

BIFURCATION AND CONTROL OF A PREDATOR-PREY SYSTEM WITH TWO TIME DELAYS

Jinting Lin¹, Changjin Xu^{2,†}, Yingyan Zhao¹, Yicheng Pang¹, Zixin Liu¹
and Jianwei Shen³

Abstract To describe the dynamic behavior within ecosystems, we develop a novel predator-prey model that incorporates two distinct time delays in this research. Firstly, by employing fixed-point theorem, inequality technique, and construct an appropriate function, the trait (e.g., existence, uniqueness, non-negativity) of the solution of this delayed predator-prey model is proved. Secondly, the critical value of time delay of this system's stability is sought and the stability and Hopf bifurcation of this model are investigated. A new delay-independent criterion for the stability and bifurcation of this model is presented. Lastly, three delay feedback controllers are designed to adjust the stability region and the onset of bifurcation phenomena of this model. The parameter conditions of this system to maintain stability under the delay feedback controller and the bifurcation phenomenon are discussed. Furthermore, the correctness of the conclusions derived from this study is verified through simulation experiments.

Keywords Predator-prey model, time delay, Hopf bifurcation, stability, delay feedback controller.

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1. Introduction

Predator-prey model stands as a pivotal category within population dynamics, it delineates the intricate interplay between predators and prey. It has aroused great interest in the disciplines of ecology and biomathematics. Time delay is an ubiquitous phenomenon in the real world, which has an important effect on the dynamical properties of system. In predator-prey model, time delay may be manifested as delay in predator's response time to the prey, delay in prey growth and reproduction, and so on. Both domestically and internationally, numerous scholars have explored the dynamic behavior of predator-prey model with time delay and obtained some valuable results. For example, Zhang and Huang [39] investigated the Hopf bifurcation phenomenon of a rigid guide rail maglev system with time delay feedback, they clarified the feedback gain condition required for the stability of this system without time delay, analyzed

[†]The corresponding author.

¹School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

²Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550025, China

³School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450046, China

Email: 15285013512@163.com(J. Lin), xcj403@126.com(C. Xu), 15185709351@163.com(Y. Zhao), ypanggy@outlook.com(Y. Pang), xinxin905@126.com(Z. Liu), phdshen@126.com(J. Shen)

the stability variation curves under two times delay, distinguished stable and unstable regions and revealed the phenomenon of periodic solutions arising through Hopf bifurcation. Li et al. [16] proposed a novel fractional-order Lengyel-Epstein model incorporating time delay. By analyzing the distribution of the roots of its characteristic equation, the authors established the time-delay dependent criteria for stability and bifurcation. With the utilization of nonlinear time-delay feedback and hybrid controllers, the stability region and bifurcation time of the model can be effectively controlled. Gao and Li [3] investigated the sufficient conditions for system stability and Hopf bifurcation. Under the influence of the fractional-order PD controller, they studied the impact of the proposed controller on system stability and Hopf bifurcation. Li et al. [12] studied the conditions of local stability of positive equilibrium point and the conditions of Hopf bifurcation driven by delay are given. By adjusting the control parameters reasonably, the stability region of the system is enhanced effectively, and the position of Hopf bifurcation periodic solution is changed. Lan et al. [10] discussed the roots of the characteristic equation and the sufficient conditions for the system to exhibit periodic oscillation. The stability region of the system was successfully expanded by incorporating a linear delayed feedback controller into the system, Li et al. [15] established a new fractional order BAM neural network model with five neurons and time delay, the authors explored the characteristics of its solutions and developed a new sufficient criterion for the onset of Hopf bifurcation stability in the fractional-order BAM neural network model. The bifurcation time and stability region of fractional-order delayed BAM neural network model are controlled by a delayed feedback controller. Li et al. [17] built a new fractional delay duopoly game model for insurance market. The research indicates that, in the fractional-order delayed duopoly game model, the delay plays a crucial role in controlling the stability of the system and the time of Hopf bifurcation occurrence. For more detailed researches, one can see [2, 5, 8, 23, 24, 34–37].

In 2014, Sharma and Samanta. [30] proposed the following two prey one predator system:

$$\begin{cases} \frac{du_1}{dt} = ru_1(t) \left(1 - \frac{u_1(t)}{h}\right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = \rho u_2(t) - \delta u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{bu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)} + \gamma u_2(t)u_3(t) - nu_3(t), \end{cases} \quad (1.1)$$

where $u_1(t)$, $u_2(t)$, $u_3(t)$ represent the population densities of the first prey, the second prey, and the predator, respectively. The intrinsic growth rate and carrying capacity of the first prey are represented by r and h . The second prey's intrinsic growth rate and predation rate are denoted by ρ and δ , respectively. γ represents the efficiency at which the second prey consumed by the predator is converted into predator biomass, while n denotes the predator's mortality rate in the absence of prey. Assume that the growth of the second prey is exponential, the population of the second prey is abundant in the absence of predators. Consequently, there is no time required for the predator to search for the second prey, and all of r , h , a , b , c , n , ξ , ρ , β , δ , γ are positive numbers.

Sharma and Samanta [30] discussed the use of Lotka-Volterra function responses to describe predator-prey interactions, and the effect of time delay on model stability. In many cases, considering that the time delay in population development can affect the densities of prey and predators, introducing time delay into predator-prey models becomes crucial. Guided by this consideration, we can develop a more fitting predator-prey model that incorporates delay. Assume that the density of first prey is affected by the self-feedback delay (say τ_1) from u_1 to u_1 ,

the density of predators is affected by the feedback time delay (say τ_2) from u_1 to u_3 . In many cases, the self-feedback delay from u_1 to u_1 and the feedback delay from u_1 to u_3 are different, then we can modify the model (1.1) as follows

$$\begin{cases} \frac{du_1}{dt} = ru_1(t) \left(1 - \frac{u_1(t - \tau_1)}{h} \right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = \rho u_2(t) - \delta u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{bu_1(t - \tau_2)u_3(t)}{a + \xi\beta u_2(t) + u_1(t - \tau_2)} + \gamma u_2(t)u_3(t) - nu_3(t), \end{cases} \quad (1.2)$$

where τ_1, τ_2 are delays. In model (1.2), we divide the time delays into the following five specific scenarios: ($\tau_1 = \tau_2 = 0$; $\tau_1 = 0, \tau_2 > 0$; $\tau_2 = 0, \tau_1 > 0$; $\tau_1 = \tau_2 = \tau$; $\tau_1 = \tau_2 \neq 0$). Through these classifications, we comprehensively investigate the effect of time delays on the stability and Hopf bifurcation of this model.

Numerous studies have confirmed that time delay plays a crucial role in the dynamic characteristic system. In instances, time delay often leads to the loss of system stability, the emergence of bifurcation phenomena, and the occurrence of chaotic behavior, among others [1,4,7,11,13,14,19–22,26–29,31–33,38,40–43,45]. Particularly, delay-induced Hopf bifurcation stands out as a significant dynamic feature. In the field of biology, it effectively uncovers the balance mechanism among biological population densities. Given this, we think that it of great theoretical significance to conduct an in-depth exploration of delay-induced Hopf bifurcation in predator-prey models. Based on the above understanding, we plan to investigate the delay-induced Hopf bifurcation and its control strategies in model (1.2). Just specific, we will focus on exploring the following core issues:

(1) Analyze the existence and non-negative of the solution of system (1.2). **(2)** Investigate the stability and Hopf bifurcation phenomena of system (1.2). **(3)** Design delay feedback controller to adjust the stability region and the onset of bifurcation phenomena of system(1.2) under different delay cases.

The primary points of emphasis in this study are as follows: **(i)** Building upon previous works, a novel delay-independent criterion for the stability and bifurcation of system (1.2) is presented. **(ii)** Investigate the stability and Hopf bifurcation and the effect of two time delays on the population density of predator and prey of system (1.2). **(iii)** By virtue of three delay feedback controllers, the onset of bifurcation phenomena and stability region of system (1.2) are available adjusted.

The basic structure of this paper is as follows. The existence, uniqueness, non-negativity of system (1.2) are demonstrated in Section 2. Sections 3 discusses the stability and bifurcation phenomena of the system (1.2) by dividing the two delays into five cases. Sections 4 focuses on the bifurcation control issue of system (1.2) with $\tau_1 = \tau_2 = \tau$ by virtue of reasonable delay feedback controller. Sections 5 deals with the bifurcation control issue of system (1.2) with $\tau_1 > 0, \tau_2 = 0$ through a practical delay feedback controller. Sections 6 deals with the bifurcation control issue of system (1.2) with $\tau_1 = 0, \tau_2 > 0$ by employing a advisable delay feedback controller. Section 7 conducts computer experiments to verify the validity of the key conclusions obtained. Finally, Section 8 concludes the study with a concise summary.

2. Peculiarity of solution

In this section, we will employ the fixed point theorem, inequality techniques, and construct an appropriate function to investigate the properties of the solutions to system (1.2), including

existence, uniqueness, non-negativity.

Theorem 2.1. *Let $\Upsilon = \{(u_1, u_2, u_3) \in R^3 : \max\{|u_1|, |u_2|, |u_3|\} \leq \mathcal{U}\}$, where $\mathcal{U} > 0$ represents a constant. For every $(u_{10}, u_{20}, u_{30}) \in \Upsilon$, system (1.2) under the initial condition (u_{10}, u_{20}, u_{30}) admits a unique solution $U = (u_1, u_2, u_3) \in \Upsilon$.*

Proof. Define a mapping as follows:

$$f(U) = (f_1(U), f_2(U), f_3(U)), \quad (2.1)$$

where

$$\begin{cases} f_1(U) = ru_1(t) \left(1 - \frac{u_1(t - \tau_1)}{h}\right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)}, \\ f_2(U) = \rho u_2(t) - \delta u_2(t)u_3(t), \\ f_3(U) = \frac{bu_1(t - \tau_2)u_3(t)}{a + \xi\beta u_2(t) + u_1(t - \tau_2)} + \gamma u_2(t)u_3(t) - nu_3(t). \end{cases} \quad (2.2)$$

For every $U, \bar{U} \in \Upsilon$, we can get

$$\begin{aligned} & \|f(U) - f(\bar{U})\| \\ &= \left\| \left[ru_1(t) \left(1 - \frac{u_1(t - \tau_1)}{h}\right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)} \right] \right. \\ &\quad \left. - \left[r\bar{u}_1(t) \left(1 - \frac{\bar{u}_1(t - \tau_1)}{h}\right) - \frac{c\bar{u}_1(t)\bar{u}_3(t)}{a + \xi\beta \bar{u}_2(t) + \bar{u}_1(t)} \right] \right\| \\ &\quad + |\rho u_2(t) - \delta u_2(t)u_3(t) - [\rho \bar{u}_2(t) - \delta \bar{u}_2(t)\bar{u}_3(t)]| \\ &\quad + \left\| \left[\frac{bu_3(t)u_1(t - \tau_2)}{a + \xi\beta u_2(t) + u_1(t - \tau_2)} + \gamma u_2(t)u_3(t) - nu_3(t) \right] \right. \\ &\quad \left. - \left[\frac{b\bar{u}_3(t)\bar{u}_1(t - \tau_2)}{a + \xi\beta \bar{u}_2(t) + \bar{u}_1(t - \tau_2)} + \gamma \bar{u}_2(t)\bar{u}_3(t) - n\bar{u}_3(t) \right] \right\| \\ &\leq r|u_1 - \bar{u}_1| + \frac{r}{h}\mathcal{U}|u_1 - \bar{u}_1| + \frac{r}{h}\mathcal{U}|u_1(t - \tau_1) - \bar{u}_1(t - \tau_1)| \\ &\quad + \frac{a}{c}\mathcal{U}|u_1 - \bar{u}_1| + \frac{c}{a^2}\xi\beta\mathcal{U}^2|u_1 - \bar{u}_1| + \frac{c}{a^2}\xi\beta\mathcal{U}^2|u_2 - \bar{u}_2| \\ &\quad + \frac{c}{a}\mathcal{U}|u_3 - \bar{u}_3| + \frac{c}{a^2}\mathcal{U}^2|u_3 - \bar{u}_3| + \frac{c}{a^2}\xi\beta\mathcal{U}^2|u_3 - \bar{u}_3| \\ &\quad + \rho|u_2 - \bar{u}_2| + \delta\mathcal{U}|u_2 - \bar{u}_2| + \delta\mathcal{U}|u_3 - \bar{u}_3| \\ &\quad + \frac{b}{a}\mathcal{U}|u_1(t - \tau_2) - \bar{u}_1(t - \tau_2)| + \gamma\mathcal{U}|u_2 - \bar{u}_2| \\ &\quad + \frac{b\xi\beta}{a^2}\mathcal{U}^2|u_2 - \bar{u}_2| + \frac{b}{a}\mathcal{U}|u_3 - \bar{u}_3| + \frac{b\xi\beta}{a^2}\mathcal{U}^2|u_3 - \bar{u}_3| \\ &\quad + \frac{b}{a^2}\mathcal{U}^2|u_3 - \bar{u}_3| + \gamma\mathcal{U}|u_3 - \bar{u}_3| + n|u_3 - \bar{u}_3| \\ &\leq r|u_1 - \bar{u}_1| + 2\frac{r}{h}\mathcal{U}|u_1 - \bar{u}_1| + \frac{a}{c}\mathcal{U}|u_1 - \bar{u}_1| \\ &\quad + \frac{c}{a^2}\xi\beta\mathcal{U}^2|u_1 - \bar{u}_1| + \frac{c}{a^2}\xi\beta\mathcal{U}^2|u_2 - \bar{u}_2| \\ &\quad + \frac{c}{a}\mathcal{U}|u_3 - \bar{u}_3| + \frac{c}{a^2}\mathcal{U}^2|u_3 - \bar{u}_3| + \frac{c}{a^2}\xi\beta\mathcal{U}^2|u_3 - \bar{u}_3| \\ &\quad + \rho|u_2 - \bar{u}_2| + \delta\mathcal{U}|u_2 - \bar{u}_2| + \delta\mathcal{U}|u_3 - \bar{u}_3| \\ &\quad + \frac{b}{a}\mathcal{U}|u_1 - \bar{u}_1| + \gamma\mathcal{U}|u_2 - \bar{u}_2| + \frac{b\xi\beta}{a^2}\mathcal{U}^2|u_2 - \bar{u}_2| \end{aligned}$$

$$\begin{aligned}
& + \frac{b}{a} \mathcal{U} |u_3 - \bar{u}_3| + \frac{b\xi\beta}{a^2} \mathcal{U}^2 |u_3 - \bar{u}_3| \\
& + \frac{b}{a^2} \mathcal{U}^2 |u_3 - \bar{u}_3| + \gamma \mathcal{U} |u_3 - \bar{u}_3| + n |u_3 - \bar{u}_3| \\
& \leq \eta_1 |u_1 - \bar{u}_1| + \eta_2 |u_2 - \bar{u}_2| + \eta_3 |u_3 - \bar{u}_3|,
\end{aligned} \tag{2.3}$$

where

$$\begin{cases} \eta_1 = r + \mathcal{U} \left(2\frac{r}{h} + \frac{c}{a} + \frac{b}{a} \right) + \frac{c\xi\beta}{a^2} \mathcal{U}^2, \\ \eta_2 = \rho + \mathcal{U}(\delta + \gamma) + \mathcal{U}^2 \left(\frac{c\xi\beta}{a^2} + \frac{b\xi\beta}{a^2} \right), \\ \eta_3 = n + \mathcal{U} \left(\gamma + \delta + \frac{b}{a} + \frac{c}{a} \right) + \mathcal{U}^2 \left(\frac{b}{a^2} + \frac{c}{a^2} + \frac{b\xi\beta}{a^2} + \frac{c\xi\beta}{a^2} \right). \end{cases} \tag{2.4}$$

Let

$$\eta = \max \{ \eta_1, \eta_2, \eta_3 \}. \tag{2.5}$$

As a result of (2.3), it can be deduced that

$$\|f(U) - f(\bar{U})\| \leq \eta \|U - \bar{U}\|. \tag{2.6}$$

Therefore, the function $f(U)$ satisfies the Lipschitz condition with respect U (see [11]). By applying the fixed point theorem, it can be readily inferred that Theorem 2.1 holds valid. $\square$

Theorem 2.2. *Every solution to system (1.2) that originates from R_+^3 is non-negative.*

Proof. Based on the first equation of system (1.2), we can obtain

$$\frac{du_1}{u_1} = \left[r \left(1 - \frac{u_1(t - \tau_1)}{h} \right) - \frac{cu_3(t)}{a + \xi\beta u_2(t) + u_1(t)} \right] dt, \tag{2.7}$$

which leads to

$$\int_0^t \frac{du_1}{u_1} = \int_0^t \left[r \left(1 - \frac{u_1(s - \tau_1)}{h} \right) - \frac{cu_3(s)}{a + \xi\beta u_2(s) + u_1(s)} \right] ds, \tag{2.8}$$

then one obtains

$$\frac{u_1(t)}{u_1(0)} = \exp \left\{ \int_0^t \left[r \left(1 - \frac{u_1(s - \tau_1)}{h} \right) - \frac{cu_3(s)}{a + \xi\beta u_2(s) + u_1(s)} \right] ds \right\}. \tag{2.9}$$

Thus

$$u_1(t) = u_1(0) \exp \left\{ \int_0^t \left[r \left(1 - \frac{u_1(s - \tau_1)}{h} \right) - \frac{cu_3(s)}{a + \xi\beta u_2(s) + u_1(s)} \right] ds \right\} > 0. \tag{2.10}$$

Equally, we know

$$u_2(t) = u_2(0) \exp \left\{ \int_0^t [\rho - \delta u_3(s)] ds \right\} > 0, \tag{2.11}$$

$$u_3(t) = u_3(0) \exp \left\{ \int_0^t \left[\frac{bu_1(s - \tau_2)}{a + \xi\beta u_2(s) + u_1(s - \tau_2)} + \gamma u_2(s) + n \right] ds \right\} > 0. \tag{2.12}$$

Consequently, Theorem 2.2 is valid. $\square$

Remark 2.1. In a biological sense, non-negativity of solutions of system (1.2) implies that the population densities of the first prey, the second prey, and the predator are reasonable.

3. Exploration of bifurcation of model (1.2)

In this segment, we will delve into the bifurcation and stability issues of model (1.2). We postulate that $E(u_{1\star}, u_{2\star}, u_{3\star})$ represents the equilibrium point of model (1.2), then $u_{1\star}, u_{2\star}, u_{3\star}$ comply with the given condition:

$$\begin{cases} ru_{1\star} \left(1 - \frac{u_{1\star}}{h}\right) - \frac{cu_{1\star}u_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} = 0, \\ \rho u_{2\star} - \delta u_{2\star}u_{3\star} = 0, \\ \frac{bu_{1\star}u_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \gamma u_{2\star}u_{3\star} - nu_{2\star} = 0. \end{cases} \quad (3.1)$$

Let

$$\begin{cases} \bar{u}_1(t) = u_1(t) - u_{1\star}, \\ \bar{u}_2(t) = u_2(t) - u_{2\star}, \\ \bar{u}_3(t) = u_3(t) - u_{3\star}. \end{cases} \quad (3.2)$$

By substituting system (3.2) into system (1.2), we derive the linear system of model (1.2) at the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$:

$$\begin{cases} \frac{d\bar{u}_1}{dt} = k_1\bar{u}_1 - k_2\bar{u}_1(t - \tau_1) + k_3\bar{u}_2 - k_4\bar{u}_3, \\ \frac{d\bar{u}_2}{dt} = k_5\bar{u}_2 - k_6\bar{u}_3, \\ \frac{d\bar{u}_3}{dt} = k_7\bar{u}_1(t - \tau_2) + k_8\bar{u}_2 + k_9\bar{u}_3, \end{cases} \quad (3.3)$$

where

$$\begin{cases} k_1 = r - \frac{ru_{1\star}}{h} - \frac{cu_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \frac{cu_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ k_2 = \frac{ru_{1\star}}{h}, \\ k_3 = \frac{c\xi\beta u_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ k_4 = \frac{cu_{1\star}}{a + \xi\beta u_{2\star} + u_{1\star}}, \\ k_5 = \rho - \delta u_{3\star}, \\ k_6 = \delta u_{2\star}, \\ k_7 = \frac{bu_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} - \frac{bu_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ k_8 = \gamma u_{3\star} - \frac{b\xi\beta u_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ k_9 = \frac{bu_{1\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \gamma u_{2\star} - n. \end{cases} \quad (3.4)$$

The expression for the characteristic equation pertaining to system (3.3) is given below:

$$\det \begin{bmatrix} \lambda - k_1 + k_2 e^{-\lambda \tau_1} & -k_3 & k_4 \\ 0 & \lambda - k_5 & k_6 \\ -k_7 e^{-\lambda \tau_2} & -k_8 & \lambda k_9 \end{bmatrix} = 0, \quad (3.5)$$

which leads to

$$\lambda^3 + m_1 \lambda^2 + m_2 \lambda + m_3 + (m_4 \lambda^2 + m_5 \lambda + m_6) e^{-\lambda \tau_1} + (m_7 \lambda + m_8) e^{-\lambda \tau_2} = 0, \quad (3.6)$$

where

$$\begin{cases} m_1 = -k_1 - k_5 - k_9, \\ m_2 = k_1 k_5 + k_5 k_9 + k_1 k_9 + k_6 k_8, \\ m_3 = k_1 k_6 k_8 - k_1 k_5 k_9, \\ m_4 = k_2, \\ m_5 = -k_2 k_9 - k_2 k_5, \\ m_6 = k_2 k_5 k_9 - k_2 k_6 k_8, \\ m_7 = k_4 k_7, \\ m_8 = k_3 k_6 k_7 - k_4 k_5 k_7. \end{cases} \quad (3.7)$$

Case 1. $\tau_1 = \tau_2 = 0$. Then, Eq. (3.6) becomes

$$\lambda^3 + (m_1 + m_4) \lambda^2 + (m_2 + m_5 + m_7) \lambda + m_3 + m_6 + m_8 = 0. \quad (3.8)$$

Therefore, the necessary and sufficient condition for all roots of Eq. (3.6) to have negative real parts

$$(\mathcal{O}_1) \begin{cases} A_1 = m_1 + m_4 > 0, \\ A_2 = \det \begin{bmatrix} m_1 + m_4 & 1 \\ m_3 + m_6 + m_8 & m_2 + m_5 + m_7 \end{bmatrix} > 0, \\ A_3 = (m_3 + m_6 + m_8) A_2 > 0. \end{cases} \quad (3.9)$$

is fulfilled, then all three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq. (3.8) exhibit negative real parts. Thus the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of system (1.2) with $\tau_1 = \tau_2 = 0$ is locally asymptotically stable.

