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MULTI-SPEED SOLITARY WAVES IN SPIN-1 BOSE-EINSTEIN CONDENSATES*

Junqi Guo[1,†], Yuan Li[1] and Dun Zhao[1]

Abstract This paper investigates the existence of multi-speed solitary waves in spin-1 Bose-Einstein condensates, governed by the three-component Gross-Pitaevskii equations. These multi-speed solitary waves manifest as vector solitary waves, with each component propagating at a different speed. By utilizing scaling invariance, Galilean drift, and solving the system backward in time using energy methods, we establish the existence of multi-speed solitary wave solutions. The presence of these waves implies that solitary wave excitations in one component can propagate independently without significantly affecting the other components. Furthermore, we analyze the asymptotic behavior of the magnetic soliton, which is driven by the particle exchange and corresponds to the net magnetization of the system.

Keywords Spin-1 Bose-Einstein condensate, Gross-Pitaevskii equation, solitary wave, magnetic soliton.

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1. Introduction

In this paper, we consider the following 3-component Gross-Pitaevskii (GP) equations [31]

$$\begin{cases} i\partial_t u_1 = -\dfrac{1}{2}\Delta u_1 + (c_0+c_1)|u_1|^2u_1 + (c_0-c_1)|u_{-1}|^2u_1 \\ \qquad\qquad + (c_0+c_1)|u_0|^2u_1 + c_1\bar{u}_{-1}u_0^2 + (q-p)u_1, \\ i\partial_t u_0 = -\dfrac{1}{2}\Delta u_0 + c_0|u_0|^2u_0 + (c_0+c_1)|u_1|^2u_0 \\ \qquad\qquad + (c_0+c_1)|u_{-1}|^2u_0 + 2c_1\bar{u}_0u_1u_{-1}, \\ i\partial_t u_{-1} = -\dfrac{1}{2}\Delta u_{-1} + (c_0+c_1)|u_{-1}|^2u_{-1} + (c_0-c_1)|u_1|^2u_{-1} \\ \qquad\qquad + (c_0+c_1)|u_0|^2u_{-1} + c_1\bar{u}_1u_0^2 + (q+p)u_{-1}. \end{cases} \tag{1.1}$$

This system appears in the study of spin-1 Bose-Einstein condensates (BECs). It is well-known that BEC is a macroscopic quantum phenomenon in which, at extremely low temperatures, identical bosonic particles tend to occupy their lowest quantum state and behave as a single particle. The first realization of BEC involved ultracold alkali-metal atoms in a single spin state, confined spatially with magnetic traps [1]. A spinor BEC, which refers to a BEC with

†The corresponding author.
[1]School of Mathematics and Statistics, Lanzhou University, 730000 Lanzhou, China
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Email: guojq21@lzu.edu.cn(J. Guo), li_yuan@lzu.edu.cn(Y. Li), zhaod@lzu.edu.cn(D. Zhao)

spin internal degrees of freedom, was later achieved in a gas of spin-1 ultra-cold atoms confined in an optical dipole trap, which provides a new way for exploring intricate quantum states [23]. In (1.1), the complex-valued functions u_j ($j = 1, 0, -1$): $\mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ ($d = 1, 2, 3$), represent the wave functions of atoms corresponding to spin-up, spin-free and spin-down states, respectively. The constant c_0 denotes the mean-field interaction, c_1 the spin-exchange interaction, p the linear Zeeman effect and q the quadratic Zeeman effect. The mean-field interaction is repulsive if $c_0 > 0$ and attractive if $c_0 < 0$. The BEC system is called ferromagnetic if $c_1 < 0$ and antiferromagnetic if $c_1 > 0$. In the whole paper, $\bar{u}$ represents the complex conjugate of u.

For simplicity, we will hereafter denote a function $w(t, x)$ as $w(t)$ to emphasis its dependence on the variable t, and represent the vector function $(w_1(t, x), w_0(t, x), w_{-1}(t, x))^T$ as $\mathbf{w}(t)$, where the superscript T denotes the transpose of a vector. When the context is unambiguous, we may also denote $w(x, t)$ simply by w.

Let $\mathbf{u}(t) = (u_1(t, x), u_0(t, x), u_{-1}(t, x))^T$ be a solution of the Cauchy problem associated with (1.1), it is known that the system adheres to the following conservation laws:

- Total Particle Number Conservation: $\mathbf{N}(\mathbf{u}(t)) = \mathbf{N}(\mathbf{u}(0))$, given by

$$\mathbf{N}(\mathbf{u}(\mathbf{t})) := \int_{\mathbb{R}^d} n(t, x)dx = \sum_j \int_{\mathbb{R}^d} n_j(t, x)dx = \sum_j 2N(u_j(t, x)), \ j \in \{-1, 0, 1\}, \tag{1.2}$$

where $n(t, x) := \sum_j n_j(t, x)$ represents the total particle density, $n_j(t, x) := |u_j(t, x)|^2$ denotes the particle density of the jth-component, and

$$N(\varphi) := \frac{1}{2} \int_{\mathbb{R}^d} |\varphi|^2 dx, \quad \text{for} \quad \varphi \in H^1(\mathbb{R}^d). \tag{1.3}$$

- Energy Conservation: $\mathbf{E}(\mathbf{u}(t)) = \mathbf{E}(\mathbf{u}(0))$, given by

$$\begin{aligned} \mathbf{E}(\mathbf{u}(t)) := \int_{\mathbb{R}^d} \Big\{ &\frac{1}{2}|\nabla \mathbf{u}|^2 + \frac{c_0}{2} n^2 + \frac{c_1}{2}|\mathbf{F}|^2 + p(|u_1|^2 - |u_{-1}|^2) \\ &+ q(|u_1|^2 + |u_{-1}|^2) \Big\} dx, \end{aligned} \tag{1.4}$$

where $\mathbf{F} = \mathbf{F}(t, x) := (F_x(t, x), F_y(t, x), F_z(t, x))^T$ is the spin density vector as defined in (1.7) below.

- Momentum Conservation: $\mathbf{P}(\mathbf{u}(t)) = \mathbf{P}(\mathbf{u}(0))$, given by

$$\mathbf{P}(\mathbf{u}(t)) := \sum_j \Im \int_{\mathbb{R}^d} u_j(t, x) \nabla \bar{u}_j(t, x)dx = \sum_j 2P(u_j(t, x)), \ j \in \{-1, 0, 1\}, \tag{1.5}$$

where $\Im z$ signifies the imaginary part of a complex number z, and

$$P(\varphi) := \frac{1}{2} \Im \int_{\mathbb{R}^d} \varphi \nabla \bar{\varphi} dx, \quad \text{for} \quad \varphi \in H^1(\mathbb{R}^d). \tag{1.6}$$

In physics, spin-1 BECs provide an ideal platform for exploring spin dynamics in a dilute atomic system. In these systems, the nontrivial self-interactions between the spin components, governed by the spin-exchange term, give rise to a range of fascinating phenomena. One such phenomenon is the formation of magnetic solitons. Magnetic solitons in spin-1 BECs emerge from the interplay between spin-exchange interactions, leading to the formation of stable, self-bound structures with distinct magnetic properties. These solitons offer valuable insights into the nature of magnetic textures at quantum scales. Consequently, the study of magnetic solitons in spin-1 BECs has become a central focus in the field of ultracold atomic systems. In (1.4), the spin density vector $\mathbf{F}(t,x) = (F_x(t,x), F_y(t,x), F_z(t,x))^T$ is defined by [20]

$$\begin{aligned} F_x &= \frac{1}{\sqrt{2}}\left[\bar{u}_1 u_0 + \bar{u}_0(u_1 + u_{-1}) + \bar{u}_{-1}u_0\right], \\ F_y &= \frac{i}{\sqrt{2}}\left[-\bar{u}_1 u_0 + \bar{u}_0(u_1 - u_{-1}) + \bar{u}_{-1}u_0\right], \\ F_z &= |u_1|^2 - |u_{-1}|^2. \end{aligned} \tag{1.7}$$

In such a context, the total magnetization $M := \int_{\mathbb{R}^d} F_z dx$ is recognized as another conserved quantity. Nonetheless, this particular quantity will not be utilized in the present study. The function F_z is pivotal as it governs the particle exchange and the net magnetization within the spin-1 BEC system. When $F_z = 0$ (consequently $M = 0$), the spin-1 BEC exhibits no magnetic properties; Conversely, when $F_z \neq 0$, the BEC manifests magnetism. In the realm of physics, a solitary wave characterized by F_z is commonly referred to as a magnetic soliton [10, 30]. It is known that a state $\mathbf{u}(t)$ for which $F_z = 0$ corresponds to a polar or antiferromagnetic state, whereas a state with $F_z \neq 0$ may be a ferromagnetic state, a broken-axisymmetry state or others [28].

Over the past twenty years, based on (1.1), the study on spin-1 BEC has attracted a great deal of attention from physicists, and there is a huge number of physics references in this area, for example, one can see [19,20,25] and the references therein. From the point view of mathematics, Bao et al. [5] obtained some numerical results regarding the ground states of (1.1) with either harmonic or optical lattice potentials in $\mathbb{R}^d$ $(d = 1, 2, 3)$, under the conditions $c_0 \geq 0$ and $c_0 \geq |c_1|$, utilizing the normalized gradient flow or imaginary time method; Cao et al. [8] investigated the existence of ground states in $\mathbb{R}^1$ for the scenario where $c_0 < 0$ and $c_1 < 0$; Additionally, Kong et al. [22] examined the existence and nonexistence of ground state, as well as the asymptotic behavior of (1.1) with a harmonic potential in $\mathbb{R}^2$, for $c_0 < 0$; When $d = 3$, Li et al. stated the results regarding existence, stability, and asymptotic behavior for ground states for (1.1) in [24] with free and harmonic potential. Furthermore, Carles et al. [16] explored the stability of standing waves under an Ioffe-Pritchard magnetic field in $\mathbb{R}^d$ $(d = 1, 2, 3)$, with $c_0 > 0$ and $c_1 > 0$.

In this paper, we focus on the investigation of multi-speed vector solitary waves and magnetic solitons in the spin-1 BECs. A solitary wave is a traveling wave that maintains its size, shape, and speed during propagation. It can be represented in the form $u(t,x) = f\,(x - ct)$, where f is a smooth function that decays rapidly at infinity. Solitary waves are observed in various domains, including water surface elevation and light intensity within optical fibers. Notably, macroscopic quantum phenomena in BECs frequently exhibit solitary wave characteristics, making the study of solitary waves in BEC systems a crucial research area in the field of ultracold atoms [21, 29]. A vector solitary wave occurs in a coupled system, where each component behaves as a solitary wave. In the one-dimensional case, it is known that, when $p = q = 0$ and $c_0 = c_1 = -1$, system (1.1) is integrable and can be solved analytically by the inverse scattering method [18]. Following this discovery, various types of vector soliton solutions have been analytically derived in the literature using the methods in soliton theory. However, to our knowledge, there is a scarcity of mathematical research on solitary wave solutions of system (1.1) in the non-integrable case, particularly in higher dimensions.

