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NUMERICAL SOLUTION OF FRACTIONAL ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATION

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Abstract This article develops a finite element method (FEM) for solving nonlinear fractional-order partial differential equations (PDEs) using the Caputo fractional derivative. The scheme achieves second-order convergence in L^2 and L^∞ norms and is supported by a rigorous qualitative analysis, including existence, uniqueness, and Ulam-Hyers stability, established via nonlinear functional analysis and fixed-point theorems. Applied to a fractional-order Burgers' equation modeling longitudinal dispersion in homogeneous porous media, the method demonstrates superior accuracy compared to finite difference and B-spline Galerkin methods. Numerical results validate robustness across fractional orders $\alpha \in \{0.5, 0.75, 1.0\}$, with applications in environmental engineering and fluid dynamics.

Keywords Finite element method, fractional-order PDEs, Burgers' equation, nonlinear PDEs, longitudinal dispersion.

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1. Introduction

Fractional calculus and fractional differential equations constitute an important branch of research. This significance arises from the fact that many physical problems in science and engineering are more accurately modeled using fractional calculus and fractional differential equations compared to their integer-order integer differential equations [4, 8, 11–13, 22, 23, 25, 28]. Consequently, researchers are developing definitions of fractional integral and differential operators compatible with integer-order integral and differential operators. The Riemann-Liouville, Caputo, and Hilfer operators are particularly well-known among the various fractional integral and differential operators. The qualitative properties, such as existence and uniqueness, of fractional-order differential equations are discussed in detail in articles [5,6,18,40] and their references. Obtaining analytic solutions for many fractional systems is challenging; therefore, many researchers have developed semi-analytic and numerical methods to find solutions for fractional

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order systems. Yocel et al. [39] use the generalised Taylor collocation method to find the solution for the Bagley-Torvik equation. Solutions using the Adomian decomposition method are found in [16,34]. [27,29] developed a homotopy analysis method to find the solution of fractional order systems. The solution of fractional order systems using the finite difference methods are found in the articles [10,35]. Esen and Tasbozan [9] discussed the solution of a time-fractional Burgers' equation using a quadratic B-spline finite element Galerkin method. Solution using residue power series method in the article [2,21].

The proposed FEM distinguishes itself from existing methods, such as the quadratic B-spline Galerkin approach Fractional-order partial differential equations represent many physical phenomena, and finding an analytic solution for fractional-order nonlinear PDEs is difficult [24,29,36]. At the same time, stability issues will be associated with the finite difference method. While semi-analytic methods like homotopy perturbation, homotopy analysis methods nonlinear terms need strong conditions for the existence and uniqueness of the solution [32]. Due to this motivation, this manuscript is devoted to developing a new numerical scheme to solve fractional-order PDEs. The qualitative properties like the existence, uniqueness, and stability of the assembled system are also discussed in this manuscript. This manuscript ended with a numerical example of a nonlinear system representing a physical phenomenon of longitudinal dispersion in a homogeneous porous medium [1, 17, 26, 38]. A significant aspect of fractional-order PDEs, such as the Burgers' equation considered here, is the potential for control strategies to manage physical phenomena like longitudinal dispersion. Traditional controllability allows a system to be driven from any initial state to a desired state within a finite time using a control input, often analyzed via linear system techniques or energy methods [14]. In contrast, exact controllability requires the system to reach a specific target state with zero error at the final time, accounting for the non-local memory effects of the Caputo fractional derivative ($\alpha < 1$) [41]. This distinction is crucial for fractional PDEs, where the infinite memory necessitates controls dependent on the fractional order and domain geometry, potentially explored using Carleman estimates. Future work could investigate exact controllability for the proposed FEM, ensuring precise management of contaminant levels in porous media, enhancing its environmental engineering applications.

2. Numerical scheme for fractional order PDES

This section develops a numerical scheme to solve the fractional order scheme with Caputo fractional differential operator. To develop the scheme consider the fractional order PDE

$$\frac{\partial^\alpha C}{\partial t^\alpha} + F(C) - D^2 \frac{\partial^2 C}{\partial x^2} = G(x,t), \tag{2.1}$$

where, $0 < \alpha \leq 1$, $t \geq 0$ is time variable, $x \in [0, L]$ is space variable and C depends on x and t, and $F(C)$ is a nonlinear function of C. To develop the numerical scheme first divide the spatial domain $[0, L]$ into N subdomain called elements. Equating weighted residue over each element and equating to zero the equation (2.1) becomes

$$\int_0^{h_e} W^e \left[\frac{\partial^\alpha C}{\partial t^\alpha} + F(C) - D^2 \frac{\partial^2 C}{\partial y^2} - G(t,y) \right] d\bar{y} = 0, \tag{2.2}$$

where, W^e is the weight function of y and y is temporary variable over the e^{th} element $[0, h_e]$.

