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GLOBAL DYNAMICS OF A POLYNOMIAL LIÉNARD SYSTEM*

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Abstract When studying traveling wave solutions of a generalized Burgers–Fisher equation we obtained a Liénard system of the form $\frac{\mathrm{d}x}{\mathrm{d}\tau} = y$, $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -x(x^m - 1) - \varepsilon\,(a + x^m)\,y$, where $m \geq 1$ is an integer and ε is a small parameter. In this paper, we present a complete analysis for its global dynamics for all $m \geq 1$, giving necessary and sufficient conditions for the existence of one, two or three limit cycles and for the existence of homoclinic or double homoclinic loops by applying new methods established in this paper.

Keywords Liénard system, limit cycle, global bifurcation, homoclinic loop.

MSC(2010) 34C05, 34C07.

1. Introduction and main results

In recent years, a generalized Burgers–Fisher equation of the form (GBF equation for short)

$$u_t + \alpha u^m u_\chi + \beta u_{\chi\chi} + \gamma u(1 - u^m) = 0 \tag{1.1}$$

has been widely studied by many scholars (see [1, 3, 12, 13, 15–17, 19, 20]), where $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$ are constants and $m \in \mathbb{Z}^+$.

As in Refs. [13, 20], by making the traveling wave ansatz

$$u(\vartheta, t) = x(\varsigma), \qquad \varsigma = \vartheta - ct,$$

where c represents the speed of the wave, and then introducing suitable rescaling of variables one can obtain from (1.1)

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= -x(1 - x^m) - \varepsilon\,(a + x^m)\,y \end{aligned} \tag{1.2}$$

for $\beta\gamma > 0$ and

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= -x(x^m - 1) - \varepsilon\,(a + x^m)\,y \end{aligned} \tag{1.3}$$

for $\beta\gamma < 0$, where $0 < |\varepsilon| \ll 1$, $a \in \mathbb{R}$.

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The above two systems are both polynomial ones of degree $m+1$ of Liénard type. The problem of finding the maximum number of limit cycles of a polynomial Liénard system was posed in [18] by Smale, and is still open since it is extremely difficult when its degree is arbitrarily high. In the paper [20], the authors considered the above systems for some positive integers m. The authors [13] investigated bifurcations of limit cycles of system (1.2) and gave necessary and sufficient conditions for it to have a unique limit cycle, a homoclinic loop and a heteroclinic loop respectively for all $m \geq 1$. For system (1.3) with the cases $m = 1, 2$, there were also many studies on the limit cycle bifurcation, see [2, 4, 10, 14] and the related references therein. About the studies on the uniqueness of limit cycles appearing in codimension two bifurcations of planar systems one can see the survey articles [8, 9].

This paper is a continuation of [13], presenting a complete analysis on the global behavior of (1.3) for $|\varepsilon|$ small, including Poincaré, Hopf, homoclinic loop and double homoclinic loop bifurcations for all $m \geq 1$. Our main results are stated below.

Theorem 1.1. *Consider system (1.3) with $m \geq 1$ odd. Let $G > 1$ be a constant. Then there are a positive constant ε^* and two C^∞ functions $a = a^*_{1,m}(\varepsilon) = -1 + O(\varepsilon)$ and $a = a^*_{2,m}(\varepsilon) = -\frac{2m+4}{3m+4} + O(\varepsilon)$ such that for all $0 < |\varepsilon| < \varepsilon^*$ and $|a| \leq G$, we have the following results about system (1.3)*

(i) there is no limit cycle if $a \leq a^*_{1,m}(\varepsilon)$ or $a \geq a^*_{2,m}(\varepsilon)$;

(ii) there is a homoclinic loop if $a = a^*_{2,m}(\varepsilon)$;

(iii) there is a unique limit cycle if $a^*_{1,m}(\varepsilon) < a < a^*_{2,m}(\varepsilon)$.

Additionally, when the aforementioned limit cycle or the homoclinic loop exists, it is unstable if ε is positive and stable if ε is negative.

The bifurcation diagram of system (1.3) with ε being positive and m odd is as shown in Figure 1.

For $m \geq 2$ even system (1.3) has a saddle $(0, 0)$ and two foci $(\pm 1, 0)$ as $0 < |\varepsilon| \ll 1$. We call a limit cycle surrounding the three singular points a large limit cycle, and a limit cycle surrounding $(1, 0)$ or $(-1, 0)$ only a small limit cycle. Then we have

Theorem 1.2. *Consider system (1.3) with $m \geq 2$ even. Let $G > 1$ be a constant. Then there exist a constant $\hat{\epsilon}^* > 0$ and three C^∞ functions $a = \hat{a}^*_{1,m}(\varepsilon) = -1+O(\varepsilon)$, $a = \hat{a}^*_{3,m}(\varepsilon) = -\frac{2m+4}{3m+4} + O(\varepsilon)$ and $a = \hat{a}^*_{0,m}(\varepsilon) = a_{0,m} + O(\varepsilon)$ with $a_{0,m} \in (-\frac{2m+4}{3m+4}, 0)$, such that for all $0 < |\varepsilon| < \hat{\epsilon}^*$ and $|a| \leq G$,*

(i) system (1.3) has no periodic orbit for $a > \hat{a}^*_{0,m}(\varepsilon)$;

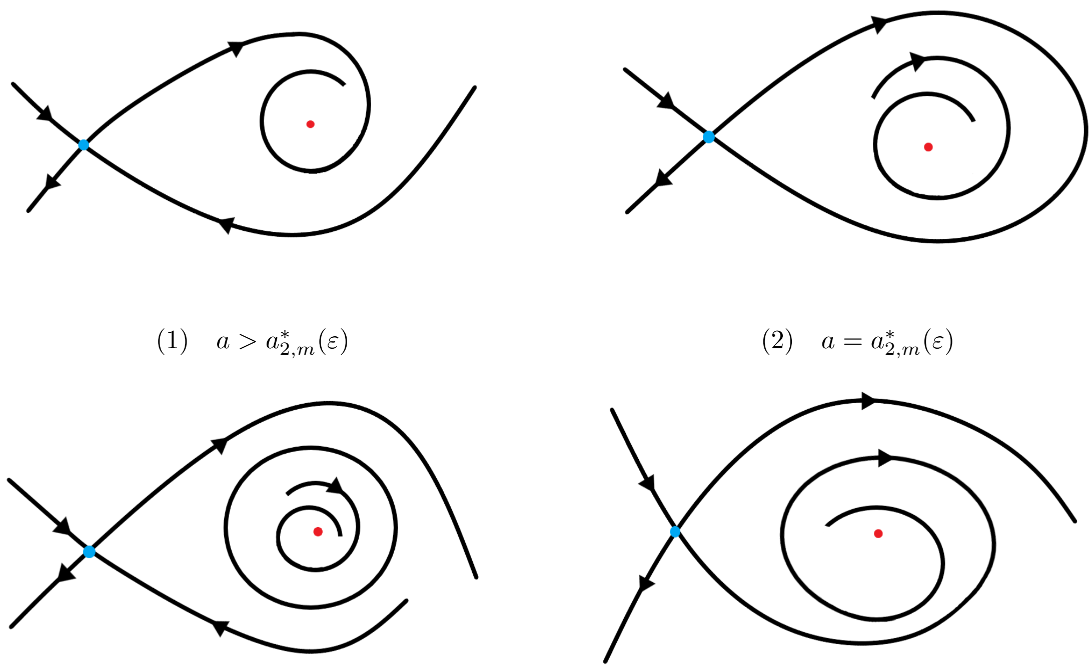
(ii) system (1.3) has a unique large limit cycle $\hat{L}^0(\varepsilon)$ for $a = \hat{a}^*_{0,m}(\varepsilon)$ which is semistable;

(iii) system (1.3) has exactly two large limit cycles $\hat{L}_1(\varepsilon, a) \subset \hat{L}_2(\varepsilon, a)$ for $\hat{a}^*_{3,m}(\varepsilon) < a < \hat{a}^*_{0,m}(\varepsilon)$; moreover, $\hat{L}_1(\varepsilon, a)$ is unstable (stable) and $\hat{L}_2(\varepsilon, a)$ is stable (unstable) as $\varepsilon > 0$ ($\varepsilon < 0$);

(iv) system (1.3) has a double homoclinic loop $\bar{L}(\varepsilon)$ and a unique large limit cycle $\hat{L}_2(\varepsilon, a)$ for $a = \hat{a}^*_{3,m}(\varepsilon)$; moreover, the double homoclinic loop is unstable (stable) and the large limit cycle $\hat{L}_2(\varepsilon, a)$ is stable (unstable) as $\varepsilon > 0$ ($\varepsilon < 0$);

(v) system (1.3) has exactly two symmetric small limit cycles $\hat{L}^+_1(\varepsilon, a)$ and $\hat{L}^-_2(\varepsilon, a)$ and a unique large limit cycle $\hat{L}_2(\varepsilon, a)$ for $\hat{a}^*_{1,m}(\varepsilon) < a < \hat{a}^*_{3,m}(\varepsilon)$; moreover, the small limit cycles $\hat{L}^+_1(\varepsilon, a)$ and $\hat{L}^-_2(\varepsilon, a)$ are unstable (stable) and the large limit cycle $\hat{L}_2(\varepsilon, a)$ is stable (unstable) as $\varepsilon > 0$ ($\varepsilon < 0$);

(vi) system (1.3) has a unique large limit cycle $\hat{L}_2(\varepsilon, a)$ for $a \leq \hat{a}^*_{1,m}(\varepsilon)$ which is stable (unstable) as $\varepsilon > 0$ ($\varepsilon < 0$).



(1) $a > a^*_{2,m}(\varepsilon)$ (2) $a = a^*_{2,m}(\varepsilon)$

(3) $a^*_{1,m}(\varepsilon) < a < a^*_{2,m}(\varepsilon)$ (4) $a \geq a^*_{1,m}(\varepsilon)$

Figure 1. Bifurcation diagram of system (1.3) with ε positive and m odd.

The bifurcation diagrams of system (1.3) with ε being positive, as stated in Theorem 1.2, are depicted in Figure 2.

In order to prove the above two theorems, we first give eight preliminary lemmas in Section 2. Then we present our proof to the two theorems in Section 3 based on these lemmas.

2. General results on limit cycle bifurcations

In this section we list three lemmas which establish certain general results on the existence of a limit cycle of general near-Hamiltonian systems in Hopf, homoclinic and double homoclinic bifurcations, respectively. Then for systems of Liénard type with two or three singular points, we present five new lemmas considering the number of limit cycles in Poincaré bifurcations.

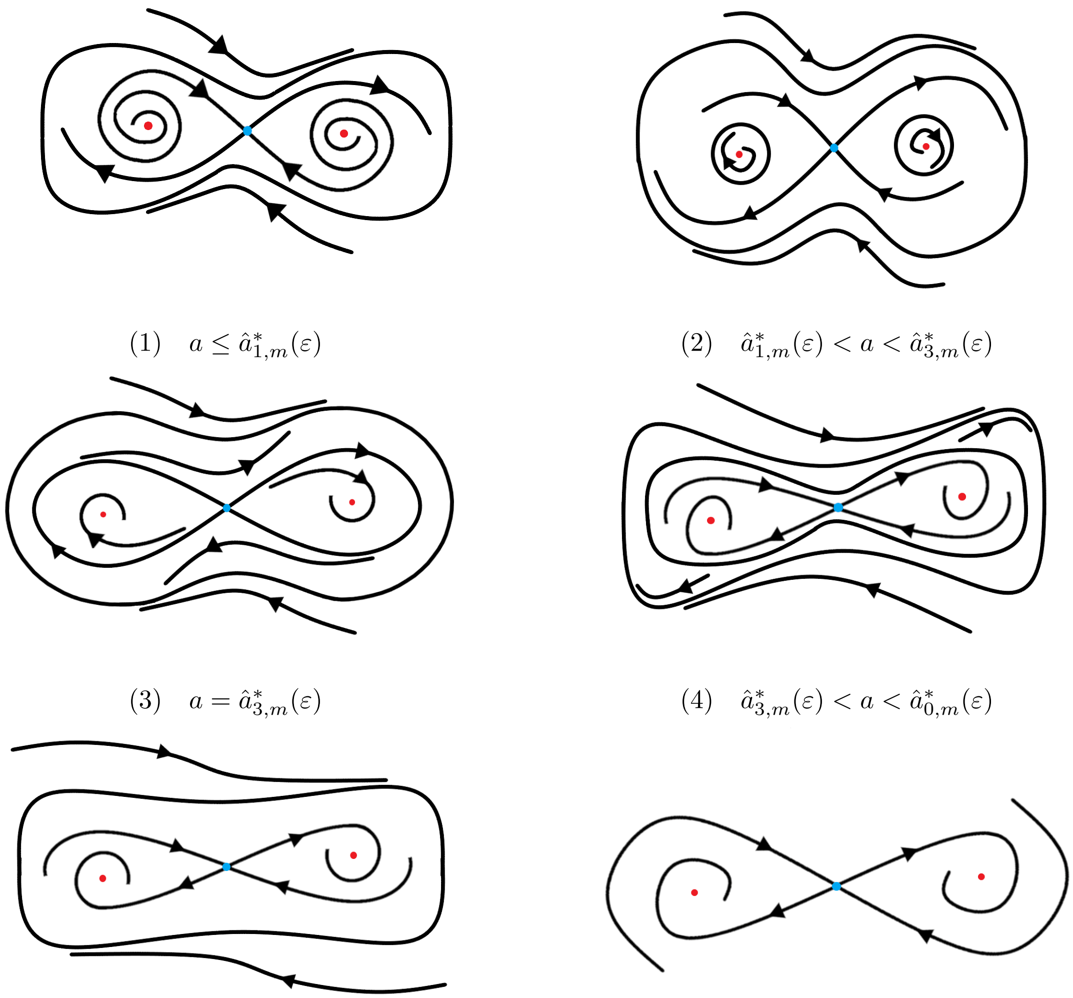
2.1. Near-Hamiltonian system

We consider the following C^∞ near-Hamiltonian system on the plane

$$\begin{aligned}\dot{x} &= H_y(x,y) + \varepsilon \bar{f}_0(x,y,\varepsilon,a),\\ \dot{y} &= -H_x(x,y) + \varepsilon \bar{g}_0(x,y,\varepsilon,a),\end{aligned} \tag{2.1}$$

where $|\varepsilon| \geq 0$ is small, H, $\bar{f}_0$ and $\bar{g}_0$ are C^∞ functions, and a belongs to D, which is a bounded subset of $\mathbb{R}$. Suppose that $(2.1)|_{\varepsilon=0}$ possesses a family of periodic orbits as the following

$$L_h : H(x,y) = h, \qquad h \in (\alpha_0, \beta_0).$$



(1) $a \le \hat{a}^*_{1,m}(\varepsilon)$ (2) $\hat{a}^*_{1,m}(\varepsilon) < a < \hat{a}^*_{3,m}(\varepsilon)$