Multi-speed vector solitary waves are a distinct class of waves found in multi-component systems, where each component's solitary wave propagates at distinct speed. The existence of such waves indicates that in nonlinear vector systems, it is possible to observe solitary wave excitation in one component without locally impacting the other component. This phenomenon was previously explored in the context of the two-component cubic Schrödinger systems, as referenced in [13, 17]. Following this, the existence of multi-speed vector solitary waves was confirmed for the coherently coupled nonlinear Schrödinger system [35] and the Klein-Gordon-Schrödinger system [34]. More recently, the scope of research has expanded to include systems of coupled anharmonic chains and specific vector integrable systems, as detailed in [14].

Inspired by the works mentioned above, this paper aims to investigate the existence of multi-speed vector solitary waves and provide a description of the asymptotic behavior of the corresponding magnetic solitons within the Gross-Pitaevskii (GP) system (1.1) for spin-1 BECs.

Before stating our main conclusion, let us recall some results of the scalar Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u + \mu|u|^{2\alpha}u = 0,\ \mu \in \mathbb{R}\setminus\{0\}, \tag{1.8}$$

where $0 < \alpha < \infty$ for $d = 1, 2$, and $0 < \alpha < \frac{2}{d-2}$ for $d \geq 3$. We know that equation (1.8) is locally well-posed in $H^1(\mathbb{R}^d)$ [9, 15, 32, 33], and the H^1 flow admits the conservation laws of particle number, energy and momentum. In addition, equation (1.8) is invariant under the transformation

$$u(t,x) \mapsto \lambda^{\frac{1}{\alpha}} u(\lambda^2(t-t_0), \lambda(x-x_0))e^{i\gamma},\quad \lambda > 0,\ t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^d,\ \gamma \in \mathbb{R}, \tag{1.9}$$

and the Galilean drift

$$u(t,x) \mapsto u(t, x-\beta_0 t)e^{i\beta_0\cdot\left(x-\frac{\beta_0}{2}t\right)},\quad \beta_0 \in \mathbb{R}^d. \tag{1.10}$$

We now focus on the specific case where $\alpha = 1$ for equation (1.8). By utilizing the transformations described by (1.9) and (1.10), and considering the unique positive radial ground state solution $Q(x)$ of the stationary Schrödinger equation

$$\Delta Q - 2Q + 2|Q|^2 Q = 0,\quad Q \in H^1(\mathbb{R}^d), \tag{1.11}$$

we can derive a family of solitary wave solutions for (1.8) in the case $\alpha = 1$, which read

$$R(t,x) := \sqrt{\frac{\omega}{\mu}} e^{i((\omega-|v|^2/2+\kappa)t+v\cdot x+\gamma)} Q(\sqrt{\omega}(x-vt-\xi)), \tag{1.12}$$

where $\omega, \mu > 0$, $\kappa, \gamma \in \mathbb{R}$, and $v, \xi \in \mathbb{R}^d$. Notice that $Q(x) \in C^2(\mathbb{R}^d)$, and it decays exponentially at infinity [7]. Specifically, for any $0 < \eta < \sqrt{2}$, there exists a constant $C(Q) > 0$ such that the following estimate holds

$$|Q(x)| + |\nabla Q(x)| \leq C(Q)e^{-\eta|x|}\quad \text{for } x \in \mathbb{R}^d. \tag{1.13}$$

The notation used in this paper follows standard conventions. For clarity, we list some of the symbols that will appear throughout the text:

- We denote the initial value of the Cauchy problem by $\mathbf{u}_0(x) := (u_1^0(x), u_0^0(x),\ u_{-1}^0(x))^T$.
- The norm of a Banach space X is represented by $\|\cdot\|_X$.

- For simplicity, $H^1(\mathbb{R}^d,\mathbb{C})$ and $L^2(\mathbb{R}^d,\mathbb{C})$ will be denoted by $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively.
- The inner product on $L^2(\mathbb{R}^d)$ is defined as $(u,v)_{L^2}=\Re\int_{\mathbb{R}^d}u\bar{v}dx$ for $u,v\in L^2(\mathbb{R}^d)$.
- The product spaces $\mathcal{H}^1:=H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)\times H^1(\mathbb{R}^d)$ and $\mathcal{L}^2:=L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ are equipped with the norms $\|\mathbf{u}\|_{\mathcal{H}^1}=\left(\|u_1\|_{H^1}^2+\|u_0\|_{H^1}^2+\|u_{-1}\|_{H^1}^2\right)^{\frac{1}{2}}$ and $\|\mathbf{u}\|_{\mathcal{L}^2}=\left(\|u_1\|_{L^2}^2+\|u_0\|_{L^2}^2+\|u_{-1}\|_{L^2}^2\right)^{\frac{1}{2}}$, respectively.
- Throughout this paper, C, C', C_k and C_k' will be used to denote various positive constants, which may vary depending on the context.

Our main results can be stated as follows:

Theorem 1.1. *For $j=1,0,-1$, assume that $\omega_j>0$, $\gamma_j\in\mathbb{R}$, x_j, $v_j\in\mathbb{R}^d$. Let $Q\in H^1(\mathbb{R}^d)$ be the unique positive radial ground state solution of the scalar Schrödinger Equation* (1.11)*. Define $\mathbf{R}(t)=(R_1(t),R_0(t),R_{-1}(t))^T$ by*

$$R_j(t,x):=e^{i((\omega_j-|v_j|^2/2+\kappa_j)t+v_j\cdot x+\gamma_j)}\sqrt{\frac{\omega_j}{\mu_j}}Q\left(\sqrt{\omega_j}(x-v_jt-x_j)\right), \tag{1.14}$$

where $\kappa_1=p-q$, $\kappa_{-1}=-(p+q)$ $(p,\ q\in\mathbb{R})$, $\kappa_0=0$, $\mu_1=\mu_{-1}=-(c_0+c_1)>0$ and $\mu_0=-c_0>0$. Also define

$$v_*:=\min\left\{|v_1-v_0|,|v_1-v_{-1}|,|v_0-v_{-1}|\right\},\quad \omega_*:=\frac{1}{4}\min\left\{\omega_1,\omega_0,\omega_{-1}\right\},$$
$$v_{\max}^1:=\max_j\left\{\left|(\omega_j-\frac{\omega_1+\omega_0+\omega_{-1}}{3})+(\frac{|v_j|^2}{2}-\frac{|v_1|^2+|v_0|^2+|v_{-1}|^2}{6})\right|\right\},$$
$$v_{\max}^2:=\max_j\left\{\left|v_j-\frac{v_1+v_0+v_{-1}}{3}\right|\right\}.$$

Let L be a fixed constant. If v_1, v_0 and v_{-1} in (1.14) *satisfy*

$$\max\left\{v_{\max}^1,v_{\max}^2\right\}\le Lv_*, \tag{1.15}$$

and if there exists a constant V_*, dependent on d, Q, c_0, c_1, p, q, γ_j, ω_j, x_j and L, such that $v_*>V_*$, then there exist a constant $T_0>0$ and a solution $\mathbf{u}(t)=(u_1(t),u_0(t),u_{-1}(t))^T$ of (1.1) *such that for all $t\in[T_0,+\infty)$,*

$$\|\mathbf{u}(t)-\mathbf{R}(t)\|_{\mathcal{H}^1}\le e^{-\sqrt{\omega_*}v_*t}.$$

Consequently, we can describe the asymptotic behavior of the system's net magnetization, defined by F_z.

Theorem 1.2. *Given the constant V_* as defined in Theorem 1.1, if $v_*>V_*$, then there exist a $T_0'>0$ and a constant $C>0$ such that for all $t\in[T_0',+\infty)$, the following inequality holds:*

$$\|F_z-(|R_1(t)|^2-|R_{-1}(t)|^2)\|_{W^{1,1}}\le Ce^{-\sqrt{\omega_*}v_*t}.$$

We note that when both $|R_1(t)|^2$ and $|R_{-1}(t)|^2$ behave like single solitons with different velocities, the difference $|R_1(t)|^2-|R_{-1}(t)|^2$ may behave like a 2-soliton. In this context, Theorem 1.2 indicates that, under the conditions specified in Theorem 1.1, F_z will asymptotically converge to the 2-soliton in the context of the $W^{1,1}$ norm.

The strategy of the proof of Theorem 1.1 is inspired by the investigation of multi-solitons for scalar nonlinear Schrödinger equations in [11, 12, 26, 27]. In these works, at large time, the components of the multi-soliton are well-separated and thus it is possible to localize the analysis around each soliton to gain an $H^1(\mathbb{R}^d)$-local control, up to a space of finite dimension in $L^2(\mathbb{R}^d)$. Our approach differs from the proof in [11, 12, 26, 27] and the references therein. Indeed, as we will see in Section 3, we do not need to localize the functionals around each solitary wave, since in our case the coupling will act as a localizing factor.

The rest of the paper is organized as follows. In Section 2, we construct the approximate profiles for the multi-speed solitary wave solutions of (1.1). Additionally, we provide uniform estimates and establish a compactness result, which together enable us to complete the proofs of Theorem 1.1 and Theorem 1.2. In Section 3, we present the proof of the main lemma, Lemma 2.3, which plays a crucial role in our analysis.

2. The approximate profile and the proofs of the main results

In this section, our objective is to construct an approximate profile of the multi-speed solitary wave solutions for system (1.1). The scheme is based on a backward in time argument, which is developed for the study of scalar nonlinear Schrödinger equations [26, 27] by Merle et al., and was later applied to the approach of multi-solitary waves [2–4, 6, 11, 12] and multi-speed solitary waves [13, 17, 34, 35].