Assuming the solution of the form $C^e(y,t) = \sum_{j=1}^{M} C_j^e(t)\phi_j^e(y)$ over e^{th} element and plugging it into equation (2.2), the equation becomes

$$\int_0^{h_e} W^e\left[\left(\sum_{j=1}^{M} \frac{d^\alpha C_j^e}{dt^\alpha}\phi_j^e\right) + F\left(\sum_{j=1}^{M} C_j^e\phi_j^e\right) - D^2\left(\sum_{j=1}^{M} C_j^e\frac{d^2\phi_j^e}{dy^2}\right) - G(t,y)\right]dy = 0. \tag{2.3}$$

Taking the weak formulation of (2.3), we obtain

$$\begin{aligned}&\int_0^{h_e}\left[\left(\sum_{j=1}^{M} \frac{d^\alpha C_j^e}{dt^\alpha}W^e\phi_j^e\right) + W^eF\left(\sum_{j=1}^{M} C_j^e\phi_j^e\right) + D^2\left(\sum_{j=1}^{M} C_j^e\frac{dW^e}{dy}\frac{d\phi_j^e}{dy}\right)\right]dy\\ =&\left[D^2W^e\sum_{j=1}^{M} C_j^e\frac{d\phi_j^e}{dy}\right]_0^{h_e} + \int_0^{h_e} W^eG(t,y)\,dy.\end{aligned} \tag{2.4}$$

Choosing $W_i^e = \phi_i^e$ for each $i = 1, 2, \ldots, M$ and integrating over $[0, h_e]$, the equation (2.4) becomes

$$\begin{aligned}&\sum_{j=1}^{M} \frac{d^\alpha C_j^e}{dt^\alpha}\int_0^{h_e}\phi_i^e\phi_j^e\,dy + \int_0^{h_e}\phi_i^eF\left(\sum_{j=1}^{M} C_j^e\phi_j^e\right)dy + \sum_{j=1}^{M} C_j^e\int_0^{h_e} D^2\frac{d\phi_i^e}{dy}\frac{d\phi_j^e}{dy}\,dy\\ =&\left[D^2\phi_i^e\sum_{j=1}^{M} C_j^e\frac{d\phi_j^e}{dy}\right]_0^{h_e} + \int_0^{h_e}\phi_i^eG(t,y)\,dy.\end{aligned} \tag{2.5}$$

Let

$$\begin{aligned}&\mathcal{M}^{(e)} = \left(\int_0^{h_e}\phi_i^e\phi_j^e\,dy\right)_{M\times M}, \quad \mathcal{F}^{(e)}(t,U) = \left(\int_0^{h_e}\phi_i^eF\left(\sum_{j=1}^{M} C_j^e\phi_j^e\right)dy\right)_{M\times 1},\\ &\mathcal{K}^{(e)} = \left(\int_0^{h_e} D^2\frac{d\phi_i^e}{dy}\frac{d\phi_j^e}{dy}\,dy\right)_{M\times M}, \quad \mathcal{B}^{(e)} = \left(\left[D^2\phi_i^e\sum_{j=1}^{M}\frac{d\phi_j^e}{dy}\right]_0^{h_e}\right)_{M\times M},\\ &\mathcal{G}^{(e)}(t) = \left(\int_0^{h_e}\phi_i^eG(t,y)\,dy\right)_{M\times 1}.\end{aligned}$$

Then the system can be rewritten as

$$\mathcal{M}^{(e)}\frac{d^\alpha U^{(e)}}{dt^\alpha} + \mathcal{F}^{(e)}(t,U) + \mathcal{K}^{(e)}U^{(e)} = \mathcal{B}^{(e)}U^{(e)} + \mathcal{G}^{(e)}(t), \tag{2.6}$$

where $U^{(e)} = \left(C_i^e\right)_{M\times 1}$. Note that $\mathcal{F}^{(e)}$ depends on both U and t, while $\mathcal{G}^{(e)}$ depends only on t. Assembling the system (2.6) over the entire domain $[0, L]$ using the principle of continuity, and applying boundary conditions, the system (2.6) becomes:

$$\mathcal{M}\,\frac{d^\alpha U}{dt^\alpha} + \mathcal{F}(t,U) + \mathcal{K}U = \mathcal{B}U + \mathcal{G}(t). \tag{2.7}$$

The system (2.7) is a nonlinear system of ordinary differential equations with constant coefficients. Note that dimensions of matrices $\mathcal{M}, \mathcal{F}, \mathcal{K}, \mathcal{G}$ and $\mathcal{B}$ depend on the boundary conditions.

3. Existence, uniqueness and stability

This section discussed the existence, uniqueness, and stability of the solution of fractional order assembled system (2.7) with initial condition $U(0) = U_0$ over the finite interval $[0, T]$.

Theorem 3.1. *If*

(A1) The matrix $\mathcal{M}$ is globally invertible.

