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ON RIEMANN-LIOUVILLE INTEGRAL INEQUALITIES VIA QUASI CONVEX WITH RESPECT TO STRICTLY MONOTONE FUNCTIONS

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Abstract This paper aims to present new estimates of generalized Riemann-Liouville (RL) fractional integrals via an increasing function. A new class of functions containing several convexities is considered in establishing fractional integral inequalities. Some special cases are deduced by considering specific functions. Also, a Hermite-Hadamard inequality is proved for RL fractional integrals of $\Xi - (h, \vartheta; \alpha)$-convex function.

Keywords Convex function, Riemann-Liouville fractional integrals, Hermite-Hadamard inequality.

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1. Introduction

Fractional derivatives and fractional integrals are useful generalizations of ordinary derivative and Riemann integral. For instance Riemann-Lioville fractional integrals, Caputo fractional derivatives, Weyl fractional integrals are considered very classical. Nowadays there have

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been developed many types of fractional derivatives and integrals; these include local fractional integrals/derivatives, conformable fractional integrals [6,9,19] and q-integrals/derivatives, see [2, 13, 25, 26] and references therein. Almost all fields of science and engineering have been updated and generalized due to fractional order integrals and derivatives. Fractional differential equations are more effective in mathematical modeling of real world problems than integer order differential equations. For a detailed study we refer the readers to [12, 14]. Very commonly used definition of fractional order integral/derivative is the Riemann-Liouville fractional integral/derivative.

Definition 1.1. [11] Let $\Xi \in L_1[a,b]$. Then the Riemann-Liouville fractional integrals of Ξ of order $\sigma > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\sigma}\Xi(t) = \frac{1}{\Gamma(\sigma)}\int_a^t (t-x)^{\sigma-1}\Xi(x)dx,\ t > a$$

and

$$I_{b_-}^{\sigma}\Xi(t) = \frac{1}{\Gamma(\sigma)}\int_t^b (x-t)^{\sigma-1}\Xi(x)dx,\ \ t < b.$$

Motivated by the above integral representations of fractional order integrals, authors have extended/generalized the RL integrals via different kinds of modifications in the integrands of above integrals. Fractional integrals containing Mittag-Leffler function were introduced in [19, 21, 23]. For instance RL integrals of a function with respect to another function are given in the next definition.

Definition 1.2. [10, p. 99-100] Let $f \in L_1[a,b]$. Also, let $\vartheta \in C^{(2)}[a,b]$ be positive monotonically increasing function. Fractional integrals denoted by ${}_{\vartheta}I_{a^+}^{\mu} f, {}_{\vartheta} I_{b^-}^{\mu} f$ of a function f with respect to ϑ of order $\mu > 0$ are defined by

$${}_{\vartheta}I_{a^+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (\vartheta(x)-\vartheta(t))^{\mu-1} f(t)\vartheta'(t)dt, \quad x > a, \tag{1.1}$$

and

$${}_{\vartheta}I_{b^-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (\vartheta(t)-\vartheta(x))^{\mu-1} f(t)\vartheta'(t)dt, \quad x < b. \tag{1.2}$$

Fractional integral operators are important tools in advancing the classical integral inequalities. In recent years many kinds of fractional integrals were considered to extend well known inequalities including Hölder, Minkowski [16], Ostrowski [4, 17], Grüss [8, 22], Ostrowsk-Grüss, Wirtinger [5], Jensen and Hermite-Hadamard [3, 15] inequalities. Some recent articles are given in references, see [1, 18, 20, 24] and references therein.

The aim of this article is to investigate the estimates of Riemann-Liouville fractional integrals of a generalized convex function with respect to an increasing function. In the following we give the definition of generalized convex function, which will be frequently utilized in establishing the results of this paper.

Definition 1.3. [7] Let h be a non-negative function on $J \subset \mathbb{R}$, $(0,1) \subset J$, $h \neq 0$ and let ϑ be a positive function on $I \subset \mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be quasi $\Xi - (h, \vartheta; \alpha)$-convex function if it is non-negative and if

$$f(\Xi^{-1}(\lambda\Xi(x) + (1-\lambda)\Xi(y)) \leq h(\lambda^{\alpha})f(x)\vartheta(x) + h(1-\lambda^{\alpha})f(y)\vartheta(y), \tag{1.3}$$

provided $\Xi : I \to \mathbb{R}$ is strictly monotone, where $\lambda,\ \alpha \in [0,1]\,, x, y \in I$.

Above definition is an implicit form of almost all convexities defined gradually in recent past decades. For example classes of exponentially convex functions like exponentially; h-convex, m-convex, $(h-m)$-convex, (α, m)-convex, (s, m)-convex are all hided in the above definition. Therefore results of this paper will simultaneously hold for all such classes defined in literature.

2. Estimates of bounds of generalized RL fractional integrals via quasi $\Xi-(h,\vartheta;\alpha)$-convex function

First, we give the following bound for RL fractional integrals.