Case 2. $\tau_1 > 0, \tau_2 = 0$. In this manner, Eq. (3.6) can be simplified to

$$\lambda^3 + m_1 \lambda^2 + (m_2 + m_7) \lambda + m_3 + m_8 + (m_4 \lambda^2 + m_5 \lambda + m_6) e^{-\lambda \tau_1} = 0. \quad (3.10)$$

Hypothesize that $\lambda = i\epsilon_1$ ($\epsilon_1 > 0$) is the root of Eq. (3.10), then Eq. (3.10) becomes

$$(i\epsilon_1)^3 + m_1 (i\epsilon_1)^2 + (m_2 + m_7) i\epsilon_1 + m_3 + m_8 + [m_4 (i\epsilon_1)^2 + m_5 i\epsilon_1 + m_6] e^{-i\epsilon_1 \tau_1} = 0, \quad (3.11)$$

which leads to

$$\begin{cases} d_1(\epsilon_1) \sin \epsilon_1 \tau_1 - d_2(\epsilon_1) \cos \epsilon_1 \tau_1 = d_3(\epsilon_1), \\ d_1(\epsilon_1) \cos \epsilon_1 \tau_1 + d_2(\epsilon_1) \sin \epsilon_1 \tau_1 = d_4(\epsilon_1), \end{cases} \quad (3.12)$$

where

$$\begin{cases} d_1 = m_5 \epsilon_1, \\ d_2 = m_4 \epsilon_1^2 - m_6, \\ d_3 = m_1 \epsilon_1^2 - m_3 - m_8, \\ d_4 = \epsilon_1^3 - (m_2 + m_7) \epsilon_1. \end{cases} \quad (3.13)$$

In consideration of $d_1^2 + d_2^2 = d_3^2 + d_4^2$, we gain

$$\epsilon_1^6 + E_1 \epsilon_1^4 + E_2 \epsilon_1^2 + E_3 = 0, \quad (3.14)$$

where

$$\begin{cases} E_1 = m_1 - 2(m_2 + m_7) - m_4^2, \\ E_2 = 2m_4 m_6 - 2(m_1 m_3 + m_1 m_8) - m_5^2, \\ E_3 = (m_3 + m_8)^2 - m_6^2. \end{cases} \quad (3.15)$$

Let

$$\Phi_1(\epsilon_1) = \epsilon_1^6 + E_1 \epsilon_1^4 + E_2 \epsilon_1^2 + E_3. \quad (3.16)$$

Suppose that

$$(\mathcal{O}_2) \quad E_3 < 0$$

holds true, it follows that $\Phi_1(0) < 0$ and $\lim_{\epsilon_1 \rightarrow \infty} \Phi_1(\epsilon_1) = +\infty > 0$, Eq. (3.14) possesses at least one positive real root. Hence, Eq. (3.10) admits at least one pair of purely imaginary roots. Without loss of generality, we can assume that Eq. (3.14) has six positive real roots (say $\epsilon_1(j), j = 1, 2, 3, \dots, 6$). Based on Eq. (3.12), we know

$$\tau_{1(j)}^{(k)} = \frac{1}{\epsilon_{1(j)}} \left[\arccos \left(\frac{d_{1(j)} d_{4(j)} - d_{2(j)} d_{3(j)}}{d_{1(j)}^2 + d_{2(j)}^2} \right) + 2k\pi \right], \quad (3.17)$$

where

$$\begin{cases} d_{1(j)} = m_5 \epsilon_{1(j)}, \\ d_{2(j)} = m_4 \epsilon_{1(j)}^2 - m_6, \\ d_{3(j)} = m_1 \epsilon_{1(j)}^2 - m_3 - m_8, \\ d_{4(j)} = \epsilon_{1(j)}^3 - (m_2 + m_7) \epsilon_{1(j)}, \end{cases} \quad (3.18)$$

and $j = 1, 2, 3, \dots, 6; k = 0, 1, 2, \dots$. Assume that $\tau_1^* = \min_{\{j=1,2,\dots,6;k=0,1,2,\dots\}} \{\tau_{1(j)}^{(k)}\}$ and given that when $\tau_1 = \tau_1^*$, Eq. (3.10) possesses a pair of imaginary roots $\pm i\epsilon_{1(0)}$.

Next we put forward the following assumption:

$$(\mathcal{O}_3) \quad F_{1R} F_{2R} + F_{1I} F_{2I} > 0,$$

where

$$\begin{cases} F_{1R} = -3\epsilon_{1(0)}^2 + m_2 + m_7 + 2m_4\epsilon_{1(0)} \sin \epsilon_{1(0)}\tau_1^*, \\ F_{1I} = 2m_1\epsilon_{1(0)} + 2m_4\epsilon_{1(0)} \cos \epsilon_{1(0)}\tau_1^* - m_5 \sin \epsilon_{1(0)}\tau_1^*, \\ F_{2R} = m_6\epsilon_{1(0)} \sin \epsilon_{1(0)}\tau_1^* - m_5\epsilon_{1(0)}^2 \cos \epsilon_{1(0)}\tau_1^*, \\ F_{2I} = (m_6\epsilon_{1(0)} - m_4\epsilon_{1(0)}^3) \cos \epsilon_{1(0)}\tau_1^* + (m_4\epsilon_{1(0)}^3 + m_5\epsilon_{1(0)}^2) \sin \epsilon_{1(0)}\tau_1^*. \end{cases} \quad (3.19)$$

Lemma 3.1. *Given that $\lambda(\tau_1) = \sigma_1(\tau_1) + i\sigma_2(\tau_1)$ is the root of Eq. (3.10) at $\tau_1 = \tau_1^*$ and satisfies $\sigma_1(\tau_1^*) = 0$, $\sigma_2(\tau_1^*) = \epsilon_{1(0)}$, then $\text{Re} \left(\frac{d\lambda}{d\tau_1} \right) \Big|_{\tau_1=\tau_1^*, \epsilon_1=\epsilon_{1(0)}} > 0$.*

Proof. By utilizing Eq. (3.10), we can derive the following result:

$$\begin{aligned} & [(3\lambda^2 + 2m_1\lambda + m_2 + m_7)] \frac{d\lambda}{d\tau_1} + (2m_4\lambda + m_5)e^{-\lambda\tau_1} \frac{d\lambda}{d\tau_1} \\ & - (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau_1} \left(\lambda + \frac{d\lambda}{d\tau_1} \tau_1 \right) = 0. \end{aligned} \quad (3.20)$$

It signifies that

$$\left(\frac{d\lambda}{d\tau_1} \right)^{-1} = \frac{F_1(\lambda)}{F_2(\lambda)} - \frac{\tau_1}{\lambda}, \quad (3.21)$$

where

$$\begin{cases} F_1(\lambda) = 3\lambda^2 + 2m_1\lambda + (m_2 + m_7)(2m_4\lambda + m_5)e^{-\lambda\tau_1}, \\ F_2(\lambda) = (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau_1}\lambda. \end{cases} \quad (3.22)$$

Thus

$$\text{Re} \left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau_1=\tau_1^*, \epsilon_1=\epsilon_{1(0)}} = \text{Re} \left[\frac{F_1(\lambda)}{F_2(\lambda)} \right]_{\tau_1=\tau_1^*, \epsilon_1=\epsilon_{1(0)}} = \frac{F_{1R}F_{2R} + F_{1I}F_{2I}}{F_{2R}^2 + F_{2I}^2}. \quad (3.23)$$

According to the assumption $(\mathcal{O}_3)$, we derive

$$\text{Re} \left[\left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right]_{\tau_1=\tau_1^*, \epsilon_1=\epsilon_{1(0)}} > 0. \quad (3.24)$$

This thereby proves the conclusion. $\square$

Based on the preceding discussion, the subsequent outcome can be readily inferred.

Theorem 3.1. *Assume that $(\mathcal{O}_1)$ – $(\mathcal{O}_3)$ are valid, then the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of model (1.2) maintains locally asymptotically stable state when $\tau_1 \in [0, \tau_1^*)$. Additionally, model (1.2) arises a cluster of Hopf bifurcations surrounding the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ when $\tau_1 = \tau_1^*$.*

Case 3. $\tau_1 = 0, \tau_2 > 0$. In this situation, Eq. (3.6) transforms into

$$\lambda^3 + (m_1 + m_4)\lambda^2 + (m_2 + m_5)\lambda + m_3 + m_6 + (m_7\lambda + m_8)e^{-\lambda\tau_2} = 0. \quad (3.25)$$

Assume that $\lambda = i\epsilon_2$ is the root of Eq. (3.25), subsequently, Eq. (3.25) becomes

$$(i\epsilon_2)^3 + (m_1 + m_4)(i\epsilon_2)^2 + (m_2 + m_5)i\epsilon_2 + m_3 + m_6 + [m_7(i\epsilon_2) + m_8]e^{-i\epsilon_2\tau_2} = 0, \quad (3.26)$$

which leads to

$$\begin{cases} g_1(\epsilon_2) \sin \epsilon_2 \tau_2 + g_2(\epsilon_2) \cos \epsilon_2 \tau_2 = g_3(\epsilon_2), \\ g_1(\epsilon_2) \cos \epsilon_2 \tau_2 - g_2(\epsilon_2) \sin \epsilon_2 \tau_2 = g_4(\epsilon_2), \end{cases} \quad (3.27)$$

where

$$\begin{cases} g_1 = m_7 \epsilon_2, \\ g_2 = m_8, \\ g_3 = (m_1 + m_4) \epsilon_2^2 - m_3 - m_6, \\ g_4 = \epsilon_2^3 - (m_2 + m_5) \epsilon_2. \end{cases} \quad (3.28)$$

In view of $g_1^2 + g_2^2 = g_3^2 + g_4^2$, we gain

$$\epsilon_2^6 + G_1 \epsilon_2^4 + G_2 \epsilon_2^2 + G_3 = 0, \quad (3.29)$$

where

$$\begin{cases} G_1 = (m_1 + m_2)^2 - 2(m_2 + m_5), \\ G_2 = (m_2 + m_5)^2 - m_7^2 - 2(m_1 + m_4)(m_3 + m_6), \\ G_3 = (m_3 + m_6)^2 - m_8^2. \end{cases} \quad (3.30)$$

Let

$$\Phi_2(\epsilon_2) = \epsilon_2^6 + G_1 \epsilon_2^4 + G_2 \epsilon_2^2 + G_3. \quad (3.31)$$

Assume that condition

$$(\mathcal{O}_4) \quad G_3 < 0$$

holds, it follows that $\Phi_2(0) < 0$ and $\lim_{\epsilon_2 \rightarrow \infty} \Phi_2(\epsilon_2) = +\infty > 0$. Consequently, Eq. (3.29) possesses at least one positive real root. Therefore, Eq. (3.25) has at least one pair of purely imaginary roots. Without loss of generality, we can suppose that Eq. (3.29) has six positive real roots (say $\epsilon_{2(j)}, j = 1, 2, \dots, 6$). Relying on (3.27), we know

$$\tau_{2(j)}^{(k)} = \frac{1}{\epsilon_{2(j)}} \left[\arccos \left(\frac{g_1(j)g_4(j) + g_2(j)g_3(j)}{g_1(j)^2 + g_2(j)^2} \right) + 2k\pi \right], \quad (3.32)$$

where

$$\begin{cases} g_1(j) = m_7 \epsilon_{2(j)}, \\ g_2(j) = m_8, \\ g_3(j) = (m_1 + m_4) \epsilon_{2(j)}^2 - m_3 - m_6, \\ g_4(j) = \epsilon_{2(j)}^3 - (m_2 + m_5) \epsilon_{2(j)}, \end{cases} \quad (3.33)$$

and $j = 1, 2, \dots, 6; k = 0, 1, 2, \dots$. Assume $\tau_2^* = \min_{\{j=1,2,\dots,6;k=0,1,2,\dots\}} \{\tau_{2(j)}^{(k)}\}$ and suppose that when $\tau_2 = \tau_2^*$, Eq. (3.25) has a pair of imaginary roots $\pm i\epsilon_{2(0)}$.

Moving forward, we introduce the following hypothesis:

$$(\mathcal{O}_5) \quad H_{1R}H_{2R} + H_{1I}H_{2I} > 0,$$

where

$$\begin{cases} H_{1R} = -3\epsilon_{2(0)}^2 + m_2 + m_5 + m_7 \cos \epsilon_{2(0)} \tau_2^*, \\ H_{1I} = 2(m_1 + m_4)\epsilon_{2(0)} - m_7 \sin \epsilon_{2(0)} \tau_2^*, \\ H_{2R} = m_8 \epsilon_{2(0)} \sin \epsilon_{2(0)} \tau_2^* - m_7 \epsilon_{2(0)}^2 \cos \epsilon_{2(0)} \tau_2^*, \\ H_{2I} = m_7 \epsilon_{2(0)} \sin \epsilon_{2(0)} \tau_2^* + m_8 \epsilon_{2(0)} \cos \epsilon_{2(0)} \tau_2^*. \end{cases} \quad (3.34)$$

Lemma 3.2. Assume that $\lambda(\tau_2) = \bar{\sigma}_1(\tau_2) + i\bar{\sigma}_2(\tau_2)$ is the root of Eq. (3.25) at $\tau_2 = \tau_2^*$ and meet $\bar{\sigma}_1(\tau_2^*) = 0$, $\bar{\sigma}_2(\tau_2^*) = \epsilon_{2(0)}$, then $\text{Re} \left(\frac{d\lambda}{d\tau_2} \right) \Big|_{\tau_2=\tau_2^*, \epsilon_2=\epsilon_{2(0)}} > 0$.

Proof. Employing Eq.(3.25), we can acquire

$$\begin{aligned} & [(3\lambda^2 + 2(m_1 + m_4)\lambda + m_2 + m_5)] \frac{d\lambda}{d\tau_2} + m_7 e^{-\lambda\tau_2} \frac{d\lambda}{d\tau_2} \\ & - (m_7\lambda + m_8)e^{-\lambda\tau_2} (\lambda + \frac{d\lambda}{d\tau_2} \tau_2) = 0. \end{aligned} \quad (3.35)$$

This suggests that

$$\left(\frac{d\lambda}{d\tau_2} \right)^{-1} = \frac{H_1(\lambda)}{H_2(\lambda)} - \frac{\tau_2}{\lambda}, \quad (3.36)$$

where

$$\begin{cases} H_1(\lambda) = 3\lambda^2 + 2(m_1 + m_4)\lambda + m_2 + m_5 + m_7 e^{-\lambda\tau_2}, \\ H_2(\lambda) = (m_7\lambda + m_8)e^{-\lambda\tau_2} \lambda. \end{cases} \quad (3.37)$$

Hence

$$\text{Re} \left[\left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right]_{\tau_2=\tau_2^*, \epsilon_2=\epsilon_{2(0)}} = \text{Re} \left[\frac{H_1(\lambda)}{H_2(\lambda)} \right]_{\tau_2=\tau_2^*, \epsilon_2=\epsilon_{2(0)}} = \frac{H_{1R}H_{2R} + H_{1I}H_{2I}}{H_{2R}^2 + H_{2I}^2}. \quad (3.38)$$

Under the assumption $(\mathcal{O}_5)$, we obtain

$$\text{Re} \left[\left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right]_{\tau_2=\tau_2^*, \epsilon_2=\epsilon_{2(0)}} > 0. \quad (3.39)$$

The proof is complete. $\square$

Based on the preceding discussion, the following conclusion can be easily deduced.

Theorem 3.2. Under the assumptions $(\mathcal{O}_1)$, $(\mathcal{O}_4)$, $(\mathcal{O}_5)$, then the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of model (1.2) holds locally asymptotically stable state when $\tau_2 \in [0, \tau_2^*)$. Furthermore, a cluster of Hopf bifurcations arise surrounding the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ when $\tau_2 = \tau_2^*$.

Case 4. $\tau_1 = \tau_2 = \tau$.

Under the circumstance, the corresponding characteristic Eq. (3.6) changes to

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 + [m_4\lambda^2 + (m_5 + m_7)\lambda + m_6 + m_8]e^{-\lambda\tau} = 0. \quad (3.40)$$

Assume that $\lambda = i\omega$ ($\omega > 0$) is the root of Eq. (3.40), then Eq. (3.40) becomes

$$(i\omega)^3 + m_1(i\omega)^2 + m_2i\omega + m_3 + [m_4(i\omega)^2 + (m_5 + m_7)i\omega + m_6 + m_8]e^{-i\omega\tau} = 0, \quad (3.41)$$

which leads to

$$\begin{cases} l_1(\omega) \sin \omega\tau - l_2(\omega) \cos \omega\tau = l_3(\omega), \\ l_1(\omega) \cos \omega\tau + l_2(\omega) \sin \omega\tau = l_4(\omega), \end{cases} \quad (3.42)$$

where

$$\begin{cases} l_1 = m_6 + m_8 - m_4\omega^2, \\ l_2 = (m_5 + m_7)\omega, \\ l_3 = m_2\omega - \omega^3, \\ l_4 = m_1\omega^2 - m_3. \end{cases} \quad (3.43)$$

In view of $l_1^2 + l_2^2 = l_3^2 + l_4^2$, we gain

$$\omega^6 + L_1\omega^4 + L_2\omega^2 + L_3 = 0, \quad (3.44)$$

where

$$\begin{cases} L_1 = m_1^2 - m_4^2 - 2m_2, \\ L_2 = m_2^2 + 2m_4(m_6 + m_8) - 2m_1m_3, \\ L_3 = m_3^2 - (m_6 + m_8)^2 - (m_5 + m_7)^2. \end{cases} \quad (3.45)$$

Let

$$\Phi_3(\omega) = \omega^6 + L_1\omega^4 + L_2\omega^2 + L_3. \quad (3.46)$$

Assume that

$$(\mathcal{O}_6) \quad L_3 < 0$$

is satisfied, then the $\Phi_3(0) < 0$ and $\lim_{\omega \rightarrow \infty} \Phi_3(\omega) = +\infty > 0$, then Eq. (3.44) at least one positive real root. Hence, Eq. (3.40) at least one pair of purely imaginary root. Without loss of generality, we can assume that Eq. (3.44) has six positive real roots (say $\omega_j, j = 1, 2, \dots, 6$). Relying on (3.42), we know

$$\tau_j^{(k)} = \frac{1}{\omega_j} \left[\arccos \left(\frac{l_{1(j)}l_{4(j)} - l_{2(j)}l_{3(j)}}{l_{1(j)}^2 + l_{2(j)}^2} + 2k\pi \right) \right], \quad (3.47)$$

where

$$\begin{cases} l_{1(j)} = m_6 + m_8 - m_4\omega_j, \\ l_{2(j)} = m_5 + m_7, \\ l_{3(j)} = m_2\omega_j - \omega_j^3, \\ l_{4(j)} = m_1\omega_j^2 - m_3, \end{cases} \quad (3.48)$$

and $j = 1, 2, \dots, 6; k = 0, 1, 2, \dots$. Assume $\tau^* = \min_{\{j=1,2,\dots,6;k=0,1,2,\dots\}} \{\tau_j^{(k)}\}$ and suppose that when $\tau = \tau^*$, Eq. (3.40) has a pair of imaginary roots $\pm i\omega_0$.

We present the following assumption:

$$(\mathcal{O}_7) \quad M_{1R}M_{2R} + M_{1I}M_{2I} > 0,$$

where

$$\begin{cases} M_{1R} = -3\omega_0^3 + 2m_4\omega_0 \sin \omega_0\tau^* + (m_5 + m_7) \cos \omega_0\tau^*, \\ M_{1I} = 2m_1\omega_0 + 2m_4\omega_0 \cos \omega_0\tau^* - (m_5 + m_7) \sin \omega_0\tau^*, \\ M_{2R} = [(m_5 + m_7)\omega_0^2 + (m_6 + m_8)\omega_0 - m_4\omega_0^3] \sin \omega_0\tau^* \\ \quad - (m_5 + m_7)\omega_0^2 \cos \omega_0\tau^*, \\ M_{2I} = (m_6\omega_0 + m_8\omega_0 - m_4\omega_0^3) \cos \omega_0\tau^*. \end{cases} \quad (3.49)$$

Lemma 3.3. Suppose that $\varrho(\tau) = \varrho_1(\tau) + i\varrho_2(\tau)$ is the root of Eq. (3.40) at $\tau = \tau^*$ such that $\varrho_1(\tau^*) = 0$, $\varrho_2(\tau^*) = \omega_0$, then $\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau=\tau^*, \omega=\omega_0} > 0$.

Proof. Utilizing Eq. (3.40), we can obtain

$$\begin{aligned} & [(3\lambda^2 + 2m_1\lambda + m_2)] \frac{d\lambda}{d\tau} + [2m_4\lambda + m_5 + m_7] e^{-\lambda\tau} \frac{d\lambda}{d\tau} \\ & - [m_4\lambda^2 + (m_5 + m_7)\lambda + m_6 + m_8] e^{-\lambda\tau} \left(\lambda + \frac{d\lambda}{d\tau} \tau \right) = 0. \end{aligned} \quad (3.50)$$

This implies that

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{M_1(\lambda)}{M_2(\lambda)} - \frac{\tau}{\lambda}, \quad (3.51)$$

where

$$\begin{cases} M_1(\lambda) = 3\lambda^2 + 2m_1\lambda + m_2 + (2m_4\lambda + m_5 + m_7)e^{\lambda\tau}, \\ M_2(\lambda) = [m_4\lambda^2 + (m_5 + m_7)\lambda + m_6 + m_8]e^{-\lambda\tau_2}\lambda. \end{cases} \quad (3.52)$$

Therefore

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau=\tau^*, \omega=\omega_0} = \operatorname{Re} \left[\frac{M_1(\lambda)}{M_2(\lambda)} \right]_{\tau=\tau^*, \omega=\omega_0} = \frac{M_{1R}M_{2R} + M_{1I}M_{2I}}{M_{2R}^2 + M_{2I}^2}. \quad (3.53)$$

Based on the assumption $(\mathcal{O}_7)$, we gain

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau=\tau^*, \omega=\omega_0} > 0, \quad (3.54)$$

which ends the proof. $\square$

Based on the preceding discussion, the subsequent conclusion can be readily inferred.

Theorem 3.3. Provided that $(\mathcal{O}_1)$, $(\mathcal{O}_6)$, $(\mathcal{O}_7)$ are valid, then the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ of model (1.2) maintains locally asymptotically stable state when $\tau \in [0, \tau^*)$. In addition, model (1.2) arises a cluster of Hopf bifurcations surrounding the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ when $\tau = \tau^*$.

