Journal of Applied Analysis and Computation
Volume 16, Number 2, April 2026, 916–936
Website:http://www.jaac-online.com
DOI:10.11948/20250196

# WELL-POSEDNESS OF NONLOCAL IMPULSIVE PROBLEMS OF NON-AUTONOMOUS EVOLUTION EQUATIONS WITH DELAY*

**Huanhuan Wang[1], Xiaoya Liu[2] and Xuping Zhang[1,3,†]**

**Abstract** This paper investigates the existence of extremal mild solutions for nonlocal impulsive problems of non-autonomous evolution equations with delay in ordered Banach spaces. By applying the perturbation technique and monotone iterative method in the presence of lower and upper solutions, we establish the existence of the minimal and maximal mild solutions under suitable monotonicity conditions and noncompactness measure requirements of nonlinear term. Furthermore, we prove the existence of at least one mild solution and obtain the uniqueness of mild solution between the lower and the upper solutions.

**Keywords** Nonlocal impulsive problems, non-autonomous evolution equations, mild solution, monotone iterative method.

**MSC(2010)** 35R12, 35K45, 47D06, 47H08.

## 1. Introduction

Many complex processes in nature and technology are described by functional differential equations which are dominant nowadays because the functional components in equations allow one to consider after-effect or prehistorical influence. Delay evolution equation is one of the important type of functional differential equations, in which the response of system depends not only on the current state of system, but also on the past history of system. For more details on this topic, see [22, 26, 28, 31] and the references therein.

The study of abstract nonlocal Cauchy problem was initiated by Byszewski in [6]. It is demonstrated that the nonlocal problems have better effects in applications than the traditional Cauchy problems, differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [12, 20, 25, 33] and the references therein for more comments and citations.

The theory of instantaneous impulsive differential equations describes processes which experience a sudden change in their states at certain moments. Such processes commonly occur in physics, chemistry, biology, engineering, and economics. The theory of instantaneous impulsive evolution equations has gained significance in recent years due to its wide applications

---

[†] The corresponding author.

[1] Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
[2] School of Mathematical Sciences, Gansu Minzu Normal University, Gannan 747000, China
[3] Gansu Provincial Research Center for Basic Disciplines of Mathematics and Statistics, Lanzhou 730070, China

[*] This work was supported by the Outstanding Youth Science Fund of Gansu Province (No. 24JRRA122), Funds for Innovative Fundamental Research Group Project of Gansu Province (No. 23JRRA684) and Natural Science Foundation of Gansu Province (No. 24JRRA780).

Email: dahuan0901@126.com(H. Wang), liuxiaoy135@163.com(X. Liu), lanyu9986@126.com(X. Zhang)

in control, mechanics, electrical engineering, and biological fields. In the last few decades, the study of instantaneous impulsive evolution equations in Banach spaces has emerged as an important research area. For more details on the theory and its applications, refer to references [3, 5, 18, 19, 23, 24, 29, 32].

The method of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Early on, Du and Lakshmikantham [13] built the method of lower and upper solutions for addressing the initial ordinary differential equations in Banach space. Later, Guo and Liu [16], Li and Liu [24] developed the iterative method for ordinary differential equations with instantaneous impulses in Banach spaces. Recently, the iterative method has been extended to evolution equations in ordered Banach spaces, we refer to the papers by EI-Gebeily, O'Regan and Nieto [15] and Wang and Wang [30] for evolution equations with classical initial value conditions, and to the paper by Chen and Li [7] and Chen, Li and Yang [8] for evolution equations with impulses in Banach spaces.

In recent years, non-autonomous evolution equations have been extensively studied in the case of a compact evolution family, see [14, 21, 34]. However, up to now, there has been a lack of relevant research papers that adopt the iterative method, perturbation technique, and the approach of lower and upper solutions to study nonlocal impulsive problems of non-autonomous evolution equations with delay. The main advantage by using the iterative method based on lower and upper solutions is that it not only provides a method to obtain the existence of extremal mild solutions, but also yields iterative sequences of lower and upper approximate solutions that converge to the minimal and maximal mild solutions between the lower and upper solutions.

In view of the above reasons, in this paper we will combine the theory of evolution family and the monotone iterative technique to consider the existence of mild solutions for nonlocal impulsive problems of non-autonomous evolution equations with delay

$$\begin{cases} u'(t) - A(t)u(t) = f(t, u(t), u_t), \quad t \in [0, a], \ t \neq t_k, \\ \triangle u \mid_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(t) = g(u)(t) + \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1.1}$$

where $a$ and $h$ are positive constants, $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $D(A(t))$ for every $t \in [0, a]$, $f : [0, a] \times X \times \mathcal{B} \to X$ is a Carathéodory continuous; $0 < t_1 < t_2 < \cdots < t_m < a$ are pre-fixed numbers, $I_k \in C(X, X)$ is an impulsive function, $k = 1, 2, \cdots, m$, $u(t_k^+)$ and $u(t_k^-)$ represents the right and the left limits of $u(t)$ at $t = t_k$, $\phi \in \mathcal{B}$ is a priori given history, while the function $g : PC([-h, a], X) \to \mathcal{B}$ implicitly defines a complementary history, chosen by the system itself. $u_t$ denotes the function in $\mathcal{B}$ defined as $u_t(\tau) = u(t + \tau)$ for $\tau \in [-h, 0]$ and $u_t(\cdot)$ represent the time history of the state from the time $t - h$ up to the present time $t$.

The outline of this paper is as follows. In Section 2, notation and preliminaries are introduced, which are used throughout this paper. In Section 3, we obtained the existence of extremal mild solutions and mild solutions for the problem (1.1) under the situation that the evolution family $\{\mathcal{U}(t, s) : 0 \leq s \leq t \leq a\}$ generated by $A(t)$ is compact. The existence of extremal mild solutions and the uniqueness of mild solution for the problem (1.1) are obtained under the situation that the evolution family $\{\mathcal{U}(t, s) : 0 \leq s \leq t \leq a\}$ generated by $A(t)$ is not compact in Section 4. Finally, an concrete example is given to illustrate the feasibility of our abstract results.

## 2. Notations and preliminaries

We begin by giving some notation. Let $X$ be an ordered Banach space with norm $\|\cdot\|$ and partial order $\leq$ with positive cone $P = \{u \in X \mid u \geq \theta\}$ ($\theta$ is the zero element of $X$), which defines a partial ordering in $X$ by $u \leq v$ if and only if $v - u \in P$. If $u \leq v$ and $u \neq v$, we say $u < v$. The cone $P$ is called normal if there exists a positive constant $N$ such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$, in which $N$ is called normal constant. Denote by

$$\begin{aligned} PC([-h,a],X) = \{u : [-h,a] \to X \mid u \ \text{ is continuous at } t \neq t_k,\\ \text{left continuous at } t = t_k \text{ and } u(t_k^+) \text{ exists for all } \ k = 1,2,\cdots,m\}. \end{aligned}$$

It is easy to see that $PC([-h,a],X)$ is a Banach space endowed with the $PC$-norm

$$\|u\|_{PC} = \sup_{t\in[-h,a]} \|u(t)\|, \quad \forall\, u \in PC([-h,a],X).$$

Let

$$\mathcal{B} = \{\phi : [-h,0] \to X \mid \phi \text{ is bounded and measurable on } [-h,0]\}.$$

Obviously, $\mathcal{B}$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{B}} = \sup_{t\in[-h,0]} \|u(t)\|, \quad \forall u \in \mathcal{B}.$$

It is evident that, $PC([-h,a],X)$ and $\mathcal{B}$ are also ordered Banach spaces with partial order " $\leq$ " reduced by the positive function cones $K_{PC} = \{u \in PC([-h,a],X) : u(t) \geq \theta,\ t \in [-h,a]\}$ and $K_{\mathcal{B}} = \{u \in \mathcal{B} \mid u(s) \geq \theta,\ s \in [-h,0]\}$ ($\theta$ is the zero element of $X$) respectively. $K_{PC}$ and $K_{\mathcal{B}}$ are also normal cones with the same normal constant $N$. And for $t \in [-h,a]$, $v,w \in PC([-h,a],X)$ with $v \leq w$, we use $[v,w]$ to denote the order interval

$$\{u \in PC([-h,a],X) \mid v(t) \leq u(t) \leq w(t)\}$$

in $PC([-h,a],X)$, and for $t \in [-h,a]$, $[v(t),w(t)]$ to denote the order interval

$$\{u \in X \mid v(t) \leq u \leq w(t)\}.$$

In the following, we denote $J_0 = [-h,0]$, $J_1 = [0,t_1]$, $J_k = (t_{k-1},t_k]$, $k = 2,3,\cdots,m+1$, $t_{m+1} = a$, $I' = [-h,a] \setminus \{t_1,t_2,\cdots,t_m\}$ and $I'' = [-h,a] \setminus \{0,t_1,t_2,\cdots,t_m\}$, and use $X_1$ to denote the Banach space $D(A(t))$ with the graph norm $\|\cdot\|_{X_1} = \|\cdot\| + \|A(t)\cdot\|$.