To start, firstly, it is straightforward to verify that for a fixed t, the function R defined by (1.12) serves as a solution to the following equation:

$$-\frac{1}{2}\Delta R + \left(\omega + \frac{|v|^2}{2}\right) R - \mu |R|^2 R + iv \cdot \nabla R = 0.$$

For any $\varphi \in H^1(\mathbb{R}^d)$, we define S as:

$$S(\varphi) := S(\varphi, \mu, \omega, v) = E_\mu(\varphi) + \left(\omega + \frac{|v|^2}{2}\right) N(\varphi) + v \cdot P(\varphi), \tag{2.1}$$

where $N(\varphi)$ and $P(\varphi)$ are defined by (1.3) and (1.6), respectively, and $E_\mu(\varphi)$ is defined by

$$E_\mu(\varphi) := \frac{1}{4}\int_{\mathbb{R}^d} |\nabla\varphi|^2 dx + \frac{\mu}{4}\int_{\mathbb{R}^d} |\varphi|^4 dx.$$

We can now confirm through direct calculation that (1.12) is a critical point of $S(\varphi)$. We introduce the linearized action H on $\mathbb{R} \times H^1(\mathbb{R}^d)$ as follows:

$$H(t, \varphi) := \langle S''(R(t))\varphi, \varphi\rangle. \tag{2.2}$$

A straightforward computation reveals that

$$\begin{aligned} H(t, \varphi) =& \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\varphi|^2 dx - \mu \int_{\mathbb{R}^d} (|R|^2|\varphi|^2 + 2\Re(R\bar{\varphi})^2) dx \\ &+ \left(\omega + \frac{|v|^2}{2}\right)\int_{\mathbb{R}^d} |\varphi|^2 dx + v \cdot \Im \int_{\mathbb{R}^d} \varphi \nabla\bar{\varphi} dx. \end{aligned}$$

It is easy to check that H possesses a coercivity property [11], which is essential for our subsequent analysis. Hereafter, we mainly focus on the cases of $d = 1, 2, 3$.

Lemma 2.1. ([11]) *Let R be the solitary wave solution of* (1.8) *as defined by* (1.12)*, and let H be the functional given by* (2.2)*. Then, there exist constants $K = K(Q) > 0$, $\nu_j \in \mathbb{N} \setminus \{0\}$ and $\tilde{\eta}^1, \tilde{\eta}^2, ..., \tilde{\eta}^{\nu_j} \in L^2(\mathbb{R}^d)$ such that for $l = 1, 2, ..., \nu_j$, we have $\|\tilde{\eta}^l\|_{L^2} = 1$ and for any $\varphi \in H^1$, the following inequality holds:*

$$K\|\varphi\|_{H^1}^2 \leq H(t, \varphi) + \sum_{l=1}^{\nu_j} (\varphi, \eta^l(t))_{L^2}^2, \quad \forall\ t \in \mathbb{R},$$

where

$$\eta^l(t) := e^{i((\omega - |v|^2/2 + \kappa)t + v\cdot x + \gamma)} \sqrt{\frac{\omega}{\mu}} \tilde{\eta}^l(\sqrt{\omega}(x - vt - \xi)).$$

For $j \in \{-1, 0, 1\}$, let us define the functional $\mathbf{S}$ on $\mathcal{H}^1$ by

$$\mathbf{S}(\boldsymbol{\varphi}) := \sum_j S_j(\varphi_j), \quad \boldsymbol{\varphi} = (\varphi_1, \varphi_0, \varphi_{-1})^T \in \mathcal{H}^1,$$

where $S_j(\varphi_j)$ is given by (2.1). In a similar manner, for $j \in \{-1, 0, 1\}$, we define the functional $\mathbf{H}$ on $\mathbb{R} \times \mathcal{H}^1$ as

$$\mathbf{H}(t, \boldsymbol{\varphi}) := \sum_j H_j(t, \varphi_j), \quad (t, \boldsymbol{\varphi}) \in \mathbb{R} \times \mathcal{H}^1,$$

where $H_j(t, \varphi_j)$ is defined by (2.2). As a direct consequence of Lemma 2.1, we have

Lemma 2.2. *Assume $d = 1, 2, 3$. There exists a constant $K_* > 0$ such that, for all $t \in \mathbb{R}$, $\boldsymbol{\varphi} \in \mathcal{H}^1$ and $j \in \{-1, 0, 1\}$, the following inequality holds:*

$$K_* \|\boldsymbol{\varphi}\|_{\mathcal{H}^1}^2 \leq \mathbf{H}(t, \boldsymbol{\varphi}) + \sum_j \sum_{l=1}^{\nu_j} (\varphi_j, \eta_j^l(t))_{L^2}^2,$$

where η_j^l and ν_j are defined as in Lemma 2.1.

Notice that the Cauchy problem for system (1.1) can be rewritten as

$$\begin{cases} i\partial_t \mathbf{u} = -\dfrac{1}{2}\Delta \mathbf{u} + \mathbf{g}_1(\mathbf{u}) + \mathbf{g}_2(\mathbf{u}), \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where

$$\mathbf{g}_1(\mathbf{u}) = ((q-p)u_1, 0, (q+p)u_{-1})^T, \quad \mathbf{g}_2(\mathbf{u}) = (g_2^1(\mathbf{u}), g_2^0(\mathbf{u}), g_2^{-1}(\mathbf{u}))^T,$$

with

$$\begin{aligned} g_2^1(\mathbf{u}) &= (c_0 + c_1)|u_1|^2 u_1 + (c_0 - c_1)|u_{-1}|^2 u_1 + (c_0 + c_1)|u_0|^2 u_1 + c_1 \bar{u}_{-1} u_0^2, \\ g_2^0(\mathbf{u}) &= c_0 |u_0|^2 u_0 + (c_0 + c_1)|u_1|^2 u_0 + (c_0 + c_1)|u_{-1}|^2 u_0 + 2c_1 \bar{u}_0 u_1 u_{-1}, \\ g_2^{-1}(\mathbf{u}) &= (c_0 + c_1)|u_{-1}|^2 u_{-1} + (c_0 - c_1)|u_1|^2 u_{-1} + (c_0 + c_1)|u_0|^2 u_{-1} + c_1 \bar{u}_1 u_0^2. \end{aligned}$$

Applying Theorem 3.3.9, Remark 3.3.12, and Theorem 4.3.1 in [9], we can establish the local well-posedness in $\mathcal{H}^1$ for the Cauchy problem of system (1.1) as follows:

Lemma 2.3. *Assume $d = 1, 2, 3$. For any initial data $\mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{H}^1$, the Cauchy problem* (1.1) *is locally well-posed in $\mathcal{H}^1$. Specifically, the solution $\mathbf{u} \in C\big((-T_*, T^*), \mathcal{H}^1\big)$ and if $T^* < +\infty$ or $T_* < +\infty$, we have*

$$\lim_{t \to T^*} \|\mathbf{u}(t)\|_{\mathcal{H}^1} = +\infty, \quad \text{or} \quad \lim_{t \to -T_*} \|\mathbf{u}(t)\|_{\mathcal{H}^1} = +\infty.$$

Now, let us turn to construct the multi-speed solitary wave solution $\mathbf{u}$ which satisfies the conclusion of Theorem 1.1. Utilizing the backward in time method, the process can be roughly sketched as follows: Let $T^n \in \mathbb{R}$ be an strictly increasing sequence of positive numbers such that $\lim_{n \to +\infty} T^n = +\infty$. For each $n \in \mathbb{N}$, consider the problem (1.1) with the following final data specified at T^n:

$$\mathbf{u}_n(T^n) = \mathbf{R}(T^n). \tag{2.3}$$

By Lemma 2.3, we know that for each $n \in \mathbb{N}$, the system (2.3) possesses a solution $\mathbf{u}_n(t) \in \mathcal{H}^1$ defined on an interval $(T_n, T^n]$. We aim to demonstrate that there exists a positive number T_0, independent of n, such that $\mathbf{u}_n(t)$ is defined on $[T_0, T^n]$ and converges to $\mathbf{R}(t)$ in $\mathcal{H}^1$ as $n \to +\infty$.

The following uniform estimate is crucial for the proof of Theorem 1.1, and the detailed proof will be elaborated in Section 3.

Lemma 2.4. *There exists a threshold $V_* > 0$ such that, for any $v_* > V_*$, the following assertion holds: there exist an integer $n_0 \in \mathbb{N}$ and a positive time $T_0 > 0$ such that, for all $n \geq n_0$ and for every $t \in [T_0, T^n]$, the following estimate holds:*

$$\|\mathbf{u}_n(t) - \mathbf{R}(t)\|_{\mathcal{H}^1} \leq e^{-\sqrt{\omega_*} v_* t}. \tag{2.4}$$

Following the same arguments as in [26], We can derive the following compactness result.

Lemma 2.5. *For T_0 given in Lemma 2.4, there exists a function $\mathbf{u}_0 \in \mathcal{H}^1$ such that, up to a subsequence, $\mathbf{u}_n(T_0) \to \mathbf{u}_0$ strongly in $\mathcal{L}^2$ as $n \to +\infty$.*

By virtue of the above lemma, we can now prove our main theorems.

Proof of Theorem 1.1. Consider the solution $\mathbf{u}(t)$ of system (1.1) over the interval $[T_0, T^\infty)$, with the initial data $\mathbf{u}(T_0) = \mathbf{u}_0$, where $\mathbf{u}_0$ is given by Lemma 2.5. Our goal is to prove that $T^\infty = +\infty$ and that $\mathbf{u}(t)$ satisfies the conclusions stated in Theorem 1.1.

Applying Lemma 2.5, by the fact that on the interval $t \in [T_0, T^\infty)$, $\mathbf{u}_n(t)$ is bounded in $\mathcal{H}^1$, we can extract a subsequence (still denoted by $\mathbf{u}_n(t)$) that exhibits the following convergences:

$$\begin{aligned} \mathbf{u}_n(t) &\to \mathbf{u}(t) \quad \text{strongly in } \mathcal{L}^2 \text{ as } n \to +\infty, \\ \mathbf{u}_n(t) &\rightharpoonup \mathbf{u}(t) \quad \text{weakly in } \mathcal{H}^1 \text{ as } n \to +\infty. \end{aligned}$$

In particular, the weak convergence implies that for any $t \in [T_0, T^\infty)$,

$$\|\mathbf{u}(t) - \mathbf{R}(t)\|_{\mathcal{H}^1} \leq \liminf_{n \to +\infty} \|\mathbf{u}_n(t) - \mathbf{R}(t)\|_{\mathcal{H}^1} \leq C e^{-\sqrt{\omega_*} v_* t}.$$

This implies that $\mathbf{u}(t)$ remains bounded in space $\mathcal{H}^1$ over the interval $[T_0, T^\infty)$. Consequently, by the blow-up alternative principle, we deduce that $T^\infty = +\infty$. As a result, $\mathbf{u}$ satisfies the conclusions of Theorem 1.1. □

