ds\Bigg\} \\
&- \frac{1}{\Gamma(\alpha)}\int_a^t h'(s)(h(t)-h(s))^{\alpha-1}r(s)ds.
\end{aligned}$$

Conversely, suppose that u is a solution of the integral equation (3.4). By applying the α-th order h-Caputo fractional derivative to (3.4), we recover the differential equation (3.1). In addition, one can directly confirm that the boundary conditions (3.2)-(3.3) also hold. Thus, the proof is completed. □

Let define $M := \int_a^b g(\tau)d\tau$.

Lemma 3.2. *The Green's function $G(t,s)$ presented in Lemma 3.1 satisfies the following bound,*

$$|G(t,s)| \leq \frac{(h(b)-h(a))^{n+\alpha-2}}{\Gamma(\alpha)|\Delta|}\Psi + (h(b)-h(a))^{\alpha-1} := R.$$

Here, the constant Ψ is given by $\Psi := \sum_{j=1}^{m-2}(M\beta_j+\lambda_j)+1$ and Δ is as defined in equation (3.6).

Proof. Let $s<\eta_1$ and $s\leq t$. From the Green's function $G_1(t,s)$ in Lemma 3.1, we obtain the following inequality

$$
\begin{aligned}
|G_1(t,s)| =& \Bigg| \frac{(h(t)-h(a))^{n-1}}{\Delta} \Bigg\{ -\sum_{j=1}^{m-2} \beta_j \Theta_j(s) - \sum_{j=1}^{m-2} \lambda_j (h(\eta_j)-h(s))^{\alpha-1} \\
&+ (h(b)-h(s))^{\alpha-1} \Bigg\} - (h(t)-h(s))^{\alpha-1} \Bigg|.
\end{aligned}
$$

Using triangle inequality, we get

$$
\begin{aligned}
|G_1(t,s)| \leq& \frac{(h(t)-h(a))^{n-1}}{|\Delta|} \Bigg\{ \sum_{j=1}^{m-2} \beta_j |\Theta_j(s)| + \sum_{j=1}^{m-2} \lambda_j (h(\eta_j)-h(s))^{\alpha-1} \\
&+ (h(b)-h(s))^{\alpha-1} \Bigg\} + (h(t)-h(s))^{\alpha-1}.
\end{aligned}
$$

From the definition (3.8) and the condition $g(s) \in \mathbb{R}^+$, we obtain

$$
|\Theta_j(s)| \leq \int_s^{\eta_j} |g(\tau)| \left|(h(\tau)-h(s))^{\alpha-1}\right| d\tau \leq (h(b)-h(a))^{\alpha-1} \int_a^b g(\tau) d\tau
$$

and we have

$$
\begin{aligned}
|G_1(t,s)| \leq& \frac{(h(b)-h(a))^{n-1}}{|\Delta|} \Bigg\{ \int_a^b g(\tau) d\tau \sum_{j=1}^{m-2} \beta_j (h(b)-h(a))^{\alpha-1} \\
&+ \sum_{j=1}^{m-2} \lambda_j (h(\eta_j)-h(s))^{\alpha-1} + (h(b)-h(s))^{\alpha-1} \Bigg\} + (h(b)-h(a))^{\alpha-1} \\
\leq& \frac{(h(b)-h(a))^{n-1}}{|\Delta|} \Bigg\{ M \sum_{j=1}^{m-2} \beta_j (h(b)-h(a))^{\alpha-1} + \sum_{j=1}^{m-2} \lambda_j (h(b)-h(a))^{\alpha-1} \\
&+ (h(b)-h(a))^{\alpha-1} \Bigg\} + (h(b)-h(a))^{\alpha-1} \\
=& \frac{(h(b)-h(a))^{n-1}(h(b)-h(a))^{\alpha-1}}{|\Delta|} \left(\sum_{j=1}^{m-2} (M\beta_j + \lambda_j) + 1 \right) + (h(b)-h(a))^{\alpha-1} \\
\leq& \frac{(h(b)-h(a))^{n+\alpha-2}}{|\Delta|} \Psi + (h(b)-h(a))^{\alpha-1} \\
=& \Gamma(\alpha) R.
\end{aligned}
$$