(3) $a = \hat{a}^*_{3,m}(\varepsilon)$ (4) $\hat{a}^*_{3,m}(\varepsilon) < a < \hat{a}^*_{0,m}(\varepsilon)$

(5) $a = \hat{a}^*_{0,m}(\varepsilon)$ (6) $a > \hat{a}^*_{0,m}(\varepsilon)$

Figure 2. The bifurcation diagram of system (1.3) with ε being positive and m even.

As we know, the first order Melnikov function of (2.1) has the form

$$M(h,a) = \oint_{L_h} \left(\bar{g}_0 \mathrm{d}x - \bar{f}_0 \mathrm{d}y\right)\Big|_{\varepsilon=0}, \qquad h \in (\alpha_0, \beta_0). \tag{2.2}$$

The following lemma concerns saddle-node bifurcation of limit cycles of system (2.1) which can be found in chapter 7 of Ref. [11]. A similar conclusion for a concrete system was also given in the proof of Lemma 2.8 in chapter 4 of Ref. [4].

Lemma 2.1. ([11]) (Saddle-node bifurcation) *Suppose that there exist $h_0 \in (\alpha_0, \beta_0)$ and $a_0 \in D$, such that $M(h_0, a_0) = M'_h(h_0, a_0) = 0$, $\mu_0 \equiv M''_h(h_0, a_0) M'_a(h_0, a_0) \neq 0$, where $M(h,a)$ is given in (2.2). Then there exist a constant $\varepsilon_0 > 0$, a neighborhood U_0 of L_{h_0} and a C^∞ function $a = \bar{a}_0(\varepsilon) = a_0 + O(\varepsilon)$, such that for $0 < |\varepsilon| < \varepsilon_0$ and $|a - a_0| < \varepsilon_0$, system (2.1) has two simple limit cycles (resp. a unique limit cycle with multiplicity two, no limit cycle) in U_0 for* $(\operatorname{sgn} \mu_0)(a - a_0(\varepsilon)) < 0$ *(resp.* $= 0,\ > 0$).

In order to consider Hopf bifurcation, suppose that $\lim_{h\to\alpha_0} L_h$ exists and the limit is an elementary center of system $(2.1)|_{\varepsilon=0}$. Without loss of generality, we assume that the center is at the origin and that for (x,y) near the origin

$$\begin{pmatrix} H_y(x,y)+\varepsilon\bar{f}_0(x,y,\varepsilon,a) \\ -H_x(x,y)+\varepsilon\bar{g}_0(x,y,\varepsilon,a)\end{pmatrix} = \bar{A}(\varepsilon,a)\begin{pmatrix} x \\ y\end{pmatrix} + O(|x,y|^2),$$

where

$$\bar{A}(0,a) = \begin{pmatrix} 0 & -b_0 \\ b_0 & 0\end{pmatrix}, \qquad b_0\neq 0.$$

It is easy to see that the real part of eigenvalues of $\bar{A}(\varepsilon,a)$, denoted by $\bar{a}(\varepsilon,a)$, satisfies

$$\bar{a}(\varepsilon,a) = \frac{\varepsilon}{2}\left(\bar{f}_{0x}+\bar{g}_{0y}\right)\big|_{(x,y)=(0,0)}.$$

The author [5] defined a succession function $\Delta(x,\varepsilon,a)$ of (2.1) in a neighborhood of the origin, having the following expansion

$$\Delta(x,\varepsilon,a) = \varepsilon\sum_{i\geq 1}\Delta_i(\varepsilon,a)x^i$$

for $|x|$ small such that system (2.1) has a limit cycle near the origin if and only if the function Δ has a small positive zero in x. From Ref. [9] we also know that $\Delta_1(0,a)=\frac{2\pi}{b_0}\frac{\partial\bar{a}}{\partial\varepsilon}(0,a)$ and

$$\Delta_3(0,a) = -\frac{b_0}{8}N_1(a), \quad N_1(a) = \frac{\partial^2 M(h,a)}{\partial h^2}\Big|_{h=0}$$

as $\Delta_1(0,a)=0$, where $M(h,a)$ is the function in (2.2).

By page 3 of Ref. [9] and Theorem 2.4.4 of Ref. [7], it is easy to see that if

$$H(x,y) = \frac{1}{2}\left(x^2+y^2\right) + \sum_{i+j\geq 3} h_{ij}x^iy^j$$

and

$$\left(\bar{f}_{0x}+\bar{g}_{0y}\right)\Big|_{\varepsilon=0} = \sum_{i+j\geq 0} c_{ij}x^iy^j$$

for (x,y) near $(0,0)$, then we have for the formula of N_1

$$N_1(a) = \frac{\partial^2 M(h,a)}{\partial h^2}\Big|_{h=0} = -2c_{10}\pi(h_{12}+3h_{30}) - 2c_{01}\pi(h_{21}+3h_{03}) + 2c_{20}\pi + 2c_{02}\pi.$$

On the uniqueness of limit cycles of system (2.1) near the origin, we have from Corollary 2.7 of Ref. [5] the following fundamental conclusion which can be also found in Han et al. [9].

Lemma 2.2. ([5]) (Hopf bifurcation) *Let $c_{00}(a_1)=0$ for some $a_1\in D$. If $N_1(a_1)\neq 0$, then there exist $\varepsilon_1>0$ and a neighborhood U_1 of the origin, such that for all $0<|\varepsilon|<\varepsilon_1$ and $a\in D$*

(1) *when $|a-a_1|<\varepsilon_1$, there is a unique limit cycle of system (2.1) in the neighborhood U_1 if and only if $N_1(a_1)(\bar{f}_{0x}+\bar{g}_{0y})(0,0,\varepsilon,a)<0$;*

(2) *when $|a-a_1|\geq\varepsilon_1$, there is no limit cycle for system (2.1) in the neighborhood U_1.*

Next we consider double homoclinic bifurcation. For simplicity, suppose that (2.1) is a centrally symmetric system. In other words, H, $\bar{f}$ and $\bar{g}$ in system (2.1) satisfy

$$\begin{aligned} &H(-x,-y)=H(x,y),\\ &\bar{f}_0(-x,-y,\varepsilon,a)=-\bar{f}_0(x,y,\varepsilon,a),\\ &\bar{g}_0(-x,-y,\varepsilon,a)=-\bar{g}_0(x,y,\varepsilon,a). \end{aligned} \tag{2.3}$$

Also, we suppose that the Hamiltonian system $(2.1)|_{\varepsilon=0}$ has a hyperbolic saddle O at the origin with $H(0,0)=0$ and that the equation $H(x,y)=0$ defines a double homoclinic loop $\bar{L}=\bar{L}_1\cup\bar{L}_2$ with a clockwise orientation, where $\bar{L}_1$ and $\bar{L}_2$ are two homoclinic loops through the hyperbolic saddle O of $(2.1)|_{\varepsilon=0}$ located in the right half plane and left half plane respectively. Then one can see that for some $h_0>0$ the equation $H(x,y)=h$ defines a family of large periodic orbits $\bar{L}(h)$ for $0<h<h_0$ and two families of small periodic orbits $\bar{L}_i(h)$ for $-h_0<h<0$, $i=1,2$. That is,

$$\begin{aligned} &\bar{L}(h): H(x,y)=h, \quad 0<h<h_0,\\ &\bar{L}_i(h): H(x,y)=h, \quad -h_0<h<0, \quad (-1)^i x<0, \quad i=1,2. \end{aligned}$$

Let

$$\bar{c}_0(a)=2\oint_{\bar{L}_1}\left(\bar{g}_0\mathrm{d}x-\bar{f}_0\mathrm{d}y\right)\Big|_{\varepsilon=0}, \quad \bar{c}_1(a)=2(\bar{f}_{0x}+\bar{g}_{0y})\,|_{\varepsilon=0,(x,y)=(0,0)}\,. \tag{2.4}$$

We have

Lemma 2.3. ([10, 11]) (Double homoclinic bifurcation) *Let (2.3) hold and the above assumptions be satisfied. If $\bar{c}_0(a_3)=0$, $\bar{\mu}\equiv\frac{\partial(\bar{c}_0)}{\partial a}(a_3)\bar{c}_1(a_3)\neq 0$ for some $a_3\in D$, then there exist a constant $\bar{\epsilon}>0$, a neighborhood $\bar{U}$ surrounding $\bar{L}$ and a C^∞ function $a_3^*(\varepsilon)=a_3+O(\varepsilon)$, such that for $0<|\varepsilon|<\bar{\epsilon}$, $|a-a_3|<\bar{\epsilon}$ and $a\in D$, we have the following results*

(1) *when $\bar{\mu}(a-a_3^*(\varepsilon))<0$, there is a unique large limit cycle for system (2.1) in $\bar{U}$;*
(2) *when $\bar{\mu}(a-a_3^*(\varepsilon))>0$, there is no large limit cycle for system (2.1) in $\bar{U}$;*
(3) *when $a=a_3^*(\varepsilon)$, there is a double homoclinic loop for system (2.1) in $\bar{U}$.*

We indicate that Lemma 2.3 was obtained in Ref. [10] and proved in detail in chapter 7 of Ref. [11].

The following corollary is direct from Lemma 2.8 of Ref. [13] and Lemma 2.3.

Corollary 2.1. ([11]) *Suppose that (2.3) is satisfied. Let $\bar{c}_0(a_3)=0$ for some $a_3\in D$. If $\frac{\partial\bar{c}_0}{\partial a}(a_3)\bar{c}_1(a_3)\neq 0$, then there exist $\varepsilon_3>0$, a neighborhood U_3 of $\bar{L}$ and a C^∞ function $a_3^*(\varepsilon)=a_3+O(\varepsilon)$ such that for $0<|\varepsilon|<\varepsilon_3$, $a\in D$ and $|a-a_3|<\varepsilon_3$*

(i) there is a double homoclinic loop for system (2.1) in $\bar{U}$ if and only if $a=a_3^*(\varepsilon)$;
(ii) there is a unique large limit cycle for system (2.1) in $\bar{U}$ if $\bar{\mu}(a-a_3^*(\varepsilon))<0$;
(iii) there is precisely two small limit cycles for system (2.1) in $\bar{U}$ if $\bar{\mu}(a-a_3^*(\varepsilon)>0$.

2.2. Liénard system

Consider Liénard system on the plane

$$\begin{aligned} &\dot{x}=y,\\ &\dot{y}=-g(x)-\varepsilon y f(x,a), \end{aligned} \tag{2.5}$$

where $a\in D$ and ε are as in (2.1), the functions $g(x)$ and $f(x,a)$ are analytic on the intervals (x_{10},x_{20}) and $(x_{10},x_{20})\times D$ with $-\infty\leq x_{10}<x_{20}\leq+\infty$, respectively.