Case 5. $\tau_1 \neq 0, \tau_2 \neq 0$.

(i) For $\tau_1 > 0, \tau_2 \in [0, \tau_2^*)$, in this situation, we take τ_1 as bifurcation parameter and place τ_2 within a stable interval. Let $\lambda = i\nu_1$ ($\nu_1 > 0$) be a root of Eq. (3.6), then Eq. (3.6) transforms into

$$(i\nu_1)^3 + m_1(i\nu_1)^2 + m_2i\nu_1 + m_3 + [m_4(i\nu_1)^2 + m_5i\nu_1 + m_6]e^{-i\nu_1\tau_1}$$

$$+(m_7 i \nu_1 + m_8) e^{-i \nu_1 \tau_2} = 0, \quad (3.55)$$

which leads to

$$\begin{cases} n_1(\nu_1) \sin \nu_1 \tau_1 - n_2(\nu_1) \cos \nu_1 \tau_1 = n_3(\nu_1), \\ n_1(\nu_1) \cos \nu_1 \tau_1 + n_2(\nu_1) \sin \nu_1 \tau_1 = n_4(\nu_1), \end{cases} \quad (3.56)$$

where

$$\begin{cases} n_1 = m_5 \nu_1, \\ n_2 = m_4 \nu_1^2 - m_6, \\ n_3 = m_1 \nu_1^3 - m_3 - m_7 \nu_1 \sin(\nu_1 \tau_2) - m_8 \cos \nu_1 \tau_2, \\ n_4 = \nu_1^3 - m_2 \nu_1 - m_3 \nu_1 \cos \nu_1 \tau_2 - m_8 \sin(\nu_1 \tau_2). \end{cases} \quad (3.57)$$

On account of $n_1^2 + n_2^2 = n_3^2 + n_4^2$, we gain

$$\nu_1^6 + N_1 \nu_1^4 + N_2 \nu_1^3 + N_3 \nu_1^2 + N_4 \nu_1 + N_5 = 0, \quad (3.58)$$

where

$$\begin{cases} N_1 = m_1^2 - m_4^2 - 2m_2 - 2m_7 \cos \nu_1 \tau_2, \\ N_2 = -2m_1 m_7 \sin \nu_1 \tau_2 - 2m_8 \sin \nu_1 \tau_2, \\ N_3 = m_7^2 \sin^2 \nu_1 \tau_2 + 2m_4 m_6 + (m_2 + m_7 \cos \nu_1 \tau_2)^2 - m_5^2 \\ \quad - 2m_1 m_8 \cos \nu_1 \tau_2 - 2m_1 m_3, \\ N_4 = 2m_3 m_7 \sin \nu_1 \tau_2 + 4m_7 m_8 \sin \nu_1 \tau_2 \cos \nu_1 \tau_2 + 2m_2 m_8 \sin \nu_1 \tau_2, \\ N_5 = (m_3 + m_8 \cos \nu_1 \tau_2)^2 + m_8^2 \sin^2 \nu_1 \tau_2 - m_6^2. \end{cases} \quad (3.59)$$

Let

$$\Phi_4(\nu_1) = \nu_1^6 + N_1 \nu_1^4 + N_2 \nu_1^3 + N_3 \nu_1^2 + N_4 \nu_1 + N_5. \quad (3.60)$$

Assume that

$$(\mathcal{O}_8) \quad N_5 < 0$$

is satisfied, then the $\Phi_4(0) < 0$ and $\lim_{\nu_1 \rightarrow \infty} \Phi_4(\nu_1) = +\infty > 0$, then Eq. (3.58) at least one positive real root. Hence, Eq. (3.6) possesses at least one pair of purely imaginary root. Without loss of generality, we can assume that Eq. (3.58) has six positive real roots (say $\nu_{1(i)}, i = 1, 2, \dots, 6$). Additionally, based on (3.56), we know

$$\tau_{1(i)}^{(k)} = \frac{1}{\nu_{1(i)}} \left[\arccos \left(\frac{n_{1(i)} n_{4(i)} - n_{2(i)} n_{3(i)}}{n_{1(i)}^2 + n_{2(i)}^2} + 2k\pi \right) \right], \quad (3.61)$$

where

$$\begin{cases} n_{1(i)} = m_5 \nu_{1(i)}, \\ n_{2(i)} = m_4 \nu_{1(i)}^2 - m_6, \\ n_{3(i)} = m_1 \nu_{1(i)}^3 - m_3 - m_7 \nu_{1(i)} \sin(\nu_{1(i)} \tau_2) - m_8 \cos \nu_{1(i)} \tau_2, \\ n_{4(i)} = \nu_{1(i)}^3 - m_2 \nu_{1(i)} - m_3 \nu_{1(i)} \cos \nu_{1(i)} \tau_2 - m_8 \sin(\nu_{1(i)} \tau_2), \end{cases} \quad (3.62)$$

and $i = 1, 2, \dots, 6; k = 0, 1, 2, \dots$. Assume $\tau_1^* = \min_{\{i=1,2,\dots,6;k=0,1,2,\dots\}} \{\tau_{1(i)}^{(k)}\}$ and suppose that when $\tau_1 = \tau_1^*$, Eq. (3.6) has a pair of imaginary roots $\pm i\nu_{1(0)}$.

Next, we put forward the following assumption:

$$(\mathcal{O}_9) \quad S_{1R}S_{2R} + S_{1I}S_{2I} > 0,$$

where

$$\begin{cases} S_{1R} = -3\nu_{1(0)}^3 + m_2 + 2m_4\nu_{1(0)} \sin \nu_{1(0)}\tau_1^* + m_5 \cos \nu_{1(0)}\tau_1^* \\ \quad + (m_7 - m_8\tau_2) \cos \nu_{1(0)}\tau_2 - m_7\tau_2\nu_{1(0)} \sin \nu_{1(0)}\tau_2, \\ S_{1I} = 2m_1\nu_{1(0)} + 2m_4\nu_{1(0)} \cos \nu_{1(0)}\tau_1^* - m_5 \sin \nu_{1(0)}\tau_1^* \\ \quad + (m_8\tau_2 - m_7) \sin \nu_{1(0)}\tau_2 - m_7\tau_2\nu_{1(0)} \cos \nu_{1(0)}\tau_2, \\ S_{2R} = (m_6\nu_{1(0)} - m_4\nu_{1(0)}^3) \sin \nu_{1(0)}\tau_1^* - m_5\nu_{1(0)}^2 \cos \nu_{1(0)}\tau_1^*, \\ S_{2I} = m_5\nu_{1(0)}^2 \sin \nu_{1(0)}\tau_1^* + (m_6\nu_{1(0)} - m_4\nu_{1(0)}^3) \cos \nu_{1(0)}\tau_1^*. \end{cases} \quad (3.63)$$

Lemma 3.4. Suppose that $\lambda(\tau_1) = \vartheta_1(\tau_1) + i\vartheta_2(\tau_1)$ is the root of Eq. (3.6) at $\tau_1 = \tau_1^*$, and satisfy $\vartheta_1(\tau_1^*) = 0$, $\vartheta_2(\tau_1^*) = \nu_{1(0)}$, then $\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau_1=\tau_1^*, \nu=\nu_{1(0)}} > 0$.

Proof. By employing Eq. (3.6), we can derive

$$\begin{aligned} & [(3\lambda^2 + 2m_1\lambda + m_2)] \frac{d\lambda}{d\tau_1} + (2m_4\lambda + m_5)e^{-\lambda\tau_1} \frac{d\lambda}{d\tau_1} \\ & - (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau_1} \left(\lambda + \frac{d\lambda}{d\tau_1} \tau_1 \right) + m_7e^{-\lambda\tau_2} \frac{d\lambda}{d\tau_1} \\ & - (m_7\lambda + m_8)e^{-\lambda\tau_2} \tau_2 \frac{d\lambda}{d\tau_1} = 0. \end{aligned} \quad (3.64)$$

In other words

$$\left(\frac{d\lambda}{d\tau_1} \right)^{-1} = \frac{S_1(\lambda)}{S_2(\lambda)} - \frac{\tau_1}{\lambda}, \quad (3.65)$$

where

$$\begin{cases} S_1(\lambda) = 3\lambda^2 + 2m_1\lambda + m_2 + (2m_4\lambda + m_5)e^{\lambda\tau_1} \\ \quad + m_7e^{\lambda\tau_2} - (m_7\lambda + m_8)e^{\lambda\tau_2}\tau_2, \\ S_2(\lambda) = (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau_1}\lambda. \end{cases} \quad (3.66)$$

Hence

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right]_{\tau_1=\tau_1^*, \nu_1=\nu_{1(0)}} = \operatorname{Re} \left[\frac{S_1(\lambda)}{S_2(\lambda)} \right]_{\tau_1=\tau_1^*, \nu_1=\nu_{1(0)}} = \frac{S_{1R}S_{2R} + S_{1I}S_{2I}}{S_{2R}^2 + S_{2I}^2}. \quad (3.67)$$

Through the assumption $(\mathcal{O}_9)$, we get

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right]_{\tau_1=\tau_1^*, \nu_1=\nu_{1(0)}} > 0, \quad (3.68)$$

which completes the proof. $\square$

On the basis of the above discussion, the following result can be effortlessly deduced.

Theorem 3.4. *If $(\mathcal{O}_1)$, $(\mathcal{O}_8)$, $(\mathcal{O}_9)$ hold, then the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of model (1.2) maintains locally asymptotically stable state when $\tau_1 \in [0, \tau_1^*)$. Moreover, model (1.2) arises a cluster of Hopf bifurcations surrounding the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ when $\tau_1 = \tau_1^*$.*

(ii) For $\tau_2 > 0, \tau_1 \in [0, \tau_1^*)$, we take τ_2 as bifurcation parameter and place τ_1 within a stable interval. Let $\lambda = i\nu_2$ ($\nu_2 > 0$) be a root of Eq. (3.6), then Eq. (3.6) becomes

$$(i\nu_2)^3 + m_1(i\nu_2)^2 + m_2i\nu_2 + m_3 + [m_4(i\nu_2)^2 + m_5i\nu_2 + m_6]e^{-i\nu_2\tau_1} + (m_7i\nu_2 + m_8)e^{-i\nu_2\tau_2} = 0, \quad (3.69)$$

which leads to

$$\begin{cases} m_7\nu_2 \sin \nu_2\tau_2 - m_8 \cos \nu_2\tau_2 = b_1(\nu_2), \\ m_7\nu_2 \cos \nu_2\tau_2 + m_8 \sin \nu_2\tau_2 = b_2(\nu_2), \end{cases} \quad (3.70)$$

where

$$\begin{cases} b_1 = m_1\nu_2^3 - m_3 + (m_4\nu_2^2 + m_6) \cos \nu_2\tau_1 - m_5\nu_2 \sin \nu_2\tau_1, \\ b_2 = \nu_1^3 - m_2\nu_1 - m_3\nu_1 \cos \nu_1\tau_2 - m_8 \sin \nu_1\tau_2. \end{cases} \quad (3.71)$$

Due to $m_7^2\nu_2^2 + m_8^2 = b_1^2 + b_2^2$, we gain

$$\nu_2^6 + B_1\nu_2^5 + B_2\nu_2^4 + B_3\nu_2^3 + B_4\nu_2^2 + B_5\nu_2 + B_6 = 0, \quad (3.72)$$

where

$$\begin{cases} B_1 = -2m_4 \sin \nu_2\tau_1, \\ B_2 = (m_1 + m_4 \cos \nu_2\tau_1)^2 + m_4^2 \sin^2 \nu_2\tau_1 - 2m_2 - 2m_5 \cos \nu_2\tau_1, \\ B_3 = (2m_2m_4 - 2m_1m_5 - 2m_6) \sin \nu_2\tau_1, \\ B_4 = (m_2 + m_5 \cos \nu_2\tau_1)^2 + (2m_1m_6 - 2m_3m_4) \cos \nu_2\tau_1 \\ \quad + m_5^2 \sin^2 \nu_2\tau_1 + 2m_4m_6 - 2m_1m_3 - m_7^2, \\ B_5 = (2m_4m_6 + 2m_3m_5) \sin \nu_2\tau_1, \\ B_6 = m_6^2 \sin^2 \nu_2\tau_1 + (m_3 - m_6 \cos \nu_2\tau_1)^2 - m_8^2. \end{cases} \quad (3.73)$$

Let

$$\Phi_5(\nu_2) = \nu_2^6 + B_1\nu_2^5 + B_2\nu_2^4 + B_3\nu_2^3 + B_4\nu_2^2 + B_5\nu_2 + B_6. \quad (3.74)$$

Assume that

$$(\mathcal{O}_{10}) \quad B_6 < 0$$

is fulfilled, it follows that $\Phi_5(0) < 0$ and $\lim_{\nu_2 \rightarrow \infty} \Phi_5(\nu_2) = +\infty > 0$, then Eq. (3.72) at least one positive real root. Hence, Eq. (3.6) at least one pair of purely imaginary root. Without loss of generality, we can assume that Eq. (3.72) has six positive real roots (say $\nu_{2(i)}, i = 1, 2, \dots, 6$). Depending on (3.56), we know

$$\tau_{2(i)}^{(k)} = \frac{1}{\nu_{2(i)}} \left[\arccos \left(\frac{m_8 b_{1(i)} + m_7 \nu_{2(i)} b_{2(i)}}{m_8^2 + m_7^2 \nu_{2(i)}^2} \right) + 2k\pi \right], \quad (3.75)$$

where

$$\begin{cases} b_{1(i)} = m_1\nu_{2(i)}^3 - m_3 + (m_4\nu_{2(i)}^2 + m_6) \cos \nu_{2(i)}\tau_1 - m_5\nu_{2(i)} \sin \nu_{2(i)}\tau_1, \\ b_{2(i)} = \nu_{2(i)}^3 - m_2\nu_{2(i)} - m_3\nu_{2(i)} \cos \nu_{2(i)}\tau_2 - m_8 \sin \nu_{2(i)}\tau_2, \end{cases} \quad (3.76)$$

and $i = 1, 2, \dots, 6; k = 0, 1, 2, \dots$. Assume $\tau_2^* = \min_{\{i=1,2,\dots,6;k=0,1,2,\dots\}} \{\tau_{2(i)}^{(k)}\}$ and suppose that when $\tau_2 = \tau_2^*$, Eq. (3.6) has a pair of imaginary roots $\pm i\nu_{2(0)}$.

Afterward, we propose the following assumption:

$$(\mathcal{O}_{11}) \quad T_{1R}T_{2R} + T_{1I}T_{2I} > 0,$$

where

$$\begin{cases} T_{1R} = -3\nu_{2(0)}^3 + m_2 + (2m_4\nu_{2(0)} - m_5\nu_{2(0)}) \sin \nu_{2(0)}\tau_1 \\ \quad + (m_5 + m_4\nu_{2(0)}^2 + m_6) \cos \nu_{2(0)}\tau_1 + m_7 \sin \nu_{2(0)}\tau_2^*, \\ T_{1I} = 2m_1\nu_{2(0)} + (2m_4\nu_{2(0)} - m_5\nu_{2(0)}) \cos \nu_{2(0)}\tau_1 \\ \quad - (m_5 + m_4\nu_{2(0)}^2 + m_6) \sin \nu_{2(0)}\tau_1 - m_7 \sin \nu_{2(0)}\tau_2^*, \\ T_{2R} = m_8\nu_{2(0)} \sin \nu_{2(0)}\tau_2^* - m_7\nu_{2(0)}^2 \cos \nu_{2(0)}\tau_2^*, \\ T_{2I} = m_7\nu_{2(0)}^2 \sin \nu_{2(0)}\tau_2^* + m_8\nu_{2(0)} \sin \nu_{2(0)}\tau_2^*. \end{cases} \quad (3.77)$$

Lemma 3.5. Let $\lambda(\tau_2) = \bar{\vartheta}_1(\tau_2) + i\bar{\vartheta}_2(\tau_2)$ be the root of Eq. (3.6) at $\tau_2 = \tau_2^*$, and when $\bar{\vartheta}_1(\tau_2^*) = 0$, $\bar{\vartheta}_2(\tau_2^*) = \nu_{2(0)}$, then $\text{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau_2=\tau_2^*, \nu_2=\nu_{2(0)}} > 0$.

Proof. By Eq. (3.6), we can gain

$$\begin{aligned} & [(3\lambda^2 + 2m_1\lambda + m_2)] \frac{d\lambda}{d\tau_2} + (2m_4\lambda + m_5)e^{-\lambda\tau_1} \frac{d\lambda}{d\tau_2} \\ & - (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau_1} \frac{d\lambda}{d\tau_2} \tau_1 + m_7e^{-\lambda\tau_2} \frac{d\lambda}{d\tau_2} \\ & - (m_7\lambda + m_8)e^{-\lambda\tau_2} (\lambda + \tau_2 \frac{d\lambda}{d\tau_2}) = 0, \end{aligned} \quad (3.78)$$

which indicates

$$\left(\frac{d\lambda}{d\tau_2} \right)^{-1} = \frac{T_1(\lambda)}{T_2(\lambda)} - \frac{\tau_2}{\lambda}, \quad (3.79)$$

where

$$\begin{cases} T_1(\lambda) = 3\lambda^2 + 2m_1\lambda + m_2 + (2m_4\lambda + m_5)e^{-\lambda\tau_1} \\ \quad - (m_4\lambda^2 + m_5\lambda + m_6)e^{-\lambda\tau_1} + m_7e^{-\lambda\tau_2}, \\ T_2(\lambda) = (m_7\lambda + m_8)e^{-\lambda\tau_2}\lambda. \end{cases} \quad (3.80)$$

Thus

$$\text{Re} \left[\left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right]_{\tau_2=\tau_2^*, \nu_2=\nu_{2(0)}} = \text{Re} \left[\frac{T_1(\lambda)}{T_2(\lambda)} \right]_{\tau_2=\tau_2^*, \nu_2=\nu_{2(0)}} = \frac{T_{1R}T_{2R} + T_{1I}T_{2I}}{T_{2R}^2 + T_{2I}^2}. \quad (3.81)$$

By $(\mathcal{O}_{11})$, we know

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right]_{\tau_2=\tau_2^*, \nu_2=\nu_{2(0)}} > 0, \quad (3.82)$$

which ends the proof. $\square$

According to the above analysis, we can draw the following conclusions.

Theorem 3.5. *If $(\mathcal{O}_1)$, $(\mathcal{O}_{10})$, $(\mathcal{O}_{11})$ hold, then the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of model (1.2) maintains locally asymptotically stable state when $\tau_2 \in [0, \tau_2^*)$. Besides, model (1.2) arises a cluster of Hopf bifurcations surrounding the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ when $\tau_2 = \tau_2^*$.*

Remark 3.1. In model (1.2), there exist two different delays. If there are three or more delays, it is difficult to explore the impact of different delays on the stability and bifurcation of model (1.2). We leave it for future work.

4. Control of bifurcation for model (1.2) with $\tau_1 = \tau_2 = \tau$ using delay feedback controller

In this segment, we will apply a delay feedback controller to dominate stability region and the time of generation of Hopf bifurcation of system (1.2) under the condition of $\tau_1 = \tau_2 = \tau$. By means of the ideas of [41–43], we set up the following controlled predator-prey model:

$$\begin{cases} \frac{du_1}{dt} = ru_1(t) \left(1 - \frac{u_1(t-\tau)}{h} \right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = \rho u_2(t) - \delta u_2(t)u_3(t) - \varsigma(u_2(t) - u_2(t-\tau)), \\ \frac{du_3}{dt} = \frac{bu_1(t-\tau)u_3(t)}{a + \xi\beta u_2(t) + u_1(t-\tau)} + \gamma u_2(t)u_3(t) - nu_3(t), \end{cases} \quad (4.1)$$

where ς denotes feedback gain parameter. System (4.1) owns the same equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ as that in system (1.2). Let

$$\begin{cases} u_{1*} = u_1(t) - \bar{u}_1(t), \\ u_{2*} = u_2(t) - \bar{u}_2(t), \\ u_{3*} = u_3(t) - \bar{u}_3(t). \end{cases} \quad (4.2)$$

The linear system of system (4.1) near $E(u_{1*}, u_{2*}, u_{3*})$ reads as:

$$\begin{cases} \frac{d\bar{u}_1}{dt} = a_1\bar{u}_1 - a_2\bar{u}_1(t-\tau) + a_3\bar{u}_2 - a_4\bar{u}_3, \\ \frac{d\bar{u}_2}{dt} = a_5\bar{u}_2 + a_7\bar{u}_2(t-\tau) - a_7\bar{u}_3, \\ \frac{d\bar{u}_2}{dt} = a_8\bar{u}_2(t-\tau) + a_9\bar{u}_2 + a_{10}\bar{u}_3, \end{cases} \quad (4.3)$$

where

$$\left\{ \begin{array}{l} a_1 = r - \frac{ru_{1\star}}{h} - \frac{cu_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \frac{cu_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ a_2 = \frac{ru_{1\star}}{h}, \\ a_3 = \frac{c\xi\beta u_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ a_4 = \frac{cu_{1\star}}{a + \xi\beta u_{2\star} + u_{1\star}}, \\ a_5 = \rho - \delta u_{3\star} - \varsigma, \\ a_6 = \varsigma, \\ a_7 = \delta u_{2\star}, \\ a_8 = \frac{bu_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} - \frac{bu_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ a_9 = \gamma u_{3\star} - \frac{b\xi\beta u_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ a_{10} = \frac{bu_{1\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \gamma u_{2\star} - n. \end{array} \right. \quad (4.4)$$

The characteristic equation of system (4.3) owns the following expression:

$$\det \begin{bmatrix} \lambda - a_1 + a_2 e^{-\lambda\tau} & -a_3 & a_4 \\ 0 & \lambda - a_5 - a_6 e^{-\lambda\tau} & a_7 \\ -a_8 e^{-\lambda\tau} & -a_9 & \lambda - a_{10} \end{bmatrix} = 0, \quad (4.5)$$

which results in

$$\lambda^3 + p_1 \lambda^2 + p_2 \lambda + p_3 + (p_4 \lambda^2 + p_5 \lambda + p_6) e^{-\lambda\tau} + (p_7 \lambda + p_8) e^{-2\lambda\tau} = 0, \quad (4.6)$$

where

$$\left\{ \begin{array}{l} p_1 = -a_{10} - a_5 - a_1, \\ p_2 = a_5 a_{10} + a_7 a_9 + a_1 a_{10} + a_1 a_5, \\ p_3 = a_1 a_5 a_{10} - a_1 a_7 a_{10}, \\ p_4 = a_2 - a_6, \\ p_5 = a_6 a_{10} + a_1 a_6 + a_4 a_8 - a_2 a_5 - a_2 a_{10}, \\ p_6 = a_2 a_5 a_{10} + a_2 a_7 a_9 + a_3 a_7 a_8 - a_4 a_5 a_8 - a_1 a_6 a_{10}, \\ p_7 = -a_2 a_6, \\ p_8 = a_2 a_6 a_{10} - a_4 a_6 a_8. \end{array} \right. \quad (4.7)$$

When $\tau = 0$, the formulation of Eq. (4.6) is as follows:

$$\lambda^3 + (h_1 + h_4) \lambda^2 + (h_2 + h_5) \lambda + h_3 + h_6 + h_7 = 0. \quad (4.8)$$

If

$$(\mathcal{Q}_{12}) \begin{cases} \Re_1 = p_1 + p_4 > 0, \\ \Re_2 = \det \begin{bmatrix} p_1 + h_4 & 1 \\ p_3 + p_6 + p_8 & p_2 + p_5 + p_7 \end{bmatrix} > 0, \\ \Re_3 = (p_3 + p_6 + p_8)\Re_2 > 0, \end{cases} \quad (4.9)$$

is valid, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq. (4.8) have negative real parts. Thus the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of system (4.1) with the delay $\tau = 0$ maintains locally asymptotically stability.