For $s \geq t$, it is easily seen from the rule of $G_1(t,s)$ that $|G_1(t,s)| \leq \Gamma(\alpha)R$. Hence for all $(t,s) \in [a,b] \times [a,b]$, we obtain

$$
|G_1(t,s)| \leq \Gamma(\alpha)R.
$$

Moreover, for $\eta_{\gamma-1} \leq s \leq t \leq \eta_\gamma$ and $\eta_{\gamma-1} \leq t \leq s \leq \eta_\gamma$ by using the same argument as before, we obtain from the Green's function $G_\gamma(t,s)$ in Lemma 3.1

$$
|G_\gamma(t,s)| \leq \Gamma(\alpha)R.
$$

Finally, let $\eta_{m-2} \leq s \leq t \leq b$ then we have

$$\begin{aligned}
|G_{m-1}(t,s)| &= \left| \frac{(h(t)-h(a))^{n-1}}{\Delta}(h(b)-h(s))^{\alpha-1} - (h(t)-h(s))^{\alpha-1} \right| \\
&\leq \frac{(h(t)-h(a))^{n-1}}{|\Delta|}(h(b)-h(s))^{\alpha-1} + (h(t)-h(s))^{\alpha-1} \\
&\leq \frac{(h(b)-h(a))^{n-1}}{|\Delta|}(h(b)-h(a))^{\alpha-1} + (h(b)-h(a))^{\alpha-1} \\
&= \frac{(h(b)-h(a))^{n+\alpha-2}}{|\Delta|} + (h(b)-h(a))^{\alpha-1} \\
&\leq \frac{(h(b)-h(a))^{n+\alpha-2}}{|\Delta|}\Psi + (h(b)-h(a))^{\alpha-1} \\
&= \Gamma(\alpha)R.
\end{aligned}$$

Similarly, for $\eta_{m-2} \leq t \leq s \leq b$ it is easily seen that $|G_{m-1}(t,s)| \leq \Gamma(\alpha)R$. So for all $t,s \in [a,b]\times[a,b]$, we obtain $|G(t,s)| \leq R$ since $|G_1(t,s)| \leq \Gamma(\alpha)R$, $|G_\gamma(t,s)| \leq \Gamma(\alpha)R$ and $|G_{m-1}(t,s)| \leq \Gamma(\alpha)R$. Thus the proof is completed. □

Now, let $P = \mathcal{C}[a,b]$ denote the Banach space of continuous functions on the interval $[a,b]$ equipped with the supremum norm defined by

$$||u|| = \sup_{t\in[a,b]} |u(t)|, \quad u \in \mathcal{C}[a,b].$$

Theorem 3.1. *Assume the following condition holds*

(H_1) The function $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is continuous, and there exists a constant $b_0 \in \mathbb{R}^+$ such that

$$|f(t,u_1) - f(t,u_2)| \leq b_0|u_1 - u_2|, \qquad t \in [a,b], u_1, u_2 \in \mathbb{R}.$$

If the following inequality is satisfied

$$Rb_0(h(b)-h(a)) < 1 \tag{3.9}$$

where R is the upper bound of the Green's function $G(t,s)$ as provided in Lemma 3.2, then the boundary value problem defined by equations (3.1)-(3.3) has a unique solution on the interval $[a,b]$.

Proof. Define the operator $T : P \to P$ by

$$Tu(t) = \int_a^b h'(s)G(t,s)f(s,u(s))ds \tag{3.10}$$

where $G(t,s)$ is the Green's function associated with the problem, and f is the given continuous function.