Theorem 2.1. *Let $f: I \longrightarrow \mathbb{R}$ be quasi $\Xi-(h,\vartheta;\alpha)$-convex function, where $\vartheta \in C^{(2)}[a,b]$ is strictly increasing. Then for $a,b \in I; a<b$ and $\sigma, \delta \geq 1$, generalized RL fractional integrals are bounded above*

$$\begin{aligned} &\frac{\Gamma(\sigma)_\vartheta I^\sigma_{a^+}f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma-1}}+\frac{\Gamma(\delta)_\vartheta I^\delta_{b^-}f(x)}{(\vartheta(b)-\vartheta(x))^{\delta-1}} \\ \leq & f.\vartheta(a)\int_a^x K^{x,t}_{x,a}(\Xi;h)d(\vartheta(t))+f.\vartheta(b)\int_x^b K^{t,x}_{b,x}(\Xi;h)d(\vartheta(t)) \\ &+f.\vartheta(x)\left[\int_a^x K^{t,a}_{x,a}(\Xi;h)d(\vartheta(t))+\int_x^b K^{b,t}_{b,x}(\Xi;h)d(\vartheta(t))\right], x\in(a,b), \end{aligned} \tag{2.1}$$

where $K^{u,v}_{p,q}(\Xi;h)=h\left(\frac{(\Xi(u)-\Xi(v))^\alpha}{(\Xi(p)-\Xi(q))^\alpha}\right)$.

Proof. We utilize the conditions imposed on functions f and ϑ as follows:
1. Let $x \in (a,b)$, $t\in[a,x]$, given function ϑ satisfies the following inequality:

$$\vartheta'(t)(\vartheta(x)-\vartheta(t))^{\sigma-1}\leq \vartheta'(t)(\vartheta(x)-\vartheta(a))^{\sigma-1}. \tag{2.2}$$

2. Given function f is quasi $\Xi-(h,\vartheta;\alpha)$-convex and satisfies the following inequality:

$$f(t)\leq K^{x,t}_{x,a}(\Xi;h)f.\vartheta(a)+K^{t,a}_{x,a}(\Xi;h)f.\vartheta(x). \tag{2.3}$$

The above two inequalities (2.2) and (2.3) generate the forthcoming integral inequality

$$\begin{aligned} &\frac{\int_a^x(\vartheta(x)-\vartheta(t))^{\sigma-1}\vartheta'(t)f(t)dt}{(\vartheta(x)-\vartheta(a))^{\sigma-1}} \\ \leq&\left[f.\vartheta(a)\int_a^x K^{x,t}_{x,a}(\Xi;h)d(\vartheta(t))+f.\vartheta(x)\int_a^x K^{t,a}_{x,a}(\Xi;h)d(\vartheta(t))\right]. \end{aligned}$$

The above inequality provides an upper bound of generalized left RL fractional integral

$$\frac{\Gamma(\sigma)_\vartheta I^\sigma_{a^+}f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma-1}}\leq f.\vartheta(a)\int_a^x K^{x,t}_{x,a}(\Xi;h)d(\vartheta(t))+f.\vartheta(x)\int_a^x K^{t,a}_{x,a}(\Xi;h)d(\vartheta(t)). \tag{2.4}$$

Also, if we let $x\in(a,b)$, $t\in[a,x]$, conditions imposed on functions f and ϑ constitute inequalities as follows:

$$\vartheta'(t)(\vartheta(t)-\vartheta(x))^{\delta-1}\leq\vartheta'(t)(\vartheta(b)-\vartheta(x))^{\delta-1}, \tag{2.5}$$

$$f(t) \leq K_{b,x}^{t,x}(\Xi;h)f.\vartheta(b) + K_{b,x}^{b,t}(\Xi;h)f.\vartheta(x). \tag{2.6}$$

The above two inequalities (2.5) and (2.6), generate the forthcoming integral inequality

$$\begin{aligned}&\frac{\int_x^b(\vartheta(t)-\vartheta(x))^{\delta-1}\vartheta'(t)f(t)dt}{(\vartheta(b)-\vartheta(x))^{\delta-1}}\\ \leq& f.\vartheta(b)\int_x^b K_{b,x}^{t,x}(\Xi;h)d(\vartheta(t)) + f.\vartheta(x)\int_x^b K_{b,x}^{b,t}(\Xi;h)d(\vartheta(t)).\end{aligned}$$

The following upper bound of generalized right RL-fractional integral is obtained

$$\frac{\Gamma(\delta)_\vartheta I_{b-}^\delta f(x)}{(\vartheta(b)-\vartheta(x))^{\delta-1}} \leq f.\vartheta(b)\int_x^b K_{b,x}^{t,x}(\Xi;h)d(\vartheta(t)) + f.\vartheta(x)\int_x^b K_{b,x}^{b,t}(\Xi;h)d(\vartheta(t)). \tag{2.7}$$

By adding (2.7) and (2.4), one can get (2.1). □

Some specific settings in above theorem produce interesting implications. For instance by fixing h as identity function the following result holds.

Theorem 2.2. *By keeping statement of Theorem 2.1 as it, quasi $\Xi-(I,\vartheta;\alpha)$-convex function satisfy the forthcoming result:*

$$\begin{aligned}&\frac{\Gamma(\sigma)_\vartheta I_{a+}^\sigma f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma-1}} + \frac{\Gamma(\delta)_\vartheta I_{b-}^\delta f(x)}{(\vartheta(b)-\vartheta(x))^{\delta-1}}\\ \leq&\Gamma(\alpha+1)\left[\frac{f.\vartheta(a)_\Xi I_{a+}^\alpha\vartheta(x) - f.\vartheta(x)_\Xi I_{x-}^\alpha\vartheta(a)}{(\Xi(x)-\Xi(a))^\alpha} + \frac{f.\vartheta(x)_\Xi I_{x+}^\alpha\vartheta(b) - f.\vartheta(b)_\Xi I_{b-}^\alpha\vartheta(x)}{(\Xi(b)-\Xi(x))^\alpha}\right]\\ &+\left(f(b)\vartheta^2(b) - f(a)\vartheta^2(a)\right),\ x\in(a,b)\end{aligned} \tag{2.8}$$

provided Ξ is differentiable.