Firstly, we consider limit cycles bifurcated from a period annulus surrounding a single singular point of system $(2.5)|_{\varepsilon=0}$. For the purpose, suppose that there exists $x_0 \in (x_{10}, x_{20})$ such that

$$g(x_0) = 0, \ (x - x_0)g(x) > 0, \text{ for } x \in (x_{10}, x_{20}), \ x \neq x_0. \tag{2.6}$$

Introduce

$$F(x,a) = \int_{x_0}^{x} f(u,a)\mathrm{d}u, \quad G(x) = \int_{x_0}^{x} g(u)\mathrm{d}u, \quad Z^* = \min\{G(x_{10}), G(x_{20})\},$$
$$U = \{(x,y) \mid x_{10} < x < x_{20}, |y| < \infty, (x,y) \neq (x_0, 0)\}.$$

Obviously, system $(2.5)|_{\varepsilon=0}$ has a family of periodic orbits as follows

$$L_h : G(x) + \frac{1}{2}y^2 = h, \qquad 0 < h < Z^*$$

in the region U, and the first order Melnikov function of system (2.5) has the form

$$M(h,a) = -\oint_{L_h} y f(x,a)\mathrm{d}x = \oint_{L_h} F(x,a)\mathrm{d}y, \qquad 0 < h < Z^*. \tag{2.7}$$

By (2.6), the function $Z = G(x)$ has two inverse functions $x = x_1(Z) \in (x_0, x_{20})$ and $x = x_2(Z) \in (x_{10}, x_0)$ in $Z \in (0, Z^*)$. Then, by Ref. [6] one has

$$M(h,a) = \int_{L_h^*} K(Z,a)\mathrm{d}y, \quad 0 < h < Z^*,$$

where L_h^* is the directed curve defined by the equation $Z + \frac{1}{2}y^2 = h$ for $0 \le Z \le h$ with $\mathrm{d}y < 0$ along it, and

$$K(Z,a) = F(x_1(Z),a) - F(x_2(Z),a), \qquad 0 \le Z < Z^*.$$

Obviously,

$$K(Z,a) = F(\beta(x),a) - F(x,a), \tag{2.8}$$

where $x = x_2(Z) < x_0$, $\beta(x) = x_1(G(x)) > x_0$ which satisfies $G(\beta(x)) = G(x)$ for $x \in (x_{10}, x_0)$.

Based on the proof of Theorem 2.1 in Ref. [6] and by (2.8), one can prove easily the following lemma.

Lemma 2.4. *Let (2.6) hold. Assume there exists a subset $D_0 \subset D$, where each element a in D_0 satisfies the following two conditions:*

(1) $F(\beta(x),a) - F(x,a) \ge 0$ (resp. ≤ 0) for $x \in (x_{10}, x_0)$;

(2) $F(\beta(x),a) - F(x,a) \not\equiv 0$ on any subinterval of (x_{10}, x_0).

Then $M(h,a) < 0$ (resp. > 0) for all $h \in (0, Z^*)$ and $a \in D_0$.

Considering that $K'_Z(Z,a) = \frac{f(x_1(Z),a)}{g(x_1(Z))} - \frac{f(x_2(Z),a)}{g(x_2(Z))}$ and

$$g(\beta(x)) > 0, \quad g(x) < 0, \quad x \in (x_{10}, x_0) \tag{2.9}$$

by (2.6). It follows from (2.9) that if $\left(\frac{f}{g}\right)'_x > 0$ (resp. < 0) then

$$K''_Z(Z,a) = \left(\frac{f(x_1(Z),a)}{g(x_1(Z))}\right)'_x \frac{1}{g(x_1(Z))} - \left(\frac{f(x_2(Z),a)}{g(x_2(Z))}\right)'_x \frac{1}{g(x_2(Z))} > 0 \text{ (resp. } < 0)$$

since $x_1(Z) = \beta(x)$ and $x_2(Z) = x$. Hence, by Corollary 2.1 in Ref. [6] we have

Lemma 2.5. *Let (2.6) hold. Suppose that there is a subset $D_0 \subset D$ such that $\left(\frac{f}{g}\right)'_x > 0$ (resp. $<$ 0) for each $a \in D_0$. Then $M'_h(h_0, a) < 0$ (resp. > 0) whenever $M(h_0, a) = 0$ where $a \in D_0$ and $h_0 \in (0, Z^*)$.*

Secondly, we consider bifurcation of large limit cycles for (2.5).

Let $\tilde{G}(x) = \int_0^x g(u)\mathrm{d}u$ and $\tilde{F}(x,a) = \int_0^x f(u,a)\mathrm{d}u$. Denote $Z^\star \equiv \tilde{G}(+\infty) \in (0, +\infty]$. In addition, we suppose that

$$\begin{aligned} &g(x) = -g(-x), \\ &g(x) < 0 \ (> 0) \quad \text{for } 0 < x < \bar{x}_0 \ (x > \bar{x}_0). \end{aligned} \tag{2.10}$$

Then $(2.5)|_{\varepsilon=0}$ has three singular points $(0,0)$ and $(\pm\bar{x}_0, 0)$, and a large family of periodic orbits surrounding the three singular points given by

$$L_h : H(x,y) = \tilde{G}(x) + \frac{1}{2}y^2 = h, \quad 0 < h < Z^\star.$$

Set $L_h^+ = L_h|_{x\geq 0}$, $\tilde{F}_1(x,a) = \tilde{F}(x,a) - \tilde{F}(-x,a)$. Let $\tilde{F}_1(x) = \tilde{F}_1(x,a)$, $M(h) = M(h,a)$. Then the first order Melnikov function of system (2.5) can be written as

$$M(h) = \int_{L_h^+} \tilde{F}_1(x)\mathrm{d}y, \quad 0 < h < Z^\star. \tag{2.11}$$

Clearly, the function $\tilde{F}_1(x)$ is odd in x. Suppose there exist $0 \leq \bar{x}_1 \leq \bar{x}_2$ such that

$$\tilde{F}_1(\bar{x}_2) = 0, \ (x - \bar{x}_1)\tilde{F}_1'(x) > 0 \ (x > 0, x \neq \bar{x}_1). \tag{2.12}$$

For each $h > 0$, denote by $A(0, y_1(h))$ and E the upper and lower intersection points of L_h^+ with the y-axis respectively, by $B(\bar{x}_0, y_2(h))$ and N the upper and lower intersection points of L_h^+ with the straight line $x = \bar{x}_0$ respectively, and by $C(\bar{x}_3(h), 0)$ the intersection of L_h^+ with the positive x-axis, as shown in Figure 3. Then y_1, y_2 and $\bar{x}_3$ satisfy

$$y_1(h) = \sqrt{2h}, \ y_2(h) = \sqrt{2(h - \tilde{G}(\bar{x}_0))}\,, \ \tilde{G}(\bar{x}_3(h)) = h.$$

Let

$$\tilde{M}_1(h) = \int_{\widehat{ABC}} \tilde{F}_1\mathrm{d}y, \ \tilde{M}_2(h) = \int_{\widehat{CNE}} \tilde{F}_1\mathrm{d}y. \tag{2.13}$$

It follows from (2.11) and (2.12) that

$$M(h) = \tilde{M}_1(h) + \tilde{M}_2(h), \ h > 0. \tag{2.14}$$

The following results (Lemmas 2.6 and 2.7 below) give some properties of M.

Lemma 2.6. *Let (2.10) and (2.12) hold. If $\bar{x}_0 > \bar{x}_2$, then $M(h) < 0$ for all $h \geq 0$.*

Proof. The curve segments $\widehat{AB}$ and $\widehat{BC}$ can be expressed as $x = b(y,h)$ $(y_1(h) \leq y \leq y_2(h))$ and $x = c(y,h)$ $(0 \leq y \leq y_2(h))$, respectively. There are two cases to consider below.

Case 1. $\bar{x}_1 < \bar{x}_2$. In this case, let B_1 be the intersection of the straight line $x = \bar{x}_2$ with the arc $\widehat{AB}$, and B_2 be the intersection of the arc $\widehat{BC}$ with the horizontal line passing through point

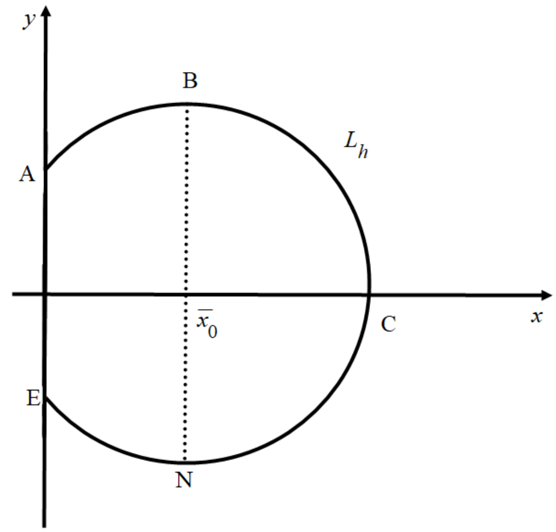


Figure 3. The part of L_h on $x \geq 0$.

B_1, see Figure 4. Hence, we have by (2.12) and (2.13)

$$\begin{aligned}\tilde{M}_1(h) &= \left(\int_{\widehat{AB_1}} + \int_{\widehat{B_2C}}\right) \tilde{F}_1 \mathrm{d}y + \int_{\widehat{B_1B_2}} \tilde{F}_1 \mathrm{d}y \\ &= \int_{y_1}^{y_{B_1}} \tilde{F}_1 \mathrm{d}y + \int_{y_{B_1}}^{0} \tilde{F}_1 \mathrm{d}y + \int_{y_{B_1}}^{y_2} \left[\tilde{F}_1(b) - \tilde{F}_1(c)\right] \mathrm{d}y \\ &< 0.\end{aligned}$$

Using a similar method as above, it is easy to get $\tilde{M}_2(h) < 0$. Thus by (2.14) we have $M(h) < 0$.

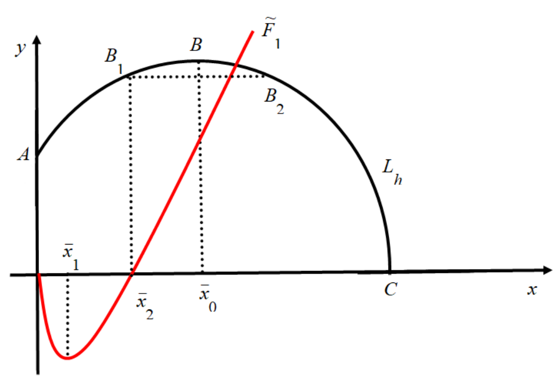


Figure 4. The part of L_h and $\tilde{F}_1$ on $x \geq 0$ for $\bar{x}_1 < \bar{x}_2$.

Case 2. $\bar{x}_1 = \bar{x}_2$. Clearly, we have $\bar{x}_1 = \bar{x}_2 = 0$ since $\tilde{F}_1(0) = 0$. Now, we suppose that A' is the intersection of the arc $\widehat{BC}$ with the horizontal line passing through the point A, as shown in Figure 5. By (2.12) and (2.13), we obtain

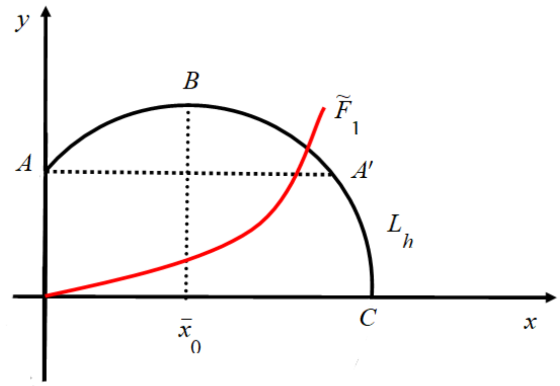


Figure 5. The part of L_h and $\tilde{F}_1$ on $x \geq 0$ for $\bar{x}_1 = \bar{x}_2$.

$$\tilde{M}_1(h) = \int_{\widehat{ABA'}} \tilde{F}_1 \mathrm{d}y + \int_{\widehat{A'C}} \tilde{F}_1 \mathrm{d}y = \int_{y_1}^{y_2} [\tilde{F}_1(b) - \tilde{F}_1(c)]\mathrm{d}y + \int_{y_1}^{0} \tilde{F}_1 \mathrm{d}y < 0.$$

Similarly, we can obtain $\tilde{M}_2(h) < 0$. Hence, by (2.14), it is easy to get $M(h) < 0$. This ends the proof. □

The following lemma is crucial in the proof of the main result Theorem 1.2.

Lemma 2.7. *Let (2.10) and (2.12) hold. Suppose that one of the following conditions is satisfied:*

(i) $\bar{x}_0 < \bar{x}_1$ *and the function* $\frac{\tilde{F}_1'(x)\sqrt{\tilde{G}(x)-\tilde{G}(\bar{x}_0)}}{g(x)}$ *is increasing in* $(\tilde{x}, +\infty)$, *where* $\tilde{x} = \min\{\bar{x}_1, x^*\}$ *with* x^* *satisfying* $\tilde{G}(x^*) = 0$ *and* $x^* > \bar{x}_0$;

(ii) $\bar{x}_0 \geq \bar{x}_1 > 0$ *and the function* $\frac{\tilde{F}_1'(x)\sqrt{\tilde{G}(x)-\tilde{G}(\bar{x}_0)}}{|g(x)|}$ *is increasing in* $(\bar{x}_1, +\infty)$.

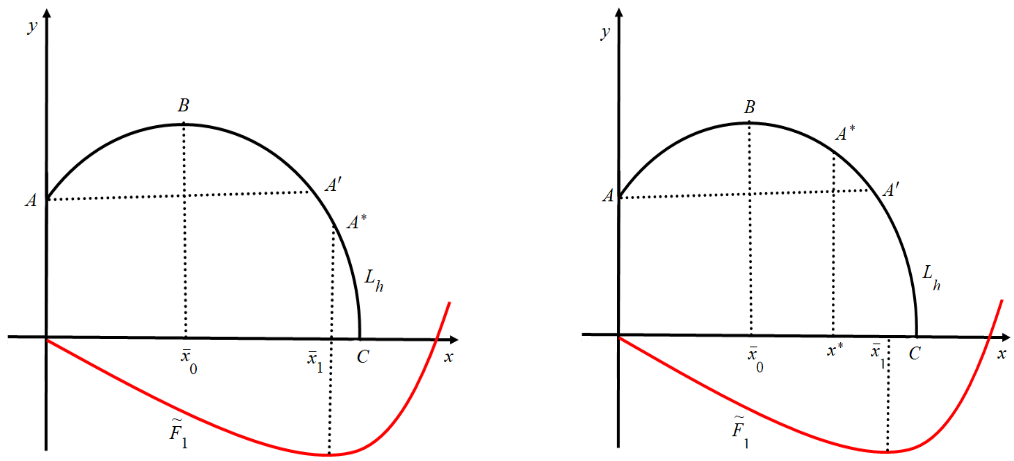
Then for all $h > 0$, $M_h''(h) < 0$.