Assume that $\lambda = i\phi$ ($\phi > 0$) is the root of Eq. (4.6), then Eq. (4.6) becomes:

$$[(i\phi)^3 + p_1(i\phi)^2 + p_2i\phi + p_3]e^{i\phi\tau} + p_4(i\phi)^2 + p_5i\phi + p_6 + (p_7i\phi + p_8)e^{-i\phi\tau} = 0. \quad (4.10)$$

It follows from Eq. (4.10) that

$$\begin{cases} \Psi_1(\phi) \sin \phi\tau + \Psi_2(\phi) \cos \phi\tau = \Psi_3(\phi), \\ \Psi_4(\phi) \sin \phi\tau + \Psi_5(\phi) \cos \phi\tau = \Psi_6(\phi), \end{cases} \quad (4.11)$$

where

$$\begin{cases} \Psi_1 = \phi^3 + p_7\phi - p_2\phi, \\ \Psi_2 = p_3 + p_8 - p_1\phi^2, \\ \Psi_3 = p_4\phi^2 - p_6, \\ \Psi_4 = p_3 - p_8 - p_1\phi^2, \\ \Psi_5 = p_2\phi - p_7\phi - \phi^3, \\ \Psi_6 = -p_5\phi. \end{cases} \quad (4.12)$$

By (4.11), one gains

$$\begin{cases} \sin \phi\tau = \frac{\Psi_2\Psi_6 - \Psi_3\Psi_5}{\Psi_2\Psi_4 - \Psi_1\Psi_5}, \\ \cos \phi\tau = \frac{\Psi_3\Psi_4 - \Psi_1\Psi_6}{\Psi_2\Psi_4 - \Psi_1\Psi_5}. \end{cases} \quad (4.13)$$

In consideration of $\cos^2 \phi\tau + \sin^2 \phi\tau = 1$, we obtain

$$\begin{aligned} & [\Psi_2(\phi\tau)\Psi_6(\phi\tau) - \Psi_3(\phi\tau)\Psi_5(\phi\tau)]^2 + [\Psi_3(\phi\tau)\Psi_4(\phi\tau) - \Psi_1(\phi\tau)\Psi_6(\phi\tau)]^2 \\ &= [\Psi_2(\phi\tau)\Psi_4(\phi\tau) - \Psi_1(\phi\tau)\Psi_5(\phi\tau)]^2. \end{aligned} \quad (4.14)$$

The result is as follows

$$\phi^{12} - W_1\phi^{10} - W_2\phi^8 - W_3\phi^6 - W_4\phi^4 - W_5\phi^2 - W_6 = 0, \quad (4.15)$$

where

$$\left\{ \begin{array}{l} W_1 = p_4^2 + 4p_2 - 2p_1^2, \\ W_2 = 4(p_1p_3 + p_1^{2p_2} + p_7^2 - p_1^2p_7) + 2(p_4^2p_7 - p_2p_4^2 - p_4p_6 - p_2^2) - p_5^2 - p_1^4, \\ W_3 = 8(p_1p_3p_7 - p_2p_7^2 - p_1p_2p_3) + 4(p_1^3p_3 + p_2^3 + p_7^3 + p_2p_4p_6 - p_4p_6p_7 \\ \quad - p_4p_5p_8) - 2(p_1p_4^2p_8 + p_5^2p_7 + p_8^2 - p_2p_4^2p_7 - p_1p_3p_4^2 - p_2p_5^2 \\ \quad - p_1p_4p_5 - p_3^2 - p_1^2p_7^2 - p_1^2p_2^2) + p_1^2p_5^2 + p_2^2p_4^2 + p_4^2p_7^2 + p_6^2 + p_1^2p_4^2, \\ W_4 = -8(p_1p_4p_6p_8 + p_1p_2p_3p_7) - 6p_1^2p_3^2 + 4(p_2p_4p_6p_7 + p_1p_3p_4p_8 \\ \quad + p_2^2p_7^2 + p_2p_3^2 + p_7p_8^2 + p_1p_3p_7^2 + p_1p_2^2p_3 + p_5p_6p_8 + p_2p_4p_5p_8 \\ \quad - p_1^2p_4p_6 - p_2p_7^3 - p_3^2p_7 - p_2p_8^2 - p_4p_5p_7p_8) + 2(p_1^2p_8^2 - p_1p_3p_5^2 \\ \quad - p_6^2p_7 - p_2^2p_4p_6 - p_4p_6p_7^2 - p_2p_6^2 - p_3p_4^2p_8 - p_2p_5^2p_7 - p_1^2p_3p_8) \\ \quad + p_3^2p_4^2 + p_4^2p_8^2 + p_5^2p_7 + p_2^2p_5^2 + p_7^4 - p_2^4, \\ W_5 = 4(p_3p - 4p_7p_8 + p_1p_3^2 - p_1p_3p_8^2 - p_2p_7p_8^2) + 2(p_5p_6p_7p_8 + p_2p_3p_5p_6 \\ \quad + p_7^2p_8^2 + p_2^2p_8^2 - p_2p_6^2p_7 - p_3^2p_4p_6 - p_4p_6p_8^2 - p_1p_3p_6^2 \\ \quad - p_4p_6p_8^2 - p_3p_5p_6p_7 - p_2p_5p_6p_8 - p_3^2p_7^2 - p_2^2p_3^2) + p_3^2p_5^2 + p_3p_5^2p_8 + p_5^2p_8^2 \\ \quad + p_2^2p_6^2 + p_6^2p_7^2 - p_3p_5^2p_8, \\ W_6 = p_3^2p_6^2 + 2p_3^2p_8^2 + 2p_5p_6p_7p_8 - 2p_3p_6^2p_8 - p_3^4 - p_8^4 - 2p_2p_5p_6p_8. \end{array} \right. \quad (4.16)$$

Define

$$\Phi_6(\phi) = \phi^{12} - W_1\phi^{10} - W_2\phi^8 - W_3\phi^6 - W_4\phi^4 - W_5\phi^2 - W_6. \quad (4.17)$$

Assume that

$$(\mathcal{O}_{13}) \quad W_6 > 0.$$

If $(\mathcal{Q}_{13})$ holds, then $\Phi_6(0) = -W_6 < 0$, notice that $\lim_{\phi \rightarrow \infty} \Phi_6(\phi) > 0$. So we know Eq. (4.15) admits at least one positive real root. Therefore Eq. (4.6) has at least one pair of purely imaginary roots. Without loss of generality, we suppose that Eq. (4.15) has six positive real roots (say $\phi_j, j = 1, 2, 3, \dots, 12$). Relying on (4.13), we know

$$\tau_j^{(k)} = \frac{1}{\phi_j} \left[\arccos \left(\frac{\Psi_3(\phi_j)\Psi_4(\phi_j) - \Psi_1(\phi_j)\Psi_6(\phi_j)}{\Psi_2(\phi_j)(\Psi_4(\phi_j) - \Psi_1(\phi_j)\Psi_5(\phi_j))} + 2k\pi \right) \right], \quad (4.18)$$

where $j = 1, 2, 3, \dots, 6; k = 0, 1, 2, \dots$. Let $\tau_{(0)} = \min_{\{j=1,2,3,\dots,12;k=0,1,2,\dots\}} \{\tau_j^{(k)}\}$ and suppose that when $\tau = \tau_{(0)}$, Eq. (4.6) has a pair of imaginary roots $\pm i\phi_0$.

Now we present the following assumption:

$$(\mathcal{O}_{14}) \quad Q_{1R}Q_{2R} + Q_{1I}Q_{2I} > 0,$$

where

$$\begin{cases} Q_{1R} = -3\phi_{(0)}^2 + 2p_4\phi_{(0)} \sin \phi_{(0)}\tau_{(0)} + p_5 \cos \phi_{(0)}\tau_{(0)} + p_7 \cos 2\phi_{(0)}\tau_{(0)}, \\ Q_{1I} = 2p_1\phi_{(0)} + 2p_4\phi_{(0)} \cos \phi_{(0)}\tau_{(0)} - p_5 \sin \phi_{(0)}\tau_{(0)} - p_7 \sin 2\phi_{(0)}\tau_{(0)}, \\ Q_{2R} = (p_5\phi_{(0)}^2 + p_6\phi_{(0)} - p_4\phi_{(0)}^3) \sin \phi_{(0)}\tau_{(0)} - p_5\phi_{(0)}^2 \cos \phi_{(0)}\tau_{(0)} \\ \quad - 2p_7\phi_{(0)}^2 \cos 2\phi_{(0)}\tau_{(0)} + 2p_8\phi_{(0)} \sin 2\phi_{(0)}\tau_{(0)}, \\ Q_{2I} = (p_6\phi_{(0)}^2 - p_4\phi_{(0)}^3) \cos \phi_{(0)}\tau_{(0)} + 2p_7\phi_{(0)}^2 \sin 2\phi_{(0)}\tau_{(0)} \\ \quad + 2p_8\phi_{(0)} \cos \phi_{(0)}\tau_{(0)}. \end{cases} \quad (4.19)$$

Lemma 4.1. *If $\lambda(\tau) = \hbar_1(\tau) + i\hbar_2(\tau)$ is the root of Eq. (4.6) near $\tau = \tau_{(0)}$ that adheres $\hbar_1(\tau_{(0)}) = 0$, $\hbar_2(\tau_{(0)}) = \phi_{(0)}$, then $\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau=\tau_{(0)}, \phi=\phi_{(0)}} > 0$.*

Proof. It follows from Eq. (4.6) that

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + (p_4\lambda^2 + p_5\lambda + p_6)e^{-\lambda\tau} + (p_7\lambda + p_8)e^{-2\lambda\tau} = 0, \quad (4.20)$$

we can get

$$\begin{aligned} & (3\lambda^2 + 2p_1\lambda + p_2) \frac{d\lambda}{d\tau} + (2\lambda p_4 + p_5)e^{-\lambda\tau} \frac{d\lambda}{d\tau} - (p_4\lambda^2 + p_5\lambda + p_6) \\ & \times e^{-\lambda\tau} \left(\frac{d\lambda}{d\tau} \tau + \lambda \right) + p_7 \frac{d\lambda}{d\tau} e^{-2\lambda\tau} - (p_7\lambda + p_8) 2e^{-2\lambda\tau} \left(\frac{d\lambda}{d\tau} \tau + \lambda \right) = 0, \end{aligned} \quad (4.21)$$

which indicates

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{Q_1(\lambda)}{Q_2(\lambda)} - \frac{\tau}{\lambda}, \quad (4.22)$$

where

$$\begin{cases} Q_1(\lambda) = 3\lambda^2 + 2p_1\lambda + p_2 + (2p_4\lambda + p_5)e^{-\lambda\tau} + p_7e^{-2\lambda\tau}, \\ Q_2(\lambda) = (p_4\lambda^2 + p_5\lambda + p_6)e^{-\lambda\tau} + 2(p_7\lambda + p_8)e^{-2\lambda\tau}\lambda. \end{cases} \quad (4.23)$$

Hence

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau=\tau_{(0)}, \phi=\phi_0} = \operatorname{Re} \left[\frac{Q_1(\lambda)}{Q_2(\lambda)} \right]_{\tau=\tau_{(0)}, \phi=\phi_0} = \frac{Q_{1R}Q_{2R} + Q_{1I}Q_{2I}}{Q_{2R}^2 + Q_{2I}^2}. \quad (4.24)$$

By $(\mathcal{O}_{14})$, we get

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\theta} \right)^{-1} \right]_{\tau=\tau_{(0)}, \phi=\phi_0} > 0. \quad (4.25)$$

The proof is concluded. $\square$

From the above discussion, the following theorem is derived.

Theorem 4.1. *If $(\mathcal{O}_{12})$ – $(\mathcal{O}_{14})$ hold, then the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ of model (4.1) maintains locally asymptotically stable state if $\tau \in [0, \tau_{(0)})$ and model (4.1) generates a cluster of Hopf bifurcations around the equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ when $\tau \in [0, \tau_{(0)})$.*

5. Control of bifurcation for model (1.2) with $\tau_1 > 0, \tau_2 = 0$ using delay feedback controller

In this segment, we will apply a delay feedback controller to dominate stability region and the time of generation of Hopf bifurcation of system (1.2) under the condition of $\tau_1 > 0, \tau_2 = 0$. By means of the ideas of [41–43], we set up the following controlled predator-prey model:

$$\begin{cases} \frac{du_1}{dt} = ru_1(t) \left(1 - \frac{u_1(t - \tau_1)}{h} \right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)} \\ \quad - \varpi(u_1(t) - u_1(t - \tau_1)), \\ \frac{du_2}{dt} = \rho u_2(t) - \delta u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{bu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)} + \gamma u_2(t)u_3(t) - nu_3(t), \end{cases} \quad (5.1)$$

where ϖ denotes feedback gain parameter. System (4.1) owns the same equilibrium point $E(u_{1\star}, u_{2\star}, u_{3\star})$ as that in system (1.2). Let

$$\begin{cases} u_{1\star} = u_1(t) - \bar{u}_1(t), \\ u_{2\star} = u_2(t) - \bar{u}_2(t), \\ u_{3\star} = u_3(t) - \bar{u}_3(t). \end{cases} \quad (5.2)$$

The linear system of system (4.1) near $E(u_{1\star}, u_{2\star}, u_{3\star})$ reads as:

$$\begin{cases} \frac{d\bar{u}_1}{dt} = v_1\bar{u}_1 - v_2\bar{u}_1(t - \tau_1) + v_3\bar{u}_2 - v_4\bar{u}_3, \\ \frac{d\bar{u}_2}{dt} = v_5\bar{u}_2 - v_6\bar{u}_3, \\ \frac{d\bar{u}_3}{dt} = v_7\bar{u}_1 + v_8\bar{u}_2 + v_9\bar{u}_3, \end{cases} \quad (5.3)$$

where

$$\begin{cases} v_1 = r - \frac{ru_{1\star}}{h} - \varpi - \frac{cu_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \frac{cu_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ v_2 = \frac{ru_{1\star}}{h} + \varpi, \\ v_3 = \frac{c\xi\beta u_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ v_4 = \frac{cu_{1\star}}{a + \xi\beta u_{2\star} + u_{1\star}}, \\ v_5 = \rho - \delta u_{3\star}, \\ v_6 = \delta u_{2\star}, \\ v_7 = \frac{bu_{3\star}}{a + \xi\beta u_{2\star} + u_{1\star}} - \frac{bu_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ v_8 = \gamma u_{3\star} - \frac{b\xi\beta u_{1\star}u_{3\star}}{(a + \xi\beta u_{2\star} + u_{1\star})^2}, \\ v_9 = \frac{bu_{1\star}}{a + \xi\beta u_{2\star} + u_{1\star}} + \gamma u_{2\star} - n. \end{cases} \quad (5.4)$$

The characteristic equation of system (5.3) owns the following expression:

$$\det \begin{bmatrix} \lambda - v_1 + v_2 e^{-\lambda \tau_1} & -v_3 & v_4 \\ 0 & \lambda - v_5 & v_6 \\ -v_7 & -v_8 & \lambda - v_9 \end{bmatrix} = 0, \quad (5.5)$$

which results in

$$\lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 + (b_4 \lambda^2 + b_5 \lambda + b_6) e^{-\lambda \tau_1} = 0, \quad (5.6)$$

where

$$\begin{cases} c_1 = -v_9 - v_5 - v_1, \\ c_2 = v_2 v_9 + v_6 v_8 + v_1 v_9 + v_1 v_5 + v_4 v_7, \\ c_3 = v_3 v_7 v_6 - v_1 v_5 v_9 - v_1 v_6 v_8 - v_4 v_5 v_7, \\ c_4 = v_2, \\ c_5 = -v_2 v_6 - v_2 v_5, \\ c_6 = v_2 v_5 v_9 - v_2 v_6 v_8. \end{cases} \quad (5.7)$$

When $\tau_1 = 0$ the formulation of Eq. (5.6) is as follows:

$$\lambda^3 + (c_1 + c_4) \lambda^2 + (c_2 + c_5) \lambda + c_3 + c_6 = 0. \quad (5.8)$$

If

$$(\mathcal{O}_{15}) \begin{cases} \Delta_1 = c_1 + c_4 > 0, \\ \Delta_2 = \det \begin{bmatrix} c_1 + c_4 & 1 \\ c_3 + c_6 & c_2 + c_5 \end{bmatrix} > 0, \\ \Delta_3 = (c_3 + c_6) \Delta_2 > 0, \end{cases} \quad (5.9)$$

is valid, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq. (5.8) have negative real parts. Thus the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of system (5.1) with the delay $\tau_1 = 0$ maintains locally asymptotically stability.

Assume that $\lambda = i\mu$ ($\mu > 0$) is the root of Eq. (5.6), then Eq. (5.6) becomes:

$$((i\mu)^3 + c_1(i\mu)^2 + c_2 i\mu + c_3 + (c_4(i\mu)^2 + c_5 i\mu + c_6) e^{-i\mu \tau_1} = 0. \quad (5.10)$$

It follows from Eq. (5.10) that

$$\begin{cases} \Theta_1(\mu) \sin \mu \tau_1 + \Theta_2(\mu) \cos \mu \tau_1 = \Theta_3(\mu), \\ \Theta_1(\mu) \cos \mu \tau_1 - \Theta_2(\mu) \sin \mu \tau_1 = \Theta_4(\mu), \end{cases} \quad (5.11)$$

where

$$\begin{cases} \Theta_1 = c_5 \mu, \\ \Theta_2 = c_6 - c_4 \mu^2, \\ \Theta_3 = c_1 \mu^2 - c_3, \\ \Theta_4 = \mu^3 - c_2 \mu. \end{cases} \quad (5.12)$$

In consideration of $\Theta_1^2 + \Theta_2^2 = \Theta_3^2 + \Theta_4^2$, we gain

$$\mu^6 + C_1\mu_2^4 + C_2\mu_2^2 + C_3 = 0, \quad (5.13)$$

where

$$\begin{cases} C_1 = c_1^2 - 2c_1 - c_4^2, \\ C_2 = c_2^2 + 2c_4c_6 - 2c_1c_3 - c_5^2, \\ C_3 = c_3^2 - c_6^2. \end{cases} \quad (5.14)$$

Let

$$\Phi_7(\mu) = \mu^6 + C_1\mu^4 + C_2\mu^2 + C_3. \quad (5.15)$$

Assume that

$$(\mathcal{O}_{16}) \quad C_3 < 0$$

holds, then $\Phi_7(0) < 0$ and $\lim_{\mu \rightarrow \infty} \Phi_7(\mu) = +\infty > 0$, then Eq. (5.13) admits at least one positive real root. Therefore Eq. (5.10) has at least one pair of purely imaginary root. Without loss of generality, we suppose that Eq. (5.13) has six positive real roots (say $\mu(j), j = 1, 2, \dots, 6$). Relying on (5.11), we know

$$\tau_{1j}^{(k)} = \frac{1}{\mu(j)} \left[\arccos \left(\frac{\Theta_{1(j)}\Theta_{4(j)} + \Theta_{2(j)}\Theta_{3(j)}}{\Theta_{1(j)}^2 + \Theta_{2(j)}^2} \right) + 2k\pi \right], \quad (5.16)$$

where

$$\begin{cases} \Theta_{1(j)} = c_5\mu(j), \\ \Theta_{2(j)} = c_6 - c_4\mu(j)^2, \\ \Theta_{3(j)} = c_1\mu(j)^2 - c_3, \\ \Theta_{4(j)} = \mu(j)^3 - c_2\mu(j), \end{cases} \quad (5.17)$$

and $j = 1, 2, \dots, 6; k = 0, 1, 2, \dots$. Assume $\tau_1^t = \min_{\{j=1,2,\dots,6;n=0,1,2,\dots\}} \{\tau_{1(j)}^{(n)}\}$ and suppose that when $\tau_1 = \tau_1^t$, Eq. (3.25) has a pair of imaginary roots $\pm i\mu_*$.

Now we present the following assumption:

$$(\mathcal{O}_{17}) \quad J_{1R}J_{2R} + J_{1I}J_{2I} > 0,$$

where

$$\begin{cases} J_{1R} = -3\mu_*^2 + c_2 + 2c_4\mu_* \sin \mu_*\tau_* + c_5 \cos \mu_*\tau_*, \\ J_{1I} = 2c_1\mu_* + 2c_4 \cos \mu_*\tau_* - c_5 \sin \mu_*\tau_*, \\ J_{2R} = (c_6\mu_* - c_4\mu_*^3) \sin \mu_*\tau_* - c_5\mu_*^2, \\ J_{2I} = c_5\mu_*^2 \sin \mu_*\tau_* + (c_6\mu_* - c_4\mu_*^3) \cos \mu_*\tau_*. \end{cases} \quad (5.18)$$

Lemma 5.1. Suppose that $\lambda(\tau_1) = j_1(\tau_1) + ij_2(\tau_1)$ is the root of Eq. (5.10) at $\tau_1 = \tau_1^t$ such that $j_1(\tau_1^t) = 0$, $j_2(\tau_1^t) = \mu_*$, then $\text{Re} \left(\frac{d\lambda}{d\tau_2} \right) \Big|_{\tau_1=\tau_1^t, \mu=\mu_*} > 0$.