Proof. By considering $h(z)=I(z)=z$, we calculate the integrals involving in right hand side of (2.1) as follows:

1.

$$\int_a^x K_{x,a}^{x,t}(\Xi;I)d(\vartheta(t)) = \frac{1}{(\Xi(x)-\Xi(a))^\alpha}\int_a^x(\Xi(x)-\Xi(t))^\alpha d(\vartheta(t)).$$

After computing integral by parts one time on the right hand side, we get

$$\int_a^x K_{x,a}^{x,t}(\Xi;I)d(\vartheta(t)) = \frac{\Gamma(\alpha+1)_\Xi I_{a+}^\alpha\vartheta(x)}{(\Xi(x)-\Xi(a))^\alpha} - \vartheta(a).$$

2.

$$\int_x^b K_{b,x}^{t,x}(\Xi;I)d(\vartheta(t)) = \frac{1}{(\Xi(b)-\Xi(x))^\alpha}\int_x^b(\Xi(t)-\Xi(x))^\alpha d(\vartheta(t)).$$

After computing integral by parts one time on the right hand side, we get

$$\int_x^b K_{b,x}^{t,x}(\Xi;I)d(\vartheta(t)) = \vartheta(b) - \frac{\Gamma(\alpha+1)_\Xi I_{b-}^\alpha\vartheta(x)}{(\Xi(b)-\Xi(x))^\alpha}.$$

3.

$$\int_a^x K_{x,a}^{t,a}(\Xi;I)d(\vartheta(t)) = \frac{1}{(\Xi(x)-\Xi(a))^\alpha}\int_a^x(\Xi(t)-\Xi(a))^\alpha d(\vartheta(t)).$$

After computing integral by parts one time on the right hand side, we get

$$\int_a^x K_{x,a}^{t,a}(\Xi;h)d(\vartheta(t)) = \Xi(x) - \frac{\Gamma(\alpha+1)_\Xi I_{x^-}^{\alpha}\vartheta(a)}{(\Xi(x)-\Xi(a))^{\alpha}}.$$

4.

$$\int_x^b K_{b,x}^{b,t}(\Xi;h)d(\vartheta(t)) = \frac{1}{(\Xi(b)-\Xi(x))^{\alpha}}\int_x^b (\Xi(x)-\Xi(t))^{\alpha}d(\vartheta(t)).$$

After computing integral by parts one time on the right hand side, we get

$$\int_x^b K_{b,x}^{b,t}(\Xi;h)d(\vartheta(t)) = \frac{\Gamma(\alpha+1)_\Xi I_{x^+}^{\alpha}\vartheta(b)}{(\Xi(b)-\Xi(x))^{\alpha}} - \vartheta(x).$$

By putting values of above computed integrals in (2.1), and after some arithmetic one can obtain the required inequality (2.8). □

By setting Ξ as power function i.e. $\Xi(x) = p(x) = x^r$, $r > 0$ in (2.8), the following result holds.

Theorem 2.3. *By keeping statement of Theorem 2.1 as it, quasi $p-(I,\vartheta;\alpha)$-convex function satisfy the forthcoming result:*

$$\begin{aligned} &\frac{\Gamma(\sigma)_\vartheta I_{a^+}^{\sigma} f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma-1}} + \frac{\Gamma(\delta)_\vartheta I_{b^-}^{\delta} f(x)}{(\vartheta(b)-\vartheta(x))^{\delta-1}} \\ \le&\Gamma(\alpha+1)\left[\frac{f.\vartheta(a)_p I_{a^+}^{\alpha}\vartheta(x) - f.\vartheta(x)_p I_{x^-}^{\alpha}\vartheta(a)}{(x^r-a^r)^{\alpha}} + \frac{f.\vartheta(x)_p I_{x^+}^{\alpha}\vartheta(b) - f.\vartheta(b)_p I_{b^-}^{\alpha}\vartheta(x)}{(b^r-x^r)^{\alpha}}\right] \\ &+ \left(f(b)\vartheta^2(b) - f(a)\vartheta^2(a)\right),\ x\in(a,b). \end{aligned} \tag{2.9}$$

Furthermore, by setting $\Xi(x) = I(x) = x$ in (2.8), the forthcoming inequality holds.