Proof. Under condition (i), we have $\bar{x}_0 < \bar{x}_1$. Then for each $h > 0$, we denote by A^* the intersection of the straight line $x = \tilde{x}$ with arc $\widehat{AC}$, and A' the intersection of the arc $\widehat{BC}$ with the horizontal line passing through A, as shown in Figure 6. It follows from (2.14) that

$$\begin{aligned} \tilde{M}_1(h) &= \int_{\widehat{ABA'}} \tilde{F}_1 \mathrm{d}y + \int_{\widehat{A'C}} \tilde{F}_1 \mathrm{d}y \\ &= \int_{y_1}^{y_2} \left[\tilde{F}_1(b) - \tilde{F}_1(c)\right] \mathrm{d}y + \int_{y_1}^{0} \tilde{F}_1(c)\mathrm{d}y \\ &\equiv \tilde{M}_{11}(h) + \tilde{M}_{12}(h). \end{aligned} \tag{2.15}$$

Note that

$$\begin{aligned} &b(y_1, h) = 0,\ c(y_1, h) \equiv x' \in (\bar{x}_0, \bar{x}_3), \\ &b(y_2, h) = c(y_2, h) = \bar{x}_0, \\ &\frac{\partial b}{\partial h} = \frac{1}{g(b)},\ \frac{\partial c}{\partial h} = \frac{1}{g(c)}. \end{aligned} \tag{2.16}$$



(a) $\tilde{x} = \bar{x}_1$ (b) $\tilde{x} = x^*$

Figure 6. The part of L_h and $\tilde{F}_1$ on $x \geq 0$ for $\bar{x}_0 < \bar{x}_1$.

In view of (2.15), together with (2.16), we derive

$$\begin{aligned}\tilde{M}'_{11h}(h) =& \tilde{F}_1\left(b(y_2,h)\right)y'_2(h) - \tilde{F}_1\left(c(y_2,h)\right)y'_2(h)\\ &- \left[\tilde{F}_1(b(y_1,h))y'_1(h) - \tilde{F}_1(c(y_1,h))y'_1(h)\right]\\ &+ \int_{y_1}^{y_2}\left[\tilde{F}'_1(b(y,h))b_h - \tilde{F}'_1(c(y,h))c_h\right]\mathrm{d}y\\ =& \int_{y_1}^{y_2}\left[\frac{\tilde{F}'_1(b)}{g(b)} - \frac{\tilde{F}'_1(c)}{g(c)}\right]\mathrm{d}y + \tilde{F}_1(x')y'_1(h)\end{aligned}$$

and

$$\begin{aligned}\tilde{M}'_{12h}(h) &= -\tilde{F}_1(c(y_1,h))y'_1(h) + \int_{y_1}^{0}\tilde{F}'_1(c(y,h))c_h\mathrm{d}y\\ &= \int_{y_1}^{0}\frac{\tilde{F}'_1(c)}{g(c)}\mathrm{d}y - \tilde{F}_1(x')y'_1(h).\end{aligned}$$

Hence,

$$\begin{aligned}\tilde{M}'_{1h}(h) &= \tilde{M}'_{11h}(h) + \tilde{M}'_{12h}(h)\\ &= \int_{\widehat{ABA'}}\frac{\tilde{F}'_1}{g}\mathrm{d}y + \int_{\widehat{A'C}}\frac{\tilde{F}'_1}{g}\mathrm{d}y\\ &= \int_{\widehat{AA^*}}\frac{\tilde{F}'_1}{g}\mathrm{d}y + \int_{\widehat{A^*C}}\frac{\tilde{F}'_1}{g}\mathrm{d}y\\ &= \int_{\widehat{AA^*}}\frac{\tilde{F}'_1}{-y}\mathrm{d}x + \int_{\widehat{A^*C}}\frac{\tilde{F}'_1}{-y}\mathrm{d}x\\ &\equiv \tilde{M}_{13}(h) + \tilde{M}_{14}(h).\end{aligned} \tag{2.17}$$

Then

$$\tilde{M}_{13}(h) = \int_0^{\tilde{x}} \frac{-\tilde{F}_1'}{\sqrt{2(h-\tilde{G}(x))}}\,\mathrm{d}x, \quad \tilde{M}_{13h}'(h) = \int_0^{\tilde{x}} \frac{\tilde{F}_1'\,\mathrm{d}x}{\left[2(h-\tilde{G}(x))\right]^{\frac{3}{2}}}.$$

Since $\tilde{F}_1'(x) < 0$ for $x \in (0, \tilde{x})$, it is easy to get $\tilde{M}_{13h}'(h) < 0$. Note that there is a value θ_0 satisfying

$$0 \le \theta_0 < \frac{\pi}{2},\ \tilde{G}(\tilde{x}) - h_0 = (h-h_0)\cos^2\theta_0,$$

where $h_0 = \tilde{G}(\bar{x}_0)$. The arc $\widehat{A^*C}$ can be expressed as

$$\tilde{G}(x) - h_0 = (h-h_0)\cos^2\theta, \quad y = \sqrt{2(h-h_0)}\sin\theta, \quad x \ge \tilde{x}, \quad 0 \le \theta \le \theta_0. \tag{2.18}$$

Introducing the integral transformation (2.18) to $\tilde{M}_{14}(h)$ in (2.17) and noting

$$g(x)\mathrm{d}x = -2(h-h_0)\cos\theta\sin\theta\mathrm{d}\theta,$$

we have

$$\tilde{M}_{14}(h) = \sqrt{2}\int_{\theta_0}^{0} \frac{\tilde{F}_1'(x)\sqrt{\tilde{G}(x)-h_0}}{g(x)}\,\mathrm{d}\theta,$$

where x satisfies the first formula of (2.18). Noting that

$$\frac{\partial x}{\partial h} = \frac{\cos^2\theta}{g(x)}, \tag{2.19}$$

it follows from condition **(i)** that

$$\tilde{M}_{14h}'(h) = \sqrt{2}\int_{\theta_0}^{0} \left(\frac{\tilde{F}_1'(x)\sqrt{\tilde{G}(x)-h_0}}{g(x)}\right)_x' \frac{\cos^2\theta}{g(x)} \le 0.$$

Thus, we have $\tilde{M}_{1h}''(h) < 0$. Similarly, we derive $\tilde{M}_{2h}''(h) < 0$. Consequently, we get $M_h''(h) < 0$ by (2.15).

Under condition (ii), we have $\bar{x}_0 \ge \bar{x}_1 > 0$. Then there exists θ_1 such that

$$\frac{\pi}{2} \le \theta_1 < \pi,\ \tilde{G}(x_1) - h_0 = (h-h_0)\cos^2\theta_1,$$

where $h_0 = \tilde{G}(\bar{x}_0)$. In this case, the point A^* is the intersection of the straight line $x = \bar{x}_1$ with arc $\widehat{AC}$, see Figure 7. As before, we have $\tilde{M}_{13h}'(h) < 0$. Introduce an integral transformation similar to (2.18) as follows

$$\tilde{G}(x) - h_0 = (h-h_0)\cos^2\theta, \quad y = \sqrt{2(h-h_0)}\sin\theta, \quad x \ge \bar{x}_1, \quad 0 \le \theta \le \theta_1. \tag{2.20}$$

It follows from (2.20) that

$$\tilde{M}_{14}(h) = \sqrt{2}\int_{\theta_1}^{0} \frac{\tilde{F}_1'(x)}{g(x)}\sqrt{\tilde{G}(x)-h_0}\,\mathrm{d}\theta,$$

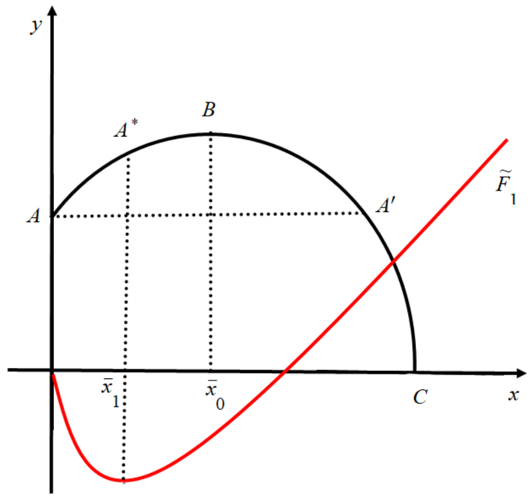


Figure 7. The part of L_h and $\tilde{F}_1$ on $x \geq 0$ for $\bar{x}_1 < \bar{x}_0$.

where x satisfies the first formula of (2.20). Combining (2.10), (2.19) and condition (ii), we get

$$\begin{aligned}\tilde{M}'_{14h}(h,a) &= \sqrt{2}\int_{\theta_1}^{0}\left(\frac{\tilde{F}'_1(x)\sqrt{\tilde{G}-h_0}}{g(x)}\right)'_x \frac{\cos^2\theta}{g(x)}\,\mathrm{d}\theta \\ &= \sqrt{2}\int_{\theta_1}^{\frac{\pi}{2}}\left(\frac{\tilde{F}'_1(x)\sqrt{\tilde{G}-h_0}}{g(x)}\right)'_x \frac{\cos^2\theta}{g(x)}\,\mathrm{d}\theta + \sqrt{2}\int_{\frac{\pi}{2}}^{0}\left(\frac{\tilde{F}'_1(x)\sqrt{\tilde{G}-h_0}}{g(x)}\right)'_x \frac{\cos^2\theta}{g(x)}\,\mathrm{d}\theta \\ &\leq 0.\end{aligned}$$

In view of (2.17), we obtain $\tilde{M}''_{1h}(h) < 0$. Analogously, we acquire $\tilde{M}''_{2h}(h) < 0$. Then $M''(h) < 0$ follows immediately from (2.14). We have thus proved Lemma 2.7. □

Lemma 2.8. *Let (2.10) and (2.12) hold. If $\bar{x}_1 = 0$, then $M(h) < 0$ for all $h \geq 0$.*

Proof. For $\bar{x}_1 = 0$, in light of (2.12) we have $\bar{x}_1 = \bar{x}_2 = 0$. It follows from (2.10) and Lemma 2.6 that $M(h) < 0$. □

3. Proof of main results

In order to prove Theorem 1.1, we consider system (1.3) with $m \geq 1$ odd. In this case, system $(1.3)|_{\varepsilon=0}$ has a family of periodic orbits surrounding an elementary center at the point $(1,0)$, denoted by

$$L_h : H(x,y) = h, \quad 0 < h < h_m, \quad x > 0.$$

The outer boundary of the family is a clockwise oriented homoclinic loop Γ, where Γ passes through the saddle $(0,0)$ and is defined by $H(x,y) = h_m,\ x > 0$ with

$$H(x,y) = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^{m+2}}{m+2} + \frac{m}{2(m+2)},\ \ h_m = H(0,0) = \frac{m}{2(m+2)}.$$

Then, by (3.9), the first order Melnikov function $M(h,a)$ in (2.7) can be simplified to

$$M(h,a) = aI_0(h) + I_m(h),\ h \in (0, h_m), \tag{3.1}$$

where $I_i(h) = -\oint_{L_h} x^i y \mathrm{d}x$, $i = 0, m$.

Now, we consider the Hopf bifurcation of system (1.3) near the center $(1,0)$. For this purpose, we first move the center to the origin, obtaining

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= -mx - \sum_{i=2}^{m+1} C_{m+1}^i x^i - \varepsilon\left[a + (x+1)^m\right] y. \end{aligned} \tag{3.2}$$

Then by using

$$x = \frac{1}{m}\bar{x}, \qquad y = \frac{1}{\sqrt{m}}\bar{y}, \qquad \tau = \frac{1}{\sqrt{m}}\bar{\tau},$$

we can get from (3.2)

$$\begin{aligned} \frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{\tau}} &= \bar{y}, \\ \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{\tau}} &= -\bar{x} - \sum_{i=2}^{m+1} C_{m+1}^i \left(\frac{\bar{x}}{m}\right)^i - \varepsilon\left[a + \left(\frac{\bar{x}}{m} + 1\right)^m\right] \frac{\bar{y}}{\sqrt{m}}. \end{aligned} \tag{3.3}$$

Therefore, we need only to study the Hopf bifurcation of system (3.3) near the origin $(0,0)$. It is evident that $(3.3)|_{\varepsilon=0}$ is Hamiltonian with

$$H(\bar{x},\bar{y}) = \frac{1}{2}(\bar{y}^2 + \bar{x}^2) + \sum_{i=2}^{m+1} C_{m+1}^i \frac{m}{i+1}(\frac{\bar{x}}{m})^{i+1}. \tag{3.4}$$

By (3.4) and applying Lemma 2.2 to (3.3), we can obtain directly the following lemma.