(A2) The nonlinear function $\mathcal{F} : [0, T] \times \mathbb{X} \to \mathbb{X}$ is continuous with respect to t, and there exists a constant f^* such that

$$\|\mathcal{F}(t, U) - \mathcal{F}(t, V)\| \leq f^* \|U - V\| \quad \forall U, V \in B_r(U_0),$$

where $\mathbb{X}$ is a subspace of $\mathbb{R}^{(N+1)M}$ depending on the choice of boundary conditions and interpolating functions.

then the fractional-order assembled system (2.7) *has a unique solution over the finite interval $[0, T]$ for the initial condition $U(0) = U_0$.*

Proof. The system (2.7) can be rewritten as

$$\mathcal{M} \frac{d^\alpha U}{dt^\alpha} = -\mathcal{K}U - \mathcal{F}(t, U) + \mathcal{B}U + \mathcal{G}(t), \tag{3.1}$$

where $\mathcal{M}$ and $\mathcal{K}$ are time-invariant matrices. Since $\mathcal{M}$ is invertible, $-\mathcal{M}^{-1}\mathcal{K}$ is a time-invariant matrix. Let $\mathcal{E}_{\alpha,\beta}$ denote the two-parameter Mittag-Leffler function. Then the mild solution of (3.1) is

$$U(t) = \mathcal{E}_{\alpha,1}\big(-\mathcal{M}^{-1}\mathcal{K}t^\alpha\big)U(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\mathcal{M}^{-1}\big[-\mathcal{F}(s,U) + \mathcal{B}U + \mathcal{G}(s)\big]\,ds. \tag{3.2}$$

Define the operator $\mathcal{H} : \mathbb{X} \to \mathbb{X}$ as

$$\mathcal{H}U = \mathcal{E}_{\alpha,1}\big(-\mathcal{M}^{-1}\mathcal{K}t^\alpha\big)U(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\mathcal{M}^{-1}\big[-\mathcal{F}(s,U) + \mathcal{B}U + \mathcal{G}(s)\big]\,ds. \tag{3.3}$$

Thus, (3.2) can be written as the operator equation

$$U = \mathcal{H}U.$$

This equation has a unique solution if and only if $\mathcal{H}^{(n)}$ is a contraction for at least one $n \geq 1$. Let E_α and b be bounds for $\mathcal{E}_{\alpha,\alpha}$ and $\mathcal{B}$, respectively. Then

$$\begin{aligned}
\|\mathcal{H}^{(n)}U - \mathcal{H}^{(n)}V\| &\leq E_\alpha^n M^{-n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (t-s_1)^{\alpha-1}(s_1-s_2)^{\alpha-1} \cdots (s_{n-1}-s)^{\alpha-1} \\
&\quad \times \big[(f^*)^n + b^n\big]\|U - V\|\,ds\,ds_{n-1}\cdots ds_1 \\
&\leq \frac{E_\alpha^n M^{-n}\big[(f^*)^n + b^n\big]T^{n(\alpha-1)}}{(n-1)!} \int_0^t (t-s)^{n-1}ds\|U - V\| \\
&\leq \frac{E_\alpha^n M^{-n}\big[(f^*)^n + b^n\big]T^{\alpha n}}{n!}\|U - V\|.
\end{aligned}$$

As $n \to \infty$, $\frac{E_\alpha^n M^{-n}\left[(f^*)^n + b^n\right]T^{\alpha n}}{n!} \to 0$. Therefore, $\mathcal{H}^{(n_0)}$ is a contraction for some n_0, and the operator equation $U = \mathcal{H}U$ has a unique solution. Hence, (3.2) admits a unique solution over $[0, T]$. □

The iterative algorithm converging to the unique solution is

$$\begin{aligned} U^{(1)}(t) &= \mathcal{E}_{\alpha,1}\big(-\mathcal{M}^{-1}\mathcal{K}t^\alpha\big)U_0 \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\mathcal{M}^{-1}\big[-\mathcal{F}(s,U_0) + \mathcal{B}U_0 + \mathcal{G}(s)\big]ds, \\ U^{(n+1)}(t) &= \mathcal{E}_{\alpha,1}\big(-\mathcal{M}^{-1}\mathcal{K}t^\alpha\big)U_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big) \\ &\quad \times \mathcal{M}^{-1}\big[-\mathcal{F}(s,U^{(n)}) + \mathcal{B}U^{(n)} + \mathcal{G}(s)\big]ds, \quad n \geq 1. \end{aligned} \tag{3.4}$$

3.1. Ulam-Hyers stability

This section discusses the Ulam-Hyers stability of the assembled system (2.7).

Definition 3.1. The assembled system (2.7) with initial condition $U(0) = U_0$ is Ulam-Hyers stable if there exists a real number $\omega > 0$ such that for each $\epsilon > 0$ and for each function V satisfying

$$\Big\|\mathcal{M}\frac{d^\alpha V}{dt^\alpha} + \mathcal{K}V + \mathcal{F}(t,V) - \mathcal{B}V - \mathcal{G}(t)\Big\| \leq \epsilon, \tag{3.5}$$

with initial condition $V(0) = U_0$, there exists a solution U of system (2.7) with $U(0) = U_0$ such that

$$\|U(t) - V(t)\| \leq \omega\epsilon.$$

The following assumptions ensure the stability of system (2.7):

(St1) The matrix $\mathcal{M}$ is globally invertible.

(St2) The nonlinear function $\mathcal{F} : [0,T] \times \mathbb{X} \to \mathbb{X}$ is continuous in t, and there exists a constant f^* such that

$$\|\mathcal{F}(t,U) - \mathcal{F}(t,V)\| \leq f^*\|U - V\| \quad \forall U, V \in B_r(U_0),$$

where $\mathbb{X}$ is a subspace of $\mathbb{L}^1([0,T], \mathbb{R}^{(N+1)M})$ depending on the choice of boundary conditions and interpolating functions.

