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IDENTIFICATION OF THE RANDOM SOURCE IN THE STOCHASTIC CAPUTO-HADAMARD TIME-FRACTIONAL DIFFUSION EQUATION*

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Abstract This paper investigates the inverse problem of identifying the unknown source term in a stochastic Caputo-Hadamard time-fractional diffusion equation, where the source term consists of a deterministic function and a stochastic process. By utilizing the statistical properties of final value data (including its expectation and variance), we recover both the deterministic and random terms. To address the inherent ill-posedness of the problem, we apply the quasi-boundary regularization method and provide both a priori and a posteriori error estimates. Finally, numerical experiments in one and two dimensions are conducted to validate the feasibility and effectiveness of the proposed method.

Keywords Caputo-Hadamard derivative, stochastic time-fractional diffusion equation, quasi-boundary regularization method, inverse random source problem.

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1. Introduction

Fractional differential equations (FDEs) have gained attention in recent decades due to their ability to model non-locality and memory-dependent processes, which are prevalent in physical and engineering systems. Compared to classical integer-order differential equations, FDEs provide a more accurate and flexible framework for describing anomalous diffusion [5], viscoelasticity materials [9], and other complex dynamics in heterogeneous media [6]. Among various fractional derivatives, the Caputo-Hadamard fractional derivative is particularly well-suited for modeling phenomena with ultra-slow diffusion and logarithmic memory effects, such as fatigue fracture [1], logarithmic creep [8], and ultra-slow dynamics [2]. This derivative incorporates a logarithmic kernel, enabling more effective modeling of processes with non-locality and long-term memory. For example, Garra et al. [8] generalized the Lomnitz creep law using Hadamard-type calculus, while Kala [14] applied stochastic inverse analysis to fatigue cracks, highlighting the feasibility of inverse modeling in such applications.

In the study of Caputo-Hadamard fractional equations, direct problems – where the solution is determined based on a given source term and boundary or initial conditions – have been extensively studied. Significant efforts have been devoted to analyzing the existence, uniqueness, and regularity of solutions, as well as developing high-accuracy numerical methods. For instance, Yang et al. [39] examined the goodness of fit and regularity of Caputo-Hadamard time-fractional

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diffusion equations. Additionally, Li et al. [3] propose a discretization scheme on nonuniform grids by combining the finite difference method with the locally discontinuous Galerkin method.

In contrast, the inverse problem for Caputo-Hadamard type equations remains less explored. These problems, which involve determining the source term or initial value from observed data, are inherently ill-posed and often exhibit issues such as non-uniqueness and instability. For example, Yang et al. [31] studied the identification of a source term in a time-fractional diffusion equation with a Caputo-Hadamard derivative. Zhang et al. [41] further explored the inverse problem of simultaneously identifying the source term and initial condition in a spherically symmetric domain governed by a Caputo-Hadamard time-fractional diffusion equation. In addition to the methods mentioned above, other regularization techniques such as quasi-inverse [18], quasi-boundary [37,40], truncated [36], Landweber iteration [33,38,41], and Tikhonov [25,26,29,30,32], as well as their modified versions [13,17,19,34,35] have also been applied in related contexts. Despite the growing body of research on direct problems, inverse problems for fractional equations, especially those incorporating uncertainties, still pose challenges.

In complex physical and engineering phenomena, the influence of external noise on the system is often inevitable. Hence, it is necessary to add the uncertainty to the mathematical models. This has led to increased interest in time-fractional stochastic diffusion equations in recent years [4, 12, 21]. However, research on the inverse problem of stochastic fractional diffusion equations is still relatively limited. For example, Niu et al. [22] analyzed the regularity of the temporal fractional stochastic diffusion equation driven by Brownian motion, while discussing the inversion of the source term and its instability. Inspired by this study, the inverse problem of stochastic time-fractional diffusion equations with random sources has also been investigated using alternative data and numerical methods [7,20]. Gong et al. [11] shifted the focus to a inverse problem of random sources perturbed by spatially dependent stochastic noise, providing a well-posedness and regularity analysis of the direct problem solution and proving the uniqueness of the random source inversion. These research efforts provide important theoretical basis and methodological guidance for the further development of the inverse problems of the fractional stochastic diffusion equation.

The inclusion of random perturbations further complicates the mathematical analysis and numerical computations, as the solutions become stochastic processes rather than deterministic functions. Such stochastic fractional diffusion equations, which account for uncertainties in the source term or initial conditions, provide a more realistic modeling framework for systems influenced by environmental noise or inherent randomness. Despite their importance, research on the inverse problems of stochastic time-fractional diffusion equations remains limited.

In this paper, we consider the following stochastic diffusion equation associated with Caputo-Hadamard fractional derivative

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}u(x,t) + \mathcal{L}u(x,t) = f(x) + \dot{\mathcal{W}}_x, & x \in \Omega, t \in (a,T], \\ u(x,t) = 0, & x \in \partial\Omega, t \in (a,T), \\ u(x,a) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d (d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. $\dot{\mathcal{W}}_x$ represents a stochastic process, precisely defined in the next section. ${}_{CH}D^{\alpha}_{a,t}$ is Caputo-Hadamard fractional derivative with order $\alpha (0 < \alpha < 1)$ which is defined by

$${}_{CH}D^{\alpha}_{a,t}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log \frac{t}{\omega}\right)^{-\alpha} \delta u(x,\omega) \frac{d\omega}{\omega}, \ 0 < \alpha < 1,$$

where $\delta = t\frac{d}{dt}$ and $\Gamma(\cdot)$ is a Gamma function. $\mathcal{L}$ denotes the symmetric strongly elliptic operator given by [27]

$$\mathcal{L}u(x) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{d} A_{ij}\frac{\partial}{\partial x_j}u(x)\Big) + c(x)u(x),\ \forall x \in \Omega,$$

where the coefficients satisfy $A_{ij} = A_{ji} \in C^1(\overline{\Omega})$ $(1 \le i, j \le d)$, $c(x) \ge 0$, $c(x) \in C(\overline{\Omega})$. There exists a constant $\nu > 0$ such that

$$\nu \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} A_{ij}(x)\xi_i\xi_j,\ \forall x \in \overline{\Omega},\ \xi \in \mathbb{R}^d.$$

In (1.1), the initial data $\varphi(x)$ is provided. When $f(x)$ and $\dot{\mathcal{W}}_x$ are known, the problem of determining $u(x,t)$ is called the forward problem. In this study, we consider the case where $f(x)$ and $\dot{\mathcal{W}}_x$ are unknown, and their identification is pursued by utilizing the statistical properties of the final value data $u(x,T)$, specifically its expectation and variance. This constitutes an inverse problem, and to address its inherent ill-posedness, the quasi-boundary regularization method is employed to identify the source term in (1.1).

This paper is organized as follows: Section 2 introduces key definitions and lemmas essential to the study. Section 3 provides the solution to problem (1.1), analyzes its ill-posedness, and establishes conditional stability. Section 4 applies the quasi-boundary regularization method to solve the inverse problem (1.1) and presents a priori and a posteriori error estimates for both deterministic and random terms. Numerical examples including both one-dimensional and two-dimensional cases are presented in Section 5, followed by a brief conclusion in the final section.

2. Preliminary

Before presenting the main results of our work, we first provide some useful definitions, lemmas, and remarks.

Definition 2.1. [27] Suppose $\lambda_n, X_n(x)$ are the Dirichlet eigenvalues and eigenfunctions of the symmetric strongly elliptic operator $\mathcal{L}$ on the domain Ω

$$\begin{cases} \mathcal{L}X_n(x) = \lambda_n X_n(x), & \text{in } \Omega, \\ X_n(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$. $\{X_n(x)\}_{n=1}^{\infty}$ can be normalized as the orthonormal basis in space $L^2(\Omega)$.

Definition 2.2. [28] The stochastic process $\dot{\mathcal{W}}_x$ can be written formally as

$$\dot{\mathcal{W}}_x = \sum_{n=1}^{\infty} \sigma_n A_n X_n(x),\ A_n \sim N(0,1),$$

where $\{A_n\}_{n=1}^{\infty}$ is a random sequence, which conforms to standard normal distribution. $\{\sigma_n\}_{n=1}^{\infty}$ is a deterministic amplitude sequence.

Remark 2.1. [28] The process $\dot{\mathcal{W}}_x$ can also be interpreted as colored noise derived from white noise $\dot{\widetilde{\mathcal{W}}}_x$. Specifically,

$$\dot{\mathcal{W}}_x = R\dot{\widetilde{\mathcal{W}}}_x = \int_{\mathbb{R}^d} r(x,y)\dot{\widetilde{\mathcal{W}}}_y dy, \ x \in \mathbb{R}^d,$$

where R is a nonnegative, symmetric, trace-class linear operator, and $r(x,y)$ captures the spatial correlation of the colored noise. The kernel is given by

$$r(x,y) = \sum_{n=1}^{\infty} \sigma_n X_n(x) X_n(y), \ \sigma_n \geq 0,$$

and the colored noise can be expressed as

$$\dot{\mathcal{W}}_x = \sum_{n=1}^{\infty} r_n A_n X_n(x), \ A_n \sim N(0,1).$$

This confirms the formal expansion given in Definition 2.2 from an operator-theoretic perspective. In particular, if σ_n is constant and considering the eigenfunction expansion of the delta function, the stochastic process $\dot{\mathcal{W}}_x$ is white noise.