Theorem 2.4. *By keeping statement of Theorem 2.1 as it, quasi $I-(I,\vartheta;\alpha)$-convex function satisfy the forthcoming result:*

$$\begin{aligned} &\frac{\Gamma(\sigma)_\vartheta I_{a^+}^{\sigma} f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma-1}} + \frac{\Gamma(\delta)_\vartheta I_{b^-}^{\delta} f(x)}{(\vartheta(b)-\vartheta(x))^{\delta-1}} \\ \le&\alpha\left(\frac{f.\vartheta(a) I_{a^+}^{\alpha}\vartheta(x) - f.\vartheta(x) I_{x^-}^{\alpha}\vartheta(a)}{(x-a)^{\alpha}} + \frac{f.\vartheta(x) I_{x^+}^{\alpha}\vartheta(b) - f.\vartheta(b) I_{b^-}^{\alpha}\vartheta(x)}{(b-x)^{\alpha}}\right) \\ &+ \left(f(b)\vartheta^2(b) - f(a)\vartheta^2(a)\right),\ x\in(a,b). \end{aligned} \tag{2.10}$$

Proof. By definition we have

$$\begin{aligned} {}_\Xi I_{a^+}^{\alpha}\vartheta(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (\Xi(x)-\Xi(t))^{\alpha-1}G'(t)\vartheta(t)dt, \\ {}_\Xi I_{b^-}^{\alpha}\vartheta(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (\Xi(t)-\Xi(x))^{\alpha-1}G'(t)\vartheta(t)dt, \\ {}_\Xi I_{x^-}^{\alpha}\vartheta(a) &= \frac{1}{\Gamma(\alpha)}\int_a^x (\Xi(t)-\Xi(a))^{\alpha-1}G'(t)\vartheta(t)dt, \end{aligned}$$

and

$${}_\Xi I_{x^+}^{\alpha}\vartheta(b) = \frac{1}{\Gamma(\alpha)}\int_x^b (\Xi(b)-\Xi(t))^{\alpha-1}G'(t)\vartheta(t)dt.$$

Setting $\varXi(t) = I(t) = t$ in the above integrals respectively, produces the following integrals:

$$
\begin{aligned}
{}_I I^{\alpha}_{a^+}\vartheta(x) = I^{\alpha}_{a^+}\vartheta(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\vartheta(t)dt,\\
{}_I I^{\alpha}_{b^-}\vartheta(x) = I^{\alpha}_{b^-}\vartheta(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\vartheta(t)dt,\\
{}_I I^{\alpha}_{x^-}\vartheta(a) = I^{\alpha}_{x^-}\vartheta(a) &= \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\vartheta(t)dt,
\end{aligned}
$$

and

$$
{}_I I^{\alpha}_{x^+}\vartheta(b) = I^{\alpha}_{x^+}\vartheta(b) = \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}\vartheta(t)dt.
$$

By putting values of above integrals in (2.8), we get the estimates required in (2.10). □

By fixing $\vartheta(x) = I(x) = x$ in above results one can get the corresponding estimates for RL-fractional integrals. We state them in the form of following corollaries and leave the proofs for readers.

Corollary 2.1. *The following RL-fractional integral inequality holds for quasi $\varXi - (h, I; \alpha)$-convex function:*

$$
\begin{aligned}
&\frac{\Gamma(\sigma)I^{\sigma}_{a^+}f(x)}{(x-a)^{\sigma-1}} + \frac{\Gamma(\delta)I^{\delta}_{b^-}f(x)}{(b-x)^{\delta-1}} \qquad (2.11)\\
\leq\, & af(a)\int_a^x K^{x,t}_{x,a}(\varXi;h)dt + bf(b)\int_x^b K^{t,x}_{b,x}(\varXi;h)dt\\
&+ xf(x)\left[\int_a^x K^{t,a}_{x,a}(\varXi;h)dt + \int_x^b K^{b,t}_{b,x}(\varXi;h)dt\right],\ x \in (a,b).
\end{aligned}
$$

Corollary 2.2. *The following RL-fractional integral inequality holds for quasi $\varXi - (I, I; \alpha)$-convex function:*

$$
\begin{aligned}
&\frac{\Gamma(\sigma)I^{\sigma}_{a^+}f(x)}{(x-a)^{\sigma-1}} + \frac{\Gamma(\delta)I^{\delta}_{b^-}f(x)}{(b-x)^{\delta-1}} \qquad (2.12)\\
\leq\, & \frac{1}{(\varXi(x)-\varXi(a))^{\alpha}}\left[af(a)\int_a^x (\varXi(x)-\varXi(t))^{\alpha}dt + xf(x)\int_a^x (\varXi(t)-\varXi(a))^{\alpha}dt\right]\\
&+ \frac{1}{(\varXi(b)-\varXi(x))^{\alpha}}\left[xf(x)\int_x^b (\varXi(b)-\varXi(t))^{\alpha}dt + bf(b)\int_x^b (\varXi(t)-\varXi(x))^{\alpha}dt\right],\\
&x \in (a,b).
\end{aligned}
$$

Corollary 2.3. *The following RL-fractional integral inequality holds for quasi $I - (I, I; \alpha)$-convex function:*

$$
\frac{\Gamma(\sigma)I^{\sigma}_{a^+}f(x)}{(x-a)^{\sigma-1}} + \frac{\Gamma(\delta)I^{\delta}_{b^-}f(x)}{(b-x)^{\delta-1}} \leq \frac{xf(x)(b-a) + af(a)(x-a) + bf(b)(b-x)}{\alpha+1},\ x \in (a,b). \qquad (2.13)
$$