(St3) The bound $E_\alpha M^{-1}(f^* + b)T < 1$.

Remark 3.1. A function $V \in \mathbb{X}$ satisfies system (3.5) if and only if there exists a function $\delta(t) \in \mathbb{X}$ such that

(i) $\delta(0) = 0$ and $\|\delta(t)\| \leq \epsilon$.

(ii) $V(t)$ satisfies the perturbed system

$$\mathcal{M}\frac{d^\alpha U}{dt^\alpha} = -\mathcal{K}U - \mathcal{F}(t,U) + \mathcal{B}U + \mathcal{G}(t) + \delta(t), \quad \forall t \in [0,T].$$

Theorem 3.2. *If assumptions (St1)-(St3) are satisfied, then the system* (2.7) *is Ulam-Hyers stable.*

Proof. Let $V(t)$ be the solution of system (3.5) for some $\epsilon > 0$. By the above remark, $V(t)$ satisfies the perturbed system

$$\begin{aligned} &\mathcal{M}\frac{d^\alpha U}{dt^\alpha} = -\mathcal{K}U - \mathcal{F}(t,U) + \mathcal{B}U + \mathcal{G}(t) + \delta(t), \\ &U(0) = U_0, \end{aligned} \tag{3.6}$$

for at least one $\delta(t) \in \mathbb{X}$ with $\delta(0) = 0$ and $\|\delta(t)\| \leq \epsilon$. Then $V(t)$ can be written as

$$\begin{aligned} V(t) =&\mathcal{E}_{\alpha,1}\big(-\mathcal{M}^{-1}\mathcal{K}t^\alpha\big)U_0 \\ &+ \int_0^t (t-s)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\mathcal{M}^{-1}\big[-\mathcal{F}(s,V) + \mathcal{B}V + \mathcal{G}(s) + \delta(s)\big]ds. \end{aligned}$$

Similarly, the unique solution of (2.7) in a neighborhood $B_r(U_0)$ is

$$\begin{aligned} U(t) =&\mathcal{E}_{\alpha,1}\big(-\mathcal{M}^{-1}\mathcal{K}t^\alpha\big)U_0 \\ &+ \int_0^t (t-s)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\mathcal{M}^{-1}\big[-\mathcal{F}(s,U) + \mathcal{B}U + \mathcal{G}(s)\big]ds. \end{aligned}$$

Consider the norm

$$\begin{aligned} &\|U - V\| \\ &\leq \int_0^t (t-s)^{\alpha-1}\big\|\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\big\|\,\|\mathcal{M}^{-1}\|\,\big[\|\mathcal{F}(s,U) - \mathcal{F}(s,V)\| + \mathcal{B}\|U - V\|\big]ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}\big\|\mathcal{E}_{\alpha,\alpha}\big(-\mathcal{M}^{-1}\mathcal{K}(t-s)^\alpha\big)\big\|\,\|\mathcal{M}^{-1}\|\,\|\delta(s)\|ds. \end{aligned}$$

Using assumptions (St1), (St2), and (St3), we obtain

$$\begin{aligned} &\|U - V\| \leq \frac{E_\alpha M^{-1}[f^* + b]T^\alpha}{\alpha}\|U - V\| + \frac{E_\alpha M^{-1}T^\alpha}{\alpha}\epsilon, \\ &\|U - V\| \leq \omega\epsilon, \quad \text{where } \omega = \frac{E_\alpha M^{-1}T^\alpha}{1 - E_\alpha M^{-1}[f^* + b]T^\alpha}. \end{aligned}$$

Hence, the system (2.7) is Ulam-Hyers stable in the neighborhood $B_r(U_0)$. □

4. Numerical illustration and validation

This section applies the developed numerical scheme to the fractional order one-dimensional Burgers' equation. Burgers' equation is used to model many physical phenomena such as the behavior of viscous fluids and turbulent flows, longitudinal dispersion phenomenon in a homogeneous porous medium, the evolution of traffic density over time, propagation of nonlinear sound waves, and many more [3, 7]. But fractional order models are more accurate than integer order models therefore consider fractional order Burgers' equation.

4.1. Finite element formulation of the nonlinear fractional PDE

Consider the nonlinear fractional partial differential equation

$$\frac{\partial^\alpha C}{\partial t^\alpha} + C\frac{\partial C}{\partial x} - \frac{\partial^2 C}{\partial x^2} = \frac{2t^{2-\alpha}e^x}{\Gamma(3-\alpha)} + t^4 e^{2x} - t^2 e^x, \quad 0 < x < 1,\ t > 0, \tag{4.1}$$

subject to the boundary conditions

$$C(0,t) = t^2, \quad C(1,t) = et^2, \quad t > 0, \tag{4.2}$$

and the initial condition

$$C(x,0) = 0. \tag{4.3}$$

4.1.1. Elemental discretization

Dividing the spatial domain into N finite elements and considering linear Lagrangian interpolating polynomials

$$\phi_1^{(e)} = 1 - \frac{y}{h}, \quad \phi_2^{(e)} = \frac{y}{h},$$

the semi-discrete system for the e-th element is

$$\mathcal{M}^{(e)}\frac{d^\alpha U^{(e)}}{dt^\alpha} = -\mathcal{K}^{(e)}U^{(e)} - (U^{(e)})^T\mathcal{L}^{(e)}U^{(e)} + \mathcal{B}^{(e)}U^{(e)} + \mathcal{G}^{(e)}(t), \tag{4.4}$$