Definition 2.3. For any $p > 0$, we define the norm

$$\|\phi\|_{\mathcal{D}(L^p)} := \Big(\sum_{n=1}^{\infty} \lambda_n^p |\langle \phi, X_n(x)\rangle|^2\Big)^{\frac{1}{2}}, \ \|\psi\|_{\mathcal{D}(\ell^p)} := \Big(\sum_{n=1}^{\infty} \lambda_n^{2p} |\langle \psi, X_n(x)\rangle|^2\Big)^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$. $\mathcal{D}(L^p)$ and $\mathcal{D}(\ell^p)$ are Hilbert spaces defined by

$$\mathcal{D}(L^p) = \Big\{\phi \in L^2(\Omega) \Big| \Big(\sum_{n=1}^{\infty} \lambda_n^p |\langle \phi, X_n(x)\rangle|^2\Big)^{\frac{1}{2}} < \infty\Big\},$$

$$\mathcal{D}(\ell^p) = \Big\{\psi \in L^2(\Omega) \Big| \Big(\sum_{n=1}^{\infty} \lambda_n^{2p} |\langle \psi, X_n(x)\rangle|^2\Big)^{\frac{1}{2}} < \infty\Big\}.$$

Definition 2.4. [15,23] The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Definition 2.5. [3,16] For a given function $f(t)$ defined on $[a, +\infty)$ $(a > 0)$, the modified Laplace transform of $f(t)$ is defined by

$$\hat{f}(s) = \mathscr{L}_m\left[f(t), s\right] = \int_a^{\infty} e^{-\log\frac{s}{a}\log\frac{t}{a}} f(t)\frac{\mathrm{d}t}{t}, \ 0 \neq s \in \mathbb{C}.$$

For the Caputo-Hadamard fractional derivative, its modified Laplace transformation is defined by

$$\mathscr{L}_m\left[{}_{CH}D^{\alpha}_{a,t} f(t), s\right] = (\log\frac{s}{a})^{\alpha}\hat{f}(s) - \sum_{k=0}^{n-1}(\log\frac{s}{a})^{\alpha-k-1}\delta^k f(a), \ n-1 < \alpha < n,$$

when $n = 1$,

$$\mathscr{L}_m\left[{}_{CH}D^{\alpha}_{a,t} f(t), s\right] = (\log\frac{s}{a})^{\alpha}\hat{f}(s) - (\log\frac{s}{a})^{\alpha-1} f(a), \ 0 < \alpha < 1.$$

Lemma 2.1. [15] *For Mittag-Leffler function, we have the following properties*

$$E_{\alpha,\beta}(z) = z \cdot E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)},\ z \in \mathbb{C}.$$

Lemma 2.2. [24] *For* $0 < \alpha < 1$, $z > 0$, *we have* $0 < E_{\alpha,1}(-z) < 1$. *Moreover,* $E_{\alpha,1}(-z)$ *is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dz^n} E_{\alpha,1}(-z) \geq 0,\ n \geq 0.$$

Lemma 2.3. [15] *For* $\lambda > 0$, $t > 0$, $0 < \alpha < 1$, *it holds that*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha).$$

Lemma 2.4. [16] *If* $\lambda > 0$, *then the following equation holds*

$$\int_a^\infty e^{-\log\frac{s}{a}\log\frac{t}{a}} (\log\frac{t}{a})^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm\lambda(\log\frac{t}{a})^\alpha)\frac{dt}{t} = \frac{k!(\log\frac{s}{a})^{\alpha-\beta}}{((\log\frac{s}{a})^\alpha \mp \lambda)^{k+1}},$$

where $\mathrm{Re}(s) > |\lambda|^{\frac{1}{\alpha}}$ *and* $E^{(k)}_{\alpha,\beta}(z) := \frac{d^k}{dz^k} E_{\alpha,\beta}(z)$.

Remark 2.2. Lemma 2.4 means that the Laplace transformation of $(\log\frac{t}{a})^{\alpha k+\beta-1}$ $E^{(k)}_{\alpha,\beta}(\pm\lambda(\log\frac{t}{a})^\alpha)$ is $\frac{k!(\log\frac{s}{a})^{\alpha-\beta}}{((\log\frac{s}{a})^\alpha \mp \lambda)^{k+1}}$.

Lemma 2.5. *For any* $p > 0, \mu > 0$ *and* $0 < \lambda_1 \leq s$, *the following inequality holds*

$$F_1(s) = \frac{\mu s^{1-\frac{p}{2}}}{C_1 + \mu s} \leq \begin{cases} C_2 \mu^{\frac{p}{2}}, & 0 < p < 2, \\ C_3 \mu, & p \geq 2, \end{cases}$$

where $C_1 := 1 - E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)$, $C_2 := \frac{1}{2}p^{\frac{p}{2}} C_1^{-\frac{p}{2}}(2-p)^{\frac{2-p}{2}}$, $C_3 := \lambda_1^{1-\frac{p}{2}}/C_1$.

Proof. If $0 < p < 2$, by computing the derivative of $F_1(s)$, we obtain

$$F_1'(s) = \frac{(1-\frac{p}{2})\mu s^{-\frac{p}{2}}(C_1 + \mu s) - \mu s^{1-\frac{p}{2}}\mu}{(C_1+\mu s)^2}.$$

If s_1 satisfies the equation $F_1'(s_1) = 0$, then $s_1 = \frac{(2-p)C_1}{p\mu}$ can be easily derived. Therefore

$$F_1(s) \leq F_1(s_1) \leq \frac{1}{2}p^{\frac{p}{2}} C_1^{-\frac{p}{2}}(2-p)^{\frac{2-p}{2}}\mu^{\frac{p}{2}}.$$

If $p \geq 2$, we can derive

$$F_1(s) = \frac{\mu s^{1-\frac{p}{2}}}{C_1 + \mu s} \leq \frac{\mu s^{1-\frac{p}{2}}}{C_1} \leq \frac{\lambda_1^{1-\frac{p}{2}}}{C_1}\mu.$$

Proof of Lemma 2.5 is complete. □

Lemma 2.6. *For any* $p > 0, \mu > 0$ *and* $0 < \lambda_1 \leq s$, *the following inequality holds*

$$F_2(s) = \frac{\mu s^{2-p}}{C_1^2 + \mu s^2} \leq \begin{cases} C_2' \mu^{\frac{p}{2}}, & 0 < p < 2, \\ C_3' \mu, & p \geq 2, \end{cases}$$

where $C_2' := \frac{1}{2}p^{\frac{p}{2}} C_1^{-p}(2-p)^{\frac{2-p}{2}}$, $C_3' := \lambda_1^{2-p}/C_1^2$.

Proof. Similar to the proof of Lemma 2.5, we shall not repeat it here. □

Lemma 2.7. *For any $p>0, \mu>0$ and $0<\lambda_1\le s$, the following inequality holds*

$$F_3(s)=\frac{\mu s^{\frac{2-p}{4}}}{C_1+\mu s}\le\begin{cases}C_4\mu^{\frac{p+2}{4}}, & 0<p<2,\\ C_5\mu, & p\ge 2,\end{cases}$$

where $C_4:=\frac{1}{4}(p+2)^{\frac{p+2}{4}}C_1^{-\frac{p+2}{4}}(2-p)^{\frac{2-p}{4}},\ C_5:=\lambda_1^{\frac{2-p}{4}}/C_1$.

Proof. As in the proof of Lemma 2.5, we omit the details here. □

Lemma 2.8. *For any $p>0, \mu>0$ and $0<\lambda_1\le s$, the following inequality holds*

$$F_4(s)=\frac{\mu s^{\frac{2-p}{2}}}{C_1^2+\mu s^2}\le\begin{cases}C_4'\mu^{\frac{p+2}{4}}, & 0<p<2,\\ C_5'\mu, & p\ge 2,\end{cases}$$

where $C_4':=\frac{1}{4}(p+2)^{\frac{p+2}{4}}C_1^{-\frac{p+2}{2}}(2-p)^{\frac{2-p}{4}},\ C_5':=\lambda_1^{\frac{2-p}{2}}/C_1^2$.

Proof. The proof follows similarly to that of Lemma 2.5. □

3. The solution, the ill-posed analysis, and the result of conditional stability

By employing eigenfunction expansions and Lemma 2.1, in conjunction with the method of separation of variables and the modified Laplace transform of the Mittag-Leffler function, the solution to (1.1) is derived as follows

$$u(x,t)=\sum_{n=1}^{\infty}\Big(\varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{t}{a})^{\alpha})+(f_n+\sigma_n A_n)\frac{1-E_{\alpha,1}(-\lambda_n(\log\frac{t}{a})^{\alpha})}{\lambda_n}\Big)X_n(x), \tag{3.1}$$

where $\varphi_n=\langle\varphi(x),X_n(x)\rangle,\ f_n=\langle f(x),X_n(x)\rangle$ are the Fourier coefficients.

In the following, we consider the inverse problem of determining both the deterministic function $f(x)$ and the stochastic process $\dot{\mathcal{W}}_x$ from the measured data, expressed as

$$u(x,T)=g(x). \tag{3.2}$$

By considering the expectation and variance of (3.1) and letting $t=T$, we obtain

$$\mathbb{E}(u_n(T))=\varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+f_n\frac{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})}{\lambda_n}, \tag{3.3}$$

$$\mathbb{V}\mathrm{ar}(u_n(T))=\frac{\sigma_n^2}{\lambda_n^2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2. \tag{3.4}$$

Thus, we can express f_n and σ_n^2 as

$$f_n=\frac{\lambda_n\big(\mathbb{E}(u_n(T))-\varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})}, \tag{3.5}$$

$$\sigma_n^2=\frac{\lambda_n^2\mathbb{V}\mathrm{ar}(u_n(T))}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2}. \tag{3.6}$$

We now introduce the notation

$$\tilde{\mathbb{V}}\mathrm{ar}(\omega(x)) = \sum_{n=1}^{\infty} \mathbb{V}\mathrm{ar}\langle \omega(x), X_n(x)\rangle X_n(x),\ \omega(x) \in L^2(\Omega),$$

where the variance of the process $\dot{\mathcal{W}}_x$ is given by $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x) = \sum_{n=1}^{\infty} \sigma_n^2 X_n(x)$.

At this point, we focus on the inverse problem of determining the function $f(x)$ and the variance of the stochastic process $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)$ using the measured data $\mathbb{E}(u(x,T))$, $\tilde{\mathbb{V}}\mathrm{ar}(u(x,T))$.

In practical applications, it is inevitable that the measured data are influenced by observational errors. Let the exact data and the measured data satisfy the following noise assumption

$$\|\mathbb{E}(u^\delta(\cdot,T)) - \mathbb{E}(u(\cdot,T))\| \le \delta,\ \|\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(\cdot,T)) - \tilde{\mathbb{V}}\mathrm{ar}(u(\cdot,T))\| \le \delta, \tag{3.7}$$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta > 0$ is the noise level.