We first demonstrate that the operator T is well-defined, i.e., $T(P) \subset P$. To this end, consider a function $u \in \mathcal{C}[a,b]$. We compute the Caputo-type fractional derivative of Tu with respect to h as follows

$$Tu(t) = \left[\int_a^b h'(s)G(t,s)f(s,u(s))ds\right]$$

$$= \left(\frac{(h(t)-h(a))^{n-1}}{\Gamma(\alpha)\Delta}\right)\Bigg\{-\sum_{j=1}^{m-2}\beta_j\int_a^{\eta_j}\Theta_j(s)f(s,u(s))ds$$
$$-\sum_{j=1}^{m-2}\lambda_j\int_a^{\eta_j}h'(s)(h(\eta_j)-h(s))^{\alpha-1}f(s,u(s))ds$$
$$+\int_a^b h'(s)(h(b)-h(s))^{\alpha-1}f(s,u(s))ds\Bigg\}-I_{a^+}^{\alpha,h}f(t,u(t)).$$

Since f, h are continuous, it follows that $Tu(t) \in \mathcal{C}[a,b]$. Let $u_1, u_2 \in P$. Then by assumption (H_1) and Lemma 3.2 for all $t \in [a,b]$, we have

$$\begin{aligned}
|Tu_1(t)-Tu_2(t)| &= \left|\int_a^b h'(s)G(t,s)f(s,u_1(s))ds - \int_a^b h'(s)G(t,s)f(s,u_2(s))ds\right| \\
&= \left|\int_a^b h'(s)G(t,s)(f(s,u_1(s))-f(s,u_2(s)))ds\right| \\
&\leq \int_a^b h'(s)|G(t,s)||f(s,u_1(s))-f(s,u_2(s))|ds \\
&\leq Rb_0\int_a^b h'(s)|u_1(s)-u_2(s)|ds \\
&= Rb_0(h(b)-h(a))\|u_1-u_2\|.
\end{aligned}$$

From the condition $Rb_0(h(b)-h(a)) < 1$, the operator T is a contraction on P. By virtue of the Banach fixed point theorem, T has a unique fixed point $u \in P$ such that $Tu(t) = u(t)$ for all $t \in [a,b]$. Therefore, u is the unique solution of the boundary value problem (3.1)-(3.3) on $[a,b]$. This completes the proof. □

Theorem 3.2. *Assume that*

(H_2) $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a positive constant b_1 such that

$$|f(t,u)| \leq b_1|u|.$$

If

$$Rb_1(h(b)-h(a)) < 1 \tag{3.11}$$

then the boundary value problem (3.1)-(3.3) has at least one solution on $[a,b]$, where R is the upper bound of $G(t,s)$ as given in Lemma 3.2.

Proof. To begin, we define the set P and the operator $T : P \to P$ as specified by (3.10). We will prove the existence of a solution to the boundary value problem (3.1)-(3.3) using Schaefer's Fixed Point Theorem. To do this, we verify the four conditions required by the theorem.

Step 1. We show that T is continuous. Let $(u_n)_{n\in\mathbb{N}} \subset P$ such that $u_n \to u$ in P, as $n \to \infty$. Then from Lemma 3.2 and for all $t \in [a,b]$, we obtain

$$|Tu_n(t)-Tu(t)| \leq \int_a^b h'(s)|G(t,s)||f(s,u_n(s))-f(s,u(s))|ds$$

$$\leq R\int_a^b h'(s)|f(s,u_n(s))-f(s,u(s))|ds$$

which implies

$$||Tu_n(t)-Tu(t)||\leq R\int_a^b h'(s)|f(s,u_n(s))-f(s,u(s))|ds.$$

By the continuity of f the right-hand side tends to zero as $n\to\infty$. Thus, T is a continuous operator.

Step 2. We will show that T maps bounded sets into bounded sets. Let be $B_{s_1}=\{u\in P: ||u||\leq s_1\}$. By assumption (H_2) and Lemma 3.2, for $u\in B_{s_1}$ we estimate

$$\begin{aligned}|Tu(t)| &\leq \int_a^b h'(s)|G(t,s)||f(s,u(s))|ds\\ &\leq b_1\|u\|R\int_a^b h'(s)ds\\ &= b_1\|u\|R(h(b)-h(a))\\ &\leq b_1s_1R(h(b)-h(a)).\end{aligned}$$

Therefore, define $s_2:=b_1s_1R(h(b)-h(a))$, and we have $||Tu||\leq s_2$, so the image of B_{s_1} under T is bounded.