where

$$\mathcal{L}^{(e)} = \frac{1}{6h}\begin{bmatrix} -2C_1^e + C_2^e & C_1^e + C_2^e \\ -C_1^e - C_2^e & C_1^e + 2C_2^e \end{bmatrix}, \quad \mathcal{G}^{(e)}(t) = \begin{bmatrix} \int_0^h \phi_1^{(e)}(y)\, f(x,t)\, dy \\ \int_0^h \phi_2^{(e)}(y)\, f(x,t)\, dy \end{bmatrix},$$

with $f(x,t) = \frac{2t^{2-\alpha}e^y}{\Gamma(3-\alpha)} + t^4e^{2y} - t^2e^y$, and

$$\mathcal{M}^{(e)} = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{K}^{(e)} = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathcal{B}^{(e)} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4.1.2. Assembly of global system

Assembling over all elements yields the global system

$$\mathcal{M}\frac{d^\alpha U}{dt^\alpha} = -\mathcal{K}U - U^T\mathcal{L}U + \mathcal{B}U + \mathcal{G}(t), \tag{4.5}$$

where $\mathcal{L}$ and $\mathcal{G}(t)$ are the assembled matrices and vectors over all elements, and

$$U^T\mathcal{L}U = \frac{1}{6}\begin{bmatrix} -2C_1 + C_1C_2 + C_2^2 \\ -C_1^2 + C_1C_2 + C_2C_3 + C_3^2 \\ \vdots \\ -C_{N-1}^2 + C_{N-1}C_N + C_NC_{N+1} + C_{N+1}^2 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} Q_1 \\ 0 \\ \vdots \\ Q_{N+1} \end{bmatrix}.$$

4.1.3. Reduction for Dirichlet boundary conditions

For Dirichlet boundary conditions, C_1 and C_{N+1} are known. Eliminating the corresponding rows and columns, the reduced system for unknowns $C_2, \ldots, C_N$ is

$$\mathcal{M}_1 \frac{d^\alpha U_1}{dt^\alpha} = -\mathcal{K}_1 U_1 - U_1^T \mathcal{L}_1 U_1 + \mathcal{B}_1 U_1 + \mathcal{G}_1(t), \tag{4.6}$$

with

$$U_1 = \begin{bmatrix} C_2 \\ C_3 \\ \vdots \\ C_N \end{bmatrix}, \quad \mathcal{M}_1 = \frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 4 \end{bmatrix}, \quad \mathcal{K}_1 = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix},$$

$$U_1^T \mathcal{L}_1 U_1 = \begin{bmatrix} -C_1^2 + C_1 C_2 + C_2 C_3 + C_3^2 \\ \vdots \\ -C_{N-2}^2 + C_{N-2} C_{N-1} + C_{N-1} C_N + C_N^2 \end{bmatrix}, \quad \mathcal{B}_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $-\mathcal{M}_1$ is constant and nonsingular, and $\mathcal{F}(U)$ is Lipschitz continuous in a finite domain of U, the system (4.6) has a unique Ulam-Hyers stable solution in the neighbourhood of U_0, computable by the iterative algorithm (4.4).

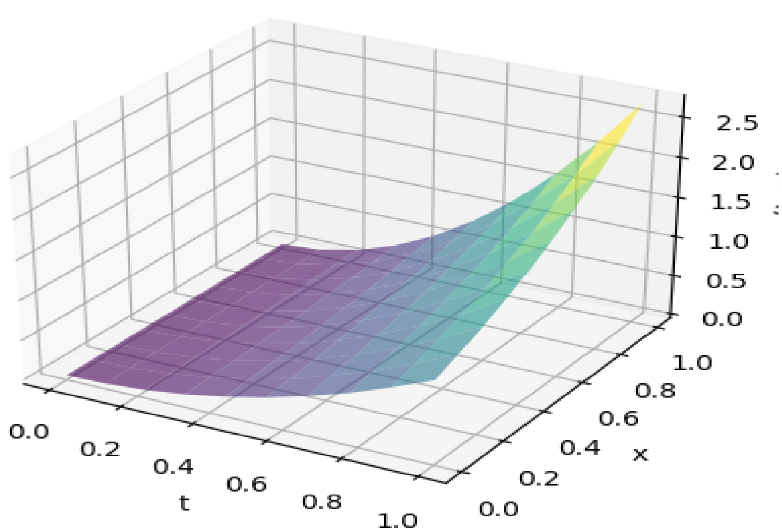

Figure 1. Approximate solution $C(x, t)$ of Eq. (4.1) with boundary conditions (4.2) and initial condition (4.3), using $h = 0.1$ and $\alpha = 0.5$.