Consider the operators K_1 and K_2 defined as follows

$$\begin{aligned} &K_1 : f(\cdot) \to \mathbb{E}(u(\cdot,T)) - \varphi(\cdot)E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha), \\ &K_2 : \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot) \to \tilde{\mathbb{V}}\mathrm{ar}(u(\cdot,T)). \end{aligned}$$

It is well-established that both K_1 and K_2 are compact, linear, and self-adjoint operators. As $n \to \infty$, $\lambda_n \to \infty$, which implies that $K_i^{-1} \to \infty$ (for $i = 1, 2$). Consequently, small variations in the input data, such as $\mathbb{E}(u(x,T))$ and $\tilde{\mathbb{V}}\mathrm{ar}(u(x,T))$, can cause significant changes in the solutions, $f(x)$ and $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)$.

This behavior indicates that the problem is ill-posed and cannot be solved with traditional methods. Instead, regularization methods are needed to find stable solutions. Below, we provide a priori boundary condition

$$\max\{\|f(\cdot)\|_{\mathcal{D}(L^p)},\ \|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|_{\mathcal{D}(\ell^p)}\} \le E, \tag{3.8}$$

where $p > 0$ and E is a positive constant.

Theorem 3.1. *If $f(x)$ and $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)$ satisfy the a priori bound condition* (3.8)*, the corresponding conditional stability can be expressed as follows*

$$\|f(\cdot)\| \le C_6 E^{\frac{2}{p+2}} \big(\|\mathbb{E}(u(\cdot,T))\| + C_7\|\varphi(\cdot)\|\big)^{\frac{p}{p+2}}, \tag{3.9}$$

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\| \le C_6' E^{\frac{2}{p+2}} \|\tilde{\mathbb{V}}\mathrm{ar}(u(\cdot,T))\|^{\frac{p}{p+2}}, \tag{3.10}$$

where $C_6 := \big(1 - E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)\big)^{-\frac{p}{p+2}}$, $C_7 := E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)$, $C_6' := \big(1 - E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)\big)^{-\frac{2p}{p+2}}$.

Proof. According to the formula (3.5) and using the Hölder inequality and the a priori bound condition (3.8), we obtain

$$\begin{aligned} \|f(\cdot)\|^2 &= \Big\| \sum_{n=1}^{\infty} \frac{\lambda_n\big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)} X_n(x)\Big\|^2 \\ &= \sum_{n=1}^{\infty} \Big(\frac{\lambda_n\big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)}\Big)^2 \end{aligned}$$

$$
\begin{aligned}
=&\sum_{n=1}^{\infty}\Big(\frac{\lambda_n^{p+2}\big(\mathbb{E}(u_n(T))-\varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^{p+2}}\Big)^{\frac{2}{p+2}}\\
&\times\big(\mathbb{E}(u_n(T))-\varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^{\frac{2p}{p+2}}\\
\le&\Big(\sum_{n=1}^{\infty}\lambda_n^p f_n^2\big(1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})\big)^{-p}\Big)^{\frac{2}{p+2}}\\
&\times\Big(\big(\sum_{n=1}^{\infty}(\mathbb{E}(u_n(T))^2\big)^{\frac12}+\big(\sum_{n=1}^{\infty}(\varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}))^2\big)^{\frac12}\Big)^{\frac{2p}{p+2}}\\
\le&\big(1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})\big)^{-\frac{2p}{p+2}}\Big(\sum_{n=1}^{\infty}\lambda_n^p f_n^2\Big)^{\frac{2}{p+2}}\big(\|\mathbb{E}(u(\cdot,T))\|+C_7\|\varphi(\cdot)\|\big)^{\frac{2p}{p+2}}\\
\le&C_6^2E^{\frac{4}{p+2}}\big(\|\mathbb{E}(u(\cdot,T))\|+C_7\|\varphi(\cdot)\|\big)^{\frac{2p}{p+2}}.
\end{aligned}
\tag{3.11}
$$

Thus

$$\|f(\cdot)\|\le C_6E^{\frac{2}{p+2}}\big(\|\mathbb{E}u(\cdot,T)\|+C_7\|\varphi(\cdot)\|\big)^{\frac{p}{p+2}},$$

where $C_6:=\big(1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})\big)^{-\frac{p}{p+2}},\ C_7:=E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})$.

Similarly,

$$
\begin{aligned}
&\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|^2\\
=&\Big\|\sum_{n=1}^{\infty}\frac{\lambda_n^2\mathbb{V}\mathrm{ar}(u_n(T))}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2}X_n(x)\Big\|^2\\
=&\sum_{n=1}^{\infty}\Big(\frac{\lambda_n^2\mathbb{V}\mathrm{ar}(u_n(T))}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2}\Big)^2\\
=&\sum_{n=1}^{\infty}\Big(\frac{\lambda_n^{2(p+2)}\big(\mathbb{V}\mathrm{ar}(u_n(T))\big)^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^{2(p+2)}}\Big)^{\frac{2}{p+2}}\big(\mathbb{V}\mathrm{ar}(u_n(T))\big)^{\frac{2p}{p+2}}\\
\le&\Big(\sum_{n=1}^{\infty}\lambda_n^{2p}(\sigma_n^2)^2\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^{-2p}\Big)^{\frac{2}{p+2}}\Big(\sum_{n=1}^{\infty}\big(\mathbb{V}\mathrm{ar}(u_n(T))\big)^2\Big)^{\frac{p}{p+2}}\\
\le&\big(1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})\big)^{-\frac{4p}{p+2}}\Big(\sum_{n=1}^{\infty}\lambda_n^{2p}(\sigma_n^2)^2\Big)^{\frac{2}{p+2}}\|\tilde{\mathbb{V}}\mathrm{ar}(u(\cdot,T))\|^{\frac{2p}{p+2}}\\
\le&(C_6')^2E^{\frac{4}{p+2}}\|\tilde{\mathbb{V}}\mathrm{ar}(u(\cdot,T))\|^{\frac{2p}{p+2}}.
\end{aligned}
\tag{3.12}
$$

Thus

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|\le C_6'E^{\frac{2}{p+2}}\|\tilde{\mathbb{V}}\mathrm{ar}(u(\cdot,T))\|^{\frac{p}{p+2}},$$

where $C_6':=\big(1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})\big)^{-\frac{2p}{p+2}}$. Therefore, we complete the proof of Theorem 3.1. □

4. The quasi-boundary regularization method and error estimates

In this section, the quasi-boundary regularization method is proposed to solve the inverse problem (1.1), and error estimates are provided under both the a priori and a posteriori parameter selection rules.

For the quasi-boundary regularization method, the main idea is to add a penalty term into the final value data, thereby obtaining a regularized solution for the inverse problem [37,40]. In particular, we focus on studying the following problem

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}u^{\delta}_{\mu}(x,t) + \mathcal{L}u^{\delta}_{\mu}(x,t) = f^{\delta}_{\mu}(x) + (\dot{\mathcal{W}}_x)^{\delta}_{\mu}, & x \in \Omega, t \in (a,T], \\ u^{\delta}_{\mu}(x,t) = 0, & x \in \partial\Omega, t \in (a,T), \\ u^{\delta}_{\mu}(x,a) = \varphi(x), & x \in \Omega, \\ u^{\delta}_{\mu}(x,T) + \mu\big(f^{\delta}_{\mu}(x) + (\dot{\mathcal{W}}_x)^{\delta}_{\mu}\big) = u^{\delta}(x,T), & x \in \Omega, \end{cases} \tag{4.1}$$

where $\mu > 0$ is a regularized parameter.

Using the method of separation of variables and the modified Laplace transform, we can obtain

$$u^{\delta}_{\mu}(x,t) = \sum_{n=1}^{\infty} \Big(\varphi_n E_{\alpha,1}(-\lambda_n(\log \frac{t}{a})^{\alpha}) + \big((f_n)^{\delta}_{\mu} + (\sigma_n)^{\delta}_{\mu} A_n\big) \frac{1 - E_{\alpha,1}(-\lambda_n(\log \frac{t}{a})^{\alpha})}{\lambda_n} \Big) X_n(x). \tag{4.2}$$

By separately determining the expectation and variance of the full expression $u^{\delta}_{\mu}(x,T) + \mu\big(f^{\delta}_{\mu}(x) + (\dot{\mathcal{W}}_x)^{\delta}_{\mu}\big) = u^{\delta}(x,T)$, the regularized solutions can be obtained as follows

$$(f_n)^{\delta}_{\mu} = \frac{\lambda_n\big(\mathbb{E}(u^{\delta}_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha})\big)}{1 - E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu\lambda_n}, \tag{4.3}$$

$$(\sigma^2_n)^{\delta}_{\mu} = \frac{\lambda^2_n \mathbb{V}\mathrm{ar}(u^{\delta}_n(T))}{\big(1 - E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha})\big)^2 + \mu\lambda^2_n}. \tag{4.4}$$

In the case of noisy free, we derive that

$$(f_n)_{\mu} = \frac{\lambda_n\big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha})\big)}{1 - E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha}) + \mu\lambda_n}, \tag{4.5}$$

$$(\sigma^2_n)_{\mu} = \frac{\lambda^2_n \mathbb{V}\mathrm{ar}(u_n(T))}{\big(1 - E_{\alpha,1}(-\lambda_n(\log \frac{T}{a})^{\alpha})\big)^2 + \mu\lambda^2_n}. \tag{4.6}$$

4.1. A priori regularization choice rule

Theorem 4.1. *Suppose that the a priori bound condition* (3.8) *and the noise assumption* (3.7) *hold,*

(i) If $0 < p < 2$, selecting $\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}$, we obtain

$$\|f^{\delta}_{\mu}(\cdot) - f(\cdot)\| \le (1 + C_2)\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}},$$

if $p \ge 2$, choosing $\mu = (\frac{\delta}{E})^{\frac{1}{2}}$, we obtain

$$\|f^{\delta}_{\mu}(\cdot) - f(\cdot)\| \le (1 + C_3)\delta^{\frac{1}{2}} E^{\frac{1}{2}},$$

where $C_2 := \frac{1}{2} p^{\frac{p}{2}} C_1^{-\frac{p}{2}} (2-p)^{\frac{2-p}{2}}$, $C_3 := \lambda_1^{1-\frac{p}{2}}/C_1$, $C_1 := 1 - E_{\alpha,1}(-\lambda_1(\log \frac{T}{a})^{\alpha})$.