Step 3. We now establish that the operator T maps bounded subsets of P into equicontinuous subsets. Let $B_{s_1}\subset P$ be a bounded set as defined in Step 2, and consider an arbitrary $u\in B_{s_1}$. For any $t_1,t_2\in[a,b]$ with $(t_1<t_2)$, we can compute the difference as follows

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &\leq \int_a^b h'(s)|G(t_2,s)-G(t_1,s)||f(s,u(s))|ds\\ &\leq b_1s_1\int_a^b h'(s)|G(t_2,s)-G(t_1,s)|ds.\end{aligned}$$

Since $G(t,s)$ is continuous in t for each fixed s, the integrand $|G(t_2,s)-G(t_1,s)|$ tends to zero as $t_1\to t_2$, and hence the integral converges to zero. This establishes that $\{Tu: u\in B_{s_1}\}$ is an equicontinuous family. Therefore, by the Arzelà-Ascoli theorem, the operator T is completely continuous on P.

Step 4. We now demonstrate that the set $S:=\{u\in P: u=\lambda Tu,\quad \lambda\in(0,1)\}$ is bounded. Let $u\in S$ and $\lambda\in(0,1)$ such that $u=\lambda Tu$. By invoking the estimate obtained in Step 2 along with the growth condition (H_2), we find that for all $t\in[a,b]$,

$$|Tu(t)|\leq b_1||u||R(h(b)-h(a)).$$

Using the relation $u=\lambda Tu$ and the fact that $\lambda\in(0,1)$, we derive

$$||u||=||\lambda Tu||=\leq\lambda||Tu||\leq\lambda b_1||u||R(h(b)-h(a))+1.$$

Rewriting, we obtain the inequality

$$||u||\leq||Tu||\leq b_1||u||R(h(b)-h(a))+1.$$

Solving for $\|u\|$ we get

$$||u|| \leq \frac{1}{1-(b_1R(h(b)-h(a))}.$$

Since the condition $(b_1R(h(b)-h(a)) < 1$ holds by assumption, the denominator is positive, and thus the norm $\|u\|$ is bounded. This confirms that the set S is bounded in P. Therefore, by Schaefer's fixed point theorem, the operator T admits at least one fixed point $u \in P$. This fixed point constitutes a solution to the boundary value problem (3.1)-(3.3) on the interval $[a,b]$, thereby completing the proof. □

4. Ulam-Hyers stability

In the study of functional equations, an important line of inquiry involves determining whether approximate solutions to a functional equation remain close to exact solutions. More specifically, researchers often investigate whether the solutions to two slightly different functional equations are nearly identical, or whether replacing a functional equation with an inequality still yields solutions that approximate those of the original equation.

This area of inquiry traces back to a talk delivered by S. Ulam in 1940 at the University of Wisconsin Mathematics Club, where she posed several open-ended questions related to the behavior and stability of functional equations [16]. These foundational questions laid the groundwork for what is now known as the theory of stability of functional equations.

In this section, we investigate the Ulam-Hyers stability of the following boundary value problem

$$\begin{aligned}
&{}^cD^{\alpha,h}_{a^+}u(t) + f(t,u(t)) = 0, \quad t \in [a,b],\\
&u^{(i)}(a) = 0, \;\; i = 0,1,2,...,n-2,\\
&u(b) = \sum_{j=1}^{m-2}\beta_j \int_a^{\eta_j} g(s)u(s)ds + \sum_{j=1}^{m-2}\lambda_j u(\eta_j),
\end{aligned}$$

where $n-1 < \alpha \leq n$ and the function $f(t,v(t)) : [a,b]\times\mathbb{R} \to \mathbb{R}$ is assumed to be continuous.