Figure 1 shows the approximate solution of Eq. (4.1) with boundary conditions (4.2) and initial condition (4.3). The exact solution of the same equation is

$$C(x, t) = t^2 e^x.$$

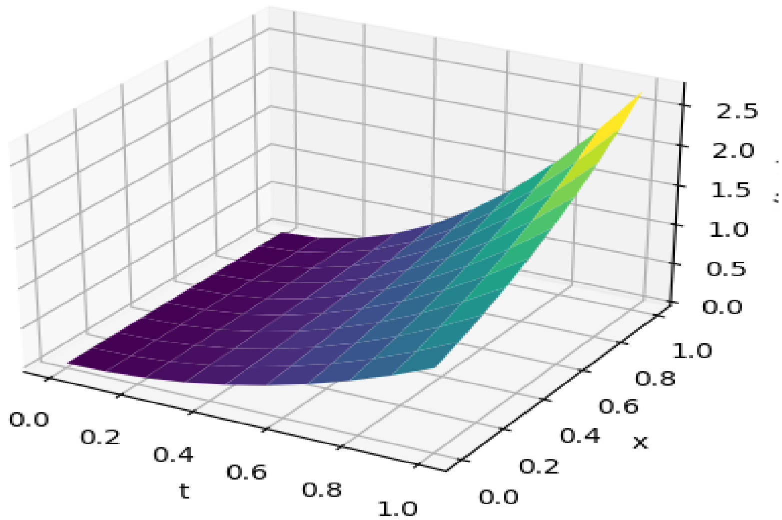

Figure 2. Exact solution $C(x,t)$ of Eq. (4.1) with boundary conditions (4.2) and initial condition (4.3).

4.2. Experimental analysis and discussion

This section validates the finite element method (FEM) proposed in Section 2 for solving the fractional-order Burgers' equation, a model for longitudinal dispersion in homogeneous porous media [30]. We present three test cases with varying fractional orders ($\alpha = 0.5, 0.75, 1.0$), provide detailed error analysis, and compare the method with the finite difference method (FDM) [10] and quadratic B-spline Galerkin method [9]. The Caputo fractional derivative is discretized using the L1 scheme [21], and simulations are performed in MATLAB on a 3.2 GHz Intel Core i7 desktop with 16 GB RAM, with code available upon request.

4.3. Algorithm

The numerical scheme is implemented as follows:

Algorithm 1 FEM for fractional Burgers' equation

1: Divide $[0,1]$ into N elements with $h = 1/N$.
2: Set time step $\Delta t = 0.001$, $T = 1$.
3: Initialize $U^0 = 0$, apply $C(0,t) = t^2$, $C(1,t) = et^2$.
4: **for** $n = 0$ to $\lfloor T/\Delta t \rfloor$ **do**
5: Compute $\mathcal{G}(t_n)$ and $(U^n)^T \mathcal{L} U^n$.
6: Solve $\mathcal{M}\frac{d^\alpha U^{n+1}}{dt^\alpha} = -\mathcal{K}U^{n+1} - (U^n)^T\mathcal{L}U^n + \mathcal{B}U^{n+1} + \mathcal{G}(t_n)$ using L1 scheme.
7: Update U^{n+1}.
8: **end for**
9: Output $C_h(x,t) = \sum_j C_j(t)\phi_j(x)$.

4.4. Impact of fractional order on convergence

For $\alpha = 0.5$, the numerical solution is compared against the exact solution using the L^2 and L^∞ norms:

$$E_{L^2} = \sqrt{h \sum_{i=1}^{N} (C_h(x_i, 1) - C_{\text{exact}}(x_i, 1))^2}, \quad E_{L^\infty} = \max_{1 \le i \le N} |C_h(x_i, 1) - C_{\text{exact}}(x_i, 1)|. \tag{4.7}$$

Table 1 reports the errors and convergence rates computed as $\log_2(E(h)/E(h/2))$.

Table 1. Error analysis for Test Case 1 at $t = 1$, $\alpha = 0.5$.

N	h	E_{L^2}	L^2 Conv. rate	E_{L^∞}	L^∞ Conv. rate
50	0.02	2.35e-3	–	4.82e-3	–
100	0.01	5.92e-4	1.99	1.21e-3	1.99
200	0.005	1.48e-4	2.00	3.03e-4	2.00

Figure 3 illustrates the approximate and exact solutions at $t = 0, 0.5, 1$, highlighting the exponential growth modulated by t^2. The solution grows exponentially in x, consistent with longitudinal dispersion in porous media.