(ii) If $0 < p < 2$, selecting $\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}$, we obtain

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\| \le (1 + C_2')\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}},$$

if $p \ge 2$, choosing $\mu = (\frac{\delta}{E})^{\frac{1}{2}}$, we obtain

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\| \le (1 + C_3')\delta^{\frac{1}{2}} E^{\frac{1}{2}},$$

where $C_2' := \frac{1}{2}p^{\frac{p}{2}} C_1^{-p}(2-p)^{\frac{2-p}{2}}$, $C_3' := \lambda_1^{2-p}/C_1^2$.

Proof. Step 1. We begin by proving part (i) of the theorem.

By applying the triangle inequality, we obtain

$$\|f^\delta_\mu(\cdot) - f(\cdot)\| \le \|f^\delta_\mu(\cdot) - f_\mu(\cdot)\| + \|f_\mu(\cdot) - f(\cdot)\|. \tag{4.7}$$

First, we estimate the first term on the right-hand side. Using (4.3) and (4.5), we deduce that

$$\begin{aligned}
\|f^\delta_\mu(\cdot) - f_\mu(\cdot)\|^2 &= \Big\|\sum_{n=1}^\infty \frac{\lambda_n\left(\mathbb{E}(u^\delta_n(T)) - \mathbb{E}(u_n(T))\right)}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha) + \mu\lambda_n} X_n(x)\Big\|^2 \\
&= \sum_{n=1}^\infty \Big(\frac{\lambda_n}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha) + \mu\lambda_n}\Big)^2 \big(\mathbb{E}(u^\delta_n(T)) - \mathbb{E}(u_n(T))\big)^2 \\
&\le \sup_{n\ge1}\Big|\frac{\lambda_n}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha) + \mu\lambda_n}\Big|^2 \sum_{n=1}^\infty \big(\mathbb{E}(u^\delta_n(T)) - \mathbb{E}(u_n(T))\big)^2 \\
&\le (\frac{\delta}{\mu})^2.
\end{aligned}$$

Thus

$$\|f^\delta_\mu(\cdot) - f_\mu(\cdot)\| \le \frac{\delta}{\mu}. \tag{4.8}$$

Next, we estimate the second term on the right-hand of (4.7). By (3.5) and (4.5), we can deduce that

$$\begin{aligned}
&\|f_\mu(\cdot) - f(\cdot)\|^2 \\
=&\Big\|\sum_{n=1}^\infty \frac{\lambda_n\big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha) + \mu\lambda_n} X_n(x) \\
&- \frac{\lambda_n\big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)} X_n(x)\Big\|^2 \\
=&\Big\|\sum_{n=1}^\infty \frac{-\mu\lambda_n^2\big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha) + \mu\lambda_n\big)\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})\big)} X_n(x)\Big\|^2 \\
=&\sum_{n=1}^\infty \Big(\frac{\mu\lambda_n}{1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha) + \mu\lambda_n}\Big)^2 f_n^2 \\
\le&\sup_{n\ge1}\Big|\frac{\mu\lambda_n^{1-\frac{p}{2}}}{1 - E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha) + \mu\lambda_n}\Big|^2 \sum_{n=1}^\infty \lambda_n^p f_n^2
\end{aligned}$$

$$\leq \sup_{n\geq 1} |A_1(\lambda_n)|^2 E^2, \tag{4.9}$$

where $A_1(\lambda_n) = \frac{\mu\lambda_n^{1-\frac{p}{2}}}{C_1+\mu\lambda_n}$, $C_1 := 1 - E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)$.

Let $s := \lambda_n$, from Lemma 2.5, we can infer

$$A_1(s) = \frac{\mu s^{1-\frac{p}{2}}}{C_1 + \mu s} \leq \begin{cases} C_2\mu^{\frac{p}{2}}, & 0 < p < 2, \\ C_3\mu, & p \geq 2, \end{cases}$$

where $C_2 := \frac{1}{2}p^{\frac{p}{2}}C_1^{-\frac{p}{2}}(2-p)^{\frac{2-p}{2}}$, $C_3 := \lambda_1^{1-\frac{p}{2}}/C_1$.

Then, we have

$$\|f_\mu(\cdot) - f(\cdot)\| \leq \begin{cases} C_2\mu^{\frac{p}{2}}E, & 0 < p < 2, \\ C_3\mu E, & p \geq 2. \end{cases} \tag{4.10}$$

Combining (4.8) with (4.10), for $0 < p < 2$, we choose the regularization parameter $\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}$, and for $p \geq 2$, we choose $\mu = (\frac{\delta}{E})^{\frac{1}{2}}$, leading to the following result

$$\|f_\mu^\delta(\cdot) - f(\cdot)\| \leq \begin{cases} (1 + C_2)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, & 0 < p < 2, \\ (1 + C_3)\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p \geq 2. \end{cases}$$

Step 2. Here, we will adopt similar techniques to prove part (ii).

By the triangle inequality, it is straightforward to verify that

$$\begin{aligned} \|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\| \leq & \|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)\| \\ & + \|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|. \end{aligned} \tag{4.11}$$

Using (4.4) and (4.6), we can obtain

$$\begin{aligned} & \|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|^2 \\ = & \Big\| \sum_{n=1}^\infty \frac{\lambda_n^2\big(\mathbb{V}\mathrm{ar}(u_n^\delta(T)) - \mathbb{V}\mathrm{ar}(u_n(T))\big)}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2 + \mu\lambda_n^2} X_n(x) \Big\|^2 \\ = & \sum_{n=1}^\infty \Big(\frac{\lambda_n^2}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2 + \mu\lambda_n^2} \Big)^2 \big(\mathbb{V}\mathrm{ar}(u_n^\delta(T)) - \mathbb{V}\mathrm{ar}(u_n(T))\big)^2 \\ \leq & \sup_{n\geq 1} \Big| \frac{\lambda_n^2}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2 + \mu\lambda_n^2} \Big|^2 \sum_{n=1}^\infty \big(\mathbb{V}\mathrm{ar}(u_n^\delta(T)) - \mathbb{V}\mathrm{ar}(u_n(T))\big)^2 \\ \leq & \Big(\frac{\delta}{\mu}\Big)^2. \end{aligned}$$

Thus

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)\| \leq \frac{\delta}{\mu}. \tag{4.12}$$

By (3.6) and (4.6), we can deduce that

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|^2$$

$$
\begin{aligned}
&= \Big\| \sum_{n=1}^{\infty} \frac{\lambda_n^2 \mathbb{V}\mathrm{ar}(u_n(T))}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}\big)^2 + \mu\lambda_n^2} X_n(x) - \frac{\lambda_n^2 \mathbb{V}\mathrm{ar}(u_n(T))}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}\big)^2} X_n(x) \Big\|^2 \\
&= \Big\| \sum_{n=1}^{\infty} \frac{-\mu\lambda_n^4 \mathbb{V}\mathrm{ar}(u_n(T))}{\big((1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}))^2 + \mu\lambda_n^2\big)\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})\big)^2} X_n(x) \Big\|^2 \\
&= \sum_{n=1}^{\infty} \Big(\frac{\mu\lambda_n^2}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2 + \mu\lambda_n^2} \Big)^2 (\sigma_n^2)^2 \\
&\leq \sup_{n\geq 1} \Big| \frac{\mu\lambda_n^{2-p}}{\big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2 + \mu\lambda_n^2} \Big|^2 \sum_{n=1}^{\infty} \lambda_n^{2p} (\sigma_n^2)^2 \\
&\leq \sup_{n\geq 1} |A_2(\lambda_n)|^2 E^2,
\end{aligned} \tag{4.13}
$$

where $A_2(\lambda_n) = \frac{\mu\lambda_n^{2-p}}{C_1^2 + \mu\lambda_n^2}$.

Let $s := \lambda_n$, from Lemma 2.6, we can infer

$$
A_2(s) = \frac{\mu s^{2-p}}{C_1^2 + \mu s^2} \leq \begin{cases} C_2' \mu^{\frac{p}{2}}, & 0 < p < 2, \\ C_3' \mu, & p \geq 2, \end{cases}
$$

where $C_2' := \frac{1}{2} p^{\frac{p}{2}} C_1^{-p} (2-p)^{\frac{2-p}{2}},\ C_3' := \lambda_1^{2-p} / C_1^2$.

Then, we obtain

$$
\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_{\mu}(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\| \leq \begin{cases} C_2' \mu^{\frac{p}{2}} E, & 0 < p < 2, \\ C_3' \mu E, & p \geq 2. \end{cases} \tag{4.14}
$$

Combining (4.12) with (4.14), for $0 < p < 2$, we choose $\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}$, and for $p \geq 2$, we choose $\mu = (\frac{\delta}{E})^{\frac{1}{2}}$, leading to the following result

$$
\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_{\mu}^{\delta}(\cdot) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\| \leq \begin{cases} (1 + C_2') \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, & 0 < p < 2, \\ (1 + C_3') \delta^{\frac{1}{2}} E^{\frac{1}{2}}, & p \geq 2. \end{cases}
$$

□

4.2. A posteriori regularization choice rule

Theorem 4.1 aims to reveal the a priori error estimate, and in the next step, we address Theorem 4.2 and Theorem 4.3 to clarify the a posteriori error estimate using the Morozov discrepancy principle.