Let the function $v \in \mathcal{C}[a,b]$ be defined as follows

$$\begin{aligned}
v(t) =&\frac{1}{\Gamma(\alpha)\Delta}\Bigg\{-\sum_{j=1}^{m-2}\beta_j\int_a^{\eta_j}\Theta_j(s)(h(t)-h(a))^{n-1}f(s,v(s))ds\\
&-\sum_{j=1}^{m-2}\lambda_j\int_a^{\eta_j}h'(s)(h(\eta_j)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}f(s,v(s))ds\\
&+\int_a^b h'(s)(h(b)-h(s))^{\alpha-1}(h(t)-h(a))^{n-1}f(s,v(s))ds\Bigg\}\\
&-\frac{1}{\Gamma(\alpha)}\int_a^t h'(s)(h(t)-h(s))^{\alpha-1}f(s,v(s))ds.
\end{aligned}$$

Now, define the operator $K:\; P \to P$ is defined by

$$Kv(t) = {}^cD^{\alpha,h}_{a^+}v(t) + f(t,v(t)).$$

Definition 4.1. (Ulam-Hyers stability) The boundary value problem (3.1)-(3.3) is said to be Hyers-Ulam stable if there exist constants $c > 0$ such that for each $\epsilon > 0$ and for each solution v of

$$||Kv|| \leq \epsilon, \tag{4.1}$$

there exists a solution $u \in \mathcal{C}[a,b]$ of the problem (3.1)-(3.3) such that

$$||u - v|| \leq c\epsilon^*, \tag{4.2}$$

where ϵ^* is a positive constant depending on ϵ.

Definition 4.2. (Generalized Ulam-Hyers stability) The boundary value problem (3.1)-(3.3) is said to be generalized Hyers-Ulam stable if there exists a function $m \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that for each $\epsilon > 0$ and for each solution v of (4.1) there exists a solution $u \in \mathcal{C}[a,b]$ of the problem (3.1)-(3.3) such that

$$||u - v|| \leq m(\epsilon). \tag{4.3}$$

Definition 4.3. (Ulam-Hyers-Rassias stability) Let $\epsilon > 0$ and let $\theta :\in [a,b] \to \mathbb{R}^+$ be a non-negative continuous function. The boundary value problem (3.1)-(3.3) is said to be Ulam-Hyers-Rassias stable if for every function $v \in P$ satisfying

$$||Kv|| \leq \epsilon\theta(t), \quad t \in [a,b], \tag{4.4}$$

where there exists a real number $c > 0$ and a solution $u \in \mathcal{C}[a,b]$ of the boundary value problem (3.1)-(3.3) such that

$$||u - v|| \leq c\epsilon_*\theta(t), \quad t \in [a,b], \tag{4.5}$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. *Assume that f satisfies the condition (H_1) in Theorem 3.1, if the inequality (3.9) is satisfied then the problem (3.1)-(3.3) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in \mathcal{C}[a,b]$ be a solution of (3.1)-(3.3), given in Lemma 3.2. Let v be any solution satisfying (4.1). Lemma 3.2 implies the equivalence between the operators K and $T - Id$ (where Id is the identity operator) for every solution v of (3.1)-(3.3) satisfying (4.1). Therefore, we deduce by the fixed-point property of the operator T that:

$$\begin{aligned}
|v(t) - u(t)| =& |v(t) - Tv(t) + Tv(t) - u(t)| \\
=& |v(t) - Tv(t) + Tv(t) - Tu(t)| \\
\leq& |Tv(t) - Tu(t)| + |Tv(t) - v(t)| \\
\leq& \|Tv - Tu\| + \|Tv - v\| \\
\leq& Rb_0(h(b) - h(a))\|v - u\| + \|(T - Id)v\| \\
\leq& Rb_0(h(b) - h(a))\|v - u\| + \|Kv\| \\
\leq& Rb_0(h(b) - h(a))||u - v|| + \epsilon.
\end{aligned}$$

Because of $Rb_0(h(b)-h(a))<1$ and $\epsilon>0$, we find

$$||u-v|| \le \frac{\epsilon}{1-Rb_0(h(b)-h(a))}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-Rb_0(h(b)-h(a))}$ and $c=1$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $m(\epsilon) = \frac{\epsilon}{1-Rb_0(h(b)-h(a))}$. □

Theorem 4.2. *Assume that (H_1) holds with (3.9), and there exists a function $\theta \in \mathcal{C}([a,b],\mathbb{R}^+)$ satisfying the condition (4.5). Then the boundary value problem (3.1)-(3.3) is Ulam-Hyers-Rassias stable with respect to θ.*

Proof. We have from the proof of Theorem 4.1,

$$|v(t)-u(t)| \le Rb_0(h(b)-h(a))||u-v|| + \epsilon\theta(t),$$

where $\epsilon_* = \dfrac{\epsilon}{1-Rb_0(h(b)-h(a))}$ and $c=1$. This completes the proof. □

In what follows, we present two examples which illustrates our results.