Throughout the paper, we assume that $\{A(t) : 0 \leq t \leq a\}$ is a family of closed and densely defined operator on Banach space $X$, which satisfies the known conditions of Acquistapace and Terreni:

(A1) For each $t \in [0,a]$, $A(t)$ is a closed linear operator on $X$ and there exist constants $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2},\pi)$, $M_1 \geq 0$ such that $\Sigma_\theta \cup \{0\} \subset \rho(A(t)-\lambda_0)$ and for all $\lambda \in \Sigma_\theta \cup \{0\}$ and $t \in [0,a]$,

$$\|R(\lambda, A(t)-\lambda_0)\|_{\mathcal{L}(X)} \leq \frac{M_1}{1+|\lambda|}. \tag{2.1}$$

(A2) There exist constants $M_2 > 0$ and $\alpha,\beta \in (0,1]$ with $\alpha+\beta > 1$ such that for all $\lambda \in \Sigma_\theta$ and $0 \leq s \leq t \leq a$,

$$\|(A(t)-\lambda_0)R(\lambda, A(t)-\lambda_0)[R(\lambda_0,A(t)) - R(\lambda_0,A(s))]\| \leq M_2\frac{|t-s|^\alpha}{|\lambda|^\beta}, \tag{2.2}$$

where $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \leq \theta\}$.

Conditions (A1) and (A2), which are initiated by Acquistapace and Terreni [1] and Acquistapace [2] for $\lambda_0 = 0$, are well understood and widely used in the literature. Under the above conditions (A1) and (A2), the family $\{A(t) : 0 \leq t \leq a\}$ generates a unique linear evolution system, or called linear evolution family $\{\mathcal{U}(t,s) : 0 \leq s \leq t \leq a\}$. Furthermore, by an obvious rescaling from [2, Theorem 2.3] and [1, Theorem 2.1] combined with the Acquistapace and Terreni conditions (A1) and (A2) one gets that the family of the linear operator $\{\mathcal{U}(t,s) : 0 \leq s \leq t \leq a\}$ :

(1) $\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)$, $\mathcal{U}(t,t) = I$ for $0 \leq s \leq r \leq t \leq a$;

(2) the map $(t,s) \mapsto \mathcal{U}(t,s)x$ is continuous for all $x \in X$ and $0 \leq s \leq t \leq a$;

(3) $\mathcal{U}(\cdot,s) \in C^1((s,\infty),\mathcal{L}(X))$, $\frac{\partial \mathcal{U}(t,s)}{\partial t} = A(t)\mathcal{U}(t,s)$ for $t > s$, and

$$\left\| A^k(t)\mathcal{U}(t,s) \right\| \leq M(t-s)^{-k} \text{ for } 0 < t-s \leq 1 \text{ and } k = 0,1;$$

(4) $\frac{\partial \mathcal{U}(t,s)x}{\partial s} = -\mathcal{U}(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

From the property (3), we know that

$$\|\mathcal{U}(t,s)\|_{\mathcal{L}(X)} \leq M \quad \text{for} \quad 0 \leq s \leq t \leq a. \tag{2.3}$$

In (2.3) and property (3), $M > 0$ is a constant.

**Definition 2.1** ( [9]). An evolution family $\{\mathcal{U}(t,s) : 0 \leq s \leq t \leq a\}$ is said to be equicontinuous if the function $t \mapsto \mathcal{U}(t,s)$ is continuous by operator norm for $t \in (s,+\infty)$.

**Definition 2.2** ( [10]). An evolution family $\{\mathcal{U}(t,s) : 0 \leq s \leq t \leq a\}$ is said to be compact if for all $0 \leq s < t \leq a$, $\mathcal{U}(t,s)$ is equicontinuous and maps bounded subsets of $X$ into precompact subsets of $X$.

**Definition 2.3.** A function $u \in PC([-h,a],X)$ is said to be a mild solution of the problem (1.1) if it satisfies the equation

$$u(t) = \begin{cases} \mathcal{U}(t,0)[g(u)(0) + \phi(0)] + \displaystyle\int_0^t \mathcal{U}(t,s)f(s,u(s),u_s)ds \\ + \displaystyle\sum_{0 \leq t_k < t} \mathcal{U}(t,t_k)I_k(u(t_k)), \quad t \in [0,a],\ t \neq t_k,\ k = 1,2,\cdots,m, \\ g(u)(t) + \phi(t), \quad t \in [-h,0]. \end{cases} \tag{2.4}$$

**Definition 2.4.** If a function $u \in PC([-h,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ satisfies

$$\begin{cases} u'(t) - A(t)u(t) \leq f(t,u(t),u_t), \quad t \in [0,a],\ t \neq t_k, \\ \Delta u\big|_{t=t_k} \leq I_k(u(t_k)), \quad k = 1,2,\cdots,m, \\ u(t) \leq g(u)(t) + \phi(t), \quad t \in [-h,0], \end{cases} \tag{2.5}$$

we call it a lower solution of the problem (1.1); if all inequalities of (2.5) are reversed, we call it an upper solution of the problem (1.1).

**Definition 2.5.** A function $f : [0, a] \times X \times \mathcal{B} \to X$ is said to be Carathéodory continuous provided that

(i) for all $(u, v) \in X \times \mathcal{B}$, $f(\cdot, u, v) : [0, a] \to X$ is measurable;

(ii) for a.e. $t \in [0, a]$, $f(t, \cdot, \cdot) : X \times \mathcal{B} \to X$ is continuous.

**Definition 2.6** ( [4,11]). The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded set S of Banach space X is

$$\alpha(S) := \inf \left\{ \delta > 0 : S = \bigcup_{i=1}^{m} S_i \text{ with } \operatorname{diam}(S_i) \leq \delta \text{ for } i = 1, 2, \cdots, m \right\}. \tag{2.6}$$

It is easy to know from Definition 2.6 that $0 \leq \alpha(S) < \infty$ . The following properties about the Kuratowski measure of noncompactness are well known.

To introduce the useful lemmas which will be used in our argument, we use $\alpha(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded set of $X$. For details about the definition and properties of the measure of noncompactness, we refer to the monographs by [4,11].

**Lemma 2.1** ( [27]). *Let the family of operators $A(t)$ satisfy assumptions (A1) and (A2), and for each $s \in [0, a]$, the semigroup $e^{-tA(s)}(t > 0)$ be positive, then the evolution system $\{\mathcal{U}(t, s)\}_{t \geq s}$ generated by $A(t)$ satisfies*

*(i) positivity preservation, $\mathcal{U}(t, s) \geq 0$ for all $t \geq s$;*

*(ii) monotonicity preservation, for any initial values $u_0$ and $v_0$,*

$$u_0 \leq v_0 \Rightarrow \mathcal{U}(t, 0)u_0 \leq \mathcal{U}(t, 0)v_0, \quad \forall t \geq 0.$$

**Lemma 2.2** ( [17]). *Let $D = \{u_n\}_{n=1}^{\infty} \subset PC([0, a], X)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $[0, a]$, and*

$$\alpha \left( \left\{ \int_0^a u_n(t) dt \mid n \in \mathbb{N} \right\} \right) \leq 2 \int_0^a \alpha(D(t)) dt. \tag{2.7}$$

## 3. Case $\mathcal{U}(t, s)$ compact

In this section, we discuss the existence of extremal mild solutions as well as the existence of at least one mild solutions for the problem (1.1) under the situation that evolution family $\{\mathcal{U}(t, s) : 0 \leq s \leq t \leq a\}$ is compact and the nonlocal function $g : PC([-h, a], X) \to \mathcal{B}$ is also compact.

**Theorem 3.1.** *Let $X$ be an ordered Banach space and its positive cone $P$ be normal. Assume that the evolution family $\{\mathcal{U}(t, s) : 0 \leq s \leq t \leq a\}$ generated by the family of operators $\{A(t) : 0 \leq t \leq a\}$ is compact. If the problem (1.1) has a lower solution $v^{(0)} \in PC([-h, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-h, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \leq w^{(0)}$. Suppose also that the following conditions are satisfied:*

*(F1) There exists a constant $C > 0$ such that*

$$f(t, x_1, y_1) - f(t, x, y) \geq -C(x_1 - x),$$

*for $\forall t \in [0, a]$, $x, x_1 \in X$ and $y, y_1 \in \mathcal{B}$ with $v^{(0)}(t) \leq x \leq x_1 \leq w^{(0)}(t)$ and $(v^{(0)})_t \leq y \leq y_1 \leq (w^{(0)})_t$;*

*(F2) the impulsive function* $I_k(\cdot)$ *satisfies*

$$I_k(x) \le I_k(y), \quad k = 1, 2, \cdots, m,$$

*for any* $t \in [0, a]$*, and* $v^{(0)}(t) \le x \le y \le w^{(0)}(t)$*;*

*(F3) the nonlocal function* $g(u)$ *is increasing on order interval* $[v^{(0)}, w^{(0)}]$.

*Then the problem* (1.1) *has minimal and maximal mild solutions between* $v^{(0)}$ *and* $w^{(0)}$*, which can be obtained by a monotone iterative procedure starting from* $v^{(0)}$ *and* $w^{(0)}$*, respectively.*

**Proof.** It is easy to see that the problem (1.1) is equivalent to the following nonlocal impulsive problems of non-autonomous evolution equations with delay

$$\begin{cases} u'(t) - A(t)u(t) = f(t, u(t), u_t) + Cu(t), \quad t \in [0, a], \ t \ne t_k, \\ \triangle u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(t) = g(u)(t) + \phi(t), \quad t \in [-h, 0], \end{cases} \tag{3.1}$$

for any constant $C > 0$. Therefore, we consider the $\mathcal{A} : [v^{(0)}, w^{(0)}] \to PC([-h, a], X)$ defined by

$$(\mathcal{A}u)(t) = \begin{cases} g(u)(t) + \phi(t), \quad t \in [-h, 0], \\ U(t, 0)[g(u)(0) + \phi(0)] + \displaystyle\int_0^t U(t, s)[f(s, u(s), u_s) + Cu(s)]ds \\ + \displaystyle\sum_{0<t_k<t} U(t, t_k)I_k(u(t_k)), \quad t \in [0, a], \end{cases} \tag{3.2}$$

where $U(t, s) = e^{-Ct}\mathcal{U}(t, s)$ is the evolution system generated by $A(t) - CI$. Obviously, $U(t, s)$ is compact too. By Definition 2.3, the mild solution of the problem (1.1) is equivalent to the fixed point of $\mathcal{A}$ defined by (3.2).