4.2.1. Deterministic term $f(x)$

Morozov discrepancy principle is used to select the regularization parameter for the a posteriori error estimate of $f(x)$ and is stated as follows

$$
\Big\| \mu(K_1 + \mu)^{-1} \big[K_1 f_{\mu}^{\delta}(\cdot) - \big(\mathbb{E}(u^{\delta}(\cdot, T)) - \varphi(\cdot) E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big) \big] \Big\| = \tau_1 \delta, \tag{4.15}
$$

where $\tau_1 > 1$ is a constant and $\big\| \mathbb{E}(u^{\delta}(\cdot, T)) - \varphi(\cdot) E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big\| > \tau_1 \delta$.

Lemma 4.1. *Let* $\rho_1(\mu) = \big\|\mu(K_1+\mu)^{-1}\big[K_1 f_\mu^\delta(\cdot) - \big(\mathbb{E}(u^\delta(\cdot,T)) - \varphi(\cdot)E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)\big]\big\|$, *then*

(i) $\rho_1(\mu)$ *is a continuous function;*

(ii) $\lim\limits_{\mu\to 0}\rho_1(\mu) = 0$*;*

(iii) $\lim\limits_{\mu\to\infty}\rho_1(\mu) = \big\|\mathbb{E}(u^\delta(\cdot,T)) - \varphi(\cdot)E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big\|$*;*

(iv) $\rho_1(\mu)$ *is a strictly monotonically increasing function for any* $\mu\in(0,+\infty)$*.*

Proof. By (4.15) , we have

$$\begin{aligned}&\rho_1(\mu)\\ =&\Big\|\mu(K_1+\mu)^{-1}\big[K_1 f_\mu^\delta(\cdot) - \big(\mathbb{E}(u^\delta(\cdot,T)) - \varphi(\cdot)E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)\big]\Big\|\\ =&\Big\|\sum_{n=1}^\infty \frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu\lambda_n}\frac{-\mu\lambda_n\big(\mathbb{E}(u_n^\delta(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu\lambda_n}X_n(x)\Big\|\\ =&\Big(\sum_{n=1}^\infty\Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu\lambda_n}\Big)^4\big(\mathbb{E}(u_n^\delta(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Clearly, the conclusions in $(a),(b),(c),(d)$ are valid. □

Theorem 4.2. *Suppose that the a priori bound condition* (3.8) *and the noise assumption* (3.7) *hold, and the regularization parameter* μ *satisfies the selection rule defined in* (4.15)*. Then, we have the following results*

(i) If $0<p<2$*, we have the error estimate*

$$\|f_\mu^\delta(\cdot)-f(\cdot)\| \le \Big(\Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}} + \Big(\frac{C_4^2}{\tau_1-1}\Big)^{\frac{2}{p+2}}\Big)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}},$$

where $C_1 := 1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)$, $C_4 := \frac{1}{4}(p+2)^{\frac{p+2}{4}}C_1^{-\frac{p+2}{4}}(2-p)^{\frac{2-p}{4}}$*.*

(ii) If $p\ge 2$*, we obtain the error estimate*

$$\|f_\mu^\delta(\cdot)-f(\cdot)\| \le \Big(\Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}} + \Big(\frac{C_5^2}{\tau_1-1}\Big)^{\frac{1}{2}}\Big)\delta^{\frac{1}{2}}E^{\frac{1}{2}},$$

where $C_5 := \lambda_1^{\frac{2-p}{4}}/C_1$*.*

Proof. By applying the triangle inequality, we have

$$\|f_\mu^\delta(\cdot)-f(\cdot)\| \le \|f_\mu^\delta(\cdot)-f_\mu(\cdot)\| + \|f_\mu(\cdot)-f(\cdot)\|. \tag{4.16}$$

The first term of (4.16) is derived from equation (4.8)

$$\|f_\mu^\delta(\cdot)-f_\mu(\cdot)\| \le \frac{\delta}{\mu}.$$

Using (4.15) and (3.7), we obtain

$$\tau_1\delta = \Big\|\sum_{n=1}^\infty\Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)+\mu\lambda_n}\Big)^2\big(\mathbb{E}(u_n^\delta(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)X_n(x)\Big\|$$

$$
\begin{aligned}
&\leq \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \Big)^2 \big(\mathbb{E}(u_n^{\delta}(T)) - \mathbb{E}(u_n(T)) \big) X_n(x) \Big\| \\
&\quad + \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \Big)^2 \big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big) X_n(x) \Big\| \\
&\leq \delta + J_1.
\end{aligned}
$$

Next, a priori bound condition (3.8) is used to estimate J_1

$$
\begin{aligned}
J_1 &= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \Big)^2 \big(\mathbb{E}(u_n(T)) - \varphi_n E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big) X_n(x) \Big\| \\
&= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \Big)^2 \lambda_n^{-1} \big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big) f_n X_n(x) \Big\| \\
&= \Big(\sum_{n=1}^{\infty} \Big(\frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \Big)^4 \big[\lambda_n^{-1} \big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big) \big]^2 (f_n)^2 \Big)^{\frac{1}{2}} \\
&\leq \sup_{n\geq 1} \Big| \frac{\mu\lambda_n}{1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \big[\lambda_n^{-1} \big(1 - E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha}) \big) \big]^{\frac{1}{2}} \lambda_n^{-\frac{p}{4}} \Big|^2 \Big(\sum_{n=1}^{\infty} \lambda_n^p f_n^2 \Big)^{\frac{1}{2}} \\
&\leq \sup_{n\geq 1} \Big| \frac{\mu\lambda_n^{\frac{2-p}{4}}}{1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})+\mu\lambda_n} \Big|^2 E \\
&= \sup_{n\geq 1} |A_3(\lambda_n)|^2 E,
\end{aligned} \tag{4.17}
$$

where $A_3(\lambda_n) = \frac{\mu\lambda_n^{\frac{2-p}{4}}}{C_1+\mu\lambda_n}$.

Let $s := \lambda_n$, according to Lemma 2.7, we obtain

$$
J_1 \leq \begin{cases} C_4^2 \mu^{\frac{p+2}{2}} E, & 0<p<2, \\ C_5^2 \mu^2 E, & p\geq 2, \end{cases}
$$

where $C_4 := \frac{1}{4}(p+2)^{\frac{p+2}{4}} C_1^{-\frac{p+2}{4}} (2-p)^{\frac{2-p}{4}},\ C_5 := \lambda_1^{\frac{2-p}{4}}/C_1$.

Thus

$$
(\tau_1-1)\delta \leq \begin{cases} C_4^2 \mu^{\frac{p+2}{2}} E, & 0<p<2, \\ C_5^2 \mu^2 E, & p\geq 2. \end{cases}
$$

Furthermore

$$
\frac{1}{\mu} \leq \begin{cases} \left(\dfrac{C_4^2}{\tau_1-1} \right)^{\frac{2}{p+2}} \left(\dfrac{E}{\delta} \right)^{\frac{2}{p+2}}, & 0<p<2, \\ \left(\dfrac{C_5^2}{\tau_1-1} \right)^{\frac{1}{2}} (\dfrac{E}{\delta})^{\frac{1}{2}}, & p\geq 2. \end{cases} \tag{4.18}
$$

Substitute (4.18) to (4.8), we have

$$
\|f_\mu^\delta(\cdot) - f_\mu(\cdot)\| \leq \frac{\delta}{\mu} \leq \begin{cases} \left(\dfrac{C_4^2}{\tau_1-1} \right)^{\frac{2}{p+2}} \delta^{\frac{2}{p+2}} E^{\frac{2}{p+2}}, & 0<p<2, \\ \left(\dfrac{C_5^2}{\tau_1-1} \right)^{\frac{1}{2}} \delta^{\frac{1}{2}} E^{\frac{1}{2}}, & p\geq 2. \end{cases} \tag{4.19}
$$

For the second term on the right-hand of (4.16), we can obtain

$$
\begin{aligned}
&\|f_\mu(\cdot)-f(\cdot)\|\\
=&\Big\|\sum_{n=1}^{\infty}\frac{-\mu f_n}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}X_n(x)\Big\|\\
=&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{\frac{p}{2}}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{1-\frac{p}{2}}\\
&\times\frac{f_n}{\big[\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)\big]^{\frac{p}{2}}}X_n(x)\Big\|\\
\le&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{\frac{p}{2}+1}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{1-\frac{p}{2}}\\
&\times\frac{f_n}{\big[\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)\big]^{\frac{p}{2}}}X_n(x)\Big\|^{\frac{p}{p+2}}\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{1-\frac{p}{2}}\\
&\times\frac{f_n}{\big[\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)\big]^{\frac{p}{2}}}X_n(x)\Big\|^{\frac{2}{p+2}}\\
\le&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{2}\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)f_nX_n(x)\Big\|^{\frac{p}{p+2}}\\
&\times\Big\|\sum_{n=1}^{\infty}\frac{f_n}{\big[\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)\big]^{\frac{p}{2}}}X_n(x)\Big\|^{\frac{2}{p+2}}\\
\le&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{2}\big(\mathbb{E}(u_n(T))-\varphi_nE_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)X_n(x)\Big\|^{\frac{p}{p+2}}\\
&\times\Big\|\sum_{n=1}^{\infty}\lambda_n^{\frac{p}{2}}f_nX_n(x)\Big\|^{\frac{2}{p+2}}C_1^{-\frac{p}{p+2}}\\
\le&\Big(\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{2}\big(\mathbb{E}(u_n(T))-\mathbb{E}(u_n^\delta(T))\big)X_n(x)\Big\|\\
&+\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)+\mu}\Big)^{2}\\
&\times\big(\mathbb{E}(u_n^\delta(T))-\varphi_nE_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)X_n(x)\Big\|\Big)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}C_1^{-\frac{p}{p+2}}\\
\le&\Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.
\end{aligned}
\tag{4.20}
$$

Combining (4.19) and (4.20), we obtain

$$
\|f_\mu^\delta(\cdot)-f(\cdot)\|\le\begin{cases}\Big(\Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}}+\Big(\frac{C_4^2}{\tau_1-1}\Big)^{\frac{2}{p+2}}\Big)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, & 0<p<2,\\ \Big(\Big(\frac{\tau_1+1}{C_1}\Big)^{\frac{p}{p+2}}+\Big(\frac{C_5^2}{\tau_1-1}\Big)^{\frac{1}{2}}\Big)\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\ge 2.\end{cases}
\tag{4.21}
$$

Hence, the proof of Theorem 4.2 is complete. □

4.2.2. Random term $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)$

We apply the Morozov discrepancy principle in the form presented below

$$\Big\|\mu(K_2+\mu)^{-1}\big(K_2\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(\cdot,T))\big)\Big\|=\tau_2\delta, \tag{4.22}$$

where $\tau_2>1$ is a constant and $\|\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(x,T))\|>\tau_2\delta$.