Proof. By Eq. (5.10), we can gain

$$\begin{aligned} & (3\lambda^2 + 2c_1\lambda + c_2)\frac{d\lambda}{d\tau_1} + (2c_4\lambda + c_5)e^{-\lambda\tau_1}\frac{d\lambda}{d\tau_1} \\ & - (c_4\lambda^2 + c_5\lambda + c_6)e^{-\lambda\tau_1}\left(\lambda + \frac{d\lambda}{d\tau_1}\tau_1\right) = 0, \end{aligned} \quad (5.19)$$

which indicates

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{J_1(\lambda)}{J_2(\lambda)} - \frac{\tau_1}{\lambda}, \quad (5.20)$$

where

$$\begin{cases} J_1(\lambda) = 3\lambda^2 + 2c_1\lambda + c_2 + (c_4\lambda + c_5)e^{-\lambda\tau_1}, \\ J_2(\lambda) = (c_4\lambda^2 + c_5\lambda + c_6)\lambda e^{-\lambda\tau_1}. \end{cases} \quad (5.21)$$

Hence

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau_1} \right)^{-1} \right]_{\tau_1=\tau_1^t, \mu_*=\mu_*} = \operatorname{Re} \left[\frac{J_1(\lambda)}{J_2(\lambda)} \right]_{\tau_1=\tau_1^t, \mu_*=\mu_*} = \frac{J_{1R}J_{2R} + J_{1I}J_{2I}}{J_{2R}^2 + J_{2I}^2}. \quad (5.22)$$

By $(\mathcal{O}_{17})$, we get

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\tau_1=\tau_1^t, \mu_*=\mu_*} > 0, \quad (5.23)$$

the proof is concluded. $\square$

From the above discussion, the following theorem is derived.

Theorem 5.1. *If $(\mathcal{O}_{15})$ – $(\mathcal{O}_{17})$ are valid, then the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of model (5.1) maintains locally asymptotically stable state if $\tau_1 \in [0, \tau_1^t)$. Moreover, model (5.1) generates a cluster of Hopf bifurcations around the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ when $\tau_1 = \tau_1^t$.*

6. Control of bifurcation for model (1.2) with $\tau_1 = 0, \tau_2 > 0$ using delay feedback controller

In this segment, we will apply a delay feedback controller to dominate stability region and the time of generation of Hopf bifurcation of system (1.2) under the condition of $\tau_1 = 0, \tau_2 > 0$. By means of the ideas of [41–43], we set up the following controlled predator-prey model:

$$\begin{cases} \frac{du_1}{dt} = ru_1(t) \left(1 - \frac{u_1(t)}{h} \right) - \frac{cu_1(t)u_3(t)}{a + \xi\beta u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = \rho u_2(t) - \delta u_2(t)u_3(t) - \chi(u_2(t) - u_2(t - \tau_2)), \\ \frac{du_3}{dt} = \frac{bu_1(t - \tau_2)u_3(t)}{a + \xi\beta u_2(t) + u_1(t - \tau_2)} + \gamma u_2(t)u_3(t) - nu_3(t), \end{cases} \quad (6.1)$$

where χ denotes feedback gain parameter. System (6.1) owns the same equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ as that in system (1.2). Let

$$\begin{cases} u_{1*} = u_1(t) - \bar{u}_1(t), \\ u_{2*} = u_2(t) - \bar{u}_2(t), \\ u_{3*} = u_3(t) - \bar{u}_3(t). \end{cases} \quad (6.2)$$

The linear system of system (6.1) near $E(u_{1*}, u_{2*}, u_{3*})$ can be expressed as follows:

$$\begin{cases} \frac{d\bar{u}_1}{dt} = z_1\bar{u}_1 - z_2\bar{u}_2 - z_3\bar{u}_3, \\ \frac{d\bar{u}_2}{dt} = z_4\bar{u}_2 + z_5\bar{u}_2(t - \tau_2) - z_6\bar{u}_3, \\ \frac{d\bar{u}_3}{dt} = z_7\bar{u}_1(t - \tau_2) + z_8\bar{u}_2 + z_9\bar{u}_3, \end{cases} \quad (6.3)$$

where

$$\begin{cases} z_1 = r - \frac{2ru_{1*}}{h} - \frac{cu_{3*}}{a + \xi\beta u_{2*} + u_{1*}} + \frac{cu_{1*}u_{3*}}{(a + \xi\beta u_{2*} + u_{1*})^2}, \\ z_2 = \frac{c\xi\beta u_{1*}u_{3*}}{(a + \xi\beta u_{2*} + u_{1*})^2}, \\ z_3 = \frac{cu_{1*}}{a + \xi\beta u_{2*} + u_{1*}}, \\ z_4 = \rho - \delta u_{3*} - \chi, \\ z_5 = \chi, \\ z_6 = \delta u_{2*}, \\ z_7 = \frac{bu_{3*}}{a + \xi\beta u_{2*} + u_{1*}} - \frac{bu_{1*}u_{3*}}{(a + \xi\beta u_{2*} + u_{1*})^2}, \\ z_8 = \gamma u_{3*} - \frac{b\xi\beta u_{1*}u_{3*}}{(a + \xi\beta u_{2*} + u_{1*})^2}, \\ z_9 = \frac{bu_{1*}}{a + \xi\beta u_{2*} + u_{1*}} + \gamma u_{2*} - n. \end{cases} \quad (6.4)$$

The characteristic equation of system (6.3) owns the following expression:

$$\det \begin{bmatrix} \lambda - z_1 & -z_3 & z_3 \\ 0 & \lambda - z_4 - z_5 e^{-\lambda\tau_2} & z_6 \\ -z_7 e^{-\lambda\tau_2} & -z_8 & \lambda - z_9 \end{bmatrix} = 0, \quad (6.5)$$

which results in

$$\lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 + (r_4\lambda^2 + r_5\lambda + r_6)e^{-\lambda\tau_2} - r_7e^{-2\lambda\tau_2} = 0, \quad (6.6)$$

where

$$\begin{cases} r_1 = -z_9 - z_4 - z_1, \\ r_2 = z_4z_9 + z_6z_8 + z_1z_4 + z_1z_9, \\ r_3 = -z_1z_4z_9 - z_1z_6z_8, \\ r_4 = -z_5, \\ r_5 = z_5z_9 + z_1z_5 + z_3z_7, \\ r_6 = z_2z_6z_7 - z_3z_4z_7 - z_1z_5z_9, \\ r_7 = z_3z_5z_7. \end{cases} \quad (6.7)$$

If $\tau_2 = 0$, then Eq. (6.6) becomes

$$\lambda^3 + (r_1 + r_4)\lambda^2 + (r_2 + r_5)\lambda + r_3 + r_6 - r_7 = 0. \quad (6.8)$$

If

$$(\mathcal{O}_{18}) \begin{cases} \nabla_1 = r_1 + r_4 > 0, \\ \nabla_2 = \det \begin{bmatrix} r_1 + r_4 & 1 \\ r_3 + r_6 - r_7 & r_2 + r_5 \end{bmatrix} > 0, \\ \nabla_3 = (r_3 + r_6 - r_7)\nabla_2 > 0, \end{cases} \quad (6.9)$$

is valid, then the three roots $\lambda_1, \lambda_2, \lambda_3$ of Eq. (6.8) have negative real parts. Thus the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of system (6.1) with the delay $\tau_2 = 0$ maintains locally asymptotically stable.

Assume that $\lambda = i\varepsilon$ ($\varepsilon > 0$) is the root of Eq. (6.6), then Eq. (6.6) becomes:

$$((i\varepsilon)^3 + r_1(i\varepsilon)^2 + r_2i\varepsilon + r_3)e^{i\varepsilon\tau} + r_4(i\varepsilon)^2 + r_5i\varepsilon + r_6 - r_7e^{-i\varepsilon\tau_2} = 0. \quad (6.10)$$

It follows from Eq. (6.10) that

$$\begin{cases} O_1(\varepsilon) \sin \varepsilon\tau_2 + O_2(\varepsilon) \cos \varepsilon\tau_2 = O_3, \\ O_1(\varepsilon) \cos \varepsilon\tau_2 - O_2(\varepsilon) \sin \varepsilon\tau_2 = O_4, \end{cases} \quad (6.11)$$

where

$$\begin{cases} O_1 = r_3 - r_1\varepsilon^2 - r_7, \\ O_2 = \varepsilon^3 - r_2\varepsilon, \\ O_3 = r_4\varepsilon^2 - r_6, \\ O_4 = -r_5\varepsilon. \end{cases} \quad (6.12)$$

In consideration of $O_1^2 + O_2^2 = O_3^2 + O_4^2$, we gain

$$\varepsilon^6 + O_1\varepsilon^4 + O_2\varepsilon^2 + O_3 = 0, \quad (6.13)$$

where

$$\begin{cases} O_1 = r_1^2 - 2r_2 - r_4^2, \\ O_2 = 2r_1r_7 + 2r_4r_6 + r_2^2 - 2r_1r_3 - r_5^2, \\ O_3 = (r_3 - r_7)^2 - r_6^2. \end{cases} \quad (6.14)$$

Let

$$\Phi_8(\varepsilon) = \varepsilon^6 + O_1\varepsilon^4 + O_2\varepsilon^2 + O_3. \quad (6.15)$$

Assume that

$$(\mathcal{O}_{19}) \quad O_3 < 0$$

is satisfied, then the $\Phi_8(0) < 0$ and $\lim_{\varepsilon \rightarrow \infty} \Phi_8(\varepsilon) = +\infty > 0$, then Eq. (6.13) at least one positive real root. Hence, Eq. (6.10) at least one pair of purely imaginary root. Without loss

of generality, we can assume that Eq. (6.13) has six positive real roots (say $\varepsilon_j, j = 1, 2, \dots, 6$). Relying on (6.11), we know

$$\tau_{2j}^{(k)} = \frac{1}{\varepsilon_j} \left[\arccos \left(\frac{O_{1(j)}O_{4(j)} - O_{2(j)}O_{3(j)}}{O_{1(j)}^2 + O_{2(j)}^2} + 2k\pi \right) \right], \quad (6.16)$$

where

$$\begin{cases} O_{1(j)} = r_3 - r_1\varepsilon_j^2 - r_7, \\ O_{2(j)} = \varepsilon_j^3 - r_2\varepsilon_j, \\ O_{3(j)} = r_4\varepsilon_j^2 - r_6, \\ O_{4(j)} = -r_5\varepsilon_j, \end{cases} \quad (6.17)$$

and $j = 1, 2, \dots, 6; k = 0, 1, 2, \dots$. Assume $\tau_2^l = \min_{\{j=1,2,\dots,6;k=0,1,2,\dots\}} \{\tau_{2(j)}^{(k)}\}$ and suppose that when $\tau_2 = \tau_2^l$, Eq. (6.10) has a pair of imaginary roots $\pm i\varepsilon_*$.

Next we present the following assumption:

$$(\mathcal{O}_{20}) \quad O_{1R}O_{2R} + O_{1I}O_{2I} > 0,$$

where

$$\begin{cases} O_{1R} = -3\varepsilon_*^3 + r_2 + 2r_4\varepsilon_* \sin \varepsilon_*\tau_2^l + r_5 \cos \varepsilon_*\tau_2^l, \\ O_{1I} = 2r_1\varepsilon_* + 2r_4\varepsilon_* \cos \varepsilon_*\tau_2^l - r_5 \sin \varepsilon_*\tau_2^l, \\ O_{2R} = (r_6\varepsilon_* - 2r_7\varepsilon_* - \varepsilon_*^3) \sin \varepsilon_*\tau_2^l - r_5\varepsilon_*^2 \cos \varepsilon_*\tau_2^l, \\ O_{2I} = (r_6\varepsilon_* - 2r_7\varepsilon_* - \varepsilon_*^3) \cos \varepsilon_*\tau_2^l + r_5\varepsilon_*^2 \sin \varepsilon_*\tau_2^l. \end{cases} \quad (6.18)$$

Lemma 6.1. Suppose that $\lambda(\tau_2) = \ell_1(\tau_2) + i\ell_2(\tau_2)$ is the root of Eq. (6.10) at $\tau_2 = \tau_2^l$ such that $\ell_1(\tau_2^l) = 0$, $\ell_2(\tau_2^l) = \varepsilon_*$, then $\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau_2=\tau_2^l, \varepsilon=\varepsilon_*} > 0$.

Proof. Using Eq. (6.10), we can gain

$$\begin{aligned} & [(3\lambda^2 + 2r_1\lambda + r_2)] \frac{d\lambda}{d\tau_2} + (2r_4\lambda + r_5)e^{-\lambda\tau_2} \frac{d\lambda}{d\tau_2} \\ & - (r_4\lambda^2 + r_5\lambda + r_6)e^{-\lambda\tau_2} (\lambda + \frac{d\lambda}{d\tau_2}\tau_2) = 0, \end{aligned} \quad (6.19)$$

which indicates

$$\left(\frac{d\lambda}{d\tau_2} \right)^{-1} = \frac{V_1(\lambda)}{V_2(\lambda)} - \frac{\tau_2}{\lambda}, \quad (6.20)$$

where

$$\begin{cases} V_1(\lambda) = 3\lambda^2 + 2r_1\lambda + r_2 + (2r_4\lambda + r_5)e^{-\lambda\tau_2}, \\ V_2(\lambda) = (r_4\lambda^2 + r_5\lambda + r_6)e^{-\lambda\tau_2}\lambda - 2r_7e^{-\lambda\tau_2}\lambda. \end{cases} \quad (6.21)$$

Therefore

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right]_{\tau_2=\tau_2^l, \varepsilon=\varepsilon_*} = \operatorname{Re} \left[\frac{V_1(\lambda)}{V_2(\lambda)} \right]_{\tau_2=\tau_2^l, \varepsilon=\varepsilon_*} = \frac{V_{1R}V_{2R} + V_{1I}V_{2I}}{V_{2R}^2 + V_{2I}^2}. \quad (6.22)$$

By the assumption $(\mathcal{O}_{20})$, we get

$$\operatorname{Re} \left[\left(\frac{d\lambda}{d\tau_2} \right)^{-1} \right]_{\tau_2=\tau_2^t, \varepsilon=\varepsilon_*} > 0, \quad (6.23)$$

the proof is concluded. $\square$

From the above discussion, the following theorem is derived.

Theorem 6.1. *Assume that $(\mathcal{O}_{18})$ – $(\mathcal{O}_{20})$ are valid, then the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ of model (6.1) holds locally asymptotically stable state if $\tau_2 \in [0, \tau_2^t)$. Moreover, model (6.1) exhibits a cluster of Hopf bifurcations around the equilibrium point $E(u_{1*}, u_{2*}, u_{3*})$ when $\tau_2 = \tau_2^t$.*

Remark 6.1. In this paper, although some assumptions are very complex, we can check the correctness of these assumptions via computer software.

Remark 6.2. Section 4 deals with the bifurcation of model (1.2) with $\tau_1 = \tau_2 = \tau$ via delay feedback controller. Also we explore the bifurcation of model (1.2) under other delay cases via delay feedback controller. We leave it for future research direction.

Remark 6.3. Section 5 deals with the bifurcation of model (1.2) with $\tau_1 > 0, \tau_2 = 0$ via delay feedback controller. Of course, we can also explore the bifurcation of model (1.2) under other delay cases via same delay feedback controller. We need to do numerical simulation and theoretical analysis to check whether the designed controller is effective or not. We leave it for future research direction.

Remark 6.4. In terms of the form of the controller, the controllers of Section 4, Section 5 and Section 6 are same. But we add this delay feedback controller to different equations under different delay cases and then obtain different critical delay values to ensure the onset of Hopf bifurcation.

7. Matlab simulations

In this section, we carry out Matlab simulation to verify the correctness of the obtained results.

Example 7.1. When $\tau_1 > 0, \tau_2 = 0$, take into account the following system that incorporates a delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t - \tau_1)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{0.9u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.1)$$

It is evident that system (7.1) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. One can readily confirm that conditions $(\mathcal{O}_1)$ – $(\mathcal{O}_3)$ outlined in Theorem 3.1 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_1^* \approx 2.4$. In order to confirm the accuracy of the outcomes obtained from Theorem 3.1, for the time-delay variable τ_1 , we select two distinct values: The first one is 2.0 and the second one is 2.7. If $\tau_1 = 2.0 < \tau_1^* \approx 2.4$, the corresponding numerical simulation diagrams are illustrated in Figure 1. In Figure 1, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.1) possesses locally asymptotically stable state. If $\tau_1 = 2.7 > \tau_1^* \approx 2.4$, the corresponding

numerical simulation diagrams are illustrated in Figure 2. In Figure 2, we can clearly observe that $u_1(t)$, $u_2(t)$, $u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerges in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.2. When $\tau_1 = 0, \tau_2 > 0$, take into account the following system that incorporates a delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{0.9u_1(t - \tau_2)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t - \tau_2)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.2)$$

It is evident that system (7.2) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. One can readily confirm that conditions $(\mathcal{O}_1)$, $(\mathcal{O}_4)$, $(\mathcal{O}_5)$ outlined in Theorem 3.2 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_2^* \approx 0.4$. In order to confirm the accuracy of the outcomes obtained from Theorem 3.2, for the time-delay variable τ_2 , we select two distinct values: The first one is 0.1 and the second one is 0.8. If $\tau_2 = 0.1 < \tau_2^* \approx 0.4$, the corresponding numerical simulation diagrams are illustrated in Figure 3. In Figure 3, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.) possesses locally asymptotically stable state. If $\tau_2 = 0.8 > \tau_2^* \approx 0.4$, the corresponding numerical simulation diagrams are illustrated in Figure 4. In Figure 4, we can clearly observe that $u_1(t)$, $u_2(t)$, $u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively. which means that, a series of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.3. When $\tau_1 = \tau_2 = \tau$, take into account the following system that incorporates a delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t - \tau)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{0.9u_1(t - \tau)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t - \tau)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.3)$$

It is evident that system (7.3) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. One can readily confirm that conditions $(\mathcal{O}_1)$, $(\mathcal{O}_6)$, $(\mathcal{O}_7)$ outlined in Theorem 3.3 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau^* \approx 0.5$. In order to confirm the accuracy of the outcomes obtained from Theorem 3.3, for the time-delay variable τ , we select two distinct values: The first one is 0.1 and the second one is 0.92. If $\tau = 0.1 < \tau^* \approx 0.5$, the corresponding numerical simulation diagrams are illustrated in Figure 5. In Figure 5, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.3) possesses locally asymptotically stable state. If $\tau = 0.92 > \tau^* \approx 0.5$, the corresponding numerical simulation diagrams are illustrated in Figure 6. In Figure , we can clearly observe that $u_1(t)$, $u_2(t)$, $u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.4. Consider the following system accompanying delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t - \tau_1)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{0.9u_1(t - \tau_2)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t - \tau_2)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.4)$$

Where $\tau_1 > 0, \tau_2 = 0.2$. It is evident that system (7.4) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. One can readily confirm that conditions $(\mathcal{O}_1)$, $(\mathcal{O}_8)$, $(\mathcal{O}_9)$ outlined in Theorem 3.4 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_1^* \approx 2.2$. In order to confirm the accuracy of the outcomes obtained from Theorem 3.4, for the time-delay variable τ_1 , we select two distinct values: The first one is 2.0 and the second one is 2.5. If $\tau_1 = 2.0 < \tau_1^* \approx 2.4$, the corresponding numerical simulation diagrams are illustrated in Figure 7. In Figure 7, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.4) possesses locally asymptotically stable state. If $\tau_1 = 2.5 > \tau_1^* \approx 2.0$, the corresponding numerical simulation diagrams are illustrated in Figure 8. In Figure 8, we can clearly observe that $u_1(t), u_2(t), u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.5. Consider the following system accompanying delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t - \tau_1)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{0.9u_1(t - \tau_2)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t - \tau_2)} + 0.56u_2(t)u_3(t) - 0.9u_3(t), \end{cases} \quad (7.5)$$

where $\tau_2 > 0, \tau_1 = 1.8$. It is evident that system (7.5) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. One can readily confirm that conditions $(\mathcal{O}_1)$, $(\mathcal{O}_{10})$, $(\mathcal{O}_{11})$ outlined in Theorem 3.5 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_2^* \approx 0.6$. In order to confirm the accuracy of the outcomes obtained from Theorem 3.5, for the time-delay variable τ_2 , we select two distinct values: The first one is 0.3 and the second one is 0.8. If $\tau_2 = 0.3 < \tau_2^* \approx 0.6$, the corresponding numerical simulation diagrams are illustrated in Figure 9. In Figure 9, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.5) possesses locally asymptotically stable state. If $\tau_2 = 0.8 > \tau_2^* \approx 0.6$, the corresponding numerical simulation diagrams are illustrated in Figure 10. In Figure 10, we can clearly observe that $u_1(t), u_2(t), u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.6. Consider the following system accompanying delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t-\tau)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t) - \varsigma[u_2(t) - u_2(t-\tau)], \\ \frac{du_3}{dt} = \frac{0.9u_1(t-\tau)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t-\tau)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.6)$$

It is evident that system (7.6) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. Let $\varsigma = 0.1$, when $\tau_1 = \tau_2 = \tau$, One can readily confirm that conditions $(\mathcal{O}_{12})$ – $(\mathcal{O}_{14})$ outlined in Theorem 4.1 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_{(0)} \approx 0.8$. In order to confirm the accuracy of the outcomes obtained from Theorem 4.1, for the time-delay variable τ , we select two distinct values: The first one is 0.55 and the second one is 2.0. If $\tau = 0.55 < \tau_1^* \approx 0.8$, the corresponding numerical simulation diagrams are illustrated in Figure 11. In Figure 11, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.6) possesses locally asymptotically stable state. If $\tau = 2.0 > \tau_{(0)} \approx 0.8$, the corresponding numerical simulation diagrams are illustrated in Figure 12. In Figure 12, we can clearly observe that $u_1(t), u_2(t), u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.7. Consider the following system accompanying delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t-\tau_1)}{0.2} \right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)} \\ \quad - \varpi[u_1(t) - u_1(t-\tau_1)], \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t), \\ \frac{du_3}{dt} = \frac{0.9u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.7)$$

It is evident that system (7.7) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. Let $\varpi = 0.02$, when $\tau_1 > 0, \tau_2 = 0$, One can readily confirm that conditions $(\mathcal{O}_{15})$ – $(\mathcal{O}_{17})$ outlined in Theorem 5.1 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_1^t \approx 2.5$. In order to confirm the accuracy of the outcomes obtained from Theorem 5.1, for the time-delay variable τ_1 , we select two distinct values: The first one is 2.0 and the second one is 2.7. If $\tau_1 = 2.0 < \tau_1^t \approx 2.5$, the corresponding numerical simulation diagrams are illustrated in Figure 13. In Figure 13, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.7) possesses locally asymptotically stable state. If $\tau_1 = 2.7 > \tau_1^t \approx 2.5$, the corresponding numerical simulation diagrams are illustrated in Figure 14. In Figure 14, we can clearly observe that $u_1(t), u_2(t), u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerges in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Example 7.8. Consider the following system accompanying delay.

$$\begin{cases} \frac{du_1}{dt} = 0.5u_1(t) \left(1 - \frac{u_1(t)}{0.2}\right) - \frac{0.3u_1(t)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t)}, \\ \frac{du_2}{dt} = 0.39u_2(t) - 0.9u_2(t)u_3(t) - \varphi[u_2(t) - u_2(t - \tau_2)], \\ \frac{du_3}{dt} = \frac{0.9u_1(t - \tau_2)u_3(t)}{0.57 + 0.1 \times 0.2u_2(t) + u_1(t - \tau_2)} + 0.56u_2(t)u_3(t) - 0.9u_3(t). \end{cases} \quad (7.8)$$

It is evident that system (7.8) possesses a unique positive equilibrium point $E(0.1283, 1.3227, 0.4333)$. Let $\varphi = 0.02$, when $\tau_1 = 0, \tau_2 > 0$, One can readily confirm that conditions $(\mathcal{O}_{18})$ - $(\mathcal{O}_{20})$ outlined in Theorem 6.1 are all valid. Numerical simulations are carried out using Matlab software, we are able to obtain $\tau_2^l \approx 0.4$. In order to confirm the accuracy of the outcomes obtained from Theorem 6.1, for the time-delay variable τ_2 , we selected two distinct values: The first one is 0.12 and the second one is 1.1. If $\tau_2 = 0.12 < \tau_2^l \approx 0.4$, the corresponding numerical simulation diagrams are illustrated in Figure 15. In Figure 15, we can clearly observe that $u_1(t) \rightarrow 0.1283, u_2(t) \rightarrow 1.3227, u_3(t) \rightarrow 0.4333$ when $t \rightarrow +\infty$. To put it another way, the equilibrium point $E(0.1283, 1.3227, 0.4333)$ of model (7.8) possesses locally asymptotically stable state. If $\tau_1 = 1.1 > \tau_1^l \approx 0.4$, the corresponding numerical simulation diagrams are illustrated in Figure 16. In Figure 16, we can clearly observe that $u_1(t), u_2(t), u_3(t)$ will keep oscillate periodically in the vicinity of the values 0.1283, 1.3227, 0.4333 respectively, which means that, a series of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium $E(0.1283, 1.3227, 0.4333)$.