Theorem 2.5. *Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is quasi $\varXi - (h, \vartheta; \alpha)$-convex, then we have the following inequality for Riemann-Liouville fractional integrals*

$$
\left|\frac{\Gamma(\sigma+1)\,{}_{\vartheta}I^{\sigma}_{a^+}f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma}} + \frac{\Gamma(\delta+1)\,{}_{\vartheta}I^{\delta}_{b^-}f(x)}{(\vartheta(b)-\vartheta(x))^{\delta}} - (f(a)+f(b))\right| \qquad (2.14)
$$

$$\leq |f'.g'(a)| \int_a^x K_{x,a}^{x,t}(\Xi;h)dt + |f'.g'(b)| \int_x^b K_{b,x}^{t,x}(\Xi;h)dt$$
$$+ |f'.\vartheta'(x)| \left(\int_a^x K_{x,a}^{t,a}(\Xi;h)dt + \int_x^b K_{b,x}^{b,t}(\Xi;h)dt \right),$$

where $a, b \in I$, $a < b$ *and* $\sigma, \delta > 0$, $x \in (a, b)$.

Proof. The following identity is useful to proceed the proof

$$\Xi(t) = \frac{\Xi(x) - \Xi(t)}{\Xi(x) - \Xi(a)}\Xi(a) + \frac{\Xi(t) - \Xi(a)}{\Xi(x) - \Xi(a)}\Xi(x). \tag{2.15}$$

$|f'|$ is quasi $\Xi - (h, \vartheta; \alpha)$-convex function for $t \in [a, x]$, we have

$$|f'(t)| \leq K_{x,a}^{x,t}(\Xi;h)|f'.g'(a)| + K_{x,a}^{t,a}(\Xi;h)|f'.\vartheta'(x)|. \tag{2.16}$$

This above inequality gives the forthcoming inequality

$$f'(t) \leq K_{x,a}^{x,t}(\Xi;h)|f'.g'(a)| + K_{x,a}^{t,a}(\Xi;h)|f'.\vartheta'(x)|. \tag{2.17}$$

Also, the following inequality holds

$$(\vartheta(x) - \vartheta(t))^\sigma \leq (\vartheta(x) - \vartheta(a))^\sigma, \ \sigma > 0. \tag{2.18}$$

The following integral inequality can be obtained from (2.17) and (2.18)

$$\begin{aligned}&\int_a^x (\vartheta(x) - \vartheta(t))^\sigma f'(t)dt \\ \leq &(\vartheta(x) - \vartheta(a))^\sigma \left(|f'.g'(a)| \int_a^x K_{x,a}^{x,t}(\Xi;h)dt \right. \\ &\left. + |f'.\vartheta'(x)| \int_a^x K_{x,a}^{t,a}(\Xi;h)dt \right).\end{aligned} \tag{2.19}$$

Also,

$$\begin{aligned}\int_a^x (\vartheta(x) - \vartheta(t))^\sigma f'(t)dt &= f(t)(\vartheta(x) - \vartheta(t))^\sigma \Big|_a^x + \sigma \int_a^x (\vartheta(x) - \vartheta(t))^{\sigma-1}\vartheta'(t)f(t)dt \\ &= -f(a)(\vartheta(x) - \vartheta(a))^\sigma + \Gamma(\sigma + 1)_\vartheta I_{a^+}^\sigma f(x).\end{aligned}$$

Therefore (2.19) takes the following form

$$\begin{aligned}&\Gamma(\sigma + 1)_\vartheta I_{a^+}^\sigma \vartheta(x) - \Xi(a)(\vartheta(x) - \vartheta(a))^\sigma \\ \leq &(\vartheta(x) - \vartheta(a))^\sigma \left(|f'.g'(a)| \int_a^x K_{x,a}^{x,t}(\Xi;h)dt + |f'.\vartheta'(x)| \int_a^x K_{x,a}^{t,a}(\Xi;h)dt \right).\end{aligned} \tag{2.20}$$

The inequality (2.16) also gives the forthcoming inequality

$$-f'(t) \leq K_{x,a}^{x,t}(\Xi;h)|f'.g'(a)| + K_{x,a}^{t,a}(\Xi;h)|f'.\vartheta'(x)|. \tag{2.21}$$

By following similar steps as we considered for (2.17) to obtain (2.20), one can get from (2.17) the following inequality:

$$f(a)(\vartheta(x) - \vartheta(a))^\sigma - \Gamma(\sigma + 1)_\vartheta I_{a^+}^\sigma f(x) \tag{2.22}$$

$$\leq (\vartheta(x)-\vartheta(a))^{\sigma}\left(|f'.g'(a)|\int_a^x K_{x,a}^{x,t}(\Xi;h)dt+|f'.\vartheta'(x)|\int_a^x K_{x,a}^{t,a}(\Xi;h)dt\right).$$

Inequalities (2.20) and (2.22) are equivalent to the following modulus inequality:

$$\begin{aligned}&\left|\frac{\Gamma(\sigma+1)_{\vartheta}I_{a^+}^{\sigma}f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma}}-f(a)\right| \\ \leq&|f'.g'(a)|\int_a^x K_{x,a}^{x,t}(\Xi;h)dt+|f'.\vartheta'(x)|\int_a^x K_{x,a}^{t,a}(\Xi;h)dt.\end{aligned} \tag{2.23}$$

The following another identity holds

$$F(t)=\frac{\Xi(t)-\Xi(x)}{\Xi(b)-\Xi(x)}F(b)+\frac{\Xi(b)-\Xi(t)}{\Xi(b)-\Xi(x)}F(x),\ t\in[x,b]. \tag{2.24}$$