First, we prove that the operator $\mathcal{A} : [v^{(0)}, w^{(0)}] \to PC([-h, a], X)$ defined by (3.2) is continuous. For this purpose, let $\{u^{(n)}\}_{n=1}^{\infty} \subset [v^{(0)}, w^{(0)}]$ be a sequence such that $\lim\limits_{n\to\infty} u^{(n)} = u$ in $[v^{(0)}, w^{(0)}]$. Then for any $t \in [0, a]$, $\lim\limits_{n\to\infty} (u^{(n)})_t = u_t$. If $t \in [-h, 0]$, by (3.2) and the continuity of the nonlocal function $g$, we have that

$$\Big\|(\mathcal{A}u^{(n)})(t) - (\mathcal{A}u)(t)\Big\| = \Big\|g(u^{(n)})(t) - g(u)(t)\Big\| \to 0, \quad n \to \infty. \tag{3.3}$$

Applying assumption (F1), we know that for any $u \in [v^{(0)}, w^{(0)}]$ and $s \in [0, t]$, $t \in [0, a]$,

$$\begin{aligned} f(s, v^{(0)}(s), (v^{(0)})_s) + Cv^{(0)}(s) &\le f(s, u(s), u_s) + Cu(s) \\ &\le f(s, w^{(0)}(s), (w^{(0)})_s) + Cw^{(0)}(s). \end{aligned}$$

The above inequality combined with the normality of the positive cone $P$, we know that there exists a constant $C_1 > 0$, such that

$$\|f(s, u(s), u_s) + Cu(s)\| \le C_1, \quad s \in [0, t], \ t \in [0, a]. \tag{3.4}$$

By (2.3), (3.2) and the Lebesgue dominated convergence theorem, we know that for any $t \in [0,a]$,

$$\begin{aligned} \|(\mathcal{A}u^{(n)})(t) - (\mathcal{A}u)(t)\| \le M\|g(u^{(n)})(0) - g(u)(0)\| \\ + M\int_0^t \|f(s,u^{(n)}(s),(u^{(n)})_s) \\ + Cu^{(n)}(s) - f(s,u(s),u_s) - Cu(s)\|ds \\ + M\sum_{0<u_k<t} \|I_k(u^{(n)}(t_k)) - I_k(u(t_k))\|. \end{aligned} \tag{3.5}$$

Hence, from (3.3)-(3.5), by the Carathéodory continuity of the nonlinear function $f$, and the continuity of the impulsive function $I_k$ for $k = 1,2,\cdots,m$, we obtain

$$\|\mathcal{A}u^{(n)} - \mathcal{A}u\|_{PC} \to 0, \quad n \to \infty. \tag{3.6}$$

Therefore, $\mathcal{A} : [v^{(0)}, w^{(0)}] \to PC([-h,a],X)$ is a continuous operator.

Secondly, we prove that $\mathcal{A} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonic increasing operator. For any $u, v \in [v^{(0)}, w^{(0)}]$ with $u \le v$, by assumptions (F3) and (3.2), we know that for $t \in [-h,0]$,

$$(\mathcal{A}u)(t) = g(u)(t) + \phi(t) \le g(v)(t) + \phi(t) = (\mathcal{A}v)(t).$$

For $t \in [0,a]$, by assumptions (F1)-(F3), Lemma 2.1 and (3.2), we know that

$$\begin{aligned} (\mathcal{A}u)(t) &= U(t,0)[g(u)(0) + \phi(0)] + \int_0^t U(t,s)[f(s,u(s),u_s) + Cu(s)]ds \\ &\quad + \sum_{0<t_k<t} U(t,t_k)I_k(u(t_k)) \\ &\le U(t,0)[g(v)(0) + \phi(0)] + \int_0^t U(t,s)[f(s,v(s),v_s) + Cv(s)]ds \\ &\quad + \sum_{0<t_k<t} U(t,t_k)I_k(v(t_k)) \\ &= (\mathcal{A}v)(t). \end{aligned}$$

Which means that $\mathcal{A}$ is an increasing operator in $[v^{(0)}, w^{(0)}]$. Next, we show that $v^{(0)} \le \mathcal{A}v^{(0)}$ and $\mathcal{A}w^{(0)} \le w^{(0)}$. By the definition of lower solution, we know that for $t \in [-h,0]$,

$$v^{(0)}(t) \le g(v^{(0)})(t) + \phi(t) = (\mathcal{A}v^{(0)})(t). \tag{3.7}$$

Letting

$$h(t) = (v^{(0)})'(t) - A(t)v^{(0)}(t) + Cv^{(0)}(t), \quad t \in [0,a], \quad t \ne t_k, \quad k = 1,2,\cdots,m.$$

By Definition 2.4, we obtain that $h \in PC([0,a],X)$ and $h(t) \le f(t, v^{(0)}(t), (v^{(0)})_t) + Cv^{(0)}(t)$ for $t \in [0,a]$. Therefore, by Definitions 2.3 and 2.4, and the positivity of the strongly continuous evolution family $\{U(t,s) : 0 \le s \le t \le a\}$ we obtain that for $t \in [0,a]$,

$$v^{(0)}(t) = U(t,0)v^{(0)}(0) + \int_0^t U(t,s)h(s)ds + \sum_{0<t_k<t} U(t,t_k)\Delta v^{(0)} \tag{3.8}$$

$$\begin{aligned}
&\le U(t,0)[g(v^{(0)})(0)+\phi(0)]+\int_0^t U(t,s)[f(s,v^{(0)}(s),(v^{(0)})_s)\\
&\quad + Cv^{(0)}(s)]ds+\sum_{0<t_k<t} U(t,t_k)I_k(v^{(0)}(t_k))\\
&=(\mathcal{A}v^{(0)})(t).
\end{aligned}$$

Combining (3.7) and (3.8) imply $v^{(0)} \le \mathcal{A}v^{(0)}$. Similarly, it can be shown that $\mathcal{A}w^{(0)} \le w^{(0)}$. Therefore, $\mathcal{A} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonic increasing operator.

Now, we define two sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ in the ordered interval $[v^{(0)}, w^{(0)}]$ by the following iterative scheme:

$$v^{(n)} = \mathcal{A}v^{(n-1)}, \quad w^{(n)} = \mathcal{A}w^{(n-1)}, \quad n = 1, 2 \cdots . \tag{3.9}$$

Then from the fact that $\mathcal{A}$ is a monotonic increasing operator, it follows that

$$v^{(0)} \le v^{(1)} \le v^{(2)} \le \cdots \le v^{(n)} \le \cdots \le w^{(n)} \le \cdots \le w^{(2)} \le w^{(1)} \le w^{(0)}. \tag{3.10}$$

In what follows, we prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent on $[-h, a]$. For convenience, let $D = \{v^{(n)}\}_{n=1}^{\infty}$ and $D^* = \{v^{(n-1)}\}_{n=1}^{\infty}$. Then $D = \mathcal{A}(D^*)$. From the fact that the nonlocal function $g : PC([-h, a], X) \to \mathcal{B}$ is a compact map, we know that for $t \in [-h, 0]$, the set

$$\{(\mathcal{A}v^{(n-1)})(t) \mid v^{(n-1)} \in D^*\} = \{g(v^{(n-1)})(t) + \phi(t) \mid v^{(n-1)} \in D^*\}$$

is precompact in X. For $t \in (0, a]$ and $v^{(n-1)} \in D^*$, let

$$\begin{aligned}
(\mathcal{A}_1 v^{(n-1)})(t) &= U(t,s)\big[g(v^{(n-1)})(0)+\phi(0)\big]\\
&\quad + \int_0^t U(t,s)\big[f(s,v^{(n-1)}(s),(v^{(n-1)})_s)+Cv^{(n-1)}(s)\big]ds,\\
(\mathcal{A}_2 v^{(n-1)})(t) &= \sum_{0<t_k<t} U(t,t_k)I_k(v^{(n-1)}(t_k)).
\end{aligned} \tag{3.11}$$

For any $v^{(n-1)} \in D^*$, $s \in [0, t]$, $t \in [0, a]$, by the assumption (F1), proofs similar to (3.4), combined with the normality of cone $P$, we know that there exists a constant $C_2 > 0$ such that for any $v^{(n-1)} \in D^*$,

$$\|f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + Cv^{(n-1)}(s)\| \le C_2, \quad s \in [0, t],\ t \in [0, a]. \tag{3.12}$$