Lemma 4.2. *Let* $\rho_2(\mu)=\big\|\mu(K_2+\mu)^{-1}\big(K_2\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(\cdot,T))\big)\big\|$, *then*

- *(i)* $\rho_2(\mu)$ *is a continuous function;*
- *(ii)* $\lim\limits_{\mu\to 0}\rho_2(\mu)=0$;
- *(iii)* $\lim\limits_{\mu\to\infty}\rho_2(\mu)=\big\|\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(\cdot,T))\big\|$;
- *(iv)* $\rho_2(\mu)$ *is a strictly monotonically increasing function for any* $\mu\in(0,+\infty)$.

Proof. After a simple calculation, this lemma is proved by the following expression

$$\rho_2(\mu)=\Big(\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu\lambda_n^2}\Big)^4\big(\mathbb{V}\mathrm{ar}(u_n^\delta(T))\big)^2\Big)^{\frac{1}{2}}.$$

□

Theorem 4.3. *Suppose that the a priori bound condition* (3.8) *and the noise assumption* (3.7) *hold, and the regularization parameter* μ *satisfies the selection rule defined in* (4.22). *Then, we have the following results*

(i) If $0<p<2$, *we have the error estimate*

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|\le\Big(\Big(\frac{\tau_2+1}{C_1^2}\Big)^{\frac{p}{p+2}}+\Big(\frac{(C_4')^2}{\tau_2-1}\Big)^{\frac{2}{p+2}}\Big)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}},$$

where $C_1:=1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^\alpha)$, $C_4':=\frac{1}{4}(p+2)^{\frac{p+2}{4}}C_1^{-\frac{p+2}{2}}(2-p)^{\frac{2-p}{4}}$.

(ii) If $p\ge 2$, *we obtain the error estimate*

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|\le\Big(\Big(\frac{\tau_2+1}{C_1^2}\Big)^{\frac{p}{p+2}}+\Big(\frac{(C_5')^2}{\tau_2-1}\Big)^{\frac{1}{2}}\Big)\delta^{\frac{1}{2}}E^{\frac{1}{2}},$$

where $C_5':=\lambda_1^{\frac{2-p}{2}}/C_1^2$.

Proof. By applying the triangle inequality, we can obtain

$$\begin{aligned}\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|\le&\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)\|\\&+\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|.\end{aligned}\tag{4.23}$$

The first term of (4.23) is derived from equation (4.12)

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)\|\le\frac{\delta}{\mu}.$$

By (4.22) and (3.7), we deduce that

$$\begin{aligned}\tau_2\delta &= \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big)^2\mathbb{V}\mathrm{ar}(u_n^{\delta}(T))X_n(x)\Big\|\\ &\le \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big)^2\big(\mathbb{V}\mathrm{ar}(u_n^{\delta}(T))-\mathbb{V}\mathrm{ar}(u_n(T))\big)X_n(x)\Big\|\\ &\quad+\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big)^2\mathbb{V}\mathrm{ar}(u_n(T))X_n(x)\Big\|\\ &\le \delta+J_2.\end{aligned}$$

Next, a priori bound condition (3.8) is used to estimate J_2

$$\begin{aligned}J_2 &= \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big)^2\mathbb{V}\mathrm{ar}(u_n(T))X_n(x)\Big\|\\ &= \Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big)^2\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2\sigma_n^2X_n(x)\Big\|\\ &= \Big(\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big)^4\big[\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)\big]^4(\sigma_n^2)^2\Big)^{\frac{1}{2}}\\ &\le \sup_{n\ge1}\Big|\frac{\mu\lambda_n^2}{\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\lambda_n^{-1}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^{\alpha})\big)\lambda_n^{-\frac{p}{2}}\Big|^2\Big(\sum_{n=1}^{\infty}\lambda_n^{2p}(\sigma_n^2)^2\Big)^{\frac{1}{2}}\\ &\le \sup_{n\ge1}\Big|\frac{\mu\lambda_n^{\frac{2-p}{2}}}{\big(1-E_{\alpha,1}(-\lambda_1(\log\frac{T}{a})^{\alpha})\big)^2+\mu\lambda_n^2}\Big|^2E\\ &= \sup_{n\ge1}|A_4(\lambda_n)|^2E,\end{aligned}\tag{4.24}$$

where $A_4(\lambda_n)=\frac{\mu\lambda_n^{\frac{2-p}{2}}}{C_1^2+\mu\lambda_n^2}$.

Let $s:=\lambda_n$, according to Lemma 2.8, we obtain

$$J_2\le\begin{cases}(C_4')^2\mu^{\frac{p+2}{2}}E, & 0<p<2,\\ (C_5')^2\mu^2E, & p\ge2,\end{cases}$$

where $C_4':=\frac{1}{4}(p+2)^{\frac{p+2}{4}}C_1^{-\frac{p+2}{2}}(2-p)^{\frac{2-p}{4}},\ C_5':=\lambda_1^{\frac{2-p}{2}}/C_1^2$.

Thus

$$(\tau_2-1)\delta\le\begin{cases}(C_4')^2\mu^{\frac{p+2}{2}}E, & 0<p<2,\\ (C_5')^2\mu^2E, & p\ge2.\end{cases}$$

Furthermore

$$\frac{1}{\mu}\le\begin{cases}\Big(\frac{(C_4')^2}{\tau_2-1}\Big)^{\frac{2}{p+2}}\Big(\frac{E}{\delta}\Big)^{\frac{2}{p+2}}, & 0<p<2,\\ \Big(\frac{(C_5')^2}{\tau_2-1}\Big)^{\frac{1}{2}}(\frac{E}{\delta})^{\frac{1}{2}}, & p\ge2.\end{cases}\tag{4.25}$$

Substitute (4.25) to (4.12), we have

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)\|\le\frac{\delta}{\mu}\le\begin{cases}\Big(\frac{(C_4')^2}{\tau_2-1}\Big)^{\frac{2}{p+2}}\delta^{\frac{2}{p+2}}E^{\frac{2}{p+2}}, & 0<p<2,\\ \Big(\frac{(C_5')^2}{\tau_2-1}\Big)^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\ge 2.\end{cases}\tag{4.26}$$

For the second term on the right-hand of (4.23), we can obtain

$$\begin{aligned}&\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|\\
=&\Big\|\sum_{n=1}^{\infty}\frac{-\mu\sigma_n^2}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}X_n(x)\Big\|\\
=&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}\Big)^{\frac{p}{2}}\Big(\frac{\mu}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}\Big)^{1-\frac{p}{2}}\\
&\times\frac{\sigma_n^2}{\big[\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2\big]^{\frac{p}{2}}}X_n(x)\Big\|\\
\le&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}\Big)^2\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2\sigma_n^2X_n(x)\Big\|^{\frac{p}{p+2}}\\
&\times\Big\|\sum_{n=1}^{\infty}\frac{\sigma_n^2}{\big[\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2\big]^{\frac{p}{2}}}X_n(x)\Big\|^{\frac{2}{p+2}}\\
\le&\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}\Big)^2\mathbb{V}\mathrm{ar}(u_n(T))X_n(x)\Big\|^{\frac{p}{p+2}}\\
&\times\Big\|\sum_{n=1}^{\infty}\lambda_n^p\sigma_n^2X_n(x)\Big\|^{\frac{2}{p+2}}C_1^{-\frac{2p}{p+2}}\\
\le&\Big(\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}\Big)^2\big(\mathbb{V}\mathrm{ar}(u_n(T))-\mathbb{V}\mathrm{ar}(u_n^\delta(T))\big)X_n(x)\Big\|\\
&+\Big\|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\lambda_n^{-2}\big(1-E_{\alpha,1}(-\lambda_n(\log\frac{T}{a})^\alpha)\big)^2+\mu}\Big)^2\mathbb{V}\mathrm{ar}(u_n^\delta(T))X_n(x)\Big\|\Big)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}C_1^{-\frac{2p}{p+2}}\\
\le&\Big(\frac{\tau_2+1}{C_1^2}\Big)^{\frac{p}{p+2}}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}.\end{aligned}\tag{4.27}$$

Combining (4.26) and (4.27), we obtain

$$\|\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)^\delta_\mu(\cdot)-\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)(\cdot)\|\le\begin{cases}\Big(\Big(\frac{\tau_2+1}{C_1^2}\Big)^{\frac{p}{p+2}}+\Big(\frac{(C_4')^2}{\tau_2-1}\Big)^{\frac{2}{p+2}}\Big)\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, & 0<p<2,\\ \Big(\Big(\frac{\tau_2+1}{C_1^2}\Big)^{\frac{p}{p+2}}+\Big(\frac{(C_5')^2}{\tau_2-1}\Big)^{\frac{1}{2}}\Big)\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\ge 2.\end{cases}\tag{4.28}$$

Hence, the proof of Theorem 4.3 is complete. □

5. Numerical experiments

In this section, in order to better demonstrate the effectiveness of the quasi-boundary regularization method in solving ill-posed problems, we present numerical examples in both one-dimensional (1D) and two-dimensional (2D) cases.

• **1D Case**

Let $\Omega = (0, l)$, and consider the following direct problem

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}u(x,t) + \mathcal{L}u(x,t) = f(x) + \dot{\mathcal{W}}_x, & x \in \Omega,\ t \in (a,T],\ 0 < \alpha < 1, \\ u(0,t) = u(l,t) = 0, & t \in (a,T), \\ u(x,a) = \varphi(x), & x \in \Omega. \end{cases} \tag{5.1}$$

The above equation is discretized by the finite difference method. The time and space steps are defined as $\Delta t = \frac{T-a}{N}$ and $\Delta x = \frac{l}{M}$, respectively. So we can obtain $t_k = a + k\Delta t\,(k = 0, 1, \cdots, N)$ and $x_i = i\Delta x\,(i = 0, 1, \cdots, M)$. The approximate value of u at each grid point is denoted by $u_i^k \approx u(x_i, t_k)$.