Example 4.1. Consider the following fractional boundary value problem

$$ {}^cD_{1^+}^{\frac{7}{2},t^3} u(t) + f(t,u(t)) = 0, \quad t\in[1,3], \tag{4.6}$$

$$ u^{(i)}(1) = 0, \ i=0,1,2, \tag{4.7}$$

$$ u(3) = \int_1^{\frac{3}{2}} s^2u(s)ds + \int_1^2 s^2u(s)ds + \int_1^{\frac{5}{2}} s^2u(s)ds + u(\frac{3}{2}) + u(2) + u(\frac{5}{2}), \tag{4.8}$$

where $f(t,u(t)) = \dfrac{1}{10^7}\arctan u(t) + t^2$. We specify the datas for the problem as

$$h(t)=t^3, \alpha = \frac{7}{2}, \ n=4, \ m=5, \ \beta_j=1, \ \lambda_j=1, \ j=1,2,3,$$
$$\eta_1 = \frac{3}{2}, \ \eta_2=2, \ \eta_3=\frac{5}{2}, \ g(s)=s^2.$$

For all $t\in[1,3]$, it is easy to see that $h'(t)=3t^2\neq 0$ and $h(t)=t^3$ is an increasing function and also $f(t,u)$ is continuous on $[1,3]\times\mathbb{R}$. We now verify that the function f satisfies condition (H_1). For any $u_1,u_2\in\mathbb{R}$, we compute

$$|f(t,u_1)-f(t,u_2)| = \frac{1}{10^7}|\arctan u_1 - \arctan u_2| \le \frac{1}{10^7}|u_1-u_2|.$$

Thus, f satisfies condition (H_1) with Lipschitz constant $b_0=\frac{1}{10^7}$. We aim to calculate the constant R which represents the upper bound of the Green's function. The expression for R is given by

$$R = \frac{(h(3)-h(1))^{\frac{11}{2}}}{\Gamma(\frac{7}{2})|\Delta|}\left(\sum_{j=1}^{3}(M+1)+1\right) + (h(3)-h(1))^{\frac{5}{2}},$$

$$h(3)-h(1) = 3^3-1 = 26, \quad \Gamma(\frac{7}{2}) = \frac{15\sqrt{\pi}}{8} \cong 3.323$$

and

$$M = \int_1^3 \tau^3 d\tau = 20, \quad \Psi = \sum_{j=1}^{3}(M+1) + 1 = 64.$$

When we calculate the number Δ then we have

$$\Delta = (h(3) - h(1))^3 - \nabla$$

where

$$\begin{aligned}\nabla = &\int_1^{\frac{3}{2}} s^2(s^3-1)^3 ds + \int_1^{2} s^2(s^3-1)^3 ds + \int_1^{\frac{5}{2}} s^2(s^3-1)^3 ds \\ &+ (\frac{27}{8} - 1)^3 + (8-1)^3 + (\frac{125}{8} - 1)^3 \\ \cong &7499.48\end{aligned}$$

then we have

$$\Delta = (h(3) - h(1))^3 - 7499.48 = 10076.52.$$

Now plug all values into the formula

$$R = \frac{26^{\frac{11}{2}}}{\frac{15\sqrt{\pi}}{8}10076,52}64 + 26^{\frac{5}{2}} \cong 119230.4197.$$

Since $Rb_0(h(b) - h(a)) = (119230.4197)\frac{26}{10^7} \cong 0.309 < 1$, the boundary value problem (4.6)-(4.8) has at least one solution on $[1,3]$ by Theorem 3.1. In addition, all conditions of Theorem 4.1 are satisfied. Thus the fractional boundary value problem (4.6)-(4.8) is both Ulam-Hyers and generalized Ulam-Hyers stable. Moreover, if there is such a continuous and positive θ function, the fractional boundary value problem (4.6)-(4.8) is Ulam-Hyers-Rassias stable.