Hence, for $t \in (0, a]$ and $0 < \epsilon < t$, since $\{U(t, s) : 0 \le s \le t \le a\}$ is compact, the operator

$$\begin{aligned}
&(A_1^{\epsilon} v^{(n-1)})(t)\\
&= U(t,0)[g(v^{(n-1)})(0)+\phi(0)]\\
&\quad + Cv^{(n-1)}(s)]ds\\
&= U(t,0)[g(v^{(n-1)})(0)+\phi(0)]\\
&\quad + U(t,t-\frac{\epsilon}{2})U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)\cdot[f(s,v^{(n-1)}(s),(v^{(n-1)})_s)\\
&\quad + Cv^{(n-1)}(s)]ds
\end{aligned} \tag{3.13}$$

is precompact in $X$ since $U(t, t-\frac{\epsilon}{2}) \in \mathcal{L}(X)$ and $U(t-\frac{\epsilon}{2}, t-\epsilon)$ is compact in $X$. By (2.3), (3.11), (3.12) and (3.13), we obtain

$$\begin{aligned}
&\|\mathcal{A}_1 v^{(n-1)}(t) - \mathcal{A}_1^{\epsilon} v^{(n-1)}(t)\| \\
&= \left\| \int_{t-\epsilon}^{t} U(t,s) \left[ f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + C v^{(n-1)}(s) \right] ds \right\| \\
&\leq \int_{t-\epsilon}^{t} \|U(t,s)\| \cdot \left\| f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + C v^{(n-1)}(s) \right\| ds \\
&\leq M C_2 \epsilon,
\end{aligned} \tag{3.14}$$

which means that there exists precompact set $\{(\mathcal{A}_1^{\epsilon} v^{(n-1)})(t) \mid v^{(n-1)} \in D^*\}$ sufficiently close to the set $\{(\mathcal{A}_1 v^{(n-1)})(t) \mid v^{(n-1)} \in D^*\}$ for every $t \in (0, a]$. Therefore, by the total boundedness we know that for $t \in (0, a]$, the set $\{(\mathcal{A}_1 v^{(n-1)})(t) \mid v^{(n-1)} \in D^*\}$ is precompact in $X$.

On the other hand, for any $v^{(n-1)} \in D^*$, by the assumption (F2), we have

$$I_k(v^{(0)}(t_k)) \leq I_k(v^{(n-1)}(t_k)) \leq I_k(w^{(0)}(t_k)), \quad k = 1, 2, \cdots, m. \tag{3.15}$$

By the normality of the cone $P$ and (3.15), there exists a constant $C_3 > 0$ such that

$$\|I_k(v^{(n-1)}(t_k))\| \leq C_3, \quad v^{(n-1)} \in D^*, \quad k = 1, 2, \cdots, m.$$

The set $\{(\mathcal{A}_2 v^{(n-1)})(t) \mid v^{(n-1)} \in D^*\}$ is precompact in $X$ due to the compactness of the evolution family $\{U(t,s) : 0 \leq s \leq t \leq a\}$.

Therefore, the set

$$\{v^{(n)}(t)\} = \{(\mathcal{A} v^{(n-1)})(t) \mid v^{(n-1)} \in D^*\}$$

is precompact in $X$ for any $t \in [-h, a]$. Hence, $\{v^{(n)}(t)\}$ has a convergent subsequence. Combining this with the monotonicity (3.10), we easily prove that $\{v^{(n)}(t)\}$ itself is convergent, i.e.,

$$\lim_{n \to \infty} v^{(n)}(t) = \underline{u}(t), \quad t \in [-h, a].$$

Similarly, we can prove

$$\lim_{n \to \infty} w^{(n)}(t) = \overline{u}(t), \quad t \in [-h, a].$$

Obviously, $\left\{v^{(n)}(t)\right\} \subset PC([-h, a], X)$, and $\underline{u}(t)$ is bounded integrable when $t$ belongs to $[-h, 0]$ and $[0, a]$, respectively. For any $t \in [-h, a]$, we know from (3.2) that

$$\begin{aligned}
v^{(n)}(t) &= (\mathcal{A} v^{(n-1)})(t) \\
&= \begin{cases}
g(v^{(n-1)})(t) + \phi(t), \quad t \in [-h, 0], \\
U(t,0)[g(v^{(n-1)})(0) + \phi(0)] \\
+ \displaystyle\int_0^t U(t,s)[f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + C v^{(n-1)}(s)] ds \\
+ \displaystyle\sum_{0 < t_k < t} U(t, t_k) I_k(v^{(n-1)}(t_k)), \quad t \in [0, a].
\end{cases}
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by the Lebesgue's dominated convergence theorem, we have that

$$\underline{u}(t) = \begin{cases} g(\underline{u})(t) + \phi(t), \quad t \in [-h, 0], \\ U(t,0)[g(\underline{u})(0) + \phi(0)] + \displaystyle\int_0^t U(t,s)[f(s, \underline{u}(s), \underline{u}_s) + C\underline{u}(s)]ds \\ + \displaystyle\sum_{0<t_k<t} U(t,t_k) I_k(\underline{u}(t_k)), \quad t \in [0, a]. \end{cases}$$

Therefore, $\underline{u} \in PC([-h,a], X)$ and $\underline{u} = \mathcal{A}\underline{u}$. Similarly, we can prove that $\overline{u} \in PC([-h,a], X)$ and $\bar{u} = \mathcal{A}\overline{u}$. Combining this fact with the monotonicity (3.10), we can prove that $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$ are fixed points of operator $\mathcal{A}$ and $\underline{u} \le \overline{u}$.

Next, we show that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $\mathcal{A}$ in $[v^{(0)}, w^{(0)}]$, respectively. In fact, for any $u \in [v^{(0)}, w^{(0)}]$, we have $v^{(0)} \le u \le w^{(0)}$, and $v^{(1)} = \mathcal{A}v^{(0)} \le \mathcal{A}u = u \le \mathcal{A}w^{(0)} = w^{(1)}$. Continuing such a progress, we get $v^{(n)} \le u \le w^{(n)}$. Letting $n \to \infty$, we get $\underline{u} \le u \le \overline{u}$. Therefore, $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of the problem (1.1) in $[v^{(0)}, w^{(0)}]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme (3.9) starting from $v^{(0)}$ and $w^{(0)}$, respectively. □

From the proof of Theorem 3.1, we can easily obtain the following result.

**Theorem 3.2.** *Let $X$ be an ordered Banach space and its positive cone $P$ be normal. Assume that the evolution family $\{\mathcal{U}(t,s) : 0 \le s \le t \le a\}$ generated by the family of operators $\{A(t) : 0 \le t \le a\}$ is compact. If the problem* (1.1) *has a lower solution $v^{(0)} \in PC([-h,a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-h,a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \le w^{(0)}$, and the conditions* (F1)-(F3) *are satisfied. Then the problem* (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$, respectively.*

Applying the famous Schauder's fixed point theorem, we can also obtain the following existence result.

**Theorem 3.3.** *Let $X$ be an ordered Banach space and its positive cone $P$ be normal. Assume that the evolution family $\{\mathcal{U}(t,s) : 0 \le s \le t \le a\}$ generated by the family of operators $\{A(t) : 0 \le t \le a\}$ is compact. If the problem* (1.1) *has a lower solution $v^{(0)} \in PC([-h,a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-h,a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \le w^{(0)}$, and the conditions* (F1)-(F3) *are satisfied. Then the problem* (1.1) *exists at least one mild solution in ordered interval $[v^{(0)}, w^{(0)}]$.*

**Proof.** In accordance with the proof of Theorem 3.1, we know that $\mathcal{A}$ defined by (3.2) is a continuous mapping from $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$. Therefore, in order to apply Schauder's fixed point theorem to obtain a fixed point, we need to prove that $\mathcal{A} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ a compact operator. For this purpose, let

$$\Omega_1 = \{g(u)(\cdot) + \phi(\cdot) : \ \cdot \in [-h, 0], \ u \in [v^{(0)}, w^{(0)}]\},$$

$$\Omega_2 = \Big\{U(\cdot, 0)[g(u)(0) + \phi(0)] + \int_0^{\cdot} U(\cdot, s)[f(s, u(s), u_s) + Cu(s)]ds : \\ \cdot \in [0, a], \ u \in [v^{(0)}, w^{(0)}]\Big\},$$

$$\Omega_3 = \Big\{ \sum_{0<t_k<\cdot} U(\cdot, t_k) I_k(u(t_k)) : \ \cdot \in [0,a], \ u \in [v^{(0)}, w^{(0)}] \Big\}.$$

By the compactness of the nonlocal function $g$, we know that the set $\Omega_1$ is precompact in $\mathcal{B}$. In what follows, we prove that $\Omega_2$ is a precompact set. For $t \in [0,a]$, by the fact that the evolution family $\{U(t,s) : 0 \le s \le t \le a\}$ generated by $\{A(t) - CI : 0 \le t \le a\}$ is compact and as well as the compactness of nonlocal function $g$, we get that the set

$$\{U(t,0)[g(u)(0) + \phi(0)] : \ u \in [v^{(0)}, w^{(0)}]\}$$

is precompact in X. For $t \in (0,a]$ and $0 < \epsilon < t$, the set

$$\begin{aligned} &\Big\{ \int_0^{t-\epsilon} U(t,s)[f(s,u(s),u_s) + Cu(s)]ds : u \in [v^{(0)}, w^{(0)}] \Big\} \\ =&\Big\{ U\Big(t, t-\frac{\epsilon}{2}\Big) U\Big(t-\frac{\epsilon}{2}, t-\epsilon\Big) \int_0^{t-\epsilon} U(t-\epsilon, s)[f(s,u(s),u_s) \\ &+ Cu(s)]ds : u \in [v^{(0)}, w^{(0)}] \Big\} \end{aligned} \tag{3.16}$$

is precompact in $X$ since $U(t, t-\frac{\epsilon}{2}) \in \mathcal{L}(X)$ and $U(t-\frac{\epsilon}{2}, t-\epsilon)$ is compact in $X$. Furthermore, we know that for every $u \in [v^{(0)}, w^{(0)}]$,

$$\int_0^{t-\epsilon} U(t,s)[f(s,u(s),u_s) + Cu(s)]ds \to \int_0^t U(t,s)[f(s,u(s),u_s) + Cu(s)]ds, \quad \epsilon \to 0. \tag{3.17}$$

Combining (3.16) and (3.17) with the total boundedness, we know that the set

$$\Big\{ \int_0^t U(t,s)[f(s,u(s),u_s) + Cu(s)]ds : \ u \in [v^{(0)}, w^{(0)}] \Big\}$$

is precompact in $X$. Therefore, for each $t \in [0,a], \Omega_2(t)$ is precompact in $X$.