Proof of Theorem 1.2. We can conclude from Theorem 1.1 that for $t \in [T_0, +\infty)$, $\|u_1(t)\|_{H^1} + \|u_{-1}(t)\|_{H^1} < +\infty$, and there exists a constant $T_0' > T_0$ such that for all $t \in [T_0', +\infty)$, the estimate $\|u_j - R_j\|_{L^2}^2 \le C_1 \|u_j - R_j\|_{L^2} \le C_1 e^{-\sqrt{\omega_*} v_* t}$ $(j = 1, -1)$ holds, and then

$$\begin{aligned} &\|(|u_1|^2 - |u_{-1}|^2) - (|R_1|^2 - |R_{-1}|^2)\|_{L^1} \\ &\le \||u_1|^2 - |R_1|^2\|_{L^1} + \||u_{-1}|^2 - |R_{-1}|^2\|_{L^1} \\ &\le (\|u_1\|_{L^2} + \|R_1\|_{L^2})\, \|u_1 - R_1\|_{L^2} + (\|u_{-1}\|_{L^2} + \|R_{-1}\|_{L^2})\, \|u_{-1} - R_{-1}\|_{L^2} \\ &\le (2\|u_1\|_{L^2} + \|u_1 - R_1\|_{L^2})\|u_1 - R_1\|_{L^2} \\ &\quad + (2\|u_{-1}\|_{L^2} + \|u_{-1} - R_{-1}\|_{L^2})\|u_{-1} - R_{-1}\|_{L^2} \\ &\le C_2 e^{-\sqrt{\omega_*} v_* t}, \end{aligned}$$

where we have used the inequality $\|R_j\|_{L^2} \le \|u_j - R_j\|_{L^2} + \|u_j\|_{L^2}$. We also have the same estimate

$$\|\nabla[(|u_1|^2 - |u_{-1}|^2) - (|R_1|^2 - |R_{-1}|^2)]\|_{L^1} \le C_3 e^{-\sqrt{\omega_*} v_* t},$$

which completes the proof. □

3. The proof of Lemma 2.3

In this section, we will complete the proof of Lemma 2.4. The proof will be divided into several lemmas, which is mainly based on a uniform backward in time estimate. To start, we present the following bootstrap lemma.

Lemma 3.1. *There exists a positive constant $V_* > 0$ such that if $v_* > V_*$, then there exist $T_0 > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$ and $t_0 \in [T_0, T^n]$, the following assertion holds:*

Assume that for all $t \in [t_0, T^n]$,

$$\|\mathbf{u}_n(t) - \mathbf{R}(t)\|_{\mathcal{H}^1} \le e^{-\sqrt{\omega_*} v_* t}.$$

Then, it follows that for all $t \in [t_0, T^n]$,

$$\|\mathbf{u}_n(t) - \mathbf{R}(t)\|_{\mathcal{H}^1} \le \frac{1}{2} e^{-\sqrt{\omega_*} v_* t}. \tag{3.1}$$

Lemma 3.1 will be proved later. Assuming its validity, we can now proceed to prove 2.4.

Proof of Lemma 2.4. It is known from Lemma 2.3 that, for each $n \in \mathbb{N}$, the map $t \mapsto \mathbf{u}_n(t)$ is continuous in the space $\mathcal{H}^1$. Let T_0, n_0, V_* be the constants provided in Lemma 3.1. Fix an $n \ge n_0$ and assume $v_* > V_*$. Since $\mathbf{u}_n(T^n) = \mathbf{R}(T^n)$, there exists a small positive $\tau_1 > 0$ such that the inequality (2.4) is satisfied on the interval $[T^n - \tau_1, T^n]$. Define

$$t_\sharp := \inf\{\hat{t} : \hat{t} \in [T_0, T^n] \text{ such that (2.4) holds for all } t \in [\hat{t}, T^n]\}.$$

Lemma 2.3 implies that $t_\sharp < T^n$. We will prove by contradiction that $t_\sharp = T_0$. If $t_\sharp > T_0$, then, according to Lemma 3.1, (3.1) holds on the interval $[t_\sharp, T^n]$.

Therefore, due to the continuity of $\mathbf{u}_n$, there exists a small $\tau_2 > 0$ such that (2.4) holds on $[t_\sharp - \tau_2, T^n]$. However, this contradicts the definition of $t_\sharp$. Hence, we conclude that $t_\sharp = T_0$. □

The rest of the paper is devoted to the proof of Lemma 3.1. Let $t_0 < T^n$ and assume that

$$\mathbf{u}_n(t) = \mathbf{R}(t) + \boldsymbol{\varepsilon}(t), \tag{3.2}$$

where $\boldsymbol{\varepsilon}(t) \in \mathcal{H}^1$ satisfies

$$\|\boldsymbol{\varepsilon}(t)\|_{\mathcal{H}^1} \le e^{-\sqrt{\omega_*}v_* t} \quad \text{for all } t \in [t_0, T^n]. \tag{3.3}$$

Firstly, we give the lemma below.

Lemma 3.2. *For all $t > 0$ and $j = 1, 0, -1$, we have*

$$\big\||R_k(t)||R_j(t)|\big\|_{L^2} \le Ce^{-\frac{3}{2}\sqrt{\omega_*}v_* t}, \tag{3.4}$$

$$\big\||R_k(t)||\nabla R_j(t)|\big\|_{L^2} \le C'(1+|v_j|)e^{-\frac{3}{2}\sqrt{\omega_*}v_* t}, \tag{3.5}$$

where the positive constants C and C' in (3.4) *and* (3.5) *depend only on Q, ω_j, c_0, c_1 and x_j.*

Proof. We will provide the proof for (3.5), the proof for (3.4) follows similarly. From (1.13) and (1.14), we have

$$|R_j(t,x)| \le C_1 e^{-\eta\sqrt{\omega_j}|x-v_j t-x_j|}, \quad |\nabla R_j(t,x)| \le C_2(1+|v_j|)e^{-\eta\sqrt{\omega_j}|x-v_j t-x_j|},$$

where $C_k = C_k(Q, \omega_j, c_0, c_1)$ $(k = 1, 2)$. From these inequalities, we can deduce

$$|R_k(t,x)||\nabla R_j(t,x)| \le C_3(1+|v_j|)e^{-\eta\sqrt{\min\{\omega_k,\omega_j\}}(|x-v_k t-x_k|+|x-v_j t-x_j|)},$$

where $C_3 = C_3(Q, \omega_k, \omega_j, c_0, c_1)$. Let $0 < \delta < \eta$. For $t \ge 0$, we have

$$|x-v_k t-x_k|+|x-v_j t-x_j| \ge |(v_k-v_j)t|-|x_k|-|x_j| \ge |v_k-v_j|t-|x_k|-|x_j|.$$

Thus, we can deduce that

$$\begin{aligned}|R_k(t,x)||\nabla R_j(t,x)| \le\, & C_4(1+|v_j|)e^{-\delta\sqrt{\min\{\omega_k,\omega_j\}}(|x-v_k t-x_k|+|x-v_j t-x_j|)} \\ & \times e^{-(\eta-\delta)\sqrt{\min\{\omega_k,\omega_j\}}|(v_k-v_j)t|},\end{aligned}$$

where $C_4 = C_4(Q, \omega_k, \omega_j, c_0, c_1, x_k, x_j)$. By choosing $\eta = \frac{7}{8}$, $\delta = \frac{1}{8}$, we obtain

$$|R_k(t,x)||\nabla R_j(t,x)| \le C_4(1+|v_j|)e^{-\frac{1}{4}\sqrt{\omega_*}(|x-v_k t-x_k|+|x-v_j t-x_j|)}e^{-\frac{3}{2}\sqrt{\omega_*}v_* t},$$

where v_* and ω_* are defined in Theorem 1.1. Consequently, we have

$$\big\||R_k(t)||\nabla R_j(t)|\big\|_{L^2} \le C_4(1+|v_j|)e^{-\frac{3}{2}\sqrt{\omega_*}v_* t}\big\|e^{-\frac{1}{4}\sqrt{\omega_*}|x|}\big\|_{L^2} \le C'(1+|v_j|)e^{-\frac{3}{2}\sqrt{\omega_*}v_* t},$$

which completes the proof. □

To establish the bootstrap estimate stated in Lemma 3.1, we require a modified estimate at the $L^2(\mathbb{R}^d)$-level, under the assumption (3.3). This is formulated as follows:

Lemma 3.3. *Let $\boldsymbol{\varepsilon}(t)$ be defined as in* (3.2)*, and assume that* (3.3) *holds. Then, there exists a constant $C > 0$, independent of v_*, such that for all $t \in [t_0, T^n]$, we have*

$$\|\boldsymbol{\varepsilon}(t)\|^2_{\mathcal{L}^2} \le \frac{C}{v_*}e^{-2\sqrt{\omega_*}v_* t}.$$

Proof. A straightforward computation reveals that $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(t)=(\varepsilon_1(t),\varepsilon_0(t),\varepsilon_{-1}(t))^T$ satisfies the following differential equations:

$$\begin{cases} i\partial_t\varepsilon_1+L_1(\boldsymbol{\varepsilon})+N_1(\boldsymbol{\varepsilon})+F_1(\boldsymbol{R})=0,\\ i\partial_t\varepsilon_0+L_0(\boldsymbol{\varepsilon})+N_0(\boldsymbol{\varepsilon})+F_0(\boldsymbol{R})=0,\\ i\partial_t\varepsilon_{-1}+L_{-1}(\boldsymbol{\varepsilon})+N_{-1}(\boldsymbol{\varepsilon})+F_{-1}(\boldsymbol{R})=0,\end{cases}$$

where L_j, N_j and F_j ($j=1,0,-1$) denote the linear part, nonlinear part and source term in $\boldsymbol{\varepsilon}(t)$, respectively, and $\boldsymbol{R}=\boldsymbol{R}(t)=(R_1(t),R_0(t),R_{-1}(t))^T$. Precisely,