The finite difference format of Caputo-Hadamard time-fractional derivative is as follows [10]

$${}_{CH}D^{\alpha}_{a,t}u(x_i,t_k) \approx -a_{1,k}u_i^0 + \sum_{s=1}^{k-1}(a_{s,k} - a_{s+1,k})\,u_i^s + a_{k,k}u_i^k,$$

where $i = 1, \cdots, M-1$; $k = 1, \cdots, N$ and

$$a_{s,k} = \frac{1}{\Gamma(2-\alpha)}\frac{(\log\frac{t_k}{t_{s-1}})^{1-\alpha} - (\log\frac{t_k}{t_s})^{1-\alpha}}{\log\frac{t_s}{t_{s-1}}}\ (s = 1, 2, \cdots, k).$$

Let $A_{ij} = 0$ and $c(x) = 0$ in (1.1), the operator $\mathcal{L}$ approximate difference scheme is given by [28]

$$\mathcal{L}u(x_i,t_k) \approx -\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2}.$$

Based on the statistical properties, we divide the problem (5.1) into

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}\mathbb{E}(u(x,t)) + \mathcal{L}\mathbb{E}(u(x,t)) = f(x), & x \in (0,l),\ t \in (a,T], \\ \mathbb{E}(u(0,t)) = \mathbb{E}(u(l,t)) = 0, & t \in (a,T), \\ \mathbb{E}(u(x,a)) = \varphi(x), & x \in (0,l), \end{cases}$$

and

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}\tilde{\mathbb{V}}\mathrm{ar}(u(x,t)) + \mathcal{L}\tilde{\mathbb{V}}\mathrm{ar}(u(x,t)) = \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x), & x \in (0,l),\ t \in (a,T], \\ \tilde{\mathbb{V}}\mathrm{ar}(u(0,t)) = \tilde{\mathbb{V}}\mathrm{ar}(u(l,t)) = 0, & t \in (a,T), \\ \tilde{\mathbb{V}}\mathrm{ar}(u(x,a)) = 0, & x \in (0,l). \end{cases}$$

First, we can derive the discrete scheme of the direct problem in the sense of expectation as follows

$$-a_{1,k}\mathbb{E}(u_i^0) + \sum_{s=1}^{k-1}(a_{s,k} - a_{s+1,k})\mathbb{E}(u_i^s) + a_{k,k}\mathbb{E}(u_i^k) = \frac{\mathbb{E}(u_{i+1}^k) - 2\mathbb{E}(u_i^k) + \mathbb{E}(u_{i-1}^k)}{(\Delta x)^2} + f_i.$$

Let $\mathbb{E}(U^k) = (\mathbb{E}(u_1^k), \mathbb{E}(u_2^k), \cdots, \mathbb{E}(u_{M-1}^k))^T$, $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_{M-1})^T$ and $F = (f_1, f_2, \cdots, f_{M-1})^T$, we can obtain

$$\begin{aligned} A^1\mathbb{E}(U^1) =& F + a_{1,1}\Phi, \\ A^k\mathbb{E}(U^k) =& F + \sum_{s=1}^{k-1}(a_{s+1,k} - a_{s,k})\mathbb{E}(U^s) + a_{1,k}\Phi. \end{aligned}$$

The tridiagonal matrix A^k is as follows

$$A^k_{(M-1)\times(M-1)} = \begin{pmatrix} d_2 & d_3 & & & \\ d_1 & d_2 & d_3 & & \\ & d_1 & d_2 & \ddots & \\ & & \ddots & \ddots & d_3 \\ & & & d_1 & d_2 \end{pmatrix},$$

where $d_1 = d_3 = -\frac{1}{(\Delta x)^2}$, $d_2 = a_{i,i} + \frac{2}{(\Delta x)^2}$.

Similarly, in the sense of variance, we can obtain

$$\begin{aligned} & - a_{1,k}\tilde{\mathbb{V}}\mathrm{ar}(u_i^0) + \sum_{s=1}^{k-1}(a_{s,k} - a_{s+1,k})\tilde{\mathbb{V}}\mathrm{ar}(u_i^s) + a_{k,k}\tilde{\mathbb{V}}\mathrm{ar}(u_i^k) \\ =& \frac{1}{(\Delta x)^2}(\tilde{\mathbb{V}}\mathrm{ar}(u_{i+1}^k) - 2\tilde{\mathbb{V}}\mathrm{ar}(u_i^k) + \tilde{\mathbb{V}}\mathrm{ar}(u_{i-1}^k)) + \sum_{n=1}^{K}\sigma_n^2 X_n(x_i). \end{aligned}$$

Let $\tilde{\mathbb{V}}\mathrm{ar}(U^k)$=$(\tilde{\mathbb{V}}\mathrm{ar}(u_1^k)$, $\tilde{\mathbb{V}}\mathrm{ar}(u_2^k)$, $\cdots$, $\tilde{\mathbb{V}}\mathrm{ar}(u_{M-1}^k))^T$ and $\Psi = (\sigma_1^2, \sigma_2^2, \cdots, \sigma_{M-1}^2)^T$, we can obtain

$$\begin{aligned} A^1\tilde{\mathbb{V}}\mathrm{ar}(U^1) =& \Psi, \\ A^k\tilde{\mathbb{V}}\mathrm{ar}(U^k) =& \Psi + \sum_{s=1}^{k-1}(a_{s+1,k} - a_{s,k})\tilde{\mathbb{V}}\mathrm{ar}(U^s). \end{aligned}$$

- **2D Case**

Let $\Omega = (0, l_1) \times (0, l_2)$ be a rectangular domain, and consider the following direct problem

$$\begin{cases} {}_{CH}D^{\alpha}_{a,t}u(x,y,t) + \mathcal{L}u(x,y,t) = f(x,y) + \dot{\mathcal{W}}_{xy}, & (x,y) \in \Omega,\ t \in (a,T], \\ u(0,y,t) = u(l_1,y,t) = u(x,0,t) = u(x,l_2,t) = 0, & t \in (a,T), \\ u(x,y,a) = \varphi(x,y), & (x,y) \in \Omega. \end{cases} \tag{5.2}$$

Let $x_i = i\Delta x\,(i = 0, 1, \cdots, M_1)$, $y_j = j\Delta y\,(j = 0, 1, \cdots, M_2)$, $t_k = a + k\Delta t\,(k = 0, 1, \cdots, N)$ and the space and time steps are defined as $\Delta x = \frac{l_1}{M_1}$, $\Delta y = \frac{l_2}{M_2}$ and $\Delta t = \frac{T-a}{N}$. The approximate value of u at each grid point is denoted by $u_{i,j}^k \approx u(x_i, y_j, t_k)$.

First, we can derive the discrete scheme of the direct problem in the sense of expectation as follows

$$- a_{1,k}\mathbb{E}(u_{i,j}^0) + \sum_{s=1}^{k-1}(a_{s,k} - a_{s+1,k})\mathbb{E}(u_{i,j}^s) + a_{k,k}\mathbb{E}(u_{i,j}^k)$$

$$=p_x\big(\mathbb{E}(u^k_{i+1,j})-2\mathbb{E}(u^k_{i,j})+\mathbb{E}(u^k_{i-1,j})\big)+p_y\big(\mathbb{E}(u^k_{i,j+1})-2\mathbb{E}(u^k_{i,j})+\mathbb{E}(u^k_{i,j-1})\big)+f_{i,j},$$

where $p_x=\frac{1}{(\Delta x)^2}$ and $p_y=\frac{1}{(\Delta y)^2}$.

Let $\mathbb{E}(U^k) = (\mathbb{E}(u^k_{1,1}),\mathbb{E}(u^k_{2,1}),\cdots,\mathbb{E}(u^k_{M_1-1,1}),\mathbb{E}(u^k_{1,2}),\mathbb{E}(u^k_{2,2}),\cdots,\mathbb{E}(u^k_{M_1-1,2}), \cdots, \mathbb{E}(u^k_{1,M_2-1}),\mathbb{E}(u^k_{2,M_2-1}),\cdots,\mathbb{E}(u^k_{M_1-1,M_2-1}))^T$, $\Phi = (\varphi_{1,1},\varphi_{2,1},\cdots,\varphi_{M_1-1,1}, \varphi_{1,2},\varphi_{2,2},\cdots, \varphi_{M_1-1,2},\cdots,\varphi_{1,M_2-1},\varphi_{2,M_2-1},\cdots,\varphi_{M_1-1,M_2-1})^T$ and $F=(f_{1,1},f_{2,1},\cdots,f_{M_1-1,1},f_{1,2},f_{2,2},\cdots, f_{M_1-1,2},\cdots,f_{1,M_2-1},f_{2,M_2-1},\cdots,f_{M_1-1,M_2-1})^T$, we can obtain

$$(A^*)^1\mathbb{E}(U^1)=F+a_{1,1}\Phi,$$
$$(A^*)^k\mathbb{E}(U^k)=F+\sum_{s=1}^{k-1}(a_{s+1,k}-a_{s,k})\mathbb{E}(U^s)+a_{1,k}\Phi.$$

The tridiagonal matrix $(A^*)^k$ is as follows

$$(A^*)^k=\begin{pmatrix} A^*_{1,1} & -p_yI & & & \\ -p_yI & A^*_{2,2} & -p_yI & & \\ & -p_yI & A^*_{3,3} & \ddots & \\ & & \ddots & \ddots & -p_yI \\ & & & -p_yI & A^*_{M_1-1,M_1-1}\end{pmatrix}\in\mathbb{R}^{(M_1-1)(M_2-1)\times(M_1-1)(M_2-1)},$$

where I is the $(M_1-1)\times(M_1-1)$ order unit matrix, and

$$A^*_{i,i}=\begin{pmatrix} h_2 & h_3 & & & \\ h_1 & h_2 & h_3 & & \\ & h_1 & h_2 & \ddots & \\ & & \ddots & \ddots & h_3 \\ & & & h_1 & h_2\end{pmatrix}\in\mathbb{R}^{(M_1-1)^2},$$

where $h_1=h_3=-p_x$, $h_2=a_{i,i}+2p_x+2p_y$.