Lemma 3.1. (Hopf bifurcation) *Let $G > 1$ be a constant. Then there exist a constant $\epsilon_1 > 0$, a neighborhood U_1 of the focus $(1,0)$ of system (1.3) (or, equivalently, $(0,0)$ of system (3.3)) and a function $a_{1,m}^*(\varepsilon) = -1 + O(\varepsilon)$ such that for $0 < |\varepsilon| < \epsilon_1$ and $|a+1| < \epsilon_1$, system (1.3) (or, equivalently, (3.3)) has a unique limit cycle in U_1 iff $a > a_{1,m}^*(\varepsilon)$; additionally, the limit cycle, when it exists, is unstable if ε is positive and stable if ε is negative. System (1.3) (or, equivalently, (3.3)) has no limit cycle in U_1 for $0 < |\varepsilon| < \epsilon_1$ and $\epsilon_1 \le |a+1| < G$.*

Consider homoclinic bifurcation. By (3.1) we have

$$\begin{aligned} M(h_m, a) &= -\oint_{H(x,y)=h_m} (a + x^m)\, y\, \mathrm{d}x \\ &= -2\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} (a + x^m)\sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\, \mathrm{d}x \\ &\equiv c_0(a). \end{aligned}$$

Letting $\frac{2}{m+2}x^m = \sin^2 \varsigma$, we derive

$$\begin{aligned}
&\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\,\mathrm{d}x \\
=&\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} x\sqrt{1 - \frac{2}{m+2}x^{m}}\,\mathrm{d}x \\
=&\frac{2}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} \int_0^{\frac{\pi}{2}} \sin^{\frac{4-m}{m}} \varsigma \cos^2 \varsigma \,\mathrm{d}\varsigma \\
=&\frac{1}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2}, \frac{2}{m}\right)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} x^m\sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\,\mathrm{d}x \\
=&\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} x^{m+1}\sqrt{1 - \frac{2}{m+2}x^{m}}\,\mathrm{d}x \\
=&\frac{2}{m}\left(\frac{m+2}{2}\right)^{\frac{m+2}{m}} \int_0^{\frac{\pi}{2}} \sin^{\frac{4+m}{m}} \varsigma \cos^2 \varsigma \,\mathrm{d}\varsigma \\
=&\frac{1}{m}\left(\frac{m+2}{2}\right)^{\frac{m+2}{m}} B\left(\frac{3}{2}, 1+\frac{2}{m}\right),
\end{aligned} \tag{3.6}$$

where $B(\cdot,\cdot)$ is the Beta function. Then,

$$\begin{aligned}
c_0(a) =& -2\left(\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} a\sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\,\mathrm{d}x + \int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} x^m\sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\,\mathrm{d}x\right) \\
=& -2\left(\frac{a}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2}, \frac{2}{m}\right) + \frac{1}{m}\left(\frac{m+2}{2}\right)^{\frac{m+2}{m}} B\left(\frac{3}{2}, 1+\frac{2}{m}\right)\right) \\
=&\frac{-2}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}}\left(aB\left(\frac{3}{2}, \frac{2}{m}\right) + \frac{m+2}{2}B\left(\frac{3}{2}, 1+\frac{2}{m}\right)\right).
\end{aligned}$$

Therefore, we have

$$c_0(a) = 0 \Leftrightarrow a = -\frac{m+2}{2}\frac{B\left(\frac{3}{2}, 1+\frac{2}{m}\right)}{B\left(\frac{3}{2}, \frac{2}{m}\right)} = -\frac{2m+4}{3m+4} \triangleq a_{2,m} \tag{3.7}$$

and

$$\frac{\partial c_0}{\partial a}(a_{2,m}) = \frac{-2}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2}, \frac{2}{m}\right) < 0. \tag{3.8}$$

By (3.7) and (3.8), we have the following lemma from Lemma 2.8 of Ref. [13] immediately.

Lemma 3.2. (Homoclinic bifurcation) *Consider system (1.3) with $m \geq 1$ odd. Then for any given constant $G > 1$ there are a constant $\epsilon_2 > 0$, a neighborhood U_2 of Γ and a unique C^∞ function $a^*_{2,m}(\varepsilon) = a_{2,m} + O(\varepsilon)$ with $a_{2,m}$ given in (3.7), such that* *(i) for* $0 < |\varepsilon| < \epsilon_2$ *and* $|a - a_{2,m}| < \epsilon_2$*, there is a homoclinic loop of system (1.3) in* U_2 *if and only if* $a = a^*_{2,m}(\varepsilon)$*; there is a unique limit cycle of system (1.3) in* U_2 *if* $a < a^*_{2,m}(\varepsilon)$*; there is no limit cycle of system (1.3) in* U_2 *if* $a > a^*_{2,m}(\varepsilon)$*; additionally, when the aforementioned limit cycle or the homoclinic loop exists, it is unstable if* ε *is positive and stable if* ε *is negative.*

(ii) for $0 < |\varepsilon| < \epsilon_2$*,* $|a - a_{2,m}| \geq \epsilon_2$ *and* $|a| \leq G$*, there is no limit cycle of system (1.3) in* U_2*.*

Further, for Poincaré bifurcation, by (1.3) and (2.5), it implies that

$$\begin{aligned} g(x) &= x(x^m - 1), \qquad & G(x) &= \int_1^x g ds = -\frac{x^2}{2} + \frac{x^{m+2}}{m+2} + \frac{m}{2(m+2)}, \\ f(x,a) &= a + x^m, \qquad & F(x,a) &= \int_1^x f ds = a(x-1) + \frac{1}{m+1}\left(x^{m+1} - 1\right). \end{aligned} \tag{3.9}$$

Since $F_x(1,a) = f(1,a) = a+1$, we have that for $\varepsilon > 0$ (< 0) the focus $(1,0)$ is unstable (stable) if $a < -1$ and is stable (unstable) if $a > -1$. Further, the point $(1,0)$ is unstable (stable) of system $(1.3)|_{a=-1}$ as $\varepsilon > 0$ (< 0). It then follows that if a is closed to -1 and $a > -1$, there is an unstable (stable) limit cycle for system (1.3) when $\varepsilon > 0$ (< 0).

It is easy to see that there exists a unique function $\beta(x) > 1 > x > 0$ satisfies $G(\beta(x)) = G(x),\ x \in (0,1)$ with $G(x)$ being found in (3.9). Then we have

$$g(\beta(x))\beta'(x) = g(x),\ \beta'(x) = \frac{g(x)}{g(\beta(x))}.$$

When $a = -1$, let $\tilde{\phi}(x) = F(\beta(x), -1) - F(x, -1)$ for $x \in (0,1)$. Then, we have

$$\tilde{\phi}'(x) = f(\beta(x), -1)\beta'(x) - f(x,-1) \triangleq g(x)\tilde{\phi}_0(x),$$

where $\tilde{\phi}_0(x) = \frac{f(\beta(x),-1)}{g(\beta(x))} - \frac{f(x,-1)}{g(x)}$ for $x \in (0,1)$. Note that

$$\tilde{\phi}_0(x) = \frac{f(\beta(x),-1)}{g(\beta(x))} - \frac{f(x,-1)}{g(x)} = \frac{1}{\beta(x)} - \frac{1}{x} = \frac{x - \beta(x)}{x\beta(x)} < 0,\ x \in (0,1).$$

Hence, $\tilde{\phi}'(x) > 0$ for $x \in (0,1)$. It follows from $\tilde{\phi}(1) = 0$ that $\tilde{\phi}(x) < 0$ for $x \in (0,1)$ and $a = -1$. Using Lemma 2.4, we get $M(h,-1) > 0,\ h \in (0, h_m)$, where M is the first order Melnikov function of system (1.3).

For $a \geq 0$, it is evident that

$$\begin{aligned} F(\beta(x),a) - F(x,a) &= a\left(\beta(x) - x\right) + \frac{1}{m+1}\left((\beta(x))^{m+1} - x^{m+1}\right) \\ &> 0,\ x \in (0,1), \end{aligned}$$

where $F(x,a)$ is given by (3.9). It follows from Lemma 2.4 that $M(h,a) < 0, \quad h \in (0,h_m)$ for $a \geq 0$, where function $M(h,a)$ is Melnikov function of system (1.3).

Based on the above analysis, system (1.3) may have a limit cycle only if $a \in (-1,0)$. In this case, we have the following lemma.

Lemma 3.3. *Consider system (1.3) with* $m \geq 1$ *odd. Then for* $a \in (-1,0)$*, we have* $\left(\frac{f(x,a)}{g(x)}\right)'_x < 0$ *with* $x > 0,\ x \neq 1$.

Proof. In fact, it is direct that

$$\begin{aligned}\left(\frac{f(x,a)}{g(x)}\right)'_x =& \frac{-1}{(g(x))^2}\left[x^{2m}+(m-1+a(m+1))\,x^m-a\right]\\ \triangleq& \frac{-1}{(g(x))^2}\omega(u),\end{aligned} \tag{3.10}$$

where

$$\omega(u)=u^2+\varrho u-a \tag{3.11}$$

with $u=x^m$ and $\varrho=(m+1)a+m-1$. The lemma will be proved if we can show $\omega(u)>0$ for $u>0$. If $\varrho\geq 0$, then it is obvious from (3.11) that $\omega(u)>0$ for $u>0$.

For the case $\varrho<0$, i.e., $a<-\frac{m-1}{m+1}$, noting $a>-1$, we have

$$\begin{aligned}&\omega(0)>0,\ \omega'(0)=\varrho<0,\\ &\omega(1)=1+\varrho-a=m(a+1)>0,\\ &\omega'(1)=2+\varrho=(m+1)(a+1)>0.\end{aligned}$$

If we can prove $\omega_{\min}(u)>0$ for $-1<a<-\frac{m-1}{m+1}$, then $\omega(u)>0$ $(u>0)$ follows immediately. Letting $\omega'(u)=2u+\varrho=0$, we get $u=-\frac{\varrho}{2}$. Then

$$\omega_{\min}(u)=\omega\left(-\frac{\varrho}{2}\right)=\frac{\varrho^2}{4}-\frac{\varrho^2}{2}-a\triangleq-\frac{1}{4}\varphi(a),$$

where $\varphi(a)=\varrho^2+4a=(m+1)^2a^2+2(m^2+1)a+(m-1)^2$. Hence, it suffices to prove $\varphi(a)<0$ for $-1<a<-\frac{m-1}{m+1}$. In fact, we have

$$\begin{aligned}&\varphi(0)=(m-1)^2\geq 0,\ \varphi'(0)=2(m^2+1)>0,\\ &\varphi(-1)=(m+1)^2-2(m^2+1)+(m-1)^2=0,\\ &\varphi'(-1)=-2(m+1)^2+2(m^2+1)=-4m<0,\\ &\varphi(-\frac{m-1}{m+1})=(m+1)^2\left(\frac{1-m}{m+1}\right)^2+2(m^2+1)\frac{1-m}{m+1}+(m-1)^2\\ &\qquad\qquad\quad=-\frac{4(m-1)}{m+1}\\ &\qquad\qquad\quad\leq 0.\end{aligned}$$

It follows that $\varphi(a)<0$ for $-1<a<-\frac{m-1}{m+1}$. Thus we arrive at the conclusion that $\omega(u)>0$ for $u>0$ if $\varrho<0$.

Therefore, we have proved $\omega(u)>0$ for $u>0$ with $a\in(-1,0)$. It follows from (3.10) that $\left(\frac{f(x,a)}{g(x)}\right)'_x<0$ for $x>0$, $x\neq 1$ with $a\in(-1,0)$. The proof is completed. □

By Lemmas 2.5 and 3.3, we have immediately

Lemma 3.4. *For $M(h,a)$ in (3.1) with $a\in(-1,0)$, it holds that $M'_h(h_0,a)>0$ if $M(h_0,a)=0$ for $h_0\in(0,h_m)$.*

By (3.1),

$$M(h,a)=I_0(h)(a-P(h)),\quad P(h)=-\frac{I_m(h)}{I_0(h)},\quad h\in(0,h_m), \tag{3.12}$$

where

$$I_0(h) = -2\int_{a^*(h)}^{b^*(h)} y(x,h)\,\mathrm{d}x < 0, \quad I_m(h) = -2\int_{a^*(h)}^{b^*(h)} x^m y(x,h)\,\mathrm{d}x. \tag{3.13}$$

It is direct that

$$I_0'(h) = -2\int_{a^*(h)}^{b^*(h)} \frac{\mathrm{d}x}{y(x,h)}, \quad I_m'(h) = -2\int_{a^*(h)}^{b^*(h)} \frac{x^m}{y(x,h)}\,\mathrm{d}x, \tag{3.14}$$

where $y(x,h) = \sqrt{x^2 - \frac{2}{m+2}x^{m+2} - \frac{m}{m+2} + 2h}$ and the values $a^*(h)$ and $b^*(h)$ satisfy $y(a^*(h),h) = y(b^*(h),h) = 0$, and $0 \le a^*(h) < 1 < b^*(h) \le \left(\frac{m+2}{2}\right)^{\frac{1}{m}}$.