$|f'|$ is quasi $\Xi-(h,\vartheta;\alpha)$-convex function for $t\in[x,b]$, we have

$$|f'(t)|\leq K_{b,x}^{t,x}(\Xi;h)|f'.g'(b)|+K_{b,x}^{b,t}(\Xi;h)|f'.\vartheta'(x)|. \tag{2.25}$$

Also, for $t\in[x,b]$ and $\delta>0$ we have

$$(\vartheta(t)-\vartheta(x))^{\delta}\leq(\vartheta(b)-\vartheta(x))^{\delta},\ \delta>0. \tag{2.26}$$

With similar way as we carried the inequalities (2.16) and (2.18), one can obtain from (2.25) and (2.26), the following inequality can be constructed:

$$\begin{aligned}&\left|\frac{\Gamma(\delta+1)_{\vartheta}I_{b^-}^{\delta}f(x)}{(\vartheta(b)-\vartheta(x))^{\delta}}-f(b)\right| \\ \leq&|f'.g'(b)|\int_x^b K_{b,x}^{t,x}(\Xi;h)dt+|f'.\vartheta'(x)|\int_x^b K_{b,x}^{b,t}(\Xi;h)dt.\end{aligned} \tag{2.27}$$

Using the triangular inequality for adding inequalities (2.23) and (2.27), one can obtain the required inequality. □

Some specific settings in above theorem produce interesting implications. For instance by fixing h as identity function the following result holds.

Theorem 2.6. *By keeping statement of Theorem 2.5 as it, quasi $\Xi-(I,\vartheta;\alpha)$-convex function satisfy the forthcoming result:*

$$\begin{aligned}&\left|\frac{\Gamma(\sigma+1)_{\vartheta}I_{a^+}^{\sigma}f(x)}{(\vartheta(x)-\vartheta(a))^{\sigma}}+\frac{\Gamma(\delta+1)_{\vartheta}I_{b^-}^{\delta}f(x)}{(\vartheta(b)-\vartheta(x))^{\delta}}-(f(a)+f(b))\right| \\ \leq&|f'.g'(a)|\int_a^x K_{x,a}^{x,t}(\Xi;I)dt+|f'.g'(b)|\int_x^b K_{b,x}^{t,x}(\Xi;I)dt \\ &+|f'.\vartheta'(x)|\left(\int_a^x K_{x,a}^{t,a}(\Xi;I)dt+\int_x^b K_{b,x}^{b,t}(\Xi;I)\right)dt.\end{aligned} \tag{2.28}$$

Furthermore, by setting $\Xi(x)=I(x)=x$ in (2.14) the forthcoming inequality holds.

Theorem 2.7. *By keeping statement of Theorem 2.5 as it, quasi $I-(I,\vartheta;\alpha)$-convex function satisfy the forthcoming result:*

$$\begin{aligned}&\left|\frac{\Gamma(\sigma+1)_\vartheta I^\sigma_{a^+}f(x)}{(\vartheta(x)-\vartheta(a))^\sigma}+\frac{\Gamma(\delta+1)_\vartheta I^\delta_{b^-}f(x)}{(\vartheta(b)-\vartheta(x))^\delta}-(f(a)+f(b))\right| \\ \leq&\frac{|f'.g'(a)|(x-a)+|f'.g'(b)|(b-x)+|f'.\vartheta'(x)|(b-a)}{\alpha+1}.\end{aligned} \tag{2.29}$$

By fixing $\Xi(x)=I(x)=x$ in above results one can get the corresponding estimates for RL-fractional integrals. We state them in the form of following corollaries and leave the proofs for readers.

Corollary 2.4. *The following RL-fractional integral inequality holds for quasi $\Xi-(h,I;\alpha)$-convex function:*

$$\begin{aligned}&\left|\frac{\Gamma(\sigma+1)I^\sigma_{a^+}f(x)}{(x-a)^\sigma}+\frac{\Gamma(\delta+1)I^\delta_{b^-}f(x)}{(b-x)^\delta}-(f(a)+f(b))\right| \\ \leq&|f'(a)|\int_a^x K^{x,t}_{x,a}(\Xi;h)dt+|f'(b)|\int_x^b K^{t,x}_{b,x}(\Xi;h)dt \\ &+|f'(x)|\left(\int_a^x K^{t,a}_{x,a}(\Xi;h)dt+\int_x^b K^{b,t}_{b,x}(\Xi;h)dt\right).\end{aligned} \tag{2.30}$$

Corollary 2.5. *The following RL-fractional integral inequality holds for quasi $\Xi-(I,I;\alpha)$-convex function:*

$$\begin{aligned}&\left|\frac{\Gamma(\sigma+1)I^\sigma_{a^+}f(x)}{(x-a)^\sigma}+\frac{\Gamma(\delta+1)I^\delta_{b^-}f(x)}{(b-x)^\delta}-(f(a)+f(b))\right| \\ \leq&|f'(a)|\int_a^x K^{x,t}_{x,a}(I;h)dt+|f'(b)|\int_x^b K^{t,x}_{b,x}(I;h)dt \\ &+|f'(x)|\left(\int_a^x K^{t,a}_{x,a}(I;h)dt+\int_x^b K^{b,t}_{b,x}(I;h)\right)dt.\end{aligned} \tag{2.31}$$