Remark 7.1. Based on the numerical simulation graphs of Example 7.3 and Example 7.6, we know that their bifurcation values are $\tau^* = 0.5$ and $\tau_{(0)} = 0.8$ respectively. This indicates that we can expand the stability region of system (7.3) through the design of an appropriate delay feedback controller. From the computer simulation graphs of Example 7.1 and Example 7.7, we are aware that the respective bifurcation values for them are $\tau_1^* = 2.4$ and $\tau_1^l = 2.5$, which indicates that we can expand the stability region of system (7.1) by designing a delay feedback controller.

8. Conclusions

This paper investigates a predator-prey model with two distinct time delays. In the first place, By using fixed point theorem, inequality transformation and construction of appropriate functions, the existence, uniqueness and non-negative of the solution of the delayed predator-prey system are proved. Next, we select time delay as the bifurcation parameter and the two time delays can be classified into five situations ($\tau_1 = \tau_2 = 0, \tau_1 > 0, \tau_2 = 0, \tau_2 > 0, \tau_1 = 0, \tau_1 = \tau_2 = \tau, \tau_1 = \tau_2 \neq 0$). Further, we explore the delay critical value that ensures the occurrence of bifurcation phenomena of this model. By utilizing bifurcation and stability theory of delayed differential equation, the stability and Hopf bifurcation of this model are studied. A new delay-independent criterion on stability and bifurcation of this model is derived. In the end, in order to effectively adjust the stability region and the onset of bifurcation phenomena of this model, three types of delay feedback controllers are designed. Then, the parametric conditions of this system remains stable under delay feedback controllers as well as the bifurcation phenomena of this model are explored. This study focuses on exploring predator-prey model with two time delays, aiming to uncover how two time delays influence the stability and periodic oscillations

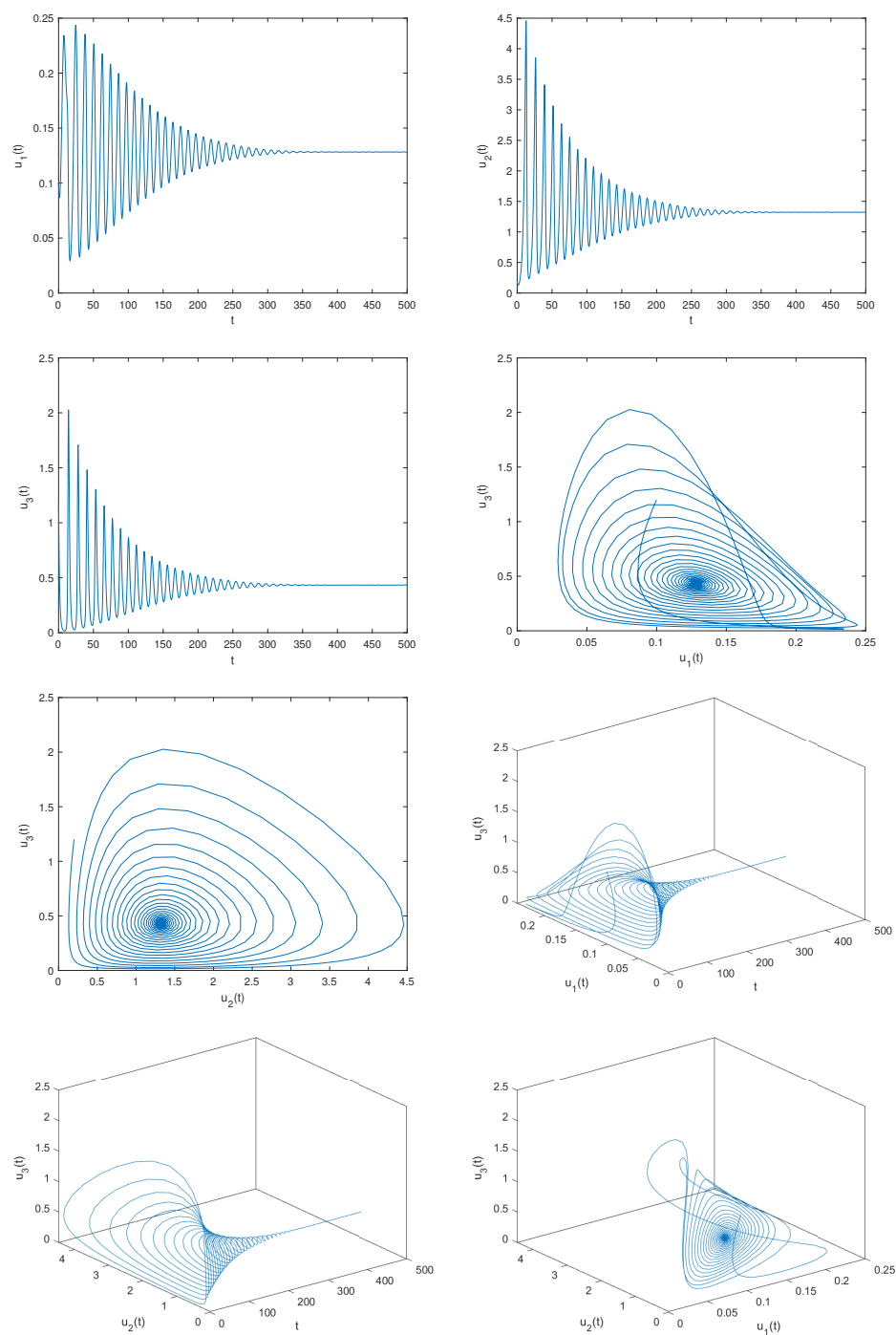


Figure 1. Matlab simulation graphs of system (7.1) under the delay $\tau_1 > 0, \tau_2 = 0$ and $\tau_1 = 2.0 < \tau_1^* = 2.4$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

of ecosystem. In addition, the dynamic behavior of the ecosystem can be further regulated by designing controllers to delay or advance the occurrence of Hopf bifurcations.

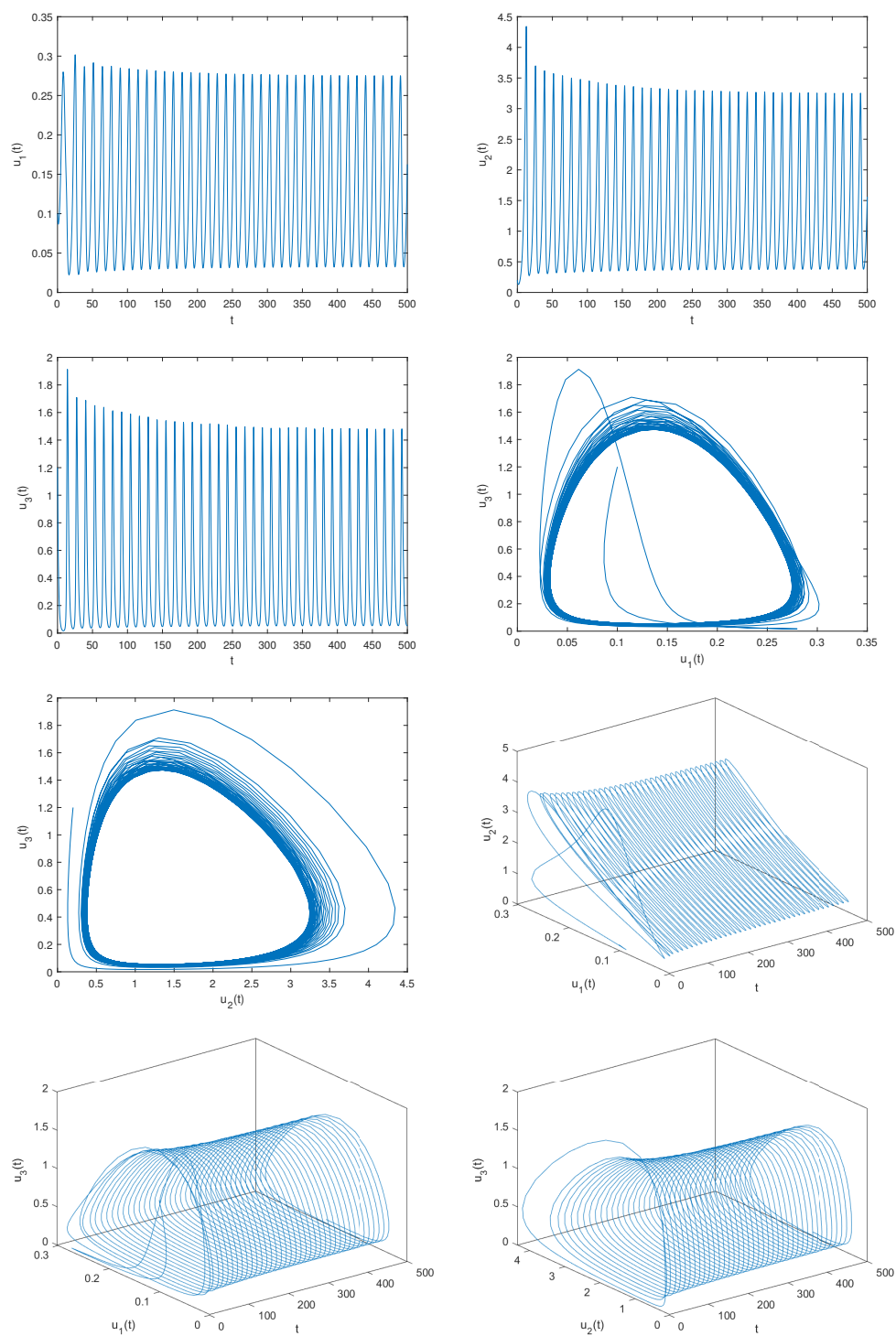


Figure 2. Matlab simulation figures of system (7.1) under the delay $\tau_1 > 0, \tau_2 = 0$ and $\tau_1 = 2.7 > \tau_1^* = 2.4$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

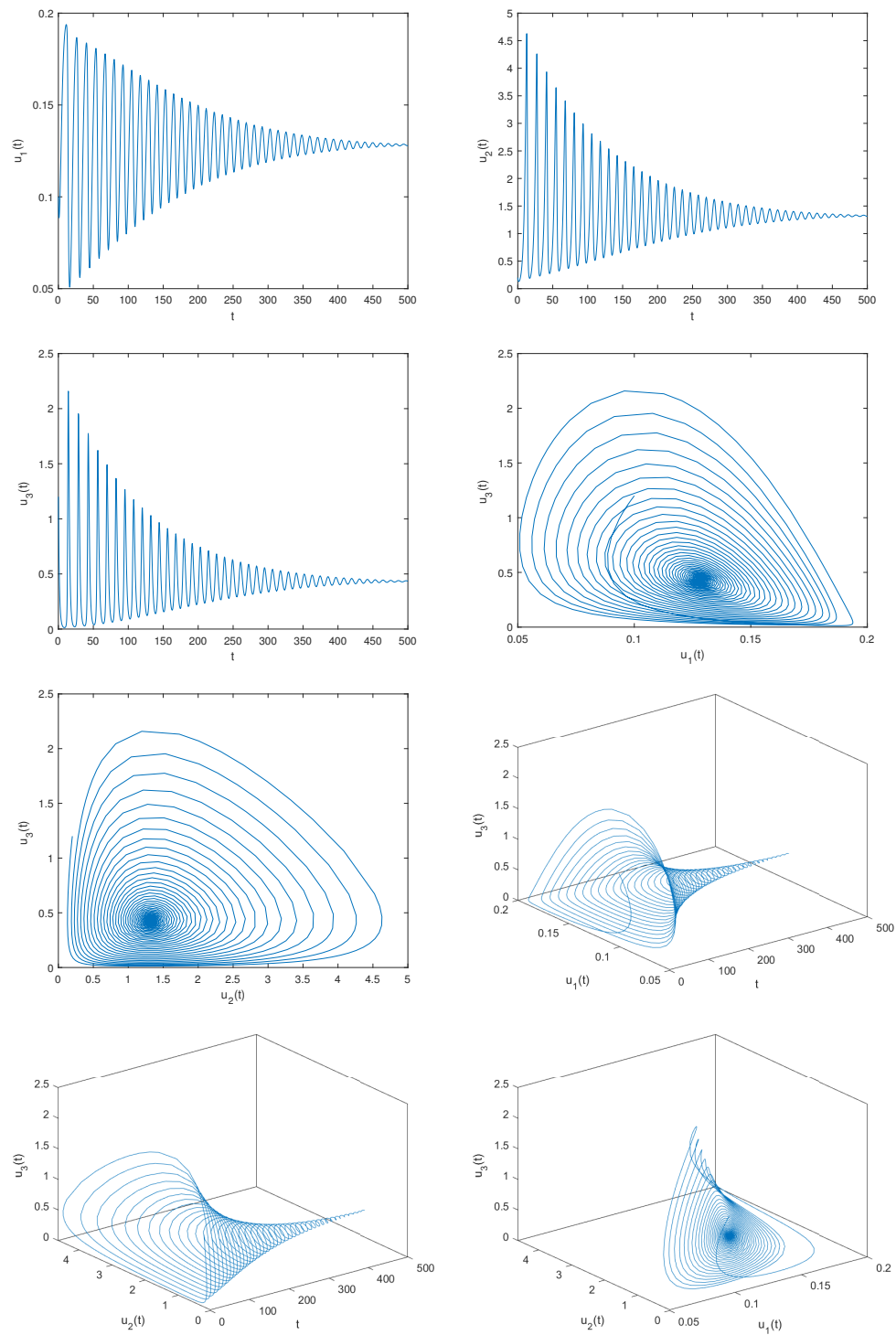


Figure 3. Matlab simulation graphs of system (7.2) under the delay $\tau_1 = 0, \tau_2 > 0$ and $\tau_2 = 0.1 < \tau_2^* = 0.4$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

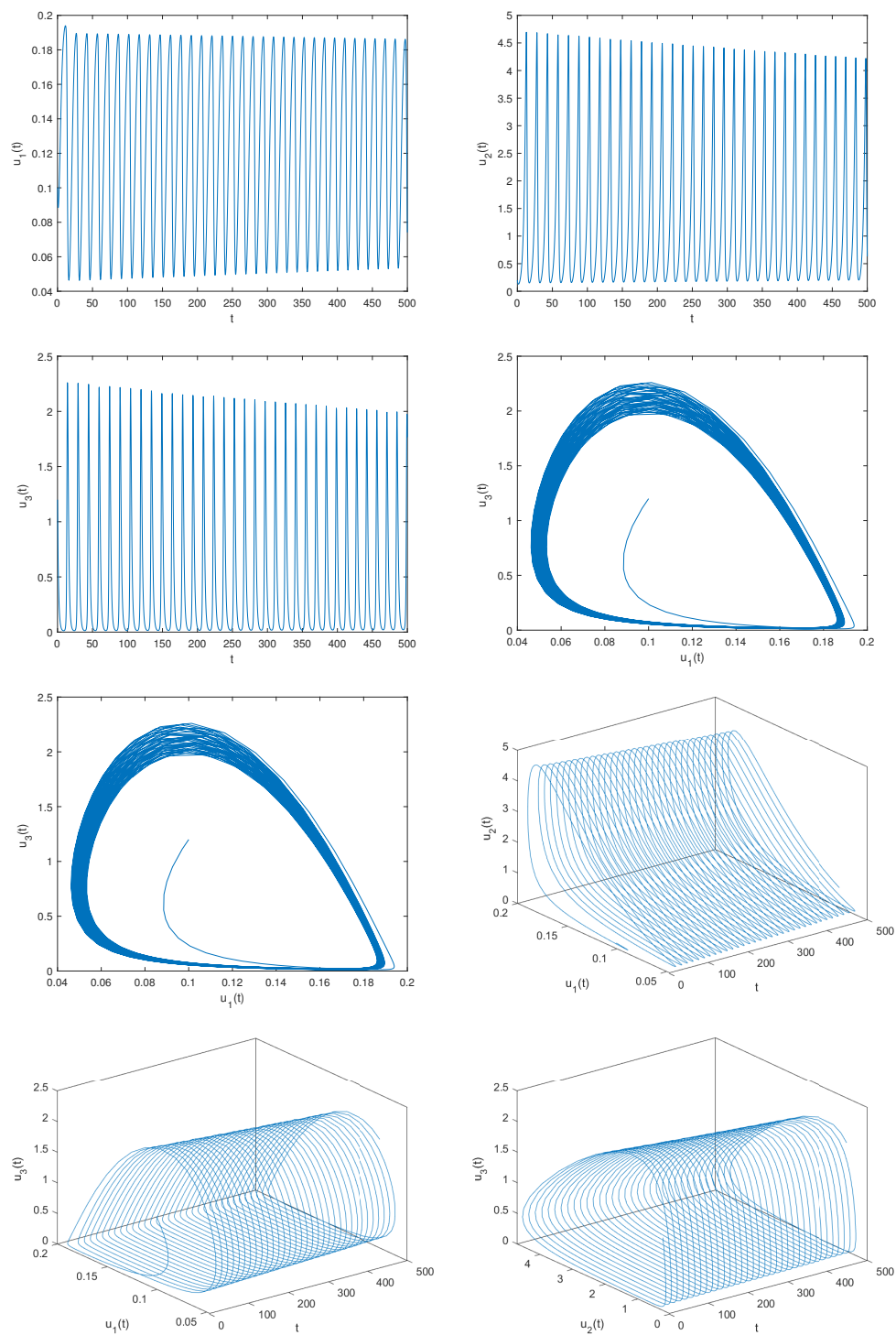


Figure 4. Matlab simulation figures of system (7.2) under the delay $\tau_1 = 0, \tau_2 > 0$ and $\tau_2 = 0.8 > \tau_2^* = 0.4$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

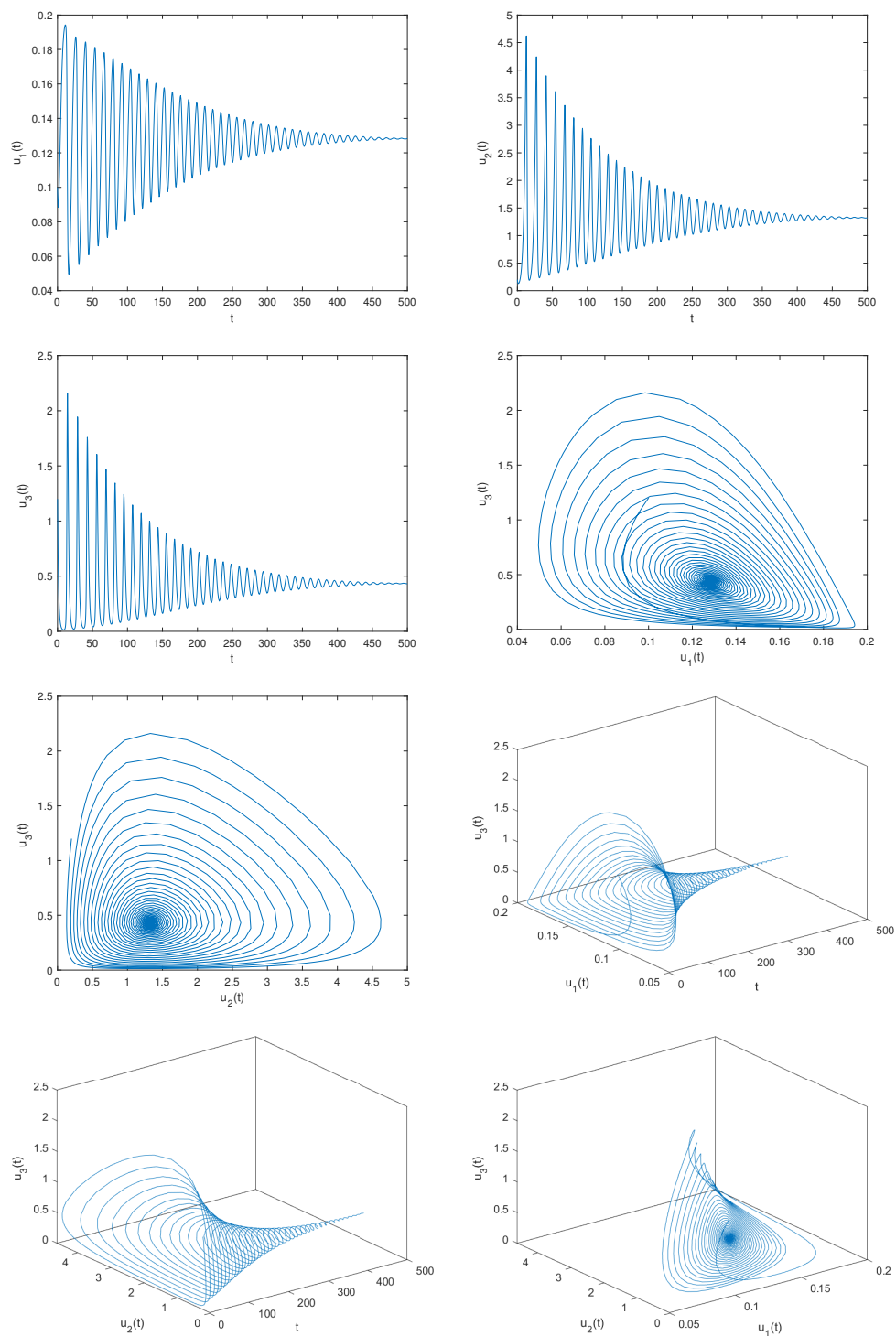


Figure 5. Matlab simulation graphs of system (7.3) under the delay $\tau_1 = \tau_2 = \tau$ and $\tau = 0.1 < \tau^* = 0.5$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