Denote by $T(h)$ the period of L_h. Then $\lim\limits_{h\to 0^+} T(h) = \frac{2\pi}{\sqrt{m}}$ by (3.3), and

$$I_0'(h) = -\oint_{L_h} \frac{\mathrm{d}x}{y} = -\int_0^T \mathrm{d}\tau \to \frac{-2\pi}{\sqrt{m}}, \quad I_m'(h) = -\int_0^T x^m \mathrm{d}\tau \to -\int_0^{\frac{2\pi}{\sqrt{m}}} \mathrm{d}\tau = \frac{-2\pi}{\sqrt{m}} \quad (h \to 0)$$

by (3.14), which give

$$\lim_{h\to 0} \frac{I_m(h)}{I_0(h)} = 1. \tag{3.15}$$

Now, we have the following result.

Lemma 3.5. *The function $P(h)$ satisfies $P'(h) > 0$ on the interval $(0, h_m)$, and*

$$P(0) = -1, \quad P(h_m) = a_{2,m}$$

with $a_{2,m}$ in (3.7).

Proof. By (3.12) and (3.15), it is easy to see that $P(0) = -1$. Using (3.12), it follows from (3.5), (3.6) and (3.7) that

$$\begin{aligned} P(h_m) &= -\frac{I_m(h_m)}{I_0(h_m)} \\ &= -\frac{\oint_{H(x,y)=h_m} x^m y\,\mathrm{d}x}{\oint_{H(x,y)=h_m} y\,\mathrm{d}x} \\ &= -\frac{2\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} x^m \sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\,dx}{2\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \sqrt{x^2 - \frac{2}{m+2}x^{m+2}}\,dx} \\ &= -\frac{\frac{2}{m}\left(\frac{m+2}{2}\right)^{\frac{m+2}{m}} B\left(\frac{3}{2}, 1 + \frac{2}{m}\right)}{\frac{2}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2}, \frac{2}{m}\right)} \\ &= a_{2,m}. \end{aligned}$$

Then noting by (3.12)

$$\frac{\partial M}{\partial h}\Big|_{M(h,a)=0} = -I_0(h)P'(h), \tag{3.16}$$

by Lemma 3.4 and (3.13) we obtain $P'(h) > 0$. This ends the proof. □

Now, we can prove Theorem 1.1. Without loss of generality, let $\varepsilon > 0$. Following the proof of Theorem 1.1 of Ref. [13] one can get the following claims:

(1) Let $G > 1$ be a constant. A positive constant ε_1^* exists such that for $\varepsilon \in (0, \varepsilon_1^*)$, there is no limit cycle of system (1.3) if $a \in [-G, a_{1,m}^*(\varepsilon)] \cup [a_{2,m}^*(\varepsilon), G]$.

(2) A positive constant ε_2^* exists such that for $\varepsilon \in (0, \varepsilon_2^*)$, there is a stable homoclinic loop near Γ of system (1.3) if and only if $a = a_{2,m}^*(\varepsilon)$.

(3) A positive constant ε_3^* exists such that for $\varepsilon \in (0, \varepsilon_3^*)$, there is a unique limit cycle of system (1.3) if $a \in (a_{1,m}^*(\varepsilon), a_{2,m}^*(\varepsilon))$. Additionally, the above limit cycle is unstable.

Then Theorem 1.1 follows immediately from the above claims (1) − (3) by letting $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}$.

To prove Theorem 1.2, we consider system (1.3) with $m \geq 2$ even. As before, we have from (1.3) and (2.5)

$$\begin{aligned} &g(x) = x(x^m - 1), \quad \tilde{G}(x) = -\frac{x^2}{2} + \frac{x^{m+2}}{m+2}, \\ &f(x,a) = a + x^m, \quad \tilde{F}(x,a) = ax + \frac{1}{m+1}x^{m+1}. \end{aligned} \tag{3.17}$$

Let

$$H(x,y) = \frac{1}{2}y^2 + \tilde{G}(x) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{m+2}x^{m+2}.$$

Then, $(1.3)|_{\varepsilon=0}$ has three families of periodic orbits given by

$$L(h): H(x,y) = h, \quad h > 0 \tag{3.18}$$

and

$$\bar{L}_i(h): H(x,y) = h, \quad (-1)^{i+1}x < 0, \quad i = 1,2, \quad \hat{h}_m < h < 0,$$

where $\hat{h}_m = \tilde{G}(\pm 1) = \frac{-m}{2(m+2)}$. Correspondingly, it yields three Melnikov functions as follows

$$\begin{aligned} M(h,a) &= -\oint_{L(h)} yf(x,a)\mathrm{d}x, \quad h > 0, \\ M_i(h,a) &= -\oint_{\bar{L}_i(h)} yf(x,a)\mathrm{d}x, \quad \hat{h}_m < h < 0, \ i = 1,2. \end{aligned} \tag{3.19}$$

By (3.17), system (1.3) is centrally symmetric. Noting (2.7) and (2.11) we have by (3.19)

$$M_1(h,a) = M_2(h,a) = [a\hat{I}_0(h) + \hat{I}_m(h)]/2, \ h \in (\hat{h}_m, 0) \tag{3.20}$$

and

$$M(h,a) = a\hat{I}_0(h) + \hat{I}_m(h), \ h \in (0, +\infty), \tag{3.21}$$

where

$$\hat{I}_j(h) = -\oint_{H(x,y)=h} x^j y \mathrm{d}x, \quad j = 0, m, \quad h \in [\hat{h}_m, +\infty). \tag{3.22}$$

For system (1.3) with $m \geq 2$ being even, we first consider the Hopf bifurcation near two singular points $(\pm 1, 0)$. Noting that system (1.3) is centrally symmetric, we obtain Lemma 3.6 as follows similar to Lemma 3.1.

Lemma 3.6. (Hopf bifurcation) *Let $G > 1$ be a constant. There are a positive constant $\hat{\epsilon}_1$, a neighborhood $\hat{U}_{i1}$ of $((-1)^{i+1}, 0)$ ($i = 1, 2$) and a function $\hat{a}^*_{1,m}(\varepsilon) = -1 + O(\varepsilon)$, such that the following results hold.*

(i) When $0 < |\varepsilon| < \hat{\epsilon}_1$ and $|a + 1| < \hat{\epsilon}_1$, there is a unique limit cycle of system (1.3) in $\hat{U}_{i1}$ if and only if $a > \hat{a}^*_{1,m}(\varepsilon)$. Additionally, when the above limit cycle exists, it is unstable if ε is positive and stable if ε is negative.

(ii) When $0 < |\varepsilon| < \hat{\epsilon}_1$, $\hat{\epsilon}_1 \le |a + 1| < G$, there is no limit cycle of system (1.3) in $\hat{U}_{i1}$ with $i = 1, 2$.

Next, for system (1.3) with $m \ge 2$ being even, we consider the double homoclinic bifurcation. It follows from (2.4), (1.3), (2.1), (3.5) and (3.6) that

$$\begin{aligned}
\bar{c}_0(a) =& -2 \oint_{H(x,y)=0,\ x>0} (a + x^m)\, y \,\mathrm{d}x \\
=& -4 \int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} (a + x^m) \sqrt{x^2 - \frac{2}{m+2} x^{m+2}} \,\mathrm{d}x \\
=& \frac{-4}{m} \left(\frac{m+2}{2}\right)^{\frac{2}{m}} \left(aB\left(\frac{3}{2}, \frac{2}{m}\right) + \frac{m+2}{2} B\left(\frac{3}{2}, 1 + \frac{2}{m}\right)\right).
\end{aligned}$$

By the above formula, it is easy to know that

$$\bar{c}_0(a) = 0 \Leftrightarrow a = -\frac{m+2}{2} \frac{B\left(\frac{3}{2}, 1 + \frac{2}{m}\right)}{B\left(\frac{3}{2}, \frac{2}{m}\right)} = -\frac{2m+4}{3m+4} \triangleq a_{3,m} \tag{3.23}$$

and

$$\frac{\partial \bar{c}_0}{\partial a}(a_{3,m}) = \frac{-4}{m} \left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2}, \frac{2}{m}\right) < 0. \tag{3.24}$$

By (2.4), (1.3), (2.1) and (3.23), we have

$$\bar{c}_1(a_{3,m}) = -2\left(a_{3,m} + 0\right) = -2a_{3,m} = \frac{4m+8}{3m+4} > 0.$$

Therefore, combining (3.24) and the above formula, we obtain by Corollary 2.1.

Lemma 3.7. (Double homoclinic bifurcation) *Let $G > 1$ be a constant. Then there exist a constant $\hat{\epsilon}_3 > 0$, a neighborhood U_3 of the loop $\bar{L}$ defined by $H(x, y) = 0$ and a C^∞ function $\hat{a}^*_{3,m}(\varepsilon) = a_{3,m} + O(\varepsilon)$ with $a_{3,m}$ given in (3.23) such that for $0 < |\varepsilon| < \hat{\epsilon}_3$, $|a - a_{3,m}| < \hat{\epsilon}_3$ and $|a| < G$, the following conclusions hold.*

(i) If $a = \hat{a}^*_{3,m}(\varepsilon)$, there is a double homoclinic loop of (1.3) in U_3.

(ii) If $a > \hat{a}^*_{3,m}(\varepsilon)$, there is a unique large limit cycle of (1.3) in U_3.

(iii) If $a < \hat{a}^*_{3,m}(\varepsilon)$, there are two small limit cycles of (1.3) in U_3.

Additionally, when a limit cycle exists, it is unstable if ε is positive and stable if ε is negative.

For Poincaré bifurcation, let $\Phi(h, \varepsilon, a)$ denote the bifurcation function of system (1.3) such that

$$\Phi(h, 0, a) = \begin{cases} M(h, a), & h \ge 0, \\ 2M_1(h, a), & \hat{h}_m \le h < 0. \end{cases}$$

We rewrite the functions $M_1(h,a)$ in (3.20) and $M(h,a)$ in (3.21) as

$$\begin{aligned} M_1(h,a) &= \frac{1}{2}\hat{I}_0(h)(a-\hat{P}(h)), \qquad h\in[\hat{h}_m,0],\\ M(h,a) &= \hat{I}_0(h)(a-\hat{P}(h)), \qquad h\in(0,+\infty) \end{aligned} \tag{3.25}$$

with

$$\hat{P}(h) = -\frac{\hat{I}_m(h)}{\hat{I}_0(h)}, \quad h\in[\hat{h}_m,+\infty). \tag{3.26}$$

By (3.5) and (3.6), we have

$$\begin{aligned} \hat{I}_0(0) &= -4\int_0^{\hat{a}(0)} y(x,0)\,\mathrm{d}x\\ &= -4\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \sqrt{x^2-\frac{2}{2+m}x^{m+2}}\,\mathrm{d}x\\ &= -\frac{4}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2},\frac{2}{m}\right) \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} \hat{I}_m(0) &= -4\int_0^{\hat{a}(0)} x^m y(x,0)\,\mathrm{d}x\\ &= -4\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} x^m\sqrt{x^2-\frac{2}{2+m}x^{m+2}}\,\mathrm{d}x\\ &= -\frac{4}{m}\left(\frac{m+2}{2}\right)^{\frac{2+m}{m}} B\left(\frac{3}{2},1+\frac{2}{m}\right). \end{aligned} \tag{3.28}$$

Then by (3.26) and (3.23) we have

$$\hat{P}(0) = -\frac{\hat{I}_m(0)}{\hat{I}_0(0)} = -\frac{m+2}{2}\frac{B\left(\frac{3}{2},1+\frac{2}{m}\right)}{B\left(\frac{3}{2},\frac{2}{m}\right)} = -\frac{2m+4}{3m+4} = a_{3,m}.$$

Then, just similar to Lemma 3.5, one can obtain:

Lemma 3.8. *For $\hat{P}(h)$ given in (3.26), we have*

$$\hat{P}(\hat{h}_m) = -1, \quad \hat{P}(0) = a_{3,m} \quad \textit{and} \quad \hat{P}'(h)>0, \;\; h\in(\hat{h}_m,0)$$

with $a_{3,m}$ in (3.23).

The following proposition is useful when considering Poincaré bifurcation of limit cycles surrounding three singular points.