Corollary 2.6. *The following RL-fractional integral inequality holds for quasi $I-(I,I;\alpha)$-convex function:*

$$\begin{aligned}&\left|\frac{\Gamma(\sigma+1)I^\sigma_{a^+}f(x)}{(x-a)^\sigma}+\frac{\Gamma(\delta+1)I^\delta_{b^-}f(x)}{(b-x)^\delta}-(f(a)+f(b))\right| \\ \leq&\frac{|f'(a)|(x-a)+|f'(b)|(b-x)+|f'(x)|(b-a)}{\alpha+1},\ x\in(a,b).\end{aligned} \tag{2.32}$$

Definition 2.1. A function $f:I\to\mathbb{R}, a,b\in I$ is said to be quasi Ξ-symmetric about $\frac{a+b}{2}$ if $f\left(\Xi^{-1}(\Xi(a)+\Xi(b)-\Xi(x))\right)=f(x)$ for all $x\in I$.

Lemma 2.1. *Let $f:[a,b]\longrightarrow\mathbb{R}$, be quasi $\Xi-(h,\vartheta;\alpha)$-convex. If f, ϑ are quasi Ξ-symmetric about $\frac{a+b}{2}$, and ϑ is increasing, then the following inequality holds*

$$\frac{f\circ\Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right)}{\vartheta(b)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)}\leq f(x),\ x\in[a,b]. \tag{2.33}$$

Proof. Ξ is strictly increasing on $[a,b]$, there exists $u \in [a,b]$ such that

$$\Xi(u) = \frac{\Xi(x) - \Xi(a)}{\Xi(b) - \Xi(a)}\Xi(b) + \frac{\Xi(b) - \Xi(x)}{\Xi(b) - \Xi(a)}\Xi(a).$$

Similarly, we have $v \in [a,b]$ such that

$$\Xi(v) = \frac{\Xi(x) - \Xi(a)}{\Xi(b) - \Xi(a)}\Xi(a) + \frac{\Xi(b) - \Xi(x)}{\Xi(b) - \Xi(a)}\Xi(b).$$

One can have

$$f \circ \Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right) = f \circ \Xi^{-1}\left(\frac{1}{2}\Xi(u) + \frac{1}{2}\Xi(v)\right). \tag{2.34}$$

f is quasi $\Xi - (h, \vartheta; \alpha)$-convex

$$\begin{aligned} & f \circ \Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right) \\ \leq & h\left(\frac{1}{2^{\alpha}}\right) f.\vartheta(u) + h\left(1 - \frac{1}{2^{\alpha}}\right) f.\vartheta(v) \\ = & h\left(\frac{1}{2^{\alpha}}\right) f.\vartheta(x) + h\left(1 - \frac{1}{2^{\alpha}}\right) f.\vartheta\left(\Xi^{-1}(\Xi(a)+\Xi(b)-\Xi(x))\right). \end{aligned} \tag{2.35}$$

Using that f, ϑ are quasi Ξ-symmetric about $\frac{a+b}{2}$ and ϑ is increasing, the required inequality (2.33) can be obtained. □

Theorem 2.8. *Let $f : [a,b] \longrightarrow \mathbb{R}$, be quasi $\Xi - (h, \vartheta; \alpha)$-convex. If f, ϑ are quasi Ξ-symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$\begin{aligned} & \frac{(\vartheta(b)-\vartheta(a))(\sigma+\delta+2)}{2(\sigma+1)(\delta+1)\vartheta(b)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)} f \circ \Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right) \\ \leq & \frac{\Gamma(\sigma+1)\,{}_{\vartheta}I_{b^-}^{\sigma+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\sigma}} + \frac{\Gamma(\delta+1)\,{}_{\vartheta}I_{b^-}^{\delta+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\delta}} \\ \leq & f.\vartheta(b)\int_a^b K_{b,a}^{x,a}(\Xi;h)\vartheta'(x)dx + f.\vartheta(a)\int_a^b K_{b,a}^{b,x}(\Xi;h)\vartheta'(x)dx. \end{aligned} \tag{2.36}$$

Proof. For $x \in [a,b]$, $\delta > 0$, the following inequality holds

$$(\vartheta(x)-\vartheta(a))^{\sigma}\vartheta'(x) \leq (\vartheta(b)-\vartheta(a))^{\sigma}\vartheta'(x). \tag{2.37}$$

f is quasi $\Xi - (h, \vartheta; \alpha)$-convex function for $x \in [a,x]$, we have

$$f(x) \leq K_{b,a}^{x,a}(\Xi;h)f.\vartheta(b) + K_{b,a}^{b,x}(\Xi;h)f.\vartheta(a). \tag{2.38}$$

The following integral inequality can be obtained from aforementioned inequalities

$$\begin{aligned} & \int_a^b (\vartheta(x)-\vartheta(a))^{\sigma}\vartheta'(x)f(x)dx \\ \leq & (\vartheta(b)-\vartheta(a))^{\sigma} f.\vartheta(b)\int_a^b K_{b,a}^{x,a}(\Xi;h)\vartheta'(x)dx \end{aligned}$$

$$+ (\vartheta(b)-\vartheta(a))^{\sigma} f.\vartheta(a)\int_a^b K_{b,a}^{b,x}(\Xi;h)\vartheta'(x)dx.$$