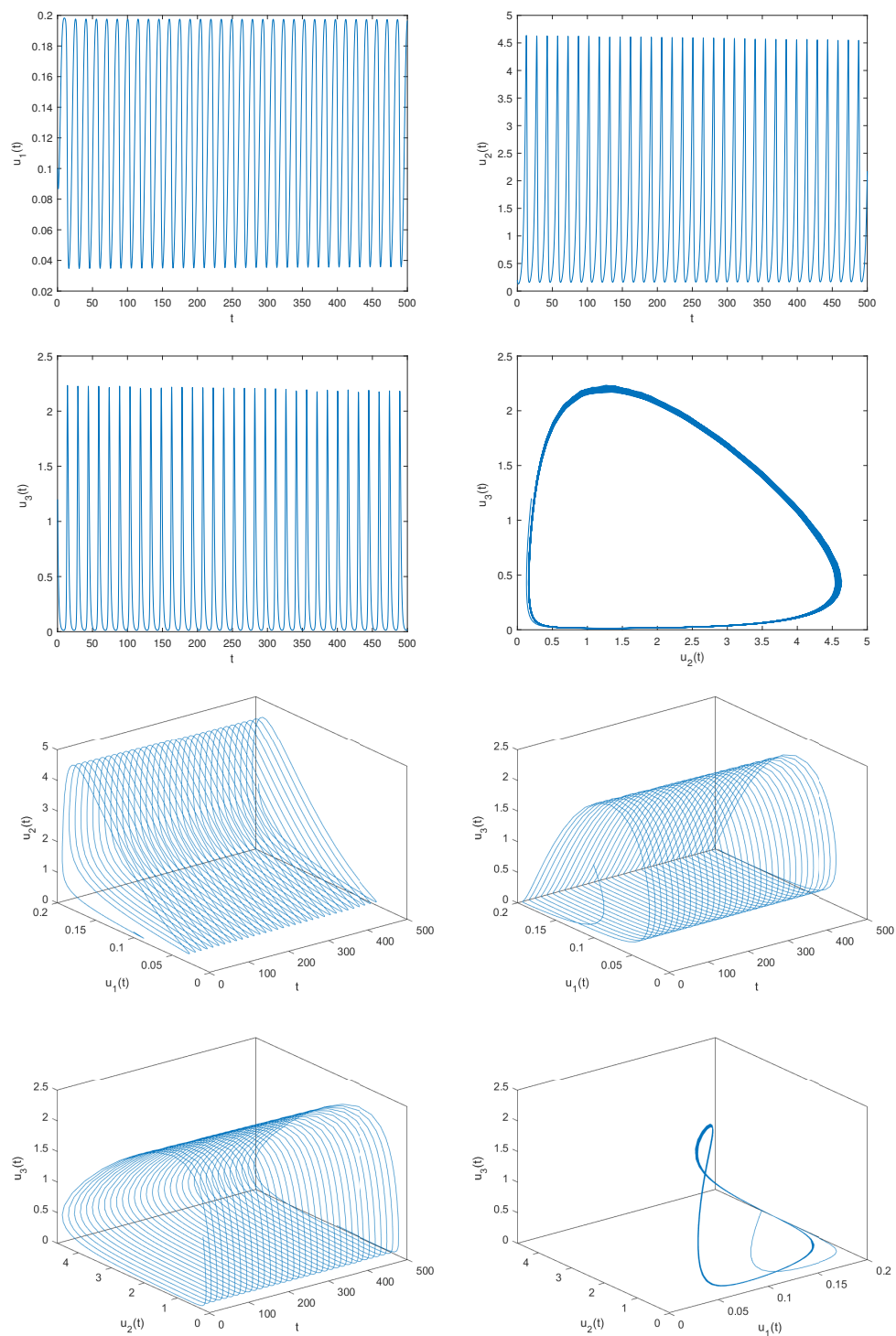


Figure 6. Matlab simulation plots of system (7.3) under the delay $\tau_1 = \tau_2 = \tau$ and $\tau = 0.92 > \tau^* = 0.5$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

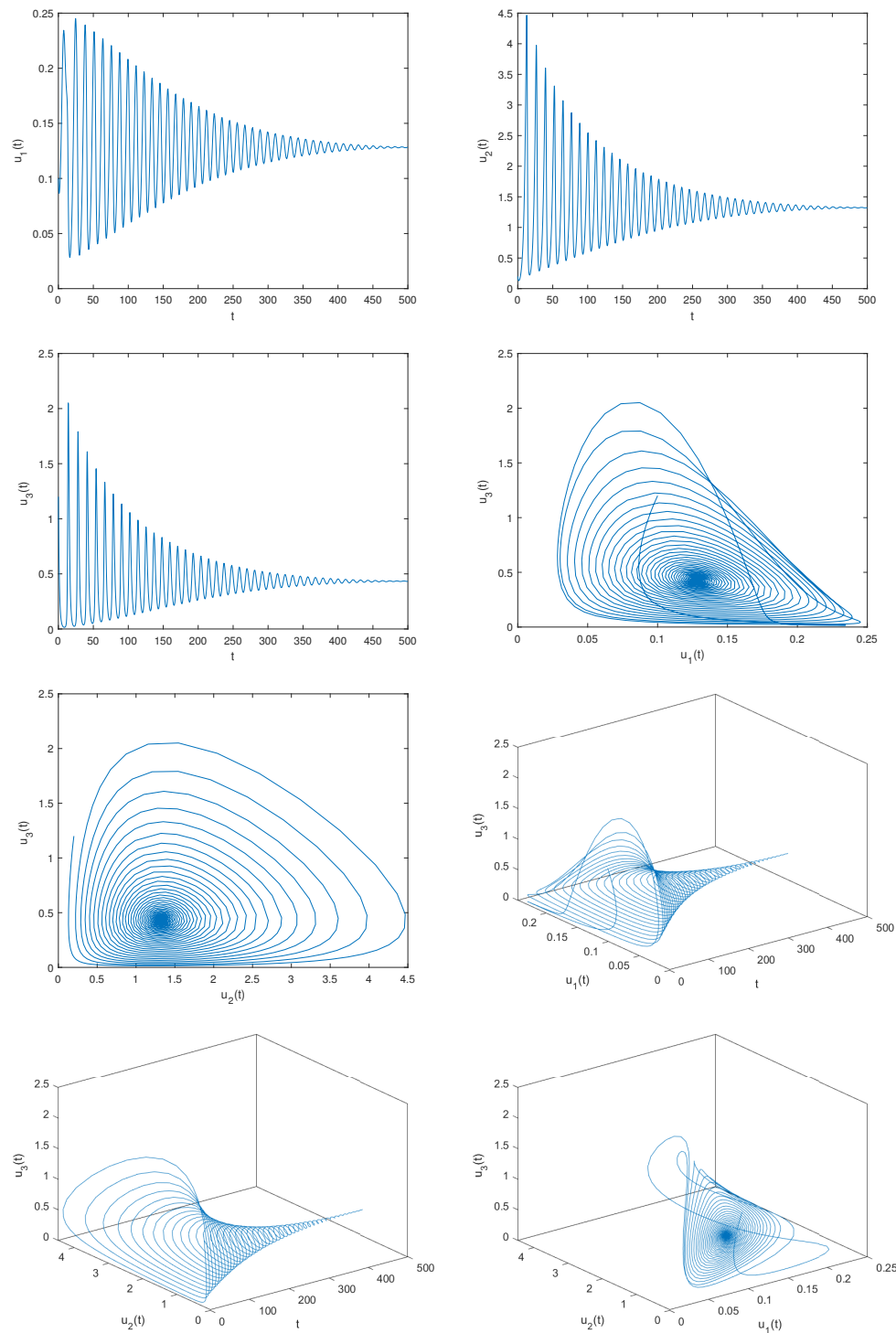


Figure 7. Matlab simulation graphs of system (7.4) under the delay $\tau_1 = 2.0, \tau_2 = 0.2$ and $\tau_1 = 2.0 < \tau_1^* = 2.2$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

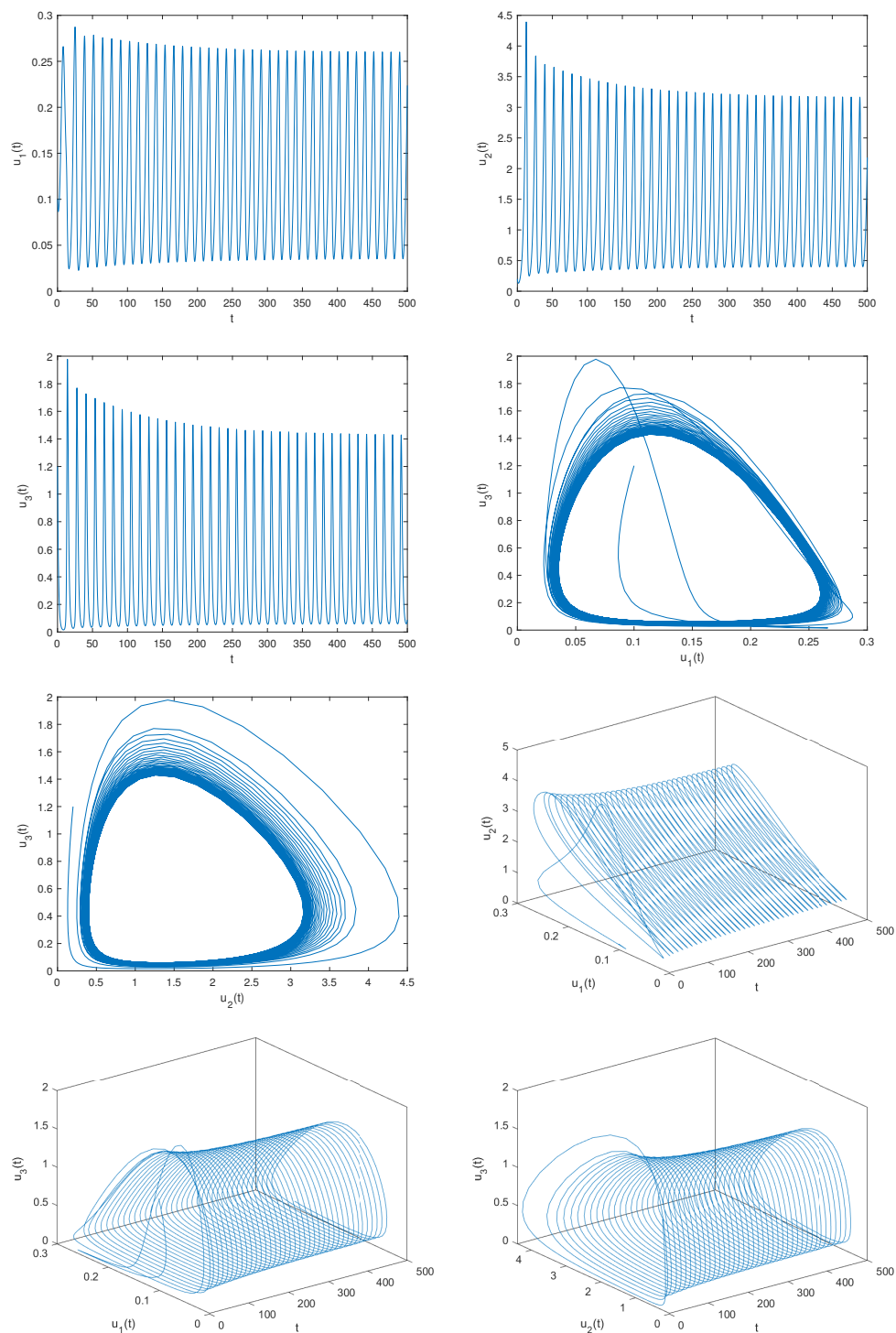


Figure 8. Matlab simulation figures of system (7.4) under the delay $\tau_1 = 2.5, \tau_2 = 0.2$ and $\tau_1 = 2.5 > \tau_1^* = 2.2$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

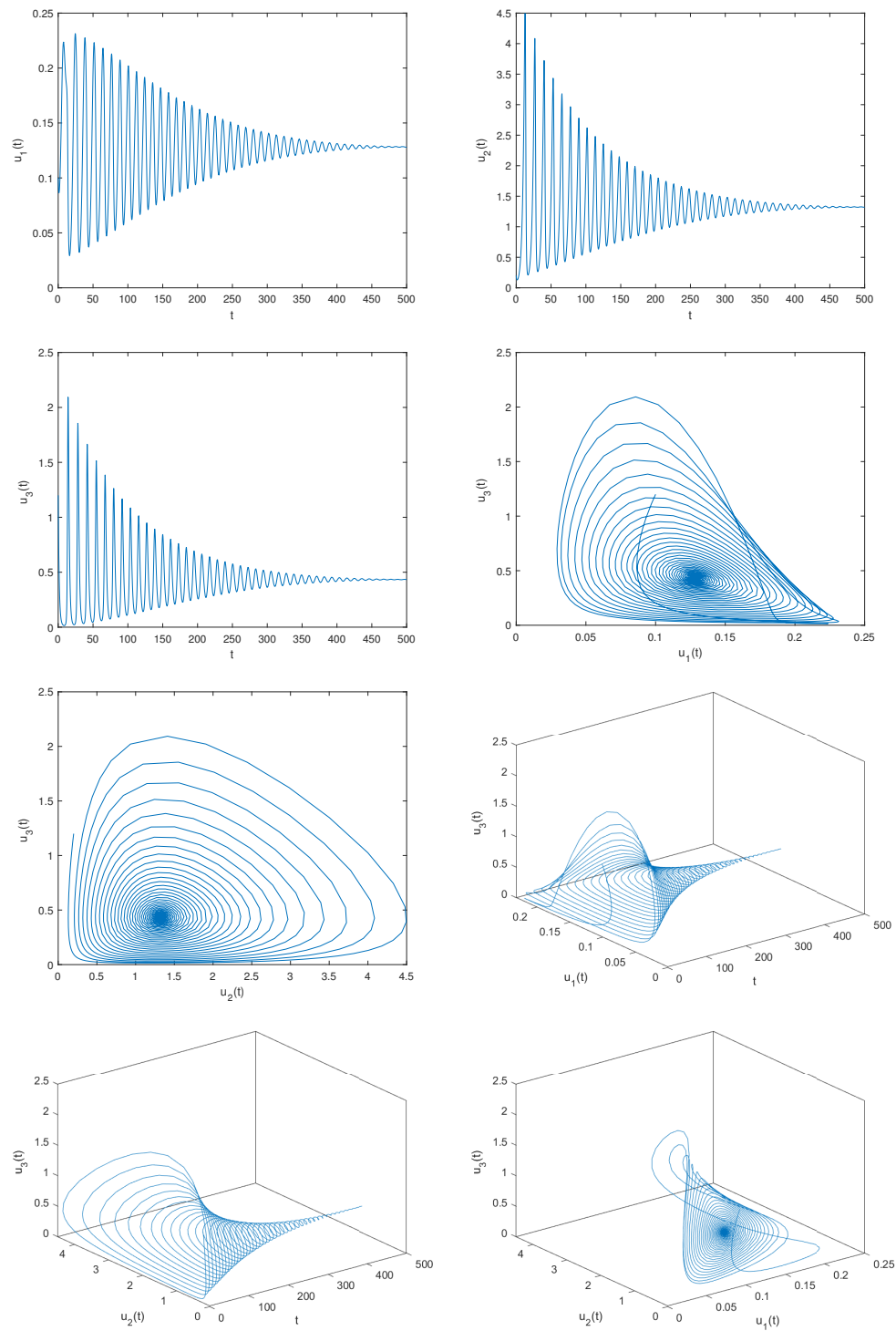


Figure 9. Matlab simulation graphs of system (7.5) under the delay $\tau_1 = 1.8, \tau_2 = 0.3$ and $\tau_2 = 0.3 < \tau_2^* = 0.6$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

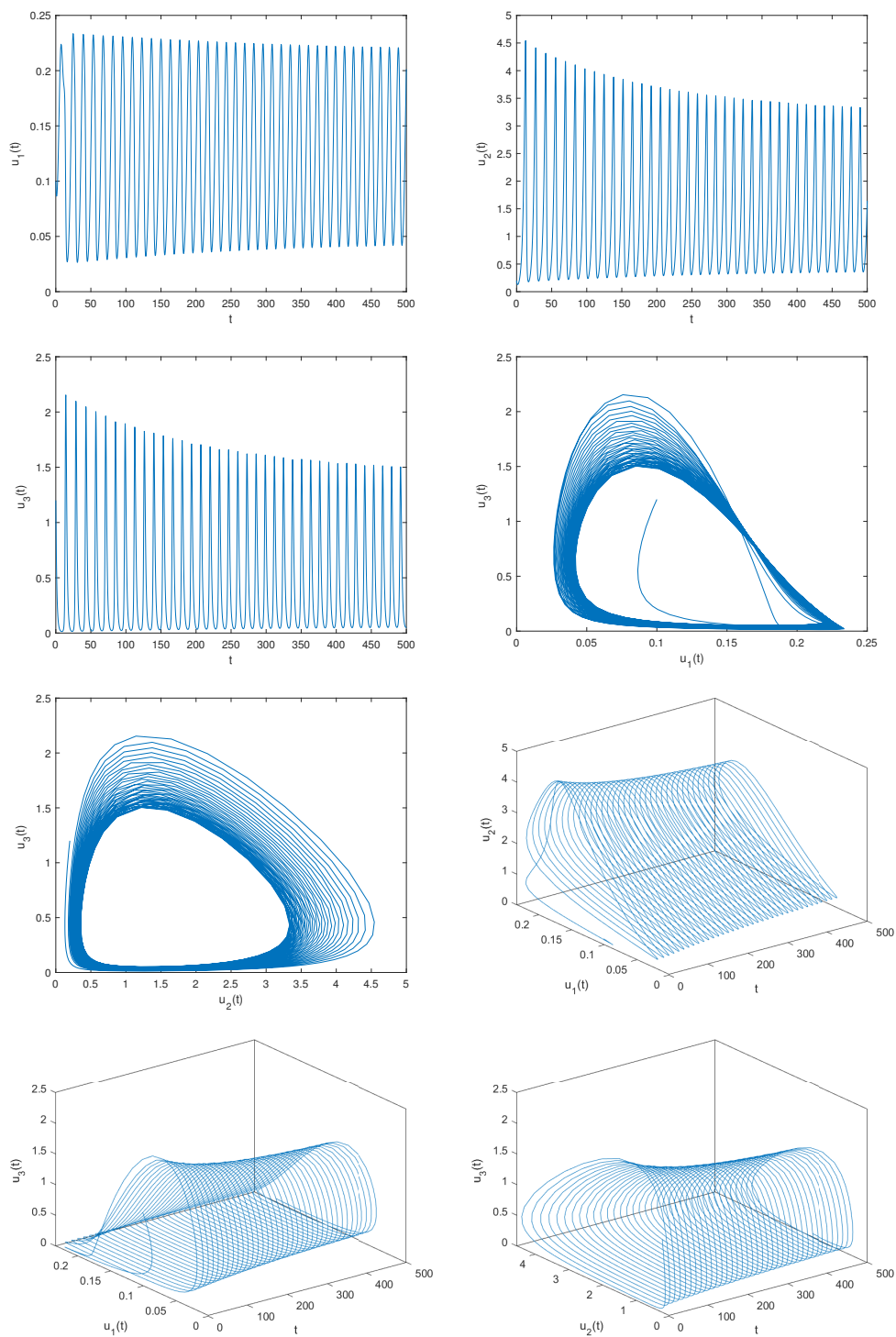


Figure 10. Matlab simulation figures of system (7.5) under the delay $\tau_1 = 1.8, \tau_2 = 0.2$ and $\tau_2 = 0.8 > \tau_2^* = 0.6$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