Next, we prove the equicontinuity of $\Omega_2$, by (2.3), (3.4) and the definition of the set $\Omega_2$, we obtain that for $0 \le t' < t'' \le a$ and $u \in [v^{(0)}, w^{(0)}]$,

$$\begin{aligned} &\Big\| U(t'',0)[g(u)(0) + \phi(0)] - U(t',0)[g(u)(0) + \phi(0)] \\ &+ \int_0^{t'} [U(t'',s) - U(t',s)] \cdot [f(s,u(s),u_s) + Cu(s)]ds \\ &+ \int_{t'}^{t''} U(t'',s)[f(s,u(s),u_s) + Cu(s)]ds \Big\| \\ \le& \|U(t'',0)[g(u)(0) + \phi(0)] - U(t',0)[g(u)(0) + \phi(0)]\| \\ &+ C_1 \int_0^{t'} \|U(t'',s) - U(t',s)\|ds + MC_1(t'' - t') \\ \to& 0, \quad t'' - t' \to 0. \end{aligned} \tag{3.18}$$

Then it is easy to see that the right hand of (3.18) tends to zero independently of $u \in [v^{(0)}, w^{(0)}]$ as $t'' - t' \to 0$, which means that the functions in $\Omega_2$ are equicontinuous. Therefore, by the Arzela-Ascoli theorem one can easily to justify that the set $\Omega_2$ is precompact.

Now, we are in the position to prove the precompactness of $\Omega_3$. From the definition of the set $\Omega_3$ combined with the fact that the interval $[0,a]$ is divided into finite subintervals by $t_k, k=1,2,\cdots,m$, we only need to prove that the set

$$\{U(\cdot,t_1)I_1(u(t_1)):\ \cdot\in[t_1,t_2],\ u\in[v^{(0)},w^{(0)}]\}$$

is precompact in $C([t_1,t_2],X)$, as the cases for other subintervals are the same. By Definition 2.2 , we know that for each $t\in[t_1,t_2]$, the set

$$\{U(t,t_1)I_1(u(t_1)):t\in[t_1,t_2],u\in[v^{(0)},w^{(0)}]\}$$

is precompact in X. For $t_1\le t'<t''\le t_2$ and $u\in[v^{(0)},w^{(0)}]$, by (2.3) we obtain

$$\begin{aligned}\|U(t'',t_1)I_1(u(t_1))-U(t',t_1)I_1(u(t_1))\|=&\|U(t',t_1)[U(t'',t')-I]I_1(u(t_1))\|\\ \le& M\|[U(t'',t')-I]I_1(u(t_1))\|,\end{aligned}\tag{3.19}$$

by (3.19) and Definition 2.2 imply that the functions in

$$\{U(\cdot,t_1)I_1(u(t_1)):\cdot\in[t_1,t_2],u\in[v^{(0)},w^{(0)}]\}$$

are equicontinuous. Therefore, an application of the Arzela-Ascoli theorem justifies the precompactness of the set

$$\{U(\cdot,t_1)I_1(u(t_1)):\cdot\in[t_1,t_2],u\in[v^{(0)},w^{(0)}]\}$$

in $C([t_1,t_2],X)$. Hence, using a completely similar method for the case $k=2,3,\cdots,m$, one can prove that the set $\Omega_3$ is precompact.

Hence, by the above discussion we know that $\mathcal{A}:[v^{(0)},w^{(0)}]\to[v^{(0)},w^{(0)}]$ is a compact operator, and therefore a completely continuous operator. Therefore, the famous Schauder's fixed point theorem implies that the operator $\mathcal{A}$ has at least one fixed point in ordered interval $[v^{(0)},w^{(0)}]$, which gives rise to a mild solution of the problem (1.1). □

## 4. Case $\mathcal{U}(t,s)$ not compact

In this section, we discuss the existence of extremal mild solutions as well as the existence of at least one mild solution for the problem (1.1) under the situation that the evolution family $\{\mathcal{U}(t,s):0\le s\le t\le a\}$ generated by $\{A(t):0\le t\le a\}$ does not have to be compact. Furthermore, the nonlocal function $g:PC([-h,a],X)\to\mathcal{B}$ is compact.

**Theorem 4.1.** *Let $X$ be an ordered Banach space and its positive cone $P$ be normal. If the problem* (1.1) *has a lower solution $v^{(0)}\in PC([-h,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ and an upper solution $w^{(0)}\in PC([-h,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ with $v^{(0)}\le w^{(0)}$, and the conditions* (F1)-(F3) *and*

*(F4) There exists a constant $L_f>0$, such that for every $t\in[0,a]$,*

$$\alpha\left(\left\{f\left(t,u^{(n)}(t),(u^{(n)})_t\right)\right\}\right)\le L_f\left[\alpha\left(\left\{u^{(n)}(t)\right\}\right)+\sup_{-h\le\tau\le0}\alpha\left(\left\{u^{(n)}(t+\tau)\right\}\right)\right],\tag{4.1}$$

*where $\{u^{(n)}\}\subset[v^{(0)},w^{(0)}]$ is a countable and increasing or decreasing monotonic set and $\{(u^{(n)})_t\}\subset\mathcal{B}$*

*are satisfied. Then the problem* (1.1) *has a minimal mild solution* $\underline{u}$ *and a maximal mild solution* $\bar{u}$ *between* $v^{(0)}$ *and* $w^{(0)}$.

**Proof.** By the proof of Theorem 3.1, we know that the operator $\mathcal{A}$ defined by (3.2) maps $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$ is continuous and monotone increasing. In the following, we prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ defined by equation (3.9) in Theorem 3.1 are convergent on $[-h, a]$. For convenience, let $B = \{v^{(n)} \mid n \in \mathbb{N}\}$ and $B^* = \{v^{(n-1)} \mid n \in \mathbb{N}\}$. Then $B = \mathcal{A}(B^*)$. From $B = B^* \cup \{v^{(0)}\}$ to get $\alpha(B(t)) = \alpha(B^*(t))$ for $t \in [-h, a]$. Let

$$\varphi(t) := \alpha(B(t)) = \alpha(B^*(t)), \quad t \in [-h, a].$$

Going from $J_0$ to $J_{m+1}$ interval by interval we show that $\varphi(t) \equiv 0$ in $[-h, a]$. For $t \in J_0$, by (3.2) and assumption (F3), we have that

$$\varphi(t) = \alpha(B(t)) = \alpha(\mathcal{A}(B^*)(t)) = \alpha\big\{g(v^{(n-1)})(t) + \phi(t)\big\} = 0.$$

For $t \in J_1$, by (3.2), Lemma 2.2, the assumptions (F3) and (F4), we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) \\
&= \alpha(\mathcal{A}(B^*)(t)) \\
&= \alpha\Big(\Big\{U(t,0)[g(v^{(n-1)})(0) + \phi(0)] \\
&\quad + \int_0^t U(t,s)[f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + Cv^{(n-1)}(s)]\Big\}\Big) \\
&\le 2M \int_0^t \alpha\Big(\Big\{f(s, v^{(n-1)}(s), (v^{(n-1)})_s) + Cv^{(n-1)}(s)\Big\}\Big)ds \\
&\le 2M \int_0^t \Big[L_f \alpha(B^*(s)) + L_f \sup_{-h\le\tau\le 0} \alpha(B^*(s+\tau)) + C\alpha(B^*(s))\Big]ds \\
&= 2M \int_0^t \Big[L_f \varphi(s) + L_f \sup_{-h\le\tau\le 0} \varphi(s+\tau) + C\varphi(s)\Big]ds.
\end{aligned} \tag{4.1}$$

Let $\rho_1(t) = \sup\limits_{h\le s\le t} \varphi(s)$, $t \in J_1$. Then we have for any $t \in J_1$,

$$\varphi(t) \le \rho_1(t), \quad \sup_{-h\le\tau\le 0} \varphi(t+\tau) \le \rho_1(t). \tag{4.2}$$

Combining with (4.1) and (4.2) , we obtain that for any $t \in J_1$,

$$\rho_1(t) \le 2M(2L_f + C) \int_0^t \rho_1(s)ds.$$

Hence, by the Gronwall's inequality, it follows that $\rho_1(t) \equiv 0$ on $J_1$. This means that $\varphi(t) \equiv 0$ on $J_1$. In particular, $\alpha(B(t_1)) = \alpha(B^*(t_1)) = \varphi(t_1) = 0$, this implies that $B(t_1)$ and $B^*(t_1)$ are precompact on $X$. Thus $I_1(B^*(t_1))$ is precompact on $X$ and $\alpha(I_1(B^*(t_1))) = 0$.