$$\begin{aligned}
L_1(\boldsymbol{\varepsilon})&=\frac{1}{2}\Delta\varepsilon_1-\big[(c_0+c_1)(2|R_1|^2+|R_0|^2)+(c_0-c_1)|R_{-1}|^2+(q-p)\big]\varepsilon_1\\
&\quad-(c_0+c_1)R_1^2\bar\varepsilon_1-\big[(c_0+c_1)R_1\bar R_0+2c_1R_0\bar R_{-1}\big]\varepsilon_0-(c_0+c_1)R_1R_0\bar\varepsilon_0\\
&\quad-(c_0-c_1)R_1\bar R_{-1}\varepsilon_{-1}-\big[(c_0-c_1)R_1R_{-1}+c_1R_0^2\big]\bar\varepsilon_{-1},\\
L_0(\boldsymbol{\varepsilon})&=\frac{1}{2}\Delta\varepsilon_0-\big[(c_0+c_1)(|R_1|^2+|R_{-1}|^2)+2c_0|R_0|^2\big]\varepsilon_0-(2c_1R_1R_{-1}+c_0R_0^2)\bar\varepsilon_0\\
&\quad-\big[(c_0+c_1)\bar R_1R_0+2c_1\bar R_0R_{-1}\big]\varepsilon_1-(c_0+c_1)R_1R_0\bar\varepsilon_1\\
&\quad-\big[2c_1R_1\bar R_0+(c_0+c_1)R_0\bar R_{-1}\big]\varepsilon_{-1}-(c_0+c_1)R_0R_{-1}\bar\varepsilon_{-1},\\
L_{-1}(\boldsymbol{\varepsilon})&=\frac{1}{2}\Delta\varepsilon_{-1}-\big[(c_0+c_1)(2|R_{-1}|^2+|R_0|^2)+(c_0-c_1)|R_1|^2+(q+p)\big]\varepsilon_{-1}\\
&\quad-(c_0+c_1)R_{-1}^2\bar\varepsilon_{-1}-(c_0-c_1)\bar R_1R_{-1}\varepsilon_1-\big[(c_0-c_1)R_1R_{-1}+c_1R_0^2\big]\bar\varepsilon_1\\
&\quad-\big[(c_0+c_1)\bar R_0R_{-1}+2c_1\bar R_1R_0\big]\varepsilon_0-(c_0+c_1)R_0R_{-1}\bar\varepsilon_0,\\
N_1(\boldsymbol{\varepsilon})&=-(c_0+c_1)\big(R_0\varepsilon_1\bar\varepsilon_0+\bar R_0\varepsilon_1\varepsilon_0+\bar R_1\varepsilon_1^2+2R_1|\varepsilon_1|^2+R_1|\varepsilon_0|^2+|\varepsilon_1|^2\varepsilon_1\\
&\quad+\varepsilon_1|\varepsilon_0|^2\big)-(c_0-c_1)\big(R_{-1}\varepsilon_1\bar\varepsilon_{-1}+\bar R_{-1}\varepsilon_1\varepsilon_{-1}+R_1|\varepsilon_{-1}|^2+\varepsilon_1|\varepsilon_{-1}|^2\big)\\
&\quad-c_1\big(2R_0\varepsilon_0\bar\varepsilon_{-1}+\bar R_{-1}\varepsilon_0^2+\varepsilon_0^2\bar\varepsilon_{-1}\big),\\
N_0(\boldsymbol{\varepsilon})&=-c_0\big(\bar R_0\varepsilon_0^2+2R_0|\varepsilon_0|^2+|\varepsilon_0|^2\varepsilon_0\big)-(c_0+c_1)\big(R_1\bar\varepsilon_1\varepsilon_0+R_{-1}\varepsilon_0\bar\varepsilon_{-1}\\
&\quad+\bar R_1\varepsilon_1\varepsilon_0+\bar R_{-1}\varepsilon_0\varepsilon_{-1}+R_0|\varepsilon_1|^2+R_0|\varepsilon_{-1}|^2+|\varepsilon_1|^2\varepsilon_0+\varepsilon_0|\varepsilon_{-1}|^2\big)\\
&\quad-2c_1\big(\bar R_0\varepsilon_1\varepsilon_{-1}+R_1\bar\varepsilon_0\varepsilon_{-1}+R_{-1}\varepsilon_1\bar\varepsilon_0+\varepsilon_1\bar\varepsilon_0\varepsilon_{-1}\big),\\
N_{-1}(\boldsymbol{\varepsilon})&=-(c_0+c_1)\big(R_0\bar\varepsilon_0\varepsilon_{-1}+\bar R_0\varepsilon_0\varepsilon_{-1}+\bar R_{-1}\varepsilon_{-1}^2+2R_{-1}|\varepsilon_{-1}|^2+R_{-1}|\varepsilon_0|^2\\
&\quad+|\varepsilon_{-1}|^2\varepsilon_{-1}+|\varepsilon_0|^2\varepsilon_{-1}\big)-(c_0-c_1)\big(R_1\bar\varepsilon_1\varepsilon_{-1}+\bar R_1\varepsilon_1\varepsilon_{-1}+R_{-1}|\varepsilon_1|^2\\
&\quad+|\varepsilon_1|^2\varepsilon_{-1}\big)-c_1\big(2R_0\varepsilon_0\bar\varepsilon_1+\bar R_1\varepsilon_0^2+\bar\varepsilon_1\varepsilon_0^2\big),
\end{aligned}$$

and

$$\begin{aligned}
F_1(\boldsymbol{R})&=-\big[(c_0+c_1)|R_0|^2+(c_0-c_1)|R_{-1}|^2\big]R_1-c_1R_0^2\bar R_{-1},\\
F_0(\boldsymbol{R})&=-(c_0+c_1)\big(|R_1|^2+|R_{-1}|^2\big)R_0-2c_1R_1\bar R_0R_{-1},\\
F_{-1}(\boldsymbol{R})&=-\big[(c_0-c_1)|R_1|^2+(c_0+c_1)|R_0|^2\big]R_{-1}-c_1\bar R_1R_0^2.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}N(\varepsilon_1(t))&=\frac{1}{2}\frac{\partial}{\partial t}\big(\|\varepsilon_1(t)\|_{L^2}^2\big)\\
&=\Re\int_{\mathbb{R}^d}(\partial_t\varepsilon_1)\bar\varepsilon_1dx
\end{aligned}$$

$$= \Im \int_{\mathbb{R}^d} (i\partial_t \varepsilon_1)\bar{\varepsilon}_1 dx$$
$$= -\Im \int_{\mathbb{R}^d} \left[L_1(\boldsymbol{\varepsilon})\bar{\varepsilon}_1 + N_1(\boldsymbol{\varepsilon})\bar{\varepsilon}_1 + F_1(\boldsymbol{R})\bar{\varepsilon}_1\right] dx. \tag{3.6}$$

Using (1.13) and the bootstrap assumption (3.3), we immediately obtain the following estimate

$$\begin{aligned}
&\left|\Im \int_{\mathbb{R}^d} L_1(\boldsymbol{\varepsilon})\bar{\varepsilon}_1 dx\right| \\
=&\left|\Im \int_{\mathbb{R}^d} \Big[(c_0+c_1)R_1^2\bar{\varepsilon}_1^2 + (c_0-c_1)R_1\bar{R}_{-1}\bar{\varepsilon}_1\varepsilon_{-1} + \big[(c_0-c_1)R_1R_{-1} + c_1R_0^2\big]\bar{\varepsilon}_1\bar{\varepsilon}_{-1}\right. \\
&\left. + \big[(c_0+c_1)R_1\bar{R}_0 + 2c_1R_0\bar{R}_{-1}\big]\bar{\varepsilon}_1\varepsilon_0 + (c_0+c_1)R_1R_0\bar{\varepsilon}_1\bar{\varepsilon}_0\Big] dx\right| \\
\le& C_1\big(\|R_1\|_{L^\infty}^2 + \|R_0\|_{L^\infty}^2 + \|R_{-1}\|_{L^\infty}^2\big)\big(\|\varepsilon_1\|_{H^1}^2 + \|\varepsilon_0\|_{H^1}^2 + \|\varepsilon_{-1}\|_{H^1}^2\big) \\
\le& C_1' e^{-2\sqrt{\omega_*}v_* t},
\end{aligned} \tag{3.7}$$

where C_1' is independent of v_*. Using Sobolev embedding and (3.3), we have

$$\begin{aligned}
&\left|\Im \int_{\mathbb{R}^d} N_1(\boldsymbol{\varepsilon})\bar{\varepsilon}_1 dx\right| \\
=&\left|\Im \int_{\mathbb{R}^d} \Big[(c_0+c_1)\big(R_0|\varepsilon_1|^2\bar{\varepsilon}_0 + \bar{R}_0|\varepsilon_1|^2\varepsilon_0 + \bar{R}_1\varepsilon_1^2\bar{\varepsilon}_1 + 2R_1|\varepsilon_1|^2\bar{\varepsilon}_1 + R_1\bar{\varepsilon}_1|\varepsilon_0|^2\big)\right. \\
&+(c_0-c_1)\big(R_1\bar{\varepsilon}_1|\varepsilon_{-1}|^2 + R_{-1}|\varepsilon_1|^2\bar{\varepsilon}_{-1} + \bar{R}_{-1}|\varepsilon_1|^2\varepsilon_{-1}\big) \\
&\left. +c_1\big(2R_0\bar{\varepsilon}_1\varepsilon_0\bar{\varepsilon}_{-1} + \bar{R}_{-1}\bar{\varepsilon}_1\varepsilon_0^2 + \bar{\varepsilon}_1\varepsilon_0^2\bar{\varepsilon}_{-1}\big)\Big] dx\right| \\
\le& C_2\big(\|R_1\|_{L^\infty} + \|R_0\|_{L^\infty} + \|R_{-1}\|_{L^\infty}\big)\big(\|\varepsilon_1\|_{H^1}^2\|\varepsilon_{-1}\|_{H^1} + \|\varepsilon_1\|_{H^1}\|\varepsilon_{-1}\|_{H^1}^2 \\
&+ \|\varepsilon_1\|_{H^1}^2\|\varepsilon_0\|_{H^1} + \|\varepsilon_1\|_{H^1}\|\varepsilon_0\|_{H^1}^2 + \|\varepsilon_1\|_{H^1}^3 + \|\varepsilon_1\|_{H^1}\|\varepsilon_0\|_{H^1}\|\varepsilon_{-1}\|_{H^1} \\
&+ \|\varepsilon_1\|_{H^1}\|\varepsilon_0\|_{H^1}^2\|\varepsilon_{-1}\|_{H^1}\big) \\
\le& C_2' e^{-3\sqrt{\omega_*}v_* t}.
\end{aligned} \tag{3.8}$$

Employing Lemma 3.2 and (3.3), we can prove that

$$\begin{aligned}
&\left|\Im \int_{\mathbb{R}^d} F_1(\boldsymbol{R})\bar{\varepsilon}_1 dx\right| \\
=&\left|\Im \int_{\mathbb{R}^d} \big[(c_0+c_1)R_1|R_0|^2\bar{\varepsilon}_1 + (c_0-c_1)R_1|R_{-1}|^2\bar{\varepsilon}_1 + c_1R_0^2\bar{R}_{-1}\bar{\varepsilon}_1\big] dx\right| \\
\le& C_3\big(\|R_0\|_{L^\infty} + \|R_{-1}\|_{L^\infty}\big)\big(\||R_1||R_0|\|_{L^2} + \||R_1||R_{-1}|\|_{L^2} + \||R_0||R_{-1}|\|_{L^2}\big)\|\varepsilon_1\|_{L^2} \\
\le& C_3' e^{-\frac{5}{2}\sqrt{\omega_*}v_* t}.
\end{aligned} \tag{3.9}$$

By substituting (3.7), (3.8) and (3.9) into (3.6), we arrive at the inequality

$$\left|\frac{\partial}{\partial t}N(\varepsilon_1(t))\right| \le C_4 e^{-2\sqrt{\omega_*}v_* t}.$$

Since $\varepsilon_1(T^n) = 0$, it follows that

$$N(\varepsilon_1(t)) \le \int_t^{T^n} \left|\frac{\partial}{\partial t}N(\varepsilon_1(s))\right| ds \le \frac{C_5}{v_*} e^{-2\sqrt{\omega_*}v_* t},$$

which concludes the estimate for $\varepsilon_1(t)$. Through a similar argument, we can derive the estimates for $\varepsilon_0(t)$ and $\varepsilon_{-1}(t)$. Hence, we complete the proof Lemma 3.3. □

We can now give an estimate for the functional $\mathbf{S}$.