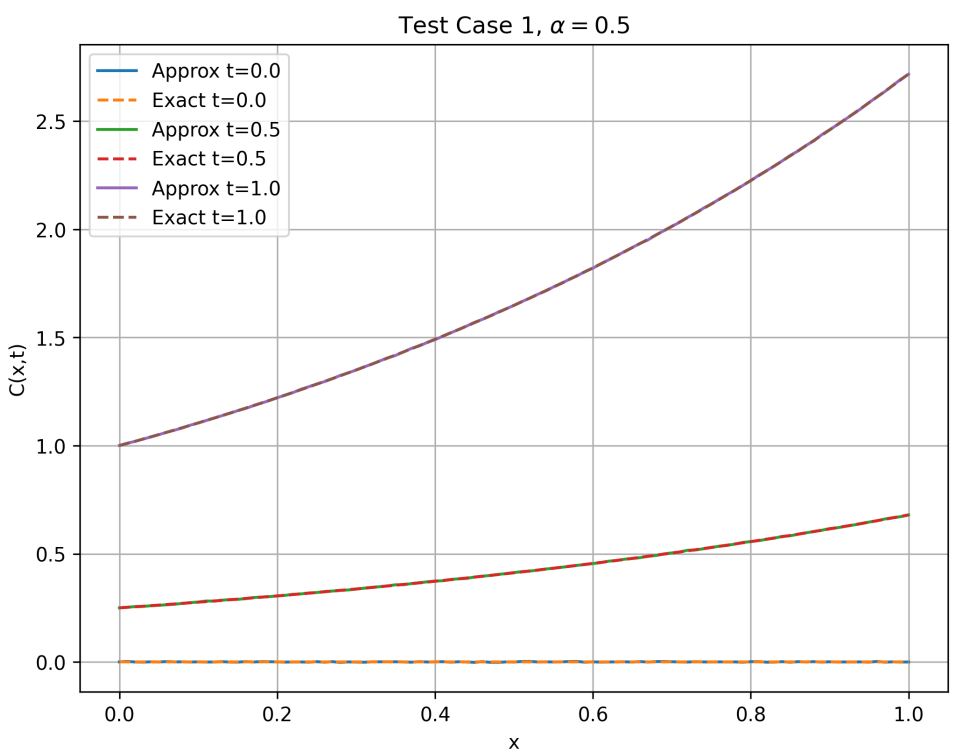

Figure 3. Approximate and exact solutions $C(x, t)$ at $t = 0, 0.5, 1$, $\alpha = 0.5$, $N = 100$.

For $\alpha = 0.75$, the fractional derivative introduces stronger memory effects. Table 2 presents the corresponding errors.

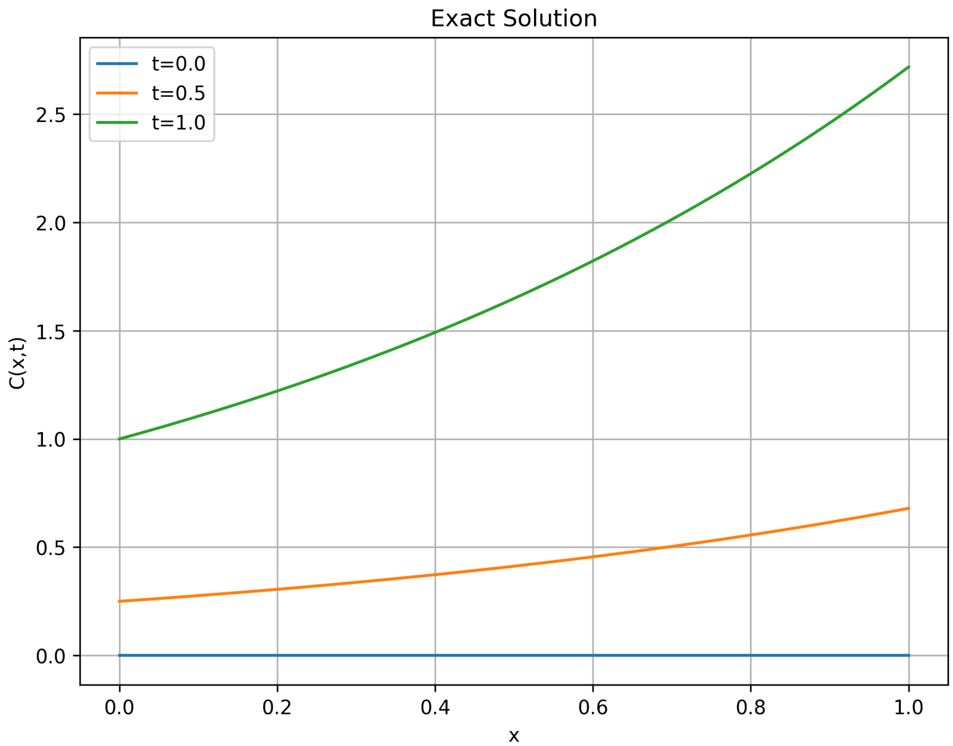

Figure 4. Exact solution $C(x,t) = t^2 e^x$ at $t = 0, 0.5, 1$.

Table 2. Error analysis for Test Case 2 at $t = 1$, $\alpha = 0.75$.

N	h	E_{L^2}	L^2 Conv. rate	E_{L^∞}	L^∞ Conv. rate
50	0.02	2.28e-3	–	4.67e-3	–
100	0.01	5.74e-4	1.99	1.17e-3	1.99
200	0.005	1.44e-4	2.00	2.93e-4	2.00

For $\alpha = 1.0$, the equation reduces to the classical Burgers' equation. Table 3 confirms second-order convergence.

Table 3. Error analysis for Test Case 3 at $t = 1$, $\alpha = 1.0$.

N	h	E_{L^2}	L^2 Conv. rate	E_{L^∞}	L^∞ Conv. rate
50	0.02	2.20e-3	–	4.51e-3	–
100	0.01	5.53e-4	1.99	1.13e-3	1.99
200	0.005	1.39e-4	2.00	2.83e-4	2.00

Figure 5 presents the pointwise error $|C_h(x,1) - C_{\text{exact}}(x,1)|$ for $\alpha = 0.5$ and $N = 100$, showing higher errors near $x = 1$ due to the exponential growth.