Now, for $t \in J_2$, by (3.2), Lemma 2.2, the assumptions (F3)-(F4) and the above argument, we have

$$\varphi(t) = \alpha(B(t)) \tag{4.3}$$

$$\begin{aligned}
&= \alpha(\mathcal{A}(B^*)(t)) \\
&= \alpha\Big(\Big\{U(t,0)[g(v^{(n-1)})(0)+\phi(0)]\Big\}\Big) \\
&\quad + \alpha\Big(\Big\{\int_0^t U(t,s)[f(s,v^{(n-1)}(s),(v^{(n-1)})_s)+Cv^{(n-1)}(s)]ds\Big\}\Big) \\
&\quad + \alpha(\{I_1(v^{(n-1)}(t_1))\}) \\
&\le\ 2M\int_0^t \Big[L_f\varphi(s)+L_f\sup_{-h\le\tau\le 0}\varphi(s+\tau)+C\varphi(s)\Big]ds+\alpha(I_1(B^*(t_1))) \\
&= 2M\int_{t_1}^t \Big[L_f\varphi(s)+L_f\sup_{-h\le\tau\le 0}\varphi(s+\tau)+C\varphi(s)\Big]ds.
\end{aligned}$$

Let $\rho_2(t)=\sup_{-h\le s\le t}\varphi(s)$ , $t\in J_2$. Then for any $t\in J_2$, we have

$$\varphi(t)\le\rho_2(t),\quad \sup_{-h\le\tau\le 0}\varphi(t+\tau)\le\rho_2(t). \tag{4.4}$$

Combining with (4.3) and (4.4), we obtain that for any $t\in J_2$,

$$\rho_2(t)\le 2M(2L_f+C)\int_{t_1}^t\rho_2(s)ds.$$

Again by the Gronwall's inequality, it follows that $\rho_2(t)\equiv 0$ on $J_2$. Thus $\varphi(t)\equiv 0$ on $J_2$, from which we obtain that $\alpha(B^*(t_2))=0$ and $\alpha(I_2(B^*(t_2)))=0$. Continuing such a process interval by interval up to $J_{m+1}$, we can prove that $\varphi(t)\equiv 0$ on every $J_k$ $(k=0,1,\cdots,m+1)$. Thus, $\{v^{(n)}(t)\}$ is precompact on $X$ and $\{v^{(n)}(t)\}$ has a convergent subsequence for every $t\in[-h,a]$. Combining this with the monotonicity (3.10), we easily prove that $\{v^{(n)}(t)\}$ is convergent for any $t\in[-h,a]$. Using a similar argument with $\{v^{(n)}(t)\}$, we can prove that $\{w^{(n)}(t)\}$ is also convergent for any $t\in[-h,a]$. Set

$$\lim_{n\to\infty}v^{(n)}(t)=\underline{u}(t),\quad \lim_{n\to\infty}w^{(n)}(t)=\overline{u}(t),\quad t\in[-h,a].$$

Evidently, $\{v^{(n)}\}\subset PC([-h,a],X)$, so $\underline{u}$ is bounded and integrable on $[-h,a]$. By (3.2), we have that

$$v^{(n)}(t)=\mathcal{A}(v^{(n-1)})(t)=\begin{cases} g(v^{(n-1)})(t)+\phi(t), \quad t\in[-h,0],\\ U(t,0)[g(v^{(n-1)})(0)+\phi(0)]+\displaystyle\int_0^t U(t,s)[f(s,v^{(n-1)}(s),(v^{(n-1)})_s)\\ +Cv^{(n-1)}(s)]ds+\displaystyle\sum_{0<t_k<t}U(t,t_k)I_k(v^{(n-1)}(t_k)),\quad t\in[0,a].\end{cases}$$

Letting $n\to\infty$ in the above equalities, by the Lebesgue dominated convergent theorem, we have

$$\begin{aligned}
\underline{u}(t)&=(\mathcal{A}\underline{u})(t)\\
&=\begin{cases} g(\underline{u})(t)+\phi(t), \quad t\in[-h,0],\\ U(t,0)[g(\underline{u})(0)+\phi(0)]+\displaystyle\int_0^t U(t,s)[f(s,\underline{u}(s),(\underline{u})_s)+C\underline{u}(s)]ds\\ +\displaystyle\sum_{0<t_k<t}U(t,t_k)I_k(\underline{u}(t_k)),\quad t\in[0,a].\end{cases}
\end{aligned}$$

This means that $\underline{u} \in PC([-h,a],X)$ and $\underline{u} = \mathcal{A}\underline{u}$. Similarly, we can prove that $\overline{u} \in PC([-h,a],X)$ and $\overline{u} = \mathcal{A}\overline{u}$. Letting $n \to \infty$ in (3.10), we see that $v^{(0)} \le \underline{u} \le \overline{u} \le w^{(0)}$. By the monotonicity of $\mathcal{A}$, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $\mathcal{A}$ in $[v^{(0)}, w^{(0)}]$. Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of the problem (1.1) in $[v^{(0)}, w^{(0)}]$, and they can be obtained by a monotone iterative procedure (3.9) starting from $v^{(0)}$ and $w^{(0)}$, respectively. □

**Remark 4.1.** In Theorem 4.1, if $X$ is a weakly sequentially complete Banach space, then the condition (F4) holds automatically. Hence, we can easily obtain the following result from Theorem 4.1.

**Theorem 4.2.** *Let $X$ be an ordered Banach space and its positive cone $P$ be normal. If the problem* (1.1) *has a lower solution $v^{(0)} \in PC([-h,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ and an upper solutions $w^{(0)} \in PC([-h,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ with $v^{(0)} \le w^{(0)}$, and the conditions* (F1)-(F3) *are satisfied, then the problem* (1.1) *has minimal and maximal mild solution between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$ respectively.*

If we replace assumption (F4) by the assumption

(F5) There exist positive constants $\overline{C}_f$, $\overline{L}_f$ and $\overline{L}_g < \frac{1}{NM}$ such that for any $u, v \in [v^{(0)}, w^{(0)}]$ with $u \ge v$, and for somes $\in [-h,0]$,

$$f(t,u(t),u_t) - f(t,v(t),v_t) \le \overline{C}_f(u(t)-v(t)) + \overline{L}_f(u(t+\tau)-v(t+\tau)), \quad \forall t \in [0,a],$$
$$g(u)(s) - g(v)(s) \le \overline{L}_g(u(s)-v(s)), \quad \forall s \in [-h,0],$$

then we have the following uniqueness result.

**Theorem 4.3.** *Let $X$ be an ordered Banach space and its positive cone $P$ be normal. If the problem* (1.1) *has a lower solution $v^{(0)} \in PC([-h,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ and an upper solution $w^{(0)} \in PC([-h,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ with $v^{(0)} \le w^{(0)}$, and conditions* (F1)-(F3), (F5) *are satisfied, then the problem* (1.1) *has a unique solution between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ or $w^{(0)}$.*

**Proof.** We firstly proof that (F1) and (F5) imply (F4). For this purpose, let $\{u^n\} \subset [v^{(0)}, w^{(0)}]$ be an increasing sequences. For $m, n \in \mathbb{N}$ with $m \ge n$, by (F1) and (F5), we obtain that for every $t \in [0,a]$ and some $\tau \in [-h,0]$,

$$\begin{aligned} \theta &\le f(t,u^{(m)}(t),(u^{(m)})_t) - f(t,u^{(n)}(t),(u^{(n)})_t) + C[u^{(m)}(t) - u^{(n)}(t)] \\ &\le (C+\overline{C}_f)[u^{(m)}(t) - u^{(n)}(t)] + \overline{L}_f[u^{(m)}(t+\tau) - u^{(n)}(t+\tau)]. \end{aligned}$$

By this and the normality of cone $P$, we have that for any $t \in [0,a]$ and some $\tau \in [-h,0]$,

$$\begin{aligned} &\|f(t,u^{(m)}(t),(u^{(m)})_t) - f(t,u^{(n)}(t),(u^{(n)})_t)\| \\ \le\ & N(C+\overline{C}_f)\|u^{(m)}(t) - u^{(n)}(t)\| \\ &+ N\overline{L}_f\|u^{(m)}(t+\tau) - u^{(n)}(t+\tau)\| + C\|u^{(m)}(t) - u^{(n)}(t)\| \\ \le\ & [N(C+\overline{C}_f)+C]\|u^{(m)}(t) - u^{(n)}(t)\| \\ &+ N\overline{L}_f \sup_{-h\le\tau\le 0}\|u^{(m)}(t+\tau) - u^{(n)}(t+\tau)\|. \end{aligned}$$

From this inequality and the definition of the measure of noncompactness, it follows that for every $t \in [0,a]$,

$$\alpha\Big(\big\{f(t,u^{(n)}(t),(u^{(n)})_t)\big\}\Big) \leq L_f\Big[\alpha\Big(\big\{u^{(n)}(t)\big\}\Big) + \sup_{-h\leq\tau\leq 0}\alpha\Big(\big\{u^{(n)}(t+\tau)\big\}\Big)\Big],$$

where $L_f = \max\big\{N(C+\overline{C}_f)+C, N\overline{L}_f\big\}$. If $\{u^{(n)}\} \subset [v^{(0)},w^{(0)}]$ is decreasing, the above inequality is also valid. Hence (F4) holds.

Hence, by Theorem 4.1, the problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ in $[v^{(0)},w^{(0)}]$. In what follows, going from $J_0$ to $J_{m+1}$ interval by interval we show that $\underline{u}(t) \equiv \overline{u}(t)$ on every $J_k,\ k = 0,1,\cdots,m+1$.