Proposition 3.1. *The function $\frac{\tilde{F}_1'(x)\sqrt{\tilde{G}(x)-\tilde{G}(1)}}{|g(x)|}$ is monotonically increasing with $x>0$.*

Proof. Let

$$\Psi_m(x) \equiv \frac{\tilde{F}_1'(x)\sqrt{\tilde{G}(x)-\tilde{G}(1)}}{|g(x)|} = \frac{(2a+2x^m)\sqrt{\frac{x^{m+2}}{m+2}-\frac{x^2}{2}-\frac{1}{m+2}+\frac{1}{2}}}{|x(x^m-1)|}. \tag{3.29}$$

Since $m \geq 2$ is even, we can set $m = 2k$ $(k \in \mathbb{Z}^+)$. It follows from (3.29) that

$$\Psi_{2k}(x) = \frac{(2a + 2x^{2k})\sqrt{\frac{x^{2k+2}}{2k+2} - \frac{x^2}{2} + \frac{k}{2k+2}}}{\mid x(x^{2k} - 1) \mid}. \tag{3.30}$$

Note that

$$\begin{aligned}
&\frac{x^{2k+2}}{2k+2} - \frac{x^2}{2} + \frac{k}{2k+2} \\
\equiv&\frac{1}{2k+2}\left(x^{2k+2} - x^2 + k - kx^2\right) \\
=&\frac{1}{2k+2}\left(x^2(x^{2k} - 1) + k(1 - x^2)\right) \\
=&\frac{1}{2k+2}\left(x^2\left(x^2 - 1\right)\sum_{i=0}^{k-1} x^{2i} - k\Big(x^2 - 1\Big)\right) \\
=&\frac{x^2 - 1}{2k+2}\left(x^2\sum_{i=0}^{k-1} x^{2i} - k\right) \\
=&\frac{x^2 - 1}{2k+2}\left(x^2\sum_{i=0}^{k-1}\left(x^{2i} - 1\right) + kx^2 - k\right) \\
=&\frac{x^2 - 1}{2k+2}\left(x^2\sum_{i=0}^{k-2}\left(x^2 - 1\right)\sum_{j=0}^{i} x^{2j} + k(x^2 - 1)\right) \\
=&\frac{(x^2 - 1)^2}{2k+2}\left(\sum_{i=1}^{k-1}\sum_{j=1}^{i} x^{2j} + k\right) \\
=&\frac{(x^2 - 1)^2}{2k+2}\sum_{i=0}^{k-1}(k - i)x^{2i}.
\end{aligned} \tag{3.31}$$

Substituting (3.31) into (3.30) gives that

$$\Psi_{2k}(x) = \frac{\left(2a + 2x^{2k}\right)\left|x^2 - 1\right|\sqrt{\sum\limits_{i=0}^{k-1}\Big(k - i\Big)x^{2i}}}{\sqrt{2k+2}x\left|x^2 - 1\right|\sum\limits_{i=0}^{k-1} x^{2i}} = \frac{\left(2a + 2x^{2k}\right)\sqrt{\sum\limits_{i=0}^{k-1}\Big(k - i\Big)x^{2i}}}{\sqrt{2k+2}x\sum\limits_{i=0}^{k-1} x^{2i}}. \tag{3.32}$$

By (3.32) we have

$$\Psi'_{2k}(x) = \frac{\left(4kx^{2k-1}\sqrt{\sum\limits_{i=0}^{k-1}\Big(k - i\Big)x^{2i}} + \left(2a + 2x^{2k}\right)\frac{\sum\limits_{i=1}^{k-1} 2i(k-i)x^{2i-1}}{2\sqrt{\sum\limits_{i=0}^{k-1}\Big(k-i\Big)x^{2i}}}\right)\sqrt{2k+2}x\sum\limits_{i=0}^{k-1} x^{2i}}{\left(2k+2\right)x^2\left(\sum\limits_{i=0}^{k-1} x^{2i}\right)^2}$$

$$-\frac{\left(2a+2x^{2k}\right)\sqrt{\sum\limits_{i=0}^{k-1}\Big(k-i\Big)x^{2i}}\sqrt{2k+2}\left(\sum\limits_{i=0}^{k-1}(2i+1)x^{2i}\right)}{\left(2k+2\right)x^{2}\left(\sum\limits_{i=0}^{k-1}x^{2i}\right)^{2}}$$

$$=\frac{\Phi_{2k}(x)}{\sqrt{2k+2}x^{2}\left(\sum\limits_{i=0}^{k-1}x^{2i}\right)^{2}\sqrt{\sum\limits_{i=0}^{k-1}\Big(k-i\Big)x^{2i}}}, \tag{3.33}$$

where

$$\Phi_{2k}(x)=\Big(4kx^{2k}\varphi_1(x)+\Big(2a+2x^{2k}\Big)\varphi_2(x)\Big)\varphi_3(x)-\Big(2a+2x^{2k}\Big)\varphi_1(x)\varphi_4(x) \tag{3.34}$$

with

$$\varphi_1(x)=\sum_{i=0}^{k-1}\Big(k-i\Big)x^{2i},\quad \varphi_2(x)=\sum_{i=1}^{k-1}i(k-i)x^{2i},$$
$$\varphi_3(x)=\sum_{i=0}^{k-1}x^{2i},\quad \varphi_4(x)=\sum_{i=0}^{k-1}(2i+1)x^{2i}.$$

By (3.33) it suffices to show $\Phi_{2k}(x)\geq 0$ for $x>0$. We first prove that

$$2a\varphi_2(x)\varphi_3(x)-2a\varphi_1(x)\varphi_4(x)>0,\quad x>0. \tag{3.35}$$

In fact, it is easy to see that

$$2a\varphi_2(x)\varphi_3(x)-2a\varphi_1(x)\varphi_4(x)=-2ak\frac{x^{4k+2}-(2k+1)x^{2k+2}+(2k+1)x^{2k}-1}{\left(x^2-1\right)^3}=-2ak\psi(u),$$

where

$$\psi(u)=\frac{u^{2k+1}-1-(2k+1)u^k(u-1)}{\left(u-1\right)^3},\ u=x^2.$$

It is direct that

$$\begin{aligned}\psi(u)=&\frac{\left(u-1\right)\sum\limits_{i=0}^{2k}u^i-\left(2k+1\right)u^k\left(u-1\right)}{\left(u-1\right)^3}\\ =&\frac{\sum\limits_{i=0}^{2k}u^i-\left(2k+1\right)u^k}{\left(u-1\right)^2}\\ =&\frac{\sum\limits_{i=1}^{k}\left(u^{k+i}-u^k\right)+\sum\limits_{i=1}^{k}\left(u^{k-i}-u^k\right)}{\left(u-1\right)^2}\\ =&\frac{\sum\limits_{i=1}^{k}\left(u^{k+i}-u^k\right)-\sum\limits_{i=1}^{k}\left(u^k-u^{k-i}\right)}{\left(u-1\right)^2}\end{aligned}$$

$$
\begin{aligned}
&=\frac{\sum_{i=1}^{k} u^k\left(u^i-1\right)-\sum_{i=1}^{k} u^{k-i}\left(u^i-1\right)}{(u-1)^2}\\
&=\frac{\sum_{i=1}^{k}\left(u^k-u^{k-i}\right)\left(u^i-1\right)}{(u-1)^2}\\
&=\frac{\sum_{i=1}^{k} u^{k-i}\left(u^i-1\right)^2}{(u-1)^2}\\
&=\frac{\sum_{i=1}^{k} u^{k-i}\left((u-1)\sum_{j=0}^{i-1} u^j\right)^2}{(u-1)^2}\\
&=\frac{(u-1)^2\sum_{i=1}^{k} u^{k-i}\left(\sum_{j=0}^{i-1} u^j\right)^2}{(u-1)^2}\\
&=\sum_{i=1}^{k} u^{k-i}\left(\sum_{j=0}^{i-1} u^j\right)^2\\
&>0.
\end{aligned}
$$

Therefore, (3.35) holds. For $x>0$, it is straightforward that

$$
\begin{aligned}
&4kx^{2k}\varphi_1(x)\varphi_3(x)+2x^{2k}\varphi_2(x)\varphi_3(x)-2x^{2k}\varphi_1(x)\varphi_4(x)\\
=\ &2x^{2k}\left(\sum_{i=0}^{k-1}\Big(k-i\Big)x^{2i}\right)\left(2k\sum_{i=0}^{k-1}x^{2i}-\sum_{i=0}^{k-1}\Big(2i+1\Big)x^{2i}\right)+2x^{2k}\left(\sum_{i=1}^{k-1}i\Big(k-i\Big)x^{2i}\right)\left(\sum_{i=0}^{k-1}x^{2i}\right)\\
=\ &2x^{2k}\left(\sum_{i=0}^{k-1}\Big(k-i\Big)x^{2i}\right)\left(\sum_{i=0}^{k-1}\Big(2(k-i)-1\Big)x^{2i}\right)+2x^{2k}\left(\sum_{i=1}^{k-1}i\Big(k-i\Big)x^{2i}\right)\left(\sum_{i=0}^{k-1}x^{2i}\right)\\
>\ &0.
\end{aligned}
$$

Then from (3.34) and (3.35) the conclusion follows. □

Lemma 3.9. *$M(h,a)$ in (3.21) satisfies that $M_h''(h,a)<0$ for all $a<0$ and $M(h,a)<0$ for all $a\geq 0$.*

Proof. In (2.12), we take $\bar{x}_1=\sqrt{-a}$ for $a<0$ and take $\bar{x}_1=\bar{x}_2=0$ for $a\geq 0$. Then Lemma 3.9 follows by Lemmas 2.7 and 2.8 and Proposition 3.1. This completes the proof. □

Let

$$
y(x,h)=\sqrt{x^2-\frac{2}{m+2}x^{m+2}+2h},\qquad 0\leq x\leq \hat{a}(h), \tag{3.36}
$$

where $(\hat{a}(h),0)$ is the right endpoint of the periodic orbit $L(h)$ in (3.18), that is, $y(\hat{a}(h),h)=0$ with $\hat{a}(h)>1$. For $h>0$, $\hat{I}_j(h)$ in (3.22) can be rewritten as

$$
\hat{I}_j(h)=-4\int_0^{\hat{a}(h)} x^j y(x,h)\mathrm{d}x,\quad j=0,m, \tag{3.37}
$$

where $y(x,h)$ is given by (3.36).

Now we can prove the following lemma.

Lemma 3.10. *The continuous function $\hat{P}(h)$ on the interval $[\hat{h}_m,+\infty)$ satisfies*
(1) $\hat{P}(\hat{h}_m)=-1$, $\hat{P}(0)=a_{3,m}$ and $\hat{P}'(h)>0$ for $h\in(\hat{h}_m,0)$;
(2) $\lim\limits_{h\to 0^+}\hat{P}'(h)=+\infty$, $\lim\limits_{h\to 0^-}\hat{P}'(h)=+\infty$, $\lim\limits_{h\to+\infty}\hat{P}(h)=-\infty$;
(3) there exists $h_0>0$ such that $\hat{P}'(h_0)=0$, $\hat{P}''(h_0)<0$ and $(h-h_0)\hat{P}'(h)<0$ for $h\in(0,h_0)\cup(h_0,+\infty)$,
where $a_{3,m}$ and $\hat{P}(h)$ are given by (3.23) and (3.26), respectively.

Proof. By Lemma 3.8, we need only to prove conclusions (2) and (3). Since $y(x,h)$ in (3.36) is monotonically decreasing with respect to x on the interval $[1,+\infty)$, we have $y(x,h)\le y(1,h)=\sqrt{\frac{m}{m+2}+2h}$, which gives for $h>0$

$$\hat{I}_0(h)=-4\int_0^{\hat{a}(h)}y(x,h)\mathrm{d}x\ge-4\int_0^{\hat{a}(h)}\sqrt{\frac{m}{m+2}+2h}\mathrm{d}x=-4\sqrt{\frac{m}{m+2}+2h}\,\hat{a}(h). \tag{3.38}$$

It is obvious that the equation $y(x,h)=\sqrt{h}$ has a unique positive root $\hat{a}_0(h)\in(1,\hat{a}(h))$ in x and that $y(x,h)\ge\sqrt{h}$ for $0\le x\le\hat{a}_0(h)$, which implies that

$$\begin{aligned}\hat{I}_m(h)&=-4\int_0^{\hat{a}(h)}x^m y(x,h)\,\mathrm{d}x\\&\le-4\int_0^{\hat{a}_0(h)}x^m\sqrt{h}\,\mathrm{d}x\\&=\frac{-4\sqrt{h}}{m+1}\left(\hat{a}_0(h)\right)^{m+1}\\&<0.\end{aligned} \tag{3.39}$$

Note that by (3.36) as $h\gg 1$

$$\hat{a}_0(h)\sim[(m+2)h/2]^{\frac{1}{m+2}},\ \hat{a}(h)\sim[(m+2)h]^{\frac{1}{m+2}}.$$

Then by (3.38) and (3.39), we obtain

$$\begin{aligned}\hat{P}(h)&=-\frac{\hat{I}_m(h)}{\hat{I}_0(h)}\\&\le-\frac{\frac{4\sqrt{h}}{m+1}\left(\hat{a}_0(h)\right)^{m+1}}{4\sqrt{\frac{m}{m+2}+2h}\,\hat{a}(h)}\\&=-\frac{\sqrt{h}}{m+1}\frac{\left(\hat{a}_0(h)\right)^{m+1}}{\sqrt{\frac{m}{m+2}+2h}\,\hat{a}(h)}\\&\to-\infty\ (h\to+\infty).\end{aligned}$$

Hence, $\lim\limits_{h\to+\infty}\hat{P}(h)=-\infty$.