Example 4.2. Consider the following fractional boundary value problem

$$^cD_{1^+}^{\frac{5}{2},\ln t}u(t) + f(t, u(t)) = 0, \quad t \in [1, e], \tag{4.9}$$

$$u(1) = 0, \ u'(1) = 0, \tag{4.10}$$

$$u(e) = \frac{1}{4}\int_1^{\frac{3}{2}} \frac{1}{s}u(s)ds + \frac{1}{2}\int_1^{2} \frac{1}{s}u(s)ds + \frac{1}{3}u(\frac{3}{2}) + \frac{1}{6}u(2), \tag{4.11}$$

where $f(t, u(t)) = \dfrac{t}{9}\sin u(t)$. Here

$$\begin{aligned}&h(t) = \ln t, \alpha = \frac{5}{2}, \ n = 3, \ m = 4, \ \beta_1 = \frac{1}{4}, \ \beta_2 = \frac{1}{2}, \ \lambda_1 = \frac{1}{3}, \ \lambda_2 = \frac{1}{6}, \\ &\eta_1 = \frac{3}{2}, \ \eta_2 = 2, \ g(s) = \frac{1}{s}.\end{aligned}$$

For all $t \in [1, e]$ it is easy to see that $h'(t) = \frac{1}{t} \neq 0$ and $h(t) = \ln t$ is an increasing function and also f is continuous on $[1, e] \times \mathbb{R}$. Now, we will show that f satisfies the condition (H_2):

$$|f(t, u(t))| = |\frac{t}{9}\sin u(t)| \leq \frac{e}{9}|\sin u(t)| \leq \frac{e}{9}|u(t)|.$$

Thus, f satisfies condition (H_2) with the constant $b_1 = \dfrac{e}{9}$. We will calculate the number R, which is the upper bound of Green's function

$$R = \frac{(h(e)-h(1))^{\frac{7}{2}}}{\Gamma(\frac{5}{2})|\Delta|}\Psi + (h(e)-h(1))^{\frac{3}{2}},$$
$$h(e)-h(1) = \ln e - \ln 1 = 1, \quad \Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}, \quad M = \int_1^e \frac{1}{s} ds = 1 \quad \Psi = \frac{9}{4}.$$

When we calculate the number Δ we obtain

$$\Delta = (h(e)-h(1))^2 - \nabla$$

and

$$\nabla = \frac{1}{4}\int_1^{\frac{3}{2}} \frac{1}{s}\ln^2 s ds + \frac{1}{2}\int_1^{2} \frac{1}{s}\ln^2 s ds + \frac{1}{3}\ln^2\frac{3}{2} + \frac{1}{6}\ln^2 2 \cong 0.1958.$$

Then we obtain

$$\Delta = 1 - 0,1958 = 0.8042, \quad R = \frac{9}{4\frac{3\sqrt{\pi}}{4}0,8042} + 1 \cong 3.1046.$$

Since $Rb_1(h(b)-h(a)) = (3.1046)\dfrac{e}{9} \cong 0.9377 < 1$, then the boundary value problem (4.9)-(4.11) has at least one solution on $[1,e]$ by Theorem 3.2.

5. Conclusion

This study has focused on the analysis of a multi-point boundary value problem involving generalized fractional derivatives with integral conditions. Through the application of the Banach and Schaefer fixed point theorems, we have successfully derived conditions that guarantee the existence and uniqueness of solutions to the problem. The inclusion of both multi-point and integral conditions adds a layer of complexity, but our approach demonstrates that such problems can be systematically addressed using advanced mathematical tools.

A significant aspect of this work is the introduction and proof of Ulam-Hyers stability for the considered boundary value problem. This stability analysis ensures that the solutions are not only mathematically valid but also resilient to small disturbances, which is crucial for practical applications where robustness is essential. The stability results further reinforce the reliability of the proposed framework.