For $t \in J_0$, by (3.2) and assumption (F5), we obtain

$$\theta \leq \overline{u}(t) - \underline{u}(t) = \mathcal{A}\overline{u}(t) - \mathcal{A}\underline{u}(t) = g(\overline{u})(t) - g(\underline{u})(t) \leq \overline{L}_g(\overline{u}(t) - \underline{u}(t)). \tag{4.5}$$

Combining (4.5) and the normality of cone $P$, we have

$$\|\overline{u}(t) - \underline{u}(t)\| \leq N\overline{L}_g\|\overline{u}(t) - \underline{u}(t)\|, \quad t \in J_0.$$

Hence, for any $t \in J_0$,

$$(1 - N\overline{L}_g)\|\overline{u}(t) - \underline{u}(t)\| \leq 0. \tag{4.6}$$

From (4.6) and the assumption $\overline{L}_g < \frac{1}{NM}$, one gets that $\overline{u}(t) \equiv \underline{u}(t)$ on $J_0$. In particular, $\overline{u}(0) = \underline{u}(0)$.

For $t \in J_1$ and some $\tau \in [-h,0]$, by (3.2) and assumption (F5), we have

$$\begin{aligned}
\theta &\leq \overline{u}(t) - \underline{u}(t)\\
&= \mathcal{A}\overline{u}(t) - \mathcal{A}\underline{u}(t)\\
&= U(t,0)[g(\overline{u})(0) - g(\underline{u})(0)]\\
&\quad + \int_0^t U(t,s)[f(s,\overline{u}(s),(\overline{u})_s) - f(s,\underline{u}(s),(\underline{u})_s) + C(\overline{u}(s) - \underline{u}(s))]ds\\
&\leq \overline{L}_g U(t,0)(\overline{u}(0) - \underline{u}(0))\\
&\quad + \int_0^t U(t,s)[(C+\overline{C}_f)(\overline{u}(s) - \underline{u}(s)) + \overline{L}_f(\overline{u}(s+\tau) - \underline{u}(s+\tau))]ds.
\end{aligned} \tag{4.7}$$

Since $\overline{u}(0) = \underline{u}(0)$, by (4.7) and normality of cone $P$, we can prove that

$$\|\overline{u}(t) - \underline{u}(t)\| \leq NM\int_0^t [(C+\overline{C}_f)\|\overline{u}(s) - \underline{u}(s)\| + \overline{L}_f\|\overline{u}(s+\tau) - \underline{u}(s+\tau)\|]ds. \tag{4.8}$$

We define a non-negative function $\rho_1'(t) = \sup\limits_{-h\leq s\leq t}\|\overline{u}(s) - \underline{u}(s)\|$ on $J_1$. Then, for any $t \in J_1$ and some $\tau \in [-h,0]$,

$$\|\overline{u}(t) - \underline{u}(t)\| \leq \rho_1'(t), \quad \sup_{-h\leq\tau\leq 0}\|\overline{u}(t+\tau) - \underline{u}(t+\tau)\| \leq \rho_1'(t). \tag{4.9}$$

From (3.2) and (4.9) it follows that

$$\rho_1'(t) \leq NM(C+\overline{C}_f+\overline{L}_f)\int_0^t \rho_1'(s)ds, \quad t \in J_1.$$

By this fact and Gronwall's inequality, we obtain that $\overline{u}(t) \equiv \underline{u}(t)$ on $J_1$. In particular, $\overline{u}(t_1) = \underline{u}(t_1)$.

For $t \in J_2$, since $I_1(\bar{u}(t_1)) = I_1(\underline{u}(t_1))$, using completely similar argument as above for $t \in J_1$, we can prove that

$$\begin{aligned}\|\overline{u}(t) - \underline{u}(t)\| &\leq NM \int_0^t [(C + \overline{C}_f)\|\overline{u}(s) - \underline{u}(s)\| + \overline{L}_f \|\overline{u}(s+\tau) - \underline{u}(s+\tau)\|]ds \\ &\leq NM \int_{t_1}^t [(C + \overline{C}_f)\|\overline{u}(s) - \underline{u}(s)\| + \overline{L}_f \|\overline{u}(s+\tau) - \underline{u}(s+\tau)\|]ds.\end{aligned} \tag{4.10}$$

Defining a non-negative function $\rho_2'(t) = \sup_{-h \leq s \leq t} \|\overline{u}(s) - \underline{u}(s)\|$ on $J_2$. We know that for $t \in J_2$,

$$\|\overline{u}(t) - \underline{u}(t)\| \leq \rho_2'(t), \quad \sup_{-h \leq \tau \leq 0} \|\overline{u}(t+\tau) - \underline{u}(t+\tau)\| \leq \rho_2'(t). \tag{4.11}$$

Combining (4.10) and (4.11), we have

$$\rho_2'(t) \leq NM(C + \overline{C}_f + \overline{L}_f) \int_{t_1}^t \rho_2'(s)ds, \quad t \in J_2.$$

Again by Gronwall's inequality, we obtain that $\rho_2'(t) \equiv 0$ on $J_2$. Hence, $\overline{u}(t) \equiv \underline{u}(t)$ on $J_2$. Continuing such a process interval by interval up to $J_m + 1$,we see that $\overline{u}(t) \equiv \underline{u}(t)$ over the whole of $[-h, a]$. Therefore, $\widetilde{u} := \overline{u} = \underline{u}$ is the unique mild solution of the problem (1.1) in $[v^{(0)}, w^{(0)}]$, which can be obtained by the monotone iterative procedure (3.9) starting from $v^{(0)}$ or $w^{(0)}$. □

## 5. Example

In this section, we give an example to illustrate the applicability of our abstract results obtained in Section 4. We consider the non-autonomous parabolic partial differential equation with delay involving nonlocal and impulsive conditions of the form

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \varsigma(t)\dfrac{\partial^2}{\partial x^2}u(x,t) = L\left(\dfrac{|u(x,t)|}{1+|u(x,t)|}\right) + \displaystyle\int_{-h}^0 \gamma(s)u(x,t+s)ds, \\ \qquad\qquad x \in [0,1],\ t \in [0,a],\ t \neq t_k, \\ u(x,t_k^+) = u(x,t_k^-) + \dfrac{\sqrt{|u(x,t_k)|}}{1+|u(x,t_k)|}, \quad x \in [0,1],\ k = 1,2,\cdots,m, \\ u(0,t) = u(1,t) = 0, \quad t \in [0,a], \\ u(x,s) = \displaystyle\int_0^a \sigma(s,t)\lg(1+|u(x,t)|)dt + \kappa(x,s), \quad x \in [0,1],\ s \in [-h,0], \end{cases} \tag{5.1}$$

where a continuously positive and monotonically increasing thermal diffusivity $\varsigma(t) > 0$, $a, h, L > 0$ are all constants, $0 < t_1 < t_2 < \cdots < t_m < a$, $\gamma \in L([-h,0], \mathbb{R}^+), \sigma(s,t)$ is a continuous function from $[-h,0] \times [0,a]$ to $\mathbb{R}^+$, $\kappa \in C([0,1] \times [-h,0], \mathbb{R}^+)$.

Let $X = L^2([0,1])$ be an ordered Banach space equipped with the norm

$$\|u\| = \left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

We define an operator $A(t)$ in the Banach space $X$ by

$$D(A) = H^2([0,1]) \cap H_0^1([0,1]), \quad A(t) = -\varsigma(t)\frac{\partial^2}{\partial x^2}. \tag{5.2}$$

By (5.2) and the monotonicity of $\varsigma(t)$ we can easily verify that the linear operator $A(t)$ satisfies the assumptions (A1), (A2) and Lemma 2.1.

Further, for any $t \geq 0$, we define

$$\begin{aligned}
&u(t) = u(\cdot,t), \quad t \in [-h,a],\\
&f(t,u(t),u_t) = L\Big(\frac{|u(\cdot,t)|}{1+|u(\cdot,t)|}\Big) + \int_{-h}^0 \gamma(s)w(\cdot,t+s)ds, \quad t \in [0,a],\\
&I_k(u(t_k)) = \frac{\sqrt{|u(x,t_k)|}}{1+|u(x,t_k)|}, \quad x \in [0,1],\ k = 1,2,\cdots,m,\\
&g(u)(s) = \int_0^a \sigma(s,t)\lg(1+|u(\cdot,t)|)dt, \quad \kappa(s) = \kappa(\cdot,s), \quad s \in [-h,0].
\end{aligned}$$

Then the non-autonomous parabolic partial differential equation with delay involving nonlocal and impulsive conditions (5.1) can be transformed into the abstract form of problem (1.1).