From which we have

$$\frac{\Gamma(\sigma+1)_{\vartheta}I_{b^-}^{\sigma+1}f(a)}{(\vartheta(b)-\vartheta(a))^{\sigma}} \leq f.\vartheta(b)\int_a^b K_{b,a}^{x,a}(\Xi;h)\vartheta'(x)dx + f.\vartheta(a)\int_a^b K_{b,a}^{b,x}(\Xi;h)\vartheta'(x)dx. \tag{2.39}$$

Also, we have

$$(\vartheta(b)-\vartheta(x))^{\delta}\vartheta'(x) \leq (\vartheta(b)-\vartheta(a))^{\delta}\vartheta'(x),\ \delta>0. \tag{2.40}$$

From (2.38) and (2.40), one can get the forthcoming inequality

$$\begin{aligned}
&\int_a^b (\vartheta(b)-\vartheta(x))^{\delta}\vartheta'(x)f(x)dx\\
\leq&(\vartheta(b)-\vartheta(a))^{\delta} f.\vartheta(b)\int_a^b K_{b,a}^{x,a}(\Xi;h)\vartheta'(x)dx\\
&+ (\vartheta(b)-\vartheta(a))^{\delta} f.\vartheta(a)\int_a^b K_{b,a}^{b,x}(\Xi;h)\vartheta'(x)dx.
\end{aligned}$$

From which we have

$$\frac{\Gamma(\delta+1)_{\vartheta}I_{a^+}^{\delta+1}f(b)}{(\vartheta(b)-\vartheta(a))^{\delta}} \leq f.\vartheta(b)\int_a^b K_{b,a}^{x,a}(\Xi;h)\vartheta'(x)dx + f.\vartheta(a)\int_a^b K_{b,a}^{b,x}(\Xi;h)\vartheta'(x)dx. \tag{2.41}$$

Summing inequalities (2.39) and (2.41), we get

$$\begin{aligned}
&\frac{\Gamma(\sigma+1)_{\vartheta}I_{b^-}^{\sigma+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\sigma}} + \frac{\Gamma(\delta+1)_{\vartheta}I_{a^+}^{\delta+1}f(b)}{2(\vartheta(b)-\vartheta(a))^{\delta}}\\
\leq& f.\vartheta(b)\int_a^b K_{b,a}^{x,a}(\Xi;h)\vartheta'(x)dx\\
&+ f.\vartheta(a)\int_a^b K_{b,a}^{b,x}(\Xi;h)\vartheta'(x)dx.
\end{aligned} \tag{2.42}$$

Using Lemma 2.1 and multiplying (2.33) with $(\vartheta(x)-\vartheta(a))^{\sigma}\vartheta'(x)$, then integrating over $[a,b]$, we get

$$\frac{f\circ\Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right)}{\vartheta(b)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)}\int_a^b(\vartheta(x)-\vartheta(a))^{\sigma}\vartheta'(x)dx \leq \int_a^b(\vartheta(x)-\vartheta(a))^{\sigma}\vartheta'(x)f(x)dx, \tag{2.43}$$

$$\frac{(\vartheta(b)-\vartheta(a))}{2\vartheta(b)(\sigma+1)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)}f\circ\Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right) \leq \frac{\Gamma(\sigma+1)_{\vartheta}I_{b^-}^{\sigma+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\sigma}}. \tag{2.44}$$

Now, multiplying (2.33) with $(\vartheta(b)-\vartheta(x))^{\delta}$, then integrating over $[a,b]$ one can get

$$\frac{(\vartheta(b)-\vartheta(a))}{2\vartheta(b)(\delta+1)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)}f\circ\Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right) \leq \frac{\Gamma(\delta+1)_{\vartheta}I_{b^-}^{\delta+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\delta}}. \tag{2.45}$$

Adding (2.44) and (2.45), we get the following inequality:

$$\frac{(\vartheta(b)-\vartheta(a))f\circ\Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right)}{2\vartheta(b)(\sigma+1)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)} + \frac{(\vartheta(b)-\vartheta(a))f\circ\Xi^{-1}\left(\frac{\Xi(a)+\Xi(b)}{2}\right)}{2\vartheta(b)(\delta+1)\left(h\left(2^{-\alpha}\right)+h\left(1-2^{-\alpha}\right)\right)} \tag{2.46}$$

$$\leq \frac{\Gamma(\sigma+1)_{\vartheta}I_{b^-}^{\sigma+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\sigma}} + \frac{\Gamma(\delta+1)_{\vartheta}I_{b^-}^{\delta+1}f(a)}{2(\vartheta(b)-\vartheta(a))^{\delta}}.$$

Inequalities (2.42) and (2.46), constitute the required inequality. □

Remark 2.1. Special cases of above theorem can be considered by the readers like the special cases of Theorem 2.1 and Theorem 2.5.

Concluding remarks

New estimates of generalized Riemann-Liouville (RL) fractional integrals via an increasing function were obtained. A new class of functions containing several convexities was considered in establishing fractional integral inequalities. The classical Hermite-Hadamard inequality was extended to the RL fractional integral case and combined with generalized convexity, broadening the application scope of traditional results. Some special cases were deduced by considering specific functions. The obtained results can provide theoretical support for stability analysis of fractional differential equations and the construction of constraints for optimization problems. The introduction of symmetry conditions (Lemma 2.1) enhances the flexibility of the results, making them applicable to a broader class of functions.

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