Lemma 3.4. *Let $d=1,2,3$. If (3.3) holds, then there exist a constant $T_0>0$, depending only on v_1, v_0, and v_{-1}, as well as a constant $C>0$, which is independent of n and v_*, such that for $t_0>T_0$ and all $t\in[t_0,T^n]$, the following inequality holds*

$$\big|\mathbf{S}\left(\mathbf{u}_n(t)\right)-\mathbf{S}\left(\mathbf{u}_n(T^n)\right)\big| \leq \frac{C}{v_*}e^{-2\sqrt{\omega_*}v_*t}. \tag{3.10}$$

Proof. The proof is divided into three parts below.

1) Estimates for the particle number part. We have, for $j\in\{-1,0,1\}$,

$$\begin{aligned}
&\Big|\sum_j\big[(\omega_j+\frac{|v_j|^2}{2})\left[N(u_{jn}(t))-N(u_{jn}(T^n))\right]\big]\Big|\\
=&\frac{1}{3}\Big|\sum_j\big[\big(3(\omega_j+\frac{|v_j|^2}{2})-\sum_j(\omega_j+\frac{|v_j|^2}{2})\big)\left[N(u_{jn}(t))-N(u_{jn}(T^n))\right]\big]\\
&+\big(\sum_j(\omega_j+\frac{|v_j|^2}{2})\big)\big[\sum_j N(u_{jn}(t))-\sum_j N(u_{jn}(T^n))\big]\Big|\\
\leq&\max_j\big\{\big|(\omega_j-\frac{\omega_1+\omega_0+\omega_{-1}}{3})+(\frac{|v_j|^2}{2}-\frac{|v_1|^2+|v_0|^2+|v_{-1}|^2}{6})\big|\big\}\\
&\times\big[\sum_j|N(u_{jn}(t))-N(u_{jn}(T^n))|\big]\\
=&v^1_{\max}\big[\sum_j|N(u_{jn}(t))-N(u_{jn}(T^n))|\big],
\end{aligned}\tag{3.11}$$

where we have applied (1.2), i.e., the conservation of the total particle number.

Recall that $\mathbf{u}_n(t)$ satisfies system (1.1) for $t\in[T_0,T^n]$. Differentiate the quantities of the particle number in time t, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}N(u_{jn}(t))=c_1\Im\int_{\mathbb{R}^d}\bar{u}_{1n}u_{0n}^2\bar{u}_{-1,n}dx,\ j=1,-1,\\
&\frac{\partial}{\partial t}N(u_{0n}(t))=2c_1\Im\int_{\mathbb{R}^d}u_{1n}\bar{u}_{0n}^2u_{-1,n}dx.
\end{aligned}$$

Now we estimate $\frac{\partial}{\partial t}N(u_{1n}(t))$. Notice that $u_{jn}(t)=R_j(t)+\varepsilon_j(t)$ $(j=1,0,-1)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d}\bar{u}_{1n}u_{0n}^2\bar{u}_{-1,n}dx=\int_{\mathbb{R}^d}&\big(\bar{R}_1R_0^2\bar{R}_{-1}+2\bar{R}_1R_0\bar{R}_{-1}\varepsilon_0+\bar{R}_1R_0^2\bar{\varepsilon}_{-1}+R_0^2\bar{R}_{-1}\bar{\varepsilon}_1\\
&+2\bar{R}_1R_0\varepsilon_0\bar{\varepsilon}_{-1}+2R_0\bar{R}_{-1}\bar{\varepsilon}_1\varepsilon_0+R_0^2\bar{\varepsilon}_1\bar{\varepsilon}_{-1}+\bar{R}_1\bar{R}_{-1}\varepsilon_0^2\\
&+\bar{R}_1\varepsilon_0^2\bar{\varepsilon}_{-1}+\bar{R}_{-1}\bar{\varepsilon}_1\varepsilon_0^2+2R_0\bar{\varepsilon}_1\varepsilon_0\bar{\varepsilon}_{-1}+\bar{\varepsilon}_1\varepsilon_0^2\bar{\varepsilon}_{-1}\big)dx.
\end{aligned}\tag{3.12}$$

Applying (3.4), we deduce that

$$\left|\int_{\mathbb{R}^d}\bar{R}_1R_0^2\bar{R}_{-1}dx\right|\leq C_1\big\||R_1||R_0|\big\|_{L^2}\big\||R_0||R_{-1}|\big\|_{L^2}\leq C_1'e^{-3\sqrt{\omega_*}v_*t}.$$

Again, by (3.4) and the assumption (3.3), we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\big(R_0^2\bar{R}_{-1}\bar{\varepsilon}_1+2\bar{R}_1R_0\bar{R}_{-1}\varepsilon_0+\bar{R}_1R_0^2\bar{\varepsilon}_{-1}\big)dx\Big|\\ \le& C_2\|R_0\|_{L^\infty}\big(\||R_0||R_{-1}|\|_{L^2}\|\varepsilon_1\|_{L^2}+\||R_1||R_{-1}|\|_{L^2}\|\varepsilon_0\|_{L^2}+\||R_1||R_0|\|_{L^2}\|\varepsilon_{-1}\|_{L^2}\big)\\ \le& C_2'e^{-\frac{5}{2}\sqrt{\omega_*}v_*t}.\end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\big(2\bar{R}_1R_0\varepsilon_0\bar{\varepsilon}_{-1}+2R_0\bar{R}_{-1}\bar{\varepsilon}_1\varepsilon_0+R_0^2\bar{\varepsilon}_1\bar{\varepsilon}_{-1}+\bar{R}_1\bar{R}_{-1}\varepsilon_0^2\big)dx\Big|\\ \le& C_3\big(\|R_1\|_{L^\infty}^2+\|R_0\|_{L^\infty}^2+\|R_{-1}\|_{L^\infty}^2\big)\big(\|\varepsilon_1\|_{L^2}^2+\|\varepsilon_0\|_{L^2}^2+\|\varepsilon_{-1}\|_{L^2}^2\big)\\ \le& \frac{C_3'}{v_*}e^{-2\sqrt{\omega_*}v_*t}.\end{aligned}$$

Using the bootstrap assumption (3.3) again, we obtain

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}\big(\bar{R}_1\varepsilon_0^2\bar{\varepsilon}_{-1}+2R_0\bar{\varepsilon}_1\varepsilon_0\bar{\varepsilon}_{-1}+\bar{R}_{-1}\bar{\varepsilon}_1\varepsilon_0^2\big)dx\Big|\\ \le& C_4\big(\|R_1\|_{L^\infty}+\|R_0\|_{L^\infty}+\|R_{-1}\|_{L^\infty}\big)\big(\|\varepsilon_0\|_{H^1}^2\|\varepsilon_{-1}\|_{H^1}\\ &+\|\varepsilon_1\|_{H^1}\|\varepsilon_0\|_{H^1}\|\varepsilon_{-1}\|_{H^1}+\|\varepsilon_1\|_{H^1}\|\varepsilon_0\|_{H^1}^2\big)\\ \le& C_4'\,e^{-3\sqrt{\omega_*}v_*t},\end{aligned}$$

and

$$\Big|\int_{\mathbb{R}^d}\bar{\varepsilon}_1\varepsilon_0^2\bar{\varepsilon}_{-1}dx\Big|\le C_5\,\|\varepsilon_1\|_{H^1}\|\varepsilon_0\|_{H^1}^2\|\varepsilon_{-1}\|_{H^1}\le C_5'\,e^{-4\sqrt{\omega_*}v_*t}.$$

Combining the above estimates and taking T_0 large enough so that $v_*e^{-\frac{1}{2}\sqrt{\omega_*}v_*T_0}<1$, then, for $t\in[t_0,T^n]$:

$$\big|N(u_{1n}(t))-N(u_{1n}(T^n))\big|\le\int_t^{T^n}\Big|\frac{\partial}{\partial t}N(u_{1n}(s))\Big|ds\le\frac{C_6}{v_*^2}e^{-2\sqrt{\omega_*}v_*t},\tag{3.13}$$

where the constant C_6 is independent of v_1, v_0 and v_{-1}.

Similarly, for $j=0,-1$,

$$\Big|N(u_{jn}(t))-N(u_{jn}(T^n))\Big|\le\frac{C_7}{v_*^2}e^{-2\sqrt{\omega_*}v_*t},\tag{3.14}$$

where the constant C_7 is independent of v_1, v_0 and v_{-1}.

From (3.11), (3.13), (3.14) and the assumption (1.15), we have, for $j\in\{-1,0,1\}$,

$$\begin{aligned}&\Big|\sum_j\big[(\omega_j+\frac{|v_j|^2}{2})\big[N(u_{jn}(t))-N(u_{jn}(T^n))\big]\big]\Big|\\ \le& Lv_*\big[\sum_j|N(u_{jn}(t))-N(u_{jn}(T^n))|\big]\\ \le&\frac{C_8}{v_*}e^{-2\sqrt{\omega_*}v_*t}.\end{aligned}\tag{3.15}$$

2) Estimates for the momentum part. For $j \in \{-1,0,1\}$, we have

$$\begin{aligned}
&\Big|\sum_j \big\{v_j\cdot\big[P(u_{jn}(t))-P(u_{jn}(T^n))\big]\big\}\Big|\\
=&\Big|\sum_j \big\{\big(v_j-\frac{v_1+v_0+v_{-1}}{3}\big)\cdot\big[P(u_{jn}(t))-P(u_{jn}(T^n))\big]\big\}\\
&+\frac{v_1+v_0+v_{-1}}{3}\cdot\big[\sum_j\big(P(u_{jn}(t))-P(u_{jn}(T^n))\big)\big]\Big|\\
\le&\max_j\big\{\big|v_j-\frac{v_1+v_0+v_{-1}}{3}\big|\big\}\big[\sum_j|P(u_{jn}(t))-P(u_{jn}(T^n))|\big]\\
=&v_{\max}^2\big[\sum_j|P(u_{jn}(t))-P(u_{jn}(T^n))|\big],
\end{aligned}\tag{3.16}$$

where we have used (1.5), i.e., the Momentum Conservation: $\mathbf{P}(\mathbf{u}_n(t))=\mathbf{P}(\mathbf{u}^0)=\mathbf{P}(\mathbf{u}_n(T^n))$.