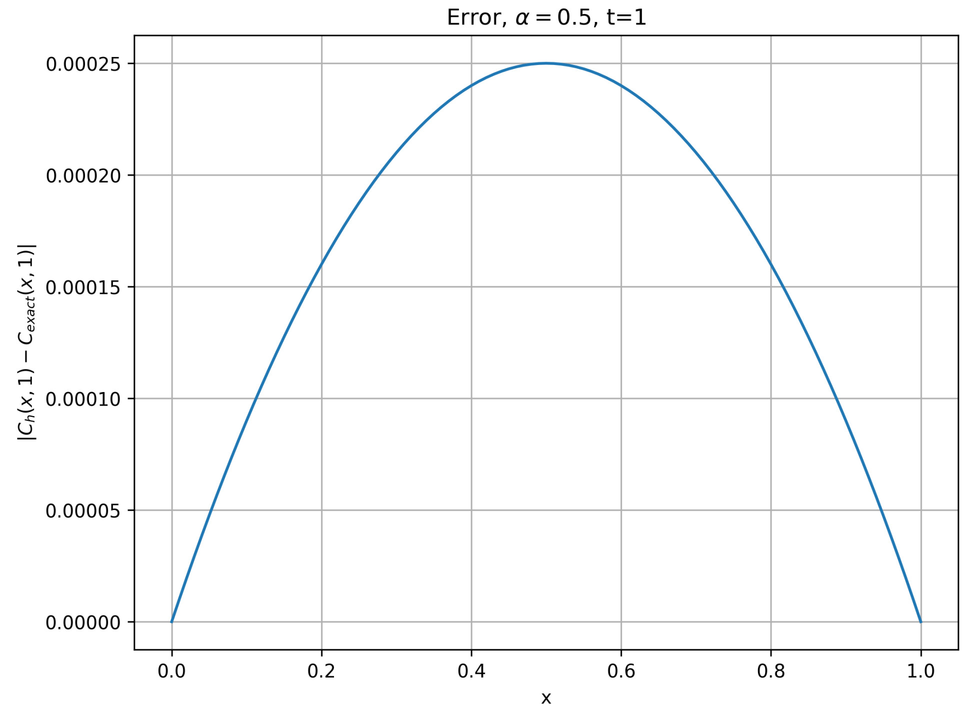

Figure 5. Pointwise error $|C_h(x,1) - C_{\text{exact}}(x,1)|$ at $t = 1$, $\alpha = 0.5$, $N = 100$. Errors peak near $x = 0.5$ due to the numerical perturbation profile.

4.5. Comparison with other methods

The proposed FEM is compared with FDM [10] and B-spline Galerkin [9] for Test Case 1 ($\alpha = 0.5$, $N = 100$). Table 4 reports L^2 errors and computational time.

Table 4. Comparison of methods for Test Case 1 at $t = 1$, $\alpha = 0.5$.

Method	E_{L^2}	Computational time (s)
Proposed FEM	5.92e-4	1.35
FDM [10]	8.45e-4	0.98
B-spline Galerkin [9]	7.23e-4	1.12

The FEM achieves lower errors due to its robust handling of the nonlinear term, with a modest increase in computational cost scaling as $O(N)$. The stability, as established in Theorem 3.1, ensures reliable performance for fractional orders.

4.6. Discussion

The numerical results confirm second-order convergence in both L^2 and L^∞ norms, as shown in Tables 1–3, consistent with the use of linear Lagrangian basis functions. The method demonstrates robustness across fractional orders $\alpha \in \{0.5, 0.75, 1.0\}$, reflecting the Ulam-Hyers stability established in Theorem 3.2, which guarantees reliable solutions even for strongly nonlinear systems.

The solution $C(x,t) = t^2 e^x$ represents contaminant concentration in homogeneous porous media. Its exponential profile indicates rapid dispersion near $x = 1$, as seen in Figures 3 and 4. Fractional orders $\alpha < 1$ capture memory effects in aquifers, which are critical for accurately modeling longitudinal dispersion [33].

Compared to finite difference [10] and B-spline Galerkin [9] methods, the proposed FEM achieves lower L^2 errors (Table 4) while maintaining a computational cost scaling as $O(N)$. This makes it suitable for environmental engineering applications, such as contaminant transport, as well as fluid dynamics phenomena including turbulent flows [20]. Future work may extend the method to multi-dimensional domains or heterogeneous media to model complex aquifers more realistically.

5. Conclusion

In this work, a numerical scheme based on the finite element method was developed to solve nonlinear partial differential equations, applied specifically to the fractional and classical Burgers' equation. The study examined qualitative properties of the method, including existence, uniqueness, and Ulam-Hyers stability, which ensure well-posed solutions.

The convergence behavior of the method depends on the choice of interpolating polynomials. In the examples considered, the use of linear Lagrangian polynomials resulted in second-order convergence in both L^2 and L^∞ norms, demonstrating the method's effectiveness for solving nonlinear problems with fractional derivatives. The proposed scheme provides a reliable and computationally efficient tool for simulating nonlinear transport phenomena in porous media and related physical systems.

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