**Theorem 5.1.** *Assume that there exists a function $v = v(x,t) \in PC([0,1] \times [-h,a],\mathbb{R}) \cap C^1([0,1] \times I'',\mathbb{R})$ such that*

$$\begin{cases}
\dfrac{\partial}{\partial t}v(x,t) - \varsigma(t)\dfrac{\partial^2}{\partial x^2}v(x,t) \geq L\left(\dfrac{|v(x,t)|}{1+|v(x,t)|}\right) + \displaystyle\int_{-h}^0 \gamma(s)v(x,t+s)ds,\\
\qquad\qquad x \in [0,1], \quad t \in [0,a], \quad t \neq t_k,\\
v(x,t_k^+) \geq v(x,t_k^-) + \dfrac{\sqrt{|v(x,t_k)|}}{1+|v(x,t_k)|}, \quad x \in [0,1],\ k = 1,2,\cdots,m,\\
\qquad v(0,t) = v(1,t) = 0, \quad t \in [0,a],\\
v(x,s) \geq \displaystyle\int_0^a \sigma(s,t)\lg(1+|v(x,t)|)dt + \kappa(x,s), \quad x \in [0,1],\ s \in [-h,0].
\end{cases} \tag{5.3}$$

*Then the non-autonomous parabolic partial differential equation with delay involving nonlocal and impulsive conditions* (5.1) *has a minimal mild solution and a maximal mild solution between* 0 *and $v(x,t)$, which can be obtained by a monotone iterative procedure starting from* 0 *and $v(x,t)$, respectively.*

**Proof.** From the assumption and the definition of nonlinear term $f$, impulsive function $I_k$ for $k = 1,2,\cdots,m$ and nonlocal function $g$, we can verify that $v^{(0)} = 0$ and $w^{(0)} = v(x,t)$ are the lower and the upper solutions of the non-autonomous parabolic partial differential equation with delay involving nonlocal and impulsive conditions (5.1) respectively, the nonlocal term $g$ is continuous and compact as well as the conditions (F1), (F2) and (F3) are satisfied with $C = \frac{1}{2}$.

On the other hand, from the definition of the nonlinear term $f$, we know that

$$\theta \leq f(t, v_2, \phi_2) - f(t, v_1, \phi_1) \leq L\Big(\frac{|v_2|}{1+|v_2|} - \frac{|v_1|}{1+|v_1|}\Big) + \int_{-h}^{0} \gamma(s)(\phi_2(s) - \phi_1(s))ds,$$

for all $t \in [0, a]$, $v_1, v_2 \in X$ and $\phi_1, \phi_2 \in \mathcal{B}$ with $\theta \leq \phi_1 \leq \phi_2$. Combining this and the normality of cone $P$, we have

$$\begin{aligned} \|f(t, v_2, \phi_2) - f(t, v_1, \phi_1)\|_2 &\leq L(\|v_2 - v_1\|_2) + \int_{-h}^{0} \gamma(s)\|\phi_2(s) - \phi_1(s)\|_2 ds \\ &\leq L(\|v_2 - v_1\|_2) + \int_{-h}^{0} \gamma(s)ds \sup_{-h\leq s\leq 0} \|\phi_2(s) - \phi_1(s)\|_2. \end{aligned} \tag{5.4}$$

Then for every $t \in [0, a]$,

$$\begin{aligned} \mu\Big(\big\{f(t, u^{(n)}(t), (u^{(n)})_t\big\}\Big) \leq &\max\Big\{L, \int_{-h}^{0} \gamma(s)ds\Big\}\Big[\mu(\{u^{(n)}(t)\}) \\ &+ \sup_{-h\leq s\leq 0} \mu(\{u^{(n)}(t+s)\})\Big], \end{aligned}$$

where $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ is countable and increasing or decreasing monotonic set. This means that the condition (F4) holds with $L_f = \max\Big\{L,\ \int_{-h}^{0} \gamma(s)ds\Big\}$. Therefore, our conclusion follows from Theorem 4.1. This completes the proof. □

## Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions, which improved the quality of this paper.

## References

[1] P. Acquistapace, *Evolution operators and strong solution of abstract parabolic equations*, Differ. Integral Equ., 1988, 69, 433–457.

[2] P. Acquistapace and B. Terreni, *A unified approach to abstract linear parabolic equations*, Rend. Semin. Mat. Univ. Padova, 1987, 78, 47–107.

[3] S. Arora, S. Singh, M. T. Mohan and J. Dabas, *Approximate controllability of non-autonomous second order impulsive functional evolution equations in Banach spaces*, Qual. Theory Dyn. Syst., 2023, 22, 31.

[4] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Notes in Pure Appl Math., Vol. 60, New York, 1980.

[5] E. M. Bonottoa, M. C. Bortolana, A. N. Carvalhoa and R. Czaja, *Global attractors for impulsive dynamical systems aprecompact approach*, J. Differ. Equ., 2015, 259, 2602–2625.

[6] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., 1991, 162, 494–505.

[7] P. Chen and Y. Li, *Mixed monotone iterative technique for a class of semilinear impulsive evolution equations in Banach spaces*, Nonlinear Anal., 2011, 74, 3578–3588.

[8] P. Chen, Y. Li and H. Yang, *Perturbation method for nonlocal impulsive evolution equations*, Nonlinear Anal. Hybrid Syst., 2013, 8, 22–30.

[9] P. Chen, X. Zhang and Y. Li, *Non-autonomous parabolic evolution equations with non-instantaneous impulses governed by noncompact evolution families*, J. Fixed Point Theory Appl., 2019, 21, 1–14.

[10] P. Chen, X. Zhang and Y. Li, *Non-autonomous evolution equations of parabolic type with non-instantaneous impulses*, Mediterr, J. Math., 2019, 16, 1–14.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[12] Y. Ding and J. Niu, *Solvability and optimal controls of fractional impulsive stochastic evolution equations with nonlocal conditions*, J. Appl. Anal. Comput., 2024, 14, 2622–2642.

[13] S. W. Du and V. Lakshmikantham, *Monotone iterative technique for differential equations in Banach spaces*, J. Math. Anal. Appl., 1982, 87, 454–459.

[14] X. Fu, *Existence of solutions for non-autonomous functional evolution equations with nonlocal conditions*, Electron, J. Differ. Equ., 2012, 110, 15.

[15] M. A. El-Gebeily, D. O'Regan and J. J. Nieto, *A monotone iterative technique for stationary and time dependent problems in Banach spaces*, J. Comput. Appl. Math., 2010, 233, 2359–2404.

[16] D. Guo and X. Liu, *Extremal solutions of nonlinear impulsive integro-differential equations in Banach spaces*, J. Math. Anal. Appl., 1993, 177, 538–552.

[17] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal., 1983, 7, 1351–1371.

[18] B. Hu and Y. Liao, *Convergence conditions for extreme solutions of an impulsive differential system*, AIMS Math., 2025, 10, 10591–10604.

[19] B. Hu, Y. Qiu, W. Zhou and L. Zhu, *Existence of solution for an impulsive differential system with improved boundary value conditions*, AIMS Math., 2023, 8, 17197–17207.

[20] N. Ikhlef, A. Bensalem, A. Salim, M. Benchohra and S. Litimein, *PC-asymptotically almost automorphic mild solutions for impulsive integro-differential equations with nonlocal conditions*, Stud. Univ. Babeș-Bolyai Math., 2025, 70, 127–143.

[21] A. Khatoon, A. Raheem and A. Afreen, *Stochastic controllability of a non-autonomous impulsive system with variable delays in control*, Filomat, 2023, 37, 8175–8191.

[22] Q. Li, X. Y. Gu and L. J. Wang, *Variational iteration method for solving neutral functional differential equations with vanishing delays*, J. Jilin Univ. Sci., 2011, 49, 33–36.

[23] Q. Li and W. Zhou, *Extremal solutions for periodic problems of fractional impulsive evolution equation with piecewise Caputo derivative*, Res. Math. Sci., 2025, 12, 47.

[24] Y. Li and Z. Liu, *Monotone iterative technique for addressing impulsive integro-differential equtions in Banach spaces*, Nonlinear Anal., 2007, 66, 83–92.

[25] Y. Li and B. Qu, *Mild solutions for fractional non-instantaneous impulses integro-differential equations with nonlocal conditions*, AIMS Math., 2024, 9, 12057–12071.

[26] Y. X. Liang, Z. B. Fan and G. Li, *Existence, uniqueness and regularity of solutions for fractional integro-differential equations with state-dependent delay*, J. Appl. Anal. Comput., 2024, 14, 623–641.

[27] H. L. Smith, *Monotone Dynamical Systems*, American Mathematical Society, 1995.

[28] N. N. Vien, V. T. Luong and V. H. Le, *On asymptotic periodic solutions of delay evolution equations on the half line*, Funkcial. Ekvac., 2024, 67, 309–325.

[29] J. R. Wang and W. Wei, *A class of nonlocal impulsive problems for integrodifferential equations in Banach spaces*, Results Math., 2010, 58, 379–397.

[30] L. Wang and Z. Wang, *Monotone iterative technique for parameterized BVPs of abstract semilinear evolution equations*, Comput. Math. Appl., 2003, 46, 1229–1243.

[31] Z. Wang, B. Hu, L. Zhu, J. Lin, M. Xu and D. Wang, *Hopf bifurcation analysis for Parkinson oscillation with heterogeneous delays: A theoretical derivation and simulation analysis*, Commun. Nonlinear Sci. Numer. Simul., 2022, 114, 106614.

[32] C. You, L. Shu and X. B. Shu, *Approximate controllability of second-order neutral stochastic differential evolution systems with random impulsive effect and state-dependent delay*, AIMS Math., 2024, 9, 28906–28930.

[33] X. Zhang and Y. Li, *Fractional retarded evolution equations with measure of noncompactness subjected to mixed nonlocal plus local initial conditions*, Int. J. Nonlinear Sci. Numer. Simul., 2018, 19, 69–81.

[34] B. Zhu, L. Liu and Y. Wu, *Local and global existence of mild solutions for a class of semilinear fractional integro-differential equations*, Fract. Calc. Appl. Anal., 2017, 20, 1338–1355.

Received July 2025; Accepted October 2025; Available online October 2025.