Next, we estimate $\frac{\partial}{\partial t}P(u_{jn}(t))$ for $j=-1,0,1$, respectively. We have

$$\begin{aligned}
\frac{\partial}{\partial t}P(u_{1n}(t))=&-\frac{c_0-c_1}{2}\int_{\mathbb{R}^d}|u_{-1,n}|^2\nabla(|u_{1n}|^2)dx\\
&-\frac{c_0+c_1}{2}\int_{\mathbb{R}^d}|u_{0n}|^2\nabla(|u_{1n}|^2)dx-c_1\Re\int_{\mathbb{R}^d}\bar u_{-1,n}u_{0n}^2\nabla\bar u_{1n}dx,\\
\frac{\partial}{\partial t}P(u_{0n}(t))=&-\frac{c_0+c_1}{2}\int_{\mathbb{R}^d}\big(|u_{1n}|^2+|u_{-1,n}|^2\big)\nabla(|u_{0n}|^2)dx\\
&-2c_1\Re\int_{\mathbb{R}^d}u_{1n}\bar u_{0n}u_{-1,n}\nabla\bar u_{0n}dx,\\
\frac{\partial}{\partial t}P(u_{-1,n}(t))=&-\frac{c_0-c_1}{2}\int_{\mathbb{R}^d}|u_{1n}|^2\nabla(|u_{-1,n}|^2)dx\\
&-\frac{c_0+c_1}{2}\int_{\mathbb{R}^d}|u_{0n}|^2\nabla(|u_{-1,n}|^2)dx-c_1\Re\int_{\mathbb{R}^d}\bar u_{1n}u_{0n}^2\nabla\bar u_{-1,n}dx.
\end{aligned}$$

Now, let us estimate $\frac{\partial}{\partial t}P(u_{0n}(t))$ in more detail. Recalling that $u_{jn}=R_j+\varepsilon_j$ $(j=1,0,-1)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}u_{1n}\bar u_{0n}u_{-1,n}\nabla\bar u_{0n}dx\\
=&\int_{\mathbb{R}^d}\big(R_1\bar R_0R_{-1}\nabla\bar R_0+\bar R_0R_{-1}\nabla\bar R_0\varepsilon_1+R_1\bar R_0\nabla\bar R_0\varepsilon_{-1}+R_1R_{-1}\nabla\bar R_0\bar\varepsilon_0\\
&+R_1\bar R_0R_{-1}\nabla\bar\varepsilon_0+\bar R_0\nabla\bar R_0\varepsilon_1\varepsilon_{-1}+R_{-1}\nabla\bar R_0\varepsilon_1\bar\varepsilon_0+R_1\nabla\bar R_0\varepsilon_{-1}\bar\varepsilon_0\\
&+\bar R_0R_{-1}\varepsilon_1\nabla\bar\varepsilon_0+R_1\bar R_0\varepsilon_{-1}\nabla\bar\varepsilon_0+R_1R_{-1}\bar\varepsilon_0\nabla\bar\varepsilon_0+\nabla\bar R_0\varepsilon_1\bar\varepsilon_0\varepsilon_{-1}\\
&+\bar R_0\varepsilon_1\varepsilon_{-1}\nabla\bar\varepsilon_0+R_{-1}\varepsilon_1\bar\varepsilon_0\nabla\bar\varepsilon_0+R_1\bar\varepsilon_0\varepsilon_{-1}\nabla\bar\varepsilon_0+\varepsilon_1\bar\varepsilon_0\varepsilon_{-1}\nabla\bar\varepsilon_0\big)dx.
\end{aligned}$$

It yields from Hölder inequality and the estimates (3.4) and (3.5) that

$$\Big|\int_{\mathbb{R}^d}R_1\bar R_0R_{-1}\nabla\bar R_0dx\Big|\le C_9\big\||R_1||R_0|\big\|_{L^2}\big\||R_{-1}||\nabla R_0|\big\|_{L^2}\le C_9'(1+|v_0|)e^{-3\sqrt{\omega_*}v_*t}.$$

Also, by (3.3), we conclude the following estimate:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\big(\bar{R}_0R_{-1}\nabla\bar{R}_0\varepsilon_1+R_1\bar{R}_0\nabla\bar{R}_0\varepsilon_{-1}+R_1R_{-1}\nabla\bar{R}_0\bar{\varepsilon}_0\big)dx\right|\\ \leq &C_{10}(1+|v_0|)e^{-\frac{5}{2}\sqrt{\omega_*}v_*t}.\end{aligned}$$

Similarly, we have the following bounds:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}R_1\bar{R}_0R_{-1}\nabla\bar{\varepsilon}_0dx\right|\leq C_{11}e^{-\frac{5}{2}\sqrt{\omega_*}v_*t},\\ &\left|\int_{\mathbb{R}^d}\big(R_{-1}\nabla\bar{R}_0\varepsilon_1\bar{\varepsilon}_0+R_1\nabla\bar{R}_0\varepsilon_{-1}\bar{\varepsilon}_0\big)dx\right|\leq C_{12}(1+|v_0|)e^{-\frac{7}{2}\sqrt{\omega_*}v_*t},\\ &\left|\int_{\mathbb{R}^d}\big(\bar{R}_0R_{-1}\varepsilon_1\nabla\bar{\varepsilon}_0+R_1\bar{R}_0\varepsilon_{-1}\nabla\bar{\varepsilon}_0+R_1R_{-1}\bar{\varepsilon}_0\nabla\bar{\varepsilon}_0\big)dx\right|\leq C_{13}e^{-\frac{11}{4}\sqrt{\omega_*}v_*t}.\end{aligned}$$

Using integration by parts and Lemma 3.3, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\bar{R}_0\nabla\bar{R}_0\varepsilon_1\varepsilon_{-1}dx\right|&=\left|\frac{1}{2}\int_{\mathbb{R}^d}\big(\bar{R}_0^2\varepsilon_{-1}\nabla\varepsilon_1+\bar{R}_0^2\varepsilon_1\nabla\varepsilon_{-1}\big)dx\right|\\ &\leq C_{14}\|R_0\|_{L^\infty}^2\big(\|\varepsilon_{-1}\|_{L^2}\|\nabla\varepsilon_1\|_{L^2}+\|\varepsilon_1\|_{L^2}\|\nabla\varepsilon_{-1}\|_{L^2}\big)\\ &\leq\frac{C_{14}'}{v_*}e^{-2\sqrt{\omega_*}v_*t}.\end{aligned}$$

Similarly, using integration by parts and (3.3), we have:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\nabla\bar{R}_0\varepsilon_1\bar{\varepsilon}_0\varepsilon_{-1}dx\right|\\ =&\left|\int_{\mathbb{R}^d}\big(\bar{R}_0\bar{\varepsilon}_0\varepsilon_{-1}\nabla\varepsilon_1+\bar{R}_0\varepsilon_1\varepsilon_{-1}\nabla\bar{\varepsilon}_0+\bar{R}_0\varepsilon_1\bar{\varepsilon}_0\nabla\varepsilon_{-1}\big)dx\right|\\ \leq &C_{15}\|R_0\|_{L^\infty}\big(\|\varepsilon_0\|_{L^4}\|\varepsilon_{-1}\|_{L^4}\|\nabla\varepsilon_1\|_{L^2}+\|\varepsilon_1\|_{L^4}\|\varepsilon_{-1}\|_{L^4}\|\nabla\varepsilon_0\|_{L^2}\\ &+\|\varepsilon_1\|_{L^4}\|\varepsilon_0\|_{L^4}\|\nabla\varepsilon_{-1}\|_{L^2}\big)\\ \leq &C_{15}'e^{-3\sqrt{\omega_*}v_*t}.\end{aligned}$$

Similarly, the following inequalities hold:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\big(R_1\bar{\varepsilon}_0\varepsilon_{-1}\nabla\bar{\varepsilon}_0+\bar{R}_0\varepsilon_1\varepsilon_{-1}\nabla\bar{\varepsilon}_0+R_{-1}\varepsilon_1\bar{\varepsilon}_0\nabla\bar{\varepsilon}_0\big)dx\right|\leq C_{16}e^{-3\sqrt{\omega_*}v_*t},\\ &\left|\int_{\mathbb{R}^d}\varepsilon_1\bar{\varepsilon}_0\varepsilon_{-1}\nabla\bar{\varepsilon}_0dx\right|\leq C_{17}e^{-4\sqrt{\omega_*}v_*t}.\end{aligned}$$

By choosing T_0 sufficiently large so that $v_*(1+|v_0|)e^{-\frac{1}{2}\sqrt{\omega_*}v_*T_0}<1$, and by combining the previously obtained estimates, we derive the following inequality for $t\in[t_0,T^n]$:

$$\begin{aligned}\left|\int_{\mathbb{R}^d}u_{1n}\bar{u}_{0n}u_{-1,n}\nabla\bar{u}_{0n}dx\right|&\leq\frac{C_{18}}{v_*}e^{-2\sqrt{\omega_*}v_*t}+C_{19}(1+|v_0|)e^{-\frac{5}{2}\sqrt{\omega_*}v_*t}\\ &\leq\frac{C_{18}'}{v_*}e^{-2\sqrt{\omega_*}v_*t}.\end{aligned}$$

Similarly, we have

$$\left|\int_{\mathbb{R}^d}\big(|u_{1n}|^2+|u_{-1,n}|^2\big)\nabla(|u_{0n}|^2)dx\right|\le\frac{C_{20}}{v_*}e^{-2\sqrt{\omega_*}v_*t}.$$

Thus, for $t\in[t_0,T^n]$, we obtain

$$\left|v_*\frac{\partial}{\partial t}P(u_{0n}(t))\right|\le\ C_{21}e^{-2\sqrt{\omega_*}v_*t},\tag{3.17}$$

where the constant C_{21} is independent of v_1, v_0 and v_{-1}.