To illustrate the theoretical findings, we have provided two concrete examples. These examples serve as practical demonstrations of the applicability of our results. The successful application of our methods to these examples underscores the versatility and strength of the proposed approach.

Looking ahead, there are several promising directions for future research. One potential avenue is the exploration of more complex boundary conditions or the inclusion of additional nonlinearities in the problem. Another interesting direction would be the development of numerical algorithms to approximate solutions for such problems, which could enhance their practical utility. Additionally, extending this work to coupled systems of fractional differential equations could open new possibilities for modeling multi-physics phenomena.

In summary, this research contributes to the ongoing development of fractional calculus by providing a robust framework for analyzing multi-point boundary value problems with integral conditions. The theoretical insights and practical applications presented here lay the groundwork for further advancements in the field, offering new opportunities for both theoretical exploration and real-world problem-solving.

6. Discussion

In recent years, significant progress has been made in the analysis of fractional differential equations (FDEs) with nonlocal, multi-point, and integral boundary conditions, owing to their broad applicability in modeling systems with memory and long-range dependencies. Among the prominent contributions, the works of Wang et al. [31] and Wahash et al. [30] have provided foundational results.

Wang et al. [31] studied a fractional boundary value problem involving a Caputo derivative of arbitrary order combined with multi-point integral boundary conditions. They derived the corresponding Green's function and applied Krasnosel'skii's and Schauder's fixed point theorems to prove the existence and multiplicity of solutions. However, their work was limited to the classical Caputo operator and did not address solution stability or generalized derivative structures.

Wahash et al. [30], on the other hand, considered a boundary value problem involving the generalized Caputo fractional derivative with respect to another function, along with integral boundary conditions. Using the method of upper and lower solutions in combination with both Banach and Schauder fixed point theorems, they established the existence and uniqueness of positive solutions. Notably, their use of the Banach contraction principle allowed them to derive uniqueness criteria, which marks a step forward in comparison to many earlier studies focusing solely on existence. However, their problem formulation featured a single integral condition and did not involve multi-point terms or a detailed investigation of Ulam-Hyers stability.

Several other studies have explored related directions. For instance, Abdo et al. [1] examined generalized Caputo operators in boundary problems with nonlinear integral conditions, while Alghanmi et al. [3] investigated coupled systems with anti-periodic conditions involving ϱ-fractional derivatives. Lachouri and Ardjouni [13] studied Hilfer-type integro differential equations and analyzed Ulam-type stability, albeit in a different operator context. Similarly, Yalçın et al. [32] worked on generalized Caputo-type problems with nonlinearities but without multi-point boundary structures.

In contrast to the aforementioned works, the present study introduces a significantly more generalized boundary value framework. First, it employs a generalized Caputo fractional derivative defined with respect to another function, thereby capturing a broader class of memory-dependent processes than classical definitions allow. Second, it integrates both multi-point and integral boundary conditions, enabling the model to reflect cumulative and distributed effects within the domain. Finally, the analysis is comprehensive, addressing not only the existence and uniqueness of positive solutions but also their Ulam-Hyers stability, which ensures robustness against small perturbations. This unified treatment distinguishes the current work from existing literature and provides a versatile foundation for future developments in the theory and applications of fractional differential equations.

The problem is first reformulated as an equivalent integral equation through the construction of a Green's function, and appropriate fixed point theorems (including Banach's and Schaefer's) are applied in the corresponding function space to obtain the main results. Furthermore, the Ulam-Hyers stability analysis addresses a gap present in many earlier works, confirming the robustness of the solutions under small perturbations in data.

To illustrate the applicability of the theoretical results, two examples are provided, showing how the abstract results can be applied to concrete cases with specific boundary structures and nonlinearities.

In summary, while earlier works such as those by Wang et al. [31] and Wahash et al. [30] have addressed important aspects of fractional boundary value problems, the present study extends and deepens the theory by treating more general operators, richer boundary conditions, and including stability in the analysis. This provides a comprehensive contribution to the literature and opens new directions for research on fractional models with nonlocal effects.

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