From (3.37), we have

$$\hat{I}_0'(h)=-4\int_0^{\hat{a}(h)}\frac{1}{y(x,h)}\,\mathrm{d}x,\quad \hat{I}_m'(h)=-4\int_0^{\hat{a}(h)}\frac{x^m}{y(x,h)}\,\mathrm{d}x. \tag{3.40}$$

Similar to (3.5), letting $\frac{2}{m+2}x^m = \sin^2 \upsilon$, we have

$$\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \frac{1}{\sqrt{x^2 - \frac{2}{2+m}x^{m+2}}}\,\mathrm{d}x = \int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \frac{1}{x\sqrt{1 - \frac{2}{2+m}x^m}}\,\mathrm{d}x = \frac{2}{m}\int_0^{\frac{\pi}{2}} \frac{1}{\sin \upsilon}\,\mathrm{d}\upsilon = +\infty$$

and

$$\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \frac{x^m}{\sqrt{x^2 - \frac{2}{2+m}x^{m+2}}}\,\mathrm{d}x = \int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \frac{x^m}{x\sqrt{1 - \frac{2}{2+m}x^m}}\,\mathrm{d}x = \frac{m+2}{m}\int_0^{\frac{\pi}{2}} \sin \upsilon\,\mathrm{d}\upsilon = \frac{m+2}{m}.$$

Combining (3.40) and the two formulas above, we have

$$\begin{aligned}\lim_{h\to 0^+} \hat{I}_0'(h) &= -4\int_0^{\hat{a}(0)} \frac{1}{y(x,0)}\,\mathrm{d}x \\ &= -4\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \frac{1}{\sqrt{x^2 - \frac{2}{2+m}x^{m+2}}}\,\mathrm{d}x \\ &= -\frac{8}{m}\int_0^{\frac{\pi}{2}} \frac{1}{\sin \upsilon}\,\mathrm{d}\upsilon \\ &= -\infty \end{aligned} \tag{3.41}$$

and

$$\lim_{h\to 0^+} \hat{I}_m'(h) = -4\int_0^{\hat{a}(0)} \frac{x^m}{y(x,0)}\mathrm{d}x = -4\int_0^{\left(\frac{m+2}{2}\right)^{\frac{1}{m}}} \frac{x^m}{\sqrt{x^2 - \frac{2}{2+m}x^{m+2}}}\,\mathrm{d}x = -\frac{4(m+2)}{m}. \tag{3.42}$$

Note that by (3.26)

$$\hat{P}'(h) = -\frac{\hat{I}_m'(h)\hat{I}_0(h) - \hat{I}_m(h)\hat{I}_0'(h)}{\hat{I}_0^2(h)} = -\frac{\hat{I}_m'(h)}{\hat{I}_0(h)} + \frac{\hat{I}_m(h)}{\hat{I}_0^2(h)}\hat{I}_0'(h).$$

Then combining (3.27), (3.28), (3.41), (3.42) and the above formula, we get

$$\begin{aligned}\lim_{h\to 0^+} \hat{P}'(h) &= \lim_{h\to 0^+}\left(-\frac{\hat{I}_m'(h)}{\hat{I}_0(h)} + \frac{\hat{I}_m(h)}{\hat{I}_0^2(h)}\hat{I}_0'(h)\right) \\ &= -\frac{1}{\hat{I}_0(0)}\lim_{h\to 0^+}\hat{I}_m'(h) + \frac{\hat{I}_m(0)}{\hat{I}_0^2(0)}\lim_{h\to 0^+}\hat{I}_0'(h) \\ &= -\frac{1}{\frac{-4}{m}\left(\frac{m+2}{m}\right)\frac{2}{m}B\left(\frac{3}{2},\frac{2}{m}\right)}\frac{-4(m+2)}{m} \\ &\quad + \frac{\frac{-4}{m}\left(\frac{m+2}{2}\right)^{\frac{m+2}{m}} B\left(\frac{3}{2}, 1+\frac{2}{m}\right)}{\left[\frac{-4}{m}\left(\frac{m+2}{2}\right)^{\frac{2}{m}} B\left(\frac{3}{2},\frac{2}{m}\right)\right]^2}\lim_{h\to 0^+}\hat{I}_0'(h) \\ &= +\infty.\end{aligned}$$

Similar to the proof of the formula above, we have $\lim_{h\to 0^-} \hat{P}'(h) = +\infty$. Then conclusion (2) follows.

Next, we prove conclusion (3). By (3.25), we have

$$M_h''(h, a_h) = -\hat{I}_0(h)\hat{P}''(h), \ \ h > 0 \tag{3.43}$$

whenever $\hat{P}'(h) = 0$, where $a_h = \hat{P}(h)$. Hence, by Lemma 3.9 we have $\hat{P}''(h) < 0$ whenever $\hat{P}'(h) = 0$, $h > 0$. This, together with conclusion (2), implies that there exists a unique $h_0 > 0$ such that $\hat{P}'(h_0) = 0$. Then by Lemma 3.9 and (3.43), we have $\hat{P}''(h_0) < 0$ and $(h-h_0)\hat{P}'(h) < 0$ for $h \in (0, h_0) \cup (h_0, +\infty)$. Therefore, conclusion (3) is proved. □

The behavior of $\hat{P}$ is shown as in Figure 8.

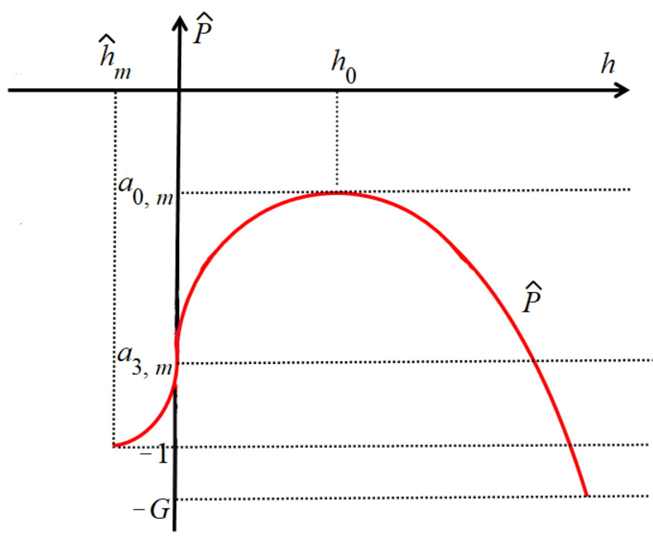


Figure 8. The behavior of $\hat{P}$.

Next, we give a detailed proof of Theorem 1.2.

Noting that system (1.3) is centrally symmetric, by Lemmas 3.6 and 3.10, there exists a constant $\tilde{\epsilon}_1 > 0$ such that for $0 < |\varepsilon| < \tilde{\epsilon}_1$, $|a + 1| < \tilde{\epsilon}_1$ and $|a| < N$, system (1.3) has two symmetric small (resp. no) limit cycles in the neighborhood $\hat{h}_m - \tilde{\epsilon}_1 < H(x, y) < \hat{h}_m + \tilde{\epsilon}_1$ for $a > \hat{a}^*_{1,m}(\varepsilon)$ (resp. $a \le \hat{a}^*_{1,m}(\varepsilon)$); moreover, limit cycles are unstable (stable) as $\varepsilon > 0$ (< 0) when they exist. From Lemmas 3.7 and 3.10, there exist a constant $\tilde{\epsilon}_3 > 0$ and a function $\hat{a}^*_{3,m}(\varepsilon)$ such that for all $0 < |\varepsilon| < \tilde{\epsilon}_3$, $|a - \hat{a}_{3,m}| < \tilde{\epsilon}_3$ and $|a| < N$, system (1.3) has in the neighborhood $-\tilde{\epsilon}_3 < H(x, y) < \tilde{\epsilon}_3$

(i) a double homoclinic loop for $a = \hat{a}^*_{3,m}(\varepsilon)$ which is unstable (stable) as $\varepsilon > 0$ (< 0);

(ii) a unique large limit cycle for $a > \hat{a}^*_{3,m}(\varepsilon)$ which is unstable (stable) as $\varepsilon > 0$ (< 0);

(iii) two small limit cycles for $a < \hat{a}^*_{3,m}(\varepsilon)$ which are unstable (stable) as $\varepsilon > 0$ (< 0).

Let $a_{0,m} = \hat{P}(h_0) \in (0, a_{3,m})$. Then by Lemma 3.10 and (3.25), we obtain that

$$M(h_0, a_{0,m}) = 0, \quad M_h'(h_0, a_{0,m}) = 0, \quad M_h''(h_0, a_{0,m}) < 0.$$

Combining Lemma 2.1 together with $M_a'(h_0, a_{0,m}) = \hat{I}_0(h_0) < 0$, there exist a constant $\tilde{\epsilon}_0 > 0$ and a function $\hat{a}^*_{0,m}(\varepsilon) = a_{0,m} + O(\varepsilon)$, such that for all $0 < |\varepsilon| < \tilde{\epsilon}_0$, $|a - a_{0,m}| < \tilde{\epsilon}_0$ and $|a| < N$, system (1.3) has two simple large limit cycles (resp. a unique double large limit cycle, no large limit cycle) in the neighborhood $h_0 - \tilde{\epsilon}_0 < H(x, y) < h_0 + \tilde{\epsilon}_0$ for $a < \hat{a}^*_{0,m}(\varepsilon)$ (resp. $= 0, > 0$).

Let h^* satisfy the equation

$$\hat{P}(h^*) = a^*, \tag{3.44}$$

i.e., $\Phi(h^*, 0, a^*) = 0$. Noting that $\frac{\partial \Phi}{\partial a}|_{(h^*,0,a^*)} = \hat{I}_0(h^*)$ for $\hat{h}_m < h < h_0$ and $h > h_0$, by the implicit function theorem, together with $\hat{P}'(h^*) \neq 0$, we see that for every a near a^* there exist $\tilde{\epsilon}^* > 0$ and $\tilde{\sigma}^* > 0$ such that for $0 < |\varepsilon| \leq \tilde{\epsilon}^*$, the equation $\Phi(h, \varepsilon, a) = 0$ has a unique solution with respect to $h \in (h^* - \tilde{\sigma}^*, h^* + \tilde{\sigma}^*)$. Therefore the number of solutions of (3.44) will determine the number of limit cycles of system (1.3).

Note that $M(h, a) \neq 0$ for $a \geq a_{0,m} + \tilde{\epsilon}_0$. Then for $0 < |\varepsilon| < \tilde{\epsilon}_0$, system (1.3) has no limit cycle for all $a > \hat{a}^*_{0,m}(\varepsilon)$ and $|a| < N$.

Obviously, there exists a constant $\hat{\epsilon}_4 > 0$ such that for $0 < |\varepsilon| < \hat{\epsilon}_4$ and $a_{3,m} + \hat{\epsilon}_3 \leq a \leq a_{0,m} - \hat{\epsilon}_0$, system (1.3) has a unique large limit cycle in the neighborhood $\hat{\epsilon}_3 \leq H(x, y) \leq h_0 - \hat{\epsilon}_0$; moreover, the large limit cycle is unstable (stable) as $\varepsilon > 0$ (< 0).

In fact, for any given $\tilde{a}_m \in (a_{3,m}, a_{0,m})$, $M(h, a)$ must have a simple root $h_1^*(\tilde{a}_m) \in (0, h_0)$ satisfying $M(h_1^*(\tilde{a}_m), \tilde{a}_m) = 0$ and $M_h'(h_1^*(\tilde{a}_m), \tilde{a}_m) = -\hat{I}_0(h_1^*(\tilde{a}_m))\hat{P}'(h_1^*(\tilde{a}_m)) < 0$ with the above analysis. Then by Theorem 1 of [9], there exists $\tilde{\varepsilon}_4 > 0$ such that $0 < |\varepsilon| < \tilde{\varepsilon}_4$ and $|a - \tilde{a}_m| < \tilde{\varepsilon}_4$, system (1.3) has a unique limit cycle near $\Gamma^{\varepsilon}_{h_1^*(\tilde{a}_m)}$. Next we prove that the limit cycle is unstable (stable) as $\varepsilon > 0$ (< 0). In fact, by Lemma 3.1.2 of [6] we have

$$\begin{aligned} I(\Gamma^{\varepsilon}_{h_1^*(\tilde{a}_m)}) &= -\varepsilon \oint_{L^{\varepsilon}_{h_1^*(\tilde{a}_m)}} f(x, a) \mathrm{d}\tau \\ &= -\varepsilon \oint_{L^{\varepsilon}_{h_1^*(\tilde{a}_m)}} (a + x^m) \, \mathrm{d}\tau \\ &= -\varepsilon \left[\oint L_{h_1^*(\tilde{a}_m)} (a + x^m) \, \mathrm{d}\tau + \varepsilon \right] \\ &= -\varepsilon \left[-M_h'(h_1^*(\tilde{a}_m), \tilde{a}_m) + \varepsilon \right] \\ &> 0 \ (< 0) \end{aligned}$$

as $\varepsilon > 0$ (< 0). It means that the limit cycle $\Gamma^{\varepsilon}_{h_1^*(\tilde{a}_m)}$ is unstable (stable) as $\varepsilon > 0$ (< 0).

In the same way, it can be proved that there exists a constant $\tilde{\epsilon}_5 > 0$ such that for $0 < |\varepsilon| < \tilde{\epsilon}_5$ and $-N < a \leq a_{0,m} - \tilde{\epsilon}_0$, system (1.3) has a unique large limit cycle in the neighborhood $h_0 + \tilde{\epsilon}_0 \leq H(x, y)$; moreover, the large limit cycle is stable (unstable) as $\varepsilon > 0$ (< 0). Similarly, there exists a constant $\tilde{\epsilon}_6 > 0$ such that for $0 < |\varepsilon| < \tilde{\epsilon}_6$ and $-1 + \tilde{\epsilon}_1 \leq a \leq a_{3,m} - \tilde{\epsilon}_3$, system (1.3) has two small limit cycles in the neighborhood $\hat{h}_m + \tilde{\epsilon}_0 \leq H(x, y) \leq -\tilde{\epsilon}_3$ which are unstable (stable) as $\varepsilon > 0$ (< 0).

Then, Theorem 1.2 follows by taking $\hat{\epsilon}^* = \min\{\tilde{\epsilon}_i, i = 0, 1, 3, 4, 5, 6\}$. This finishes the proof of Theorem 1.2.

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