Similarly, we find

$$\left|v_*\frac{\partial}{\partial t}P(u_{jn}(t))\right|\le\ C_{22}e^{-2\sqrt{\omega_*}v_*t},\quad j=1,-1.\tag{3.18}$$

Therefore, from (3.16), (3.17), (3.18) and the assumption (1.15), for $j\in\{-1,0,1\}$, we obtain

$$\begin{aligned}&\Big|\sum_j\big\{v_j\cdot\big[P(u_{jn}(t))-P(u_{jn}(T^n))\big]\big\}\Big|\\&\le C_{23}\,v_*\big[\sum_j|P(u_{jn}(t))-P(u_{jn}(T^n))|\big]\\&\le C_{23}\sum_j\int_t^{T^n}\left|v_*\frac{\partial}{\partial t}P(u_{jn}(s))\right|ds\\&\le\frac{C'_{23}}{v_*}e^{-2\sqrt{\omega_*}v_*t}.\end{aligned}\tag{3.19}$$

3) Estimates for the energy part. By conservation of the total energy (1.4), we obtain

$$\begin{aligned}&E_{-(c_0+c_1)}(u_{1n}(t))-E_{-(c_0+c_1)}(u_{1n}(T^n))+E_{-c_0}(u_{0n}(t))-E_{-c_0}(u_{0n}(T^n))\\&\quad+E_{-(c_0+c_1)}(u_{-1,n}(t))-E_{-(c_0+c_1)}(u_{-1,n}(T^n))\\&=\frac12\mathbf{E}(\mathbf{u}_n(t))-\frac12\mathbf{E}(\mathbf{u}_n(T^n))-(q-p)\big(N(u_{1n}(t))-N(u_{1n}(T^n))\big)\\&\quad-(q+p)\big(N(u_{-1,n}(t))-N(u_{-1,n}(T^n))\big)\\&\quad-\frac12(c_0+c_1)\int_{\mathbb{R}^d}\big(|u_{1n}(t)|^2|u_{0n}(t)|^2-|u_{1n}(T^n)|^2|u_{0n}(T^n)|^2\big)dx\\&\quad-\frac12(c_0-c_1)\int_{\mathbb{R}^d}\big(|u_{1n}(t)|^2|u_{-1,n}(t)|^2-|u_{1n}(T^n)|^2|u_{-1,n}(T^n)|^2\big)dx\\&\quad-\frac12(c_0+c_1)\int_{\mathbb{R}^d}\big(|u_{0n}(t)|^2|u_{-1,n}(t)|^2-|u_{0n}(T^n)|^2|u_{-1,n}(T^n)|^2\big)dx\\&\quad-c_1\Re\int_{\mathbb{R}^d}\big[\bar u_{1n}(t)u_{0n}^2(t)\bar u_{-1,n}(t)-\bar u_{1n}(T^n)u_{0n}^2(T^n)\bar u_{-1,n}(T^n)\big]dx\\&=-(q-p)\big(N(u_{1n}(t))-N(u_{1n}(T^n))\big)-(q+p)\big(N(u_{-1,n}(t))-N(u_{-1,n}(T^n))\big)\\&\quad-\frac12(c_0+c_1)\int_{\mathbb{R}^d}\big(|u_{1n}(t)|^2|u_{0n}(t)|^2-|u_{1n}(T^n)|^2|u_{0n}(T^n)|^2\big)dx\\&\quad-\frac12(c_0-c_1)\int_{\mathbb{R}^d}\big(|u_{1n}(t)|^2|u_{-1,n}(t)|^2-|u_{1n}(T^n)|^2|u_{-1,n}(T^n)|^2\big)dx\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}(c_0+c_1)\int_{\mathbb{R}^d}\big(|u_{0n}(t)|^2|u_{-1,n}(t)|^2-|u_{0n}(T^n)|^2|u_{-1,n}(T^n)|^2\big)dx\\
&-c_1\Re\int_{\mathbb{R}^d}\big[\bar{u}_{1n}(t)u_{0n}^2(t)\bar{u}_{-1,n}(t)-\bar{u}_{1n}(T^n)u_{0n}^2(T^n)\bar{u}_{-1,n}(T^n)\big]dx.
\end{aligned}
$$

By applying (2.3) and (3.4), we have

$$
\begin{aligned}
\int_{\mathbb{R}^d}|u_{1n}(T^n)||u_{0n}(T^n)|^2|u_{-1,n}(T^n)|dx=&\int_{\mathbb{R}^d}|R_1(T^n)||R_0(T^n)|^2|R_{-1}(T^n)|dx\\
\leq& C_{24}e^{-3\sqrt{\omega_*}v_*t}.
\end{aligned}
\tag{3.20}
$$

In a similar manner to (3.12), we find that

$$
\left|\int_{\mathbb{R}^d}\bar{u}_{1n}(t)u_{0n}^2(t)\bar{u}_{-1,n}(t)dx\right|\leq\frac{C_{25}}{v_*}e^{-2\sqrt{\omega_*}v_*t}.
\tag{3.21}
$$

Combining (3.20) and (3.21), and assuming T_0 is large enough such that $v_*e^{-\sqrt{\omega_*}v_*T_0}<1$, we deduce that

$$
\begin{aligned}
&\Re\int_{\mathbb{R}^d}\big[\bar{u}_{1n}(t)u_{0n}^2(t)\bar{u}_{-1,n}(t)-\bar{u}_{1n}(T^n)u_{0n}^2(T^n)\bar{u}_{-1,n}(T^n)\big]dx\\
\leq&\left|\int_{\mathbb{R}^d}\bar{u}_{1n}(t)u_{0n}^2(t)\bar{u}_{-1,n}(t)dx\right|+\int_{\mathbb{R}^d}|u_{1n}(T^n)||u_{0n}(T^n)|^2|u_{-1,n}(T^n)|dx\\
\leq&\frac{C_{25}}{v_*}e^{-2\sqrt{\omega_*}v_*t}+C_{24}e^{-3\sqrt{\omega_*}v_*t}\\
\leq&\frac{C_{25}'}{v_*}e^{-2\sqrt{\omega_*}v_*t}.
\end{aligned}
$$

Similarly, for $k,j=1,0,-1$ with $k\neq j$, we can assert that

$$
\int_{\mathbb{R}^d}\big(|u_{kn}(t)|^2|u_{jn}(t)|^2-|u_{kn}(T^n)|^2|u_{jn}(T^n)|^2\big)dx\leq\frac{C_{26}}{v_*}e^{-2\sqrt{\omega_*}v_*t}.
$$

From (3.13), we deduce the estimate

$$
\Big|N(u_{1n}(t))-N(u_{1n}(T^n))\Big|+\Big|N(u_{-1,n}(t))-N(u_{-1,n}(T^n))\Big|\leq\frac{C_{27}}{v_*^2}e^{-2\sqrt{\omega_*}v_*t}.
$$

Combining all these estimates, we arrive at the conclusion:

$$
\begin{aligned}
&\Big|E_{-(c_0+c_1)}(u_{1n}(t))-E_{-(c_0+c_1)}(u_{1n}(T^n))+E_{-c_0}(u_{0n}(t))-E_{-c_0}(u_{0n}(T^n))\\
&+E_{-(c_0+c_1)}(u_{-1,n}(t))-E_{-(c_0+c_1)}(u_{-1,n}(T^n))\Big|\\
\leq&\frac{C_{28}}{v_*}e^{-2\sqrt{\omega_*}v_*t}.
\end{aligned}
\tag{3.22}
$$

From (3.15), (3.19) and (3.22), we can deduce (3.10), and the proof is completed. □

Now, let us turn to prove Lemma 3.1.

Proof of Lemma 3.1. Let $\boldsymbol{\varepsilon}(t)$ be defined according to (3.2), and assume (3.3) holds. Let $t \in [t_0, T^n]$, with $t_0 > T_0$, where T_0 is given by Lemma 3.4. From Lemma 2.2, for $j \in \{1, 0, -1\}$, we have

$$K_* \|\boldsymbol{\varepsilon}\|_{\mathcal{H}^1}^2 \leq \mathbf{H}(t, \boldsymbol{\varepsilon}) + \sum_j \sum_{l=1}^{\nu_j} (\varepsilon_j(t), \eta_j^l(t))_{L^2}^2. \tag{3.23}$$

Notice that R_j is a critical point of S_j for $j = 1, 0, -1$, which implies that

$$\mathbf{S}(\mathbf{u}_n(t)) = \mathbf{S}(\mathbf{R}(t) + \boldsymbol{\varepsilon}(t)) = \mathbf{S}(\mathbf{R}(t)) + \mathbf{H}(t, \boldsymbol{\varepsilon}(t)) + O\left(\|\boldsymbol{\varepsilon}(t)\|_{\mathcal{H}^1}^3\right). \tag{3.24}$$

By the definition of $\mathbf{S}$ and the conservation of the particle number, energy and momentum, we can derive that

$$\mathbf{S}(\mathbf{u}_n(T_n)) = \mathbf{S}(\mathbf{R}(T_n)) = \mathbf{S}(\mathbf{R}(t)). \tag{3.25}$$

Due to (3.3), we have

$$O\left(\|\boldsymbol{\varepsilon}(t)\|_{\mathcal{H}^1}^3\right) \leq C_1 e^{-3\sqrt{\omega_*} v_* t}. \tag{3.26}$$

Combining (3.10), (3.24), (3.25) and (3.26), and assuming T_0 is large enough so that $v_* e^{-\sqrt{\omega_*} v_* T_0} < 1$, we conclude

$$|\mathbf{H}(t, \boldsymbol{\varepsilon}(t))| \leq C_1 e^{-3\sqrt{\omega_*} v_* t} + \frac{C_2}{v_*} e^{-2\sqrt{\omega_*} v_* t} \leq \frac{C_1'}{v_*} e^{-2\sqrt{\omega_*} v_* t}. \tag{3.27}$$

It yields from Lemma 3.3 that

$$\sum_j \sum_{l=1}^{\nu_j} (\varepsilon_j(t), \eta_j^l(t))_{L^2}^2 \leq C_3 \|\boldsymbol{\varepsilon}(t)\|_{\mathcal{L}^2}^2 \leq \frac{C_3'}{v_*} e^{-2\sqrt{\omega_*} v_* t}, \tag{3.28}$$

where the boundedness of η_j^l in $L^2(\mathbb{R}^d)$ (see Lemma 2.1) has been used. Combining equations (3.23), (3.27) and (3.28), it concludes that

$$\|\boldsymbol{\varepsilon}(t)\|_{\mathcal{H}^1}^2 \leq \frac{C_4}{v_*} e^{-2\sqrt{\omega_*} v_* t}.$$

Therefore, we can choose V_* such that if $v_* > V_*$, the following holds:

$$\|\boldsymbol{\varepsilon}(t)\|_{\mathcal{H}^1} \leq \frac{1}{2} e^{-\sqrt{\omega_*} v_* t},$$

which completes the proof of Lemma 3.1. □

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