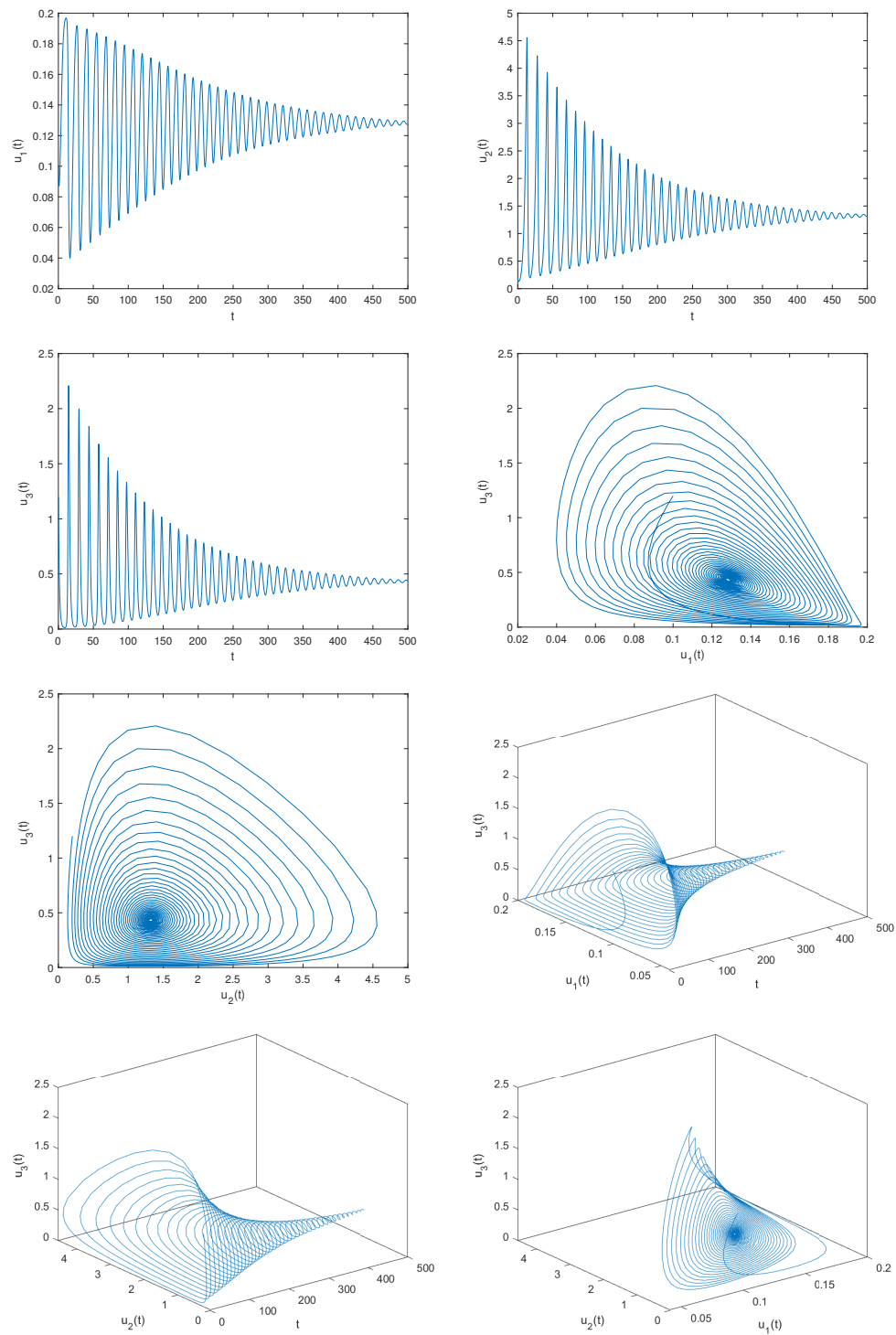


Figure 11. Matlab simulation graphs of system (7.6) under the delay $\tau_1 = \tau_2 = \tau$ and $\tau = 0.55 < \tau_{(0)} = 0.8$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

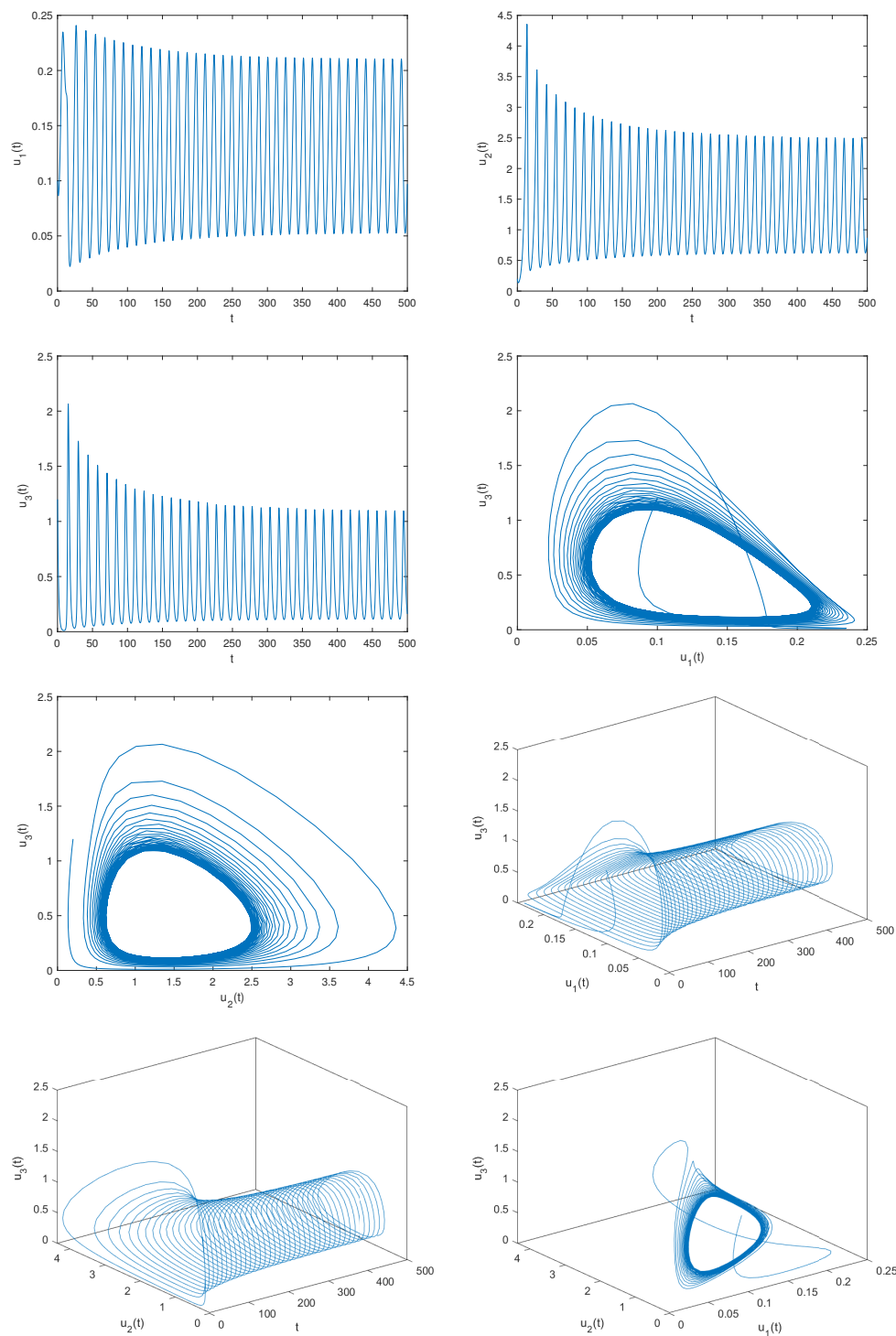


Figure 12. Matlab simulation figures of system (7.6) under the delay $\tau_1 = \tau_2 = \tau$ and $\tau = 2.0 > \tau_{(0)} = 0.8$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

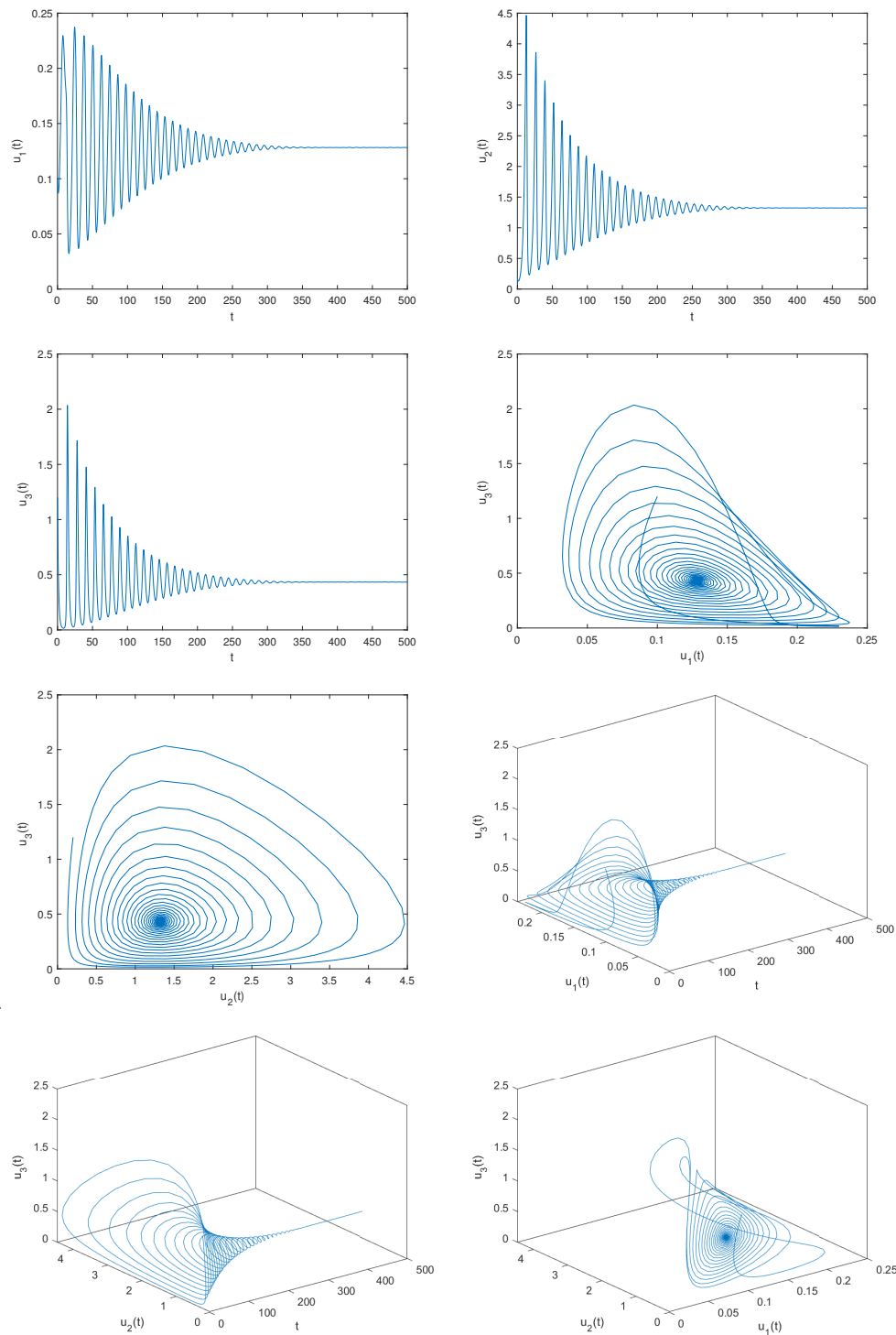


Figure 13. Matlab simulation graphs of system (7.7) under the delay $\tau_1 = 2.0, \tau_2 = 0$ and $\tau_1 = 2.0 < \tau_1^* = 2.5$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

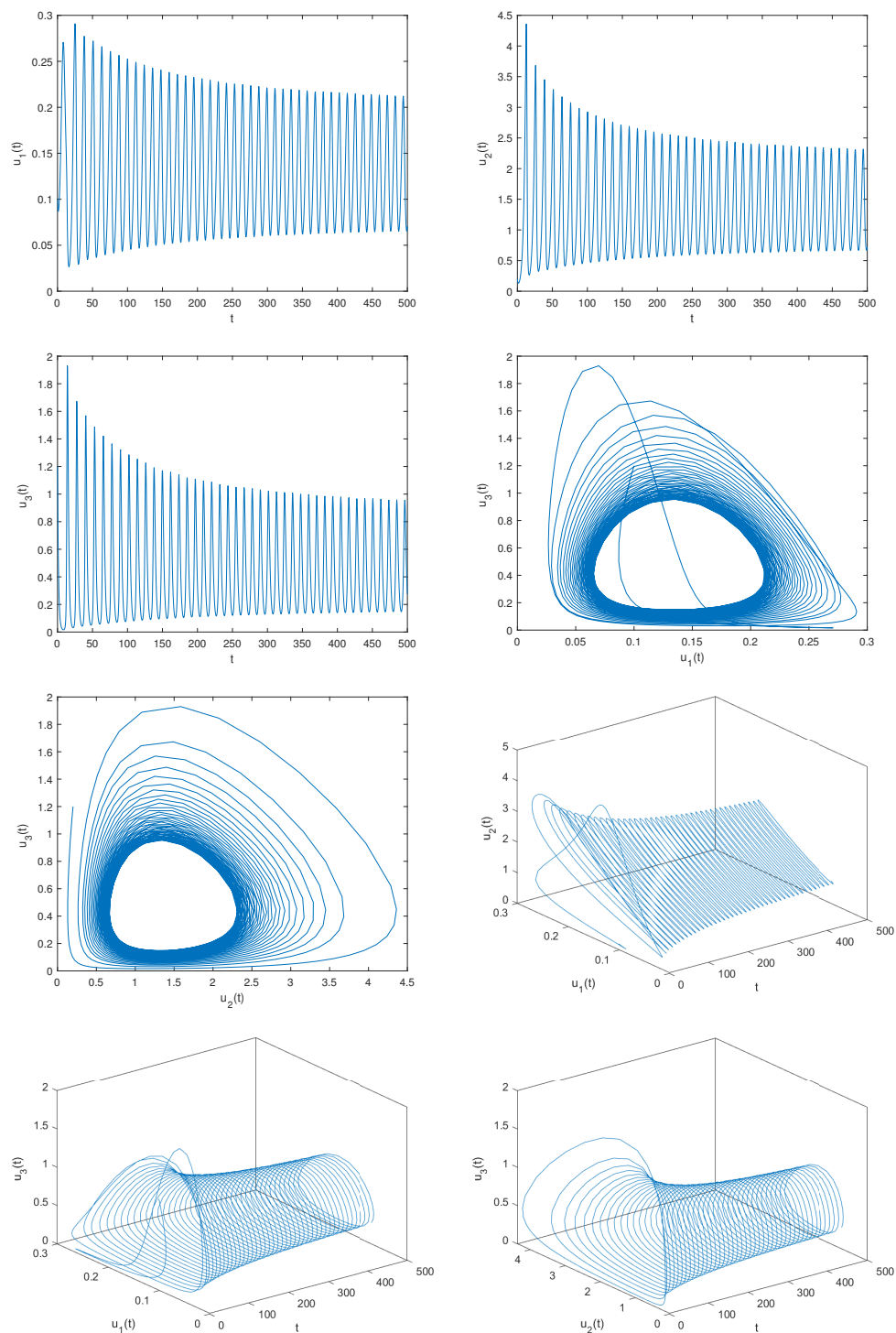


Figure 14. Matlab simulation figures of system (7.7) under the delay $\tau_1 = 2.0, \tau_2 = 0$ and $\tau_1 = 2.7 > \tau_1^c = 2.5$. A cluster of periodic solutions (namely, Hopf bifurcations) emerge in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

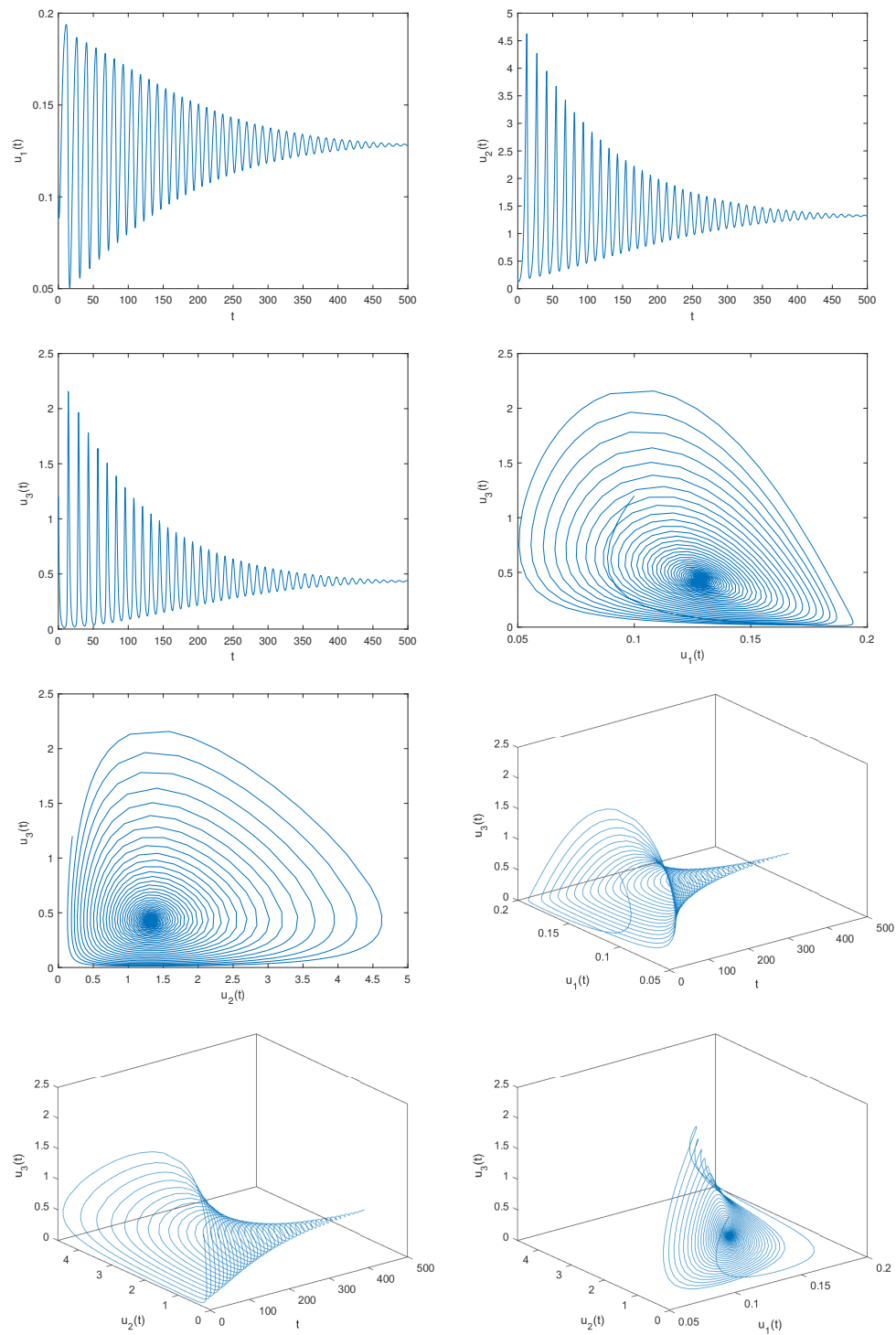


Figure 15. Matlab simulation graphs of system (7.8) under the delay $\tau_1 = 0, \tau_2 = 0.12$ and $\tau_2 = 0.12 < \tau_2^* = 0.4$. The equilibrium point $E(0.1283, 1.3227, 0.4333)$ maintains locally asymptotically stable level.

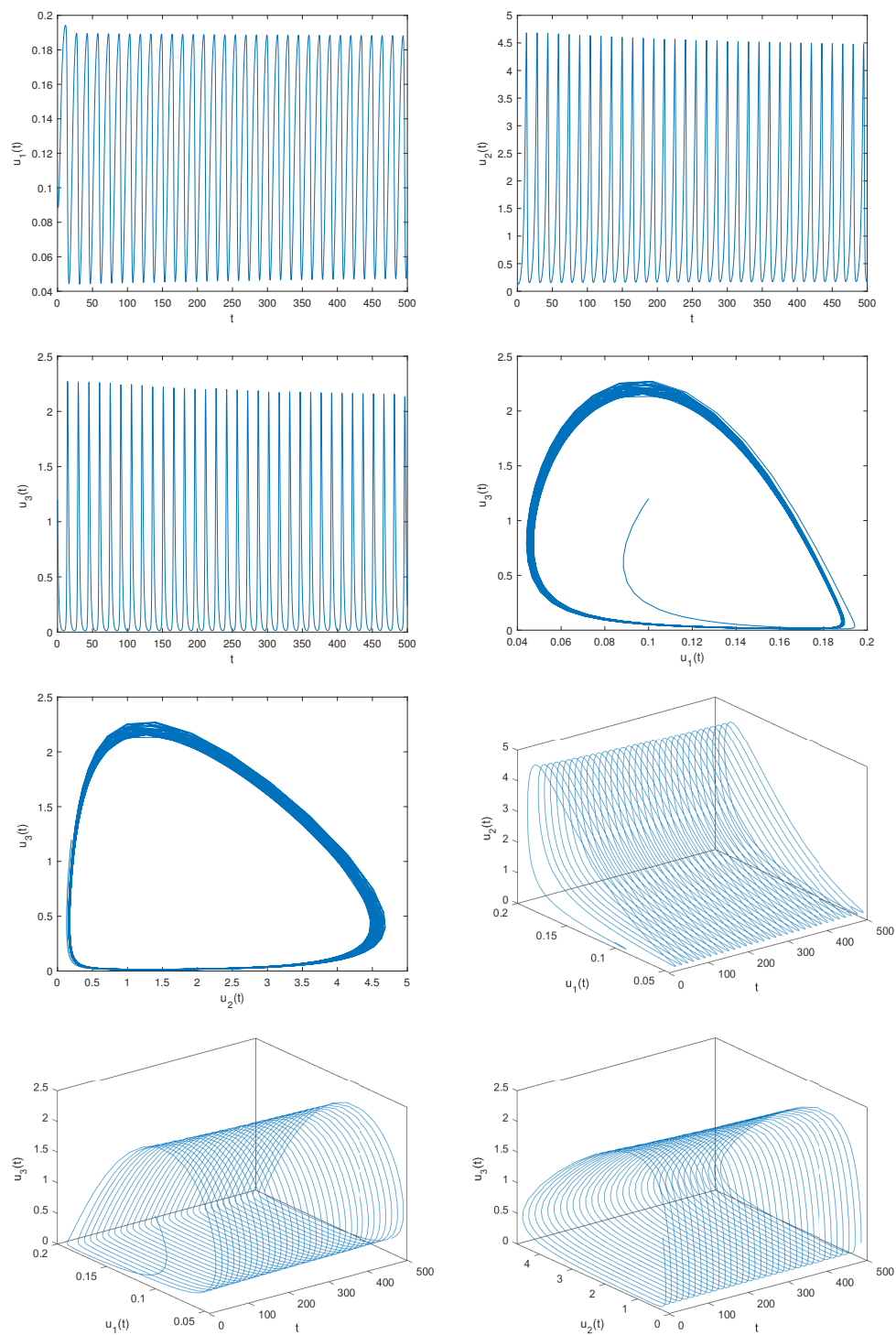


Figure 16. Matlab simulation figures of system (7.8) under the delay $\tau_1 = 0, \tau_2 = 1.1$ and $\tau_2 = 1.1 > \tau_2^L = 0.4$. A cluster of periodic solutions (namely, Hopf bifurcations) emerges in the vicinity of the equilibrium point $E(0.1283, 1.3227, 0.4333)$.

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