Similarly, in the sense of variance, we can obtain

$$\begin{aligned}&-a_{1,k}\tilde{\mathbb{V}}\mathrm{ar}(u^0_{i,j})+\sum_{s=1}^{k-1}(a_{s,k}-a_{s+1,k})\tilde{\mathbb{V}}\mathrm{ar}(u^s_{i,j})+a_{k,k}\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i,j})\\ =&p_x\big(\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i+1,j})-2\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i,j})+\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i-1,j})\big)\\ &+p_y\big(\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i,j+1})-2\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i,j})+\tilde{\mathbb{V}}\mathrm{ar}(u^k_{i,j-1})\big)+\sum_{n,m=1}^{K}\sigma^2_{n,m}X_{n,m}(x_i,y_j).\end{aligned}$$

Let $\tilde{\mathbb{V}}\mathrm{ar}(U^k)$=($\tilde{\mathbb{V}}\mathrm{ar}(u^k_{1,1})$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{2,1})$, $\cdots$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{M_1-1,1})$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{1,2})$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{2,2})$, $\cdots$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{M_1-1,2})$, $\cdots$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{1,M_2-1})$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{2,M_2-1})$, $\cdots$, $\tilde{\mathbb{V}}\mathrm{ar}(u^k_{M_1-1,M_2-1}))^T$, $\Psi=(\sigma^2_{1,1}$, $\sigma^2_{2,1}$, $\cdots$, $\sigma^2_{M_1-1,1}$, $\sigma^2_{1,2}$, $\sigma^2_{2,2}$, $\cdots$, $\sigma^2_{M_1-1,2}$, $\cdots$, σ^2_{1,M_2-1}, σ^2_{2,M_2-1}, $\cdots$, $\sigma^2_{M_1-1,M_2-1})^T$, we can obtain

$$(A^*)^1\tilde{\mathbb{V}}\mathrm{ar}(U^1)=\Psi,$$

$$(A^*)^k\tilde{\mathbb{V}}\mathrm{ar}(U^k) = \Psi + \sum_{s=1}^{k-1}(a_{s+1,k} - a_{s,k})\tilde{\mathbb{V}}\mathrm{ar}(U^s).$$

Using the previously derived difference equations, we can compute the values of $\mathbb{E}(u(\cdot, T))$ and $\tilde{\mathbb{V}}\mathrm{ar}(u(\cdot, T))$ numerically, where the spatial variable "$\cdot$" denotes either x (in 1D) or (x, y) (in 2D). These results are then used to solve the inverse problem and obtain the regularized solution. In practice, measurements used in the inverse problem are subject to noise. To simulate this, we introduce noisy data

$$\mathbb{E}(u^\delta(\cdot, T)) = \mathbb{E}(u(\cdot, T))(1 + \frac{\varepsilon}{\pi}(2\mathrm{rand}(\mathrm{size}(\mathbb{E}(u(\cdot, T)) - 1)),$$
$$\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(\cdot, T)) = \tilde{\mathbb{V}}\mathrm{ar}(u(\cdot, T))(1 + \frac{\varepsilon}{\pi}(2\mathrm{rand}(\mathrm{size}(\tilde{\mathbb{V}}\mathrm{ar}(u(\cdot, T)) - 1)),$$

where rand$(\cdot)$ generates random numbers with mean 0 and variance 1, and ε denotes the noise level.

The absolute errors between the exact and noisy data are measured as

$$\delta_1 = \|\mathbb{E}(u^\delta(\cdot, \cdot, T)) - \mathbb{E}(u(\cdot, \cdot, T))\|,\ \delta_2 = \|\tilde{\mathbb{V}}\mathrm{ar}(u^\delta(\cdot, \cdot, T)) - \tilde{\mathbb{V}}\mathrm{ar}(u(\cdot, \cdot, T))\|,$$

with the L^2 norm over the spatial grid in 1D or 2D. Specifically, for the 2D case

$$\delta_1 = \sqrt{\frac{\sum_i \sum_j (\mathbb{E}(u_{i,j}^\delta) - \mathbb{E}(u_{i,j}))^2}{(M_1 + 1)(M_2 + 1)}},\ \delta_2 = \sqrt{\frac{\sum_i \sum_j (\tilde{\mathbb{V}}\mathrm{ar}(u_{i,j}^\delta) - \tilde{\mathbb{V}}\mathrm{ar}(u_{i,j}))^2}{(M_1 + 1)(M_2 + 1)}}.$$

The relative errors in the recovered functions are given by

$$e_f = \frac{\sqrt{\sum (f - f_\mu^\delta)^2}}{\sqrt{\sum f^2}},\ e_{\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)} = \frac{\sqrt{\sum (\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x) - \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)_\mu^\delta)^2}}{\sqrt{\sum (\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x))^2}}.$$

- **1D Numerical Examples**

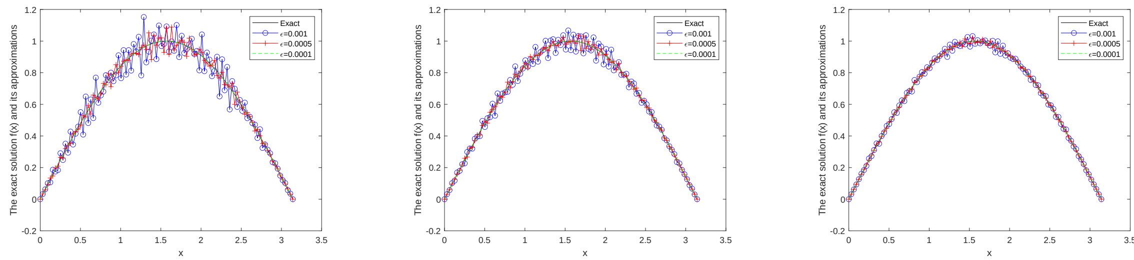
We know that it is difficult to obtain a priori boundary condition in practical applications and the selection of a priori regularization parameter is based on a priori boundary condition. Therefore, we only give the numerical results under the a posteriori regularization parameter selection rule. Let $l = \pi$, $a = 1$, $T = 5$, $\varphi(x) = x\sin(x)$, $M = 100$, $N = 100$, $\tau_1 = \tau_2 = 1.1$. Select $K = 10$ from the random term in $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x) = \sum_{n=1}^K \sigma_n^2 X_n(x)$. The eigenvalues and eigenfunctions of the operator $\mathcal{L}$ are

$$\lambda_n = n^2,\ X_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx),\ n = 1, 2, \cdots.$$

Here we give three numerical examples.

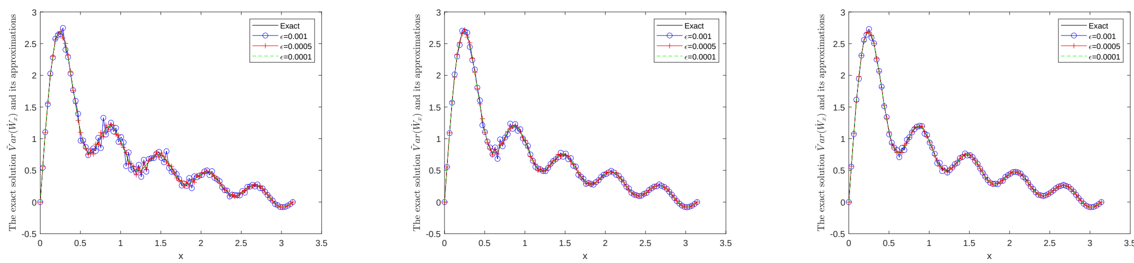
Example 5.1. Consider the following functions

$$f(x) = \sin(x),\ \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x) = \sum_{n=1}^{10} n^{-\frac{1}{2}} X_n(x).$$



(a) α=0.3 (b) α=0.5 (c) α=0.7

Figure 1. The comparison of the exact $f(x)$ and approximations $f_\mu^\delta(x)$ with $\varepsilon = 0.001, 0.0005, 0.0001$ for Example 5.1 at different α.



(a) α=0.3 (b) α=0.5 (c) α=0.7

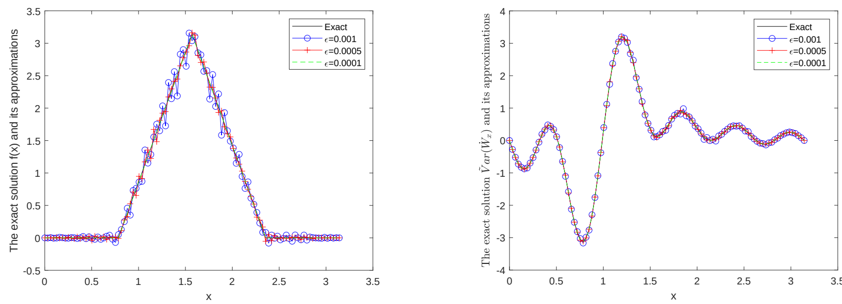
Figure 2. The comparison of the exact $\tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_x)$ and approximations $\tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_x)_\mu^\delta$ with $\varepsilon = 0.001, 0.0005, 0.0001$ for Example 5.1 at different α.

Figures 1 and 2 present the results of Example 5.1, showing the exact solutions $f(x)$ and $\tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_x)$, along with their quasi-boundary regularized solutions $f_\mu^\delta(x)$ and $\tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_x)_\mu^\delta$, under noise levels $\varepsilon = 0.001, 0.0005, 0.0001$. To assess the impact of the Caputo-Hadamard fractional order α, we test three values: 0.3, 0.5 and 0.7. As α increases, the solutions become smoother and more stable, demonstrating the stronger regularization effect of higher-order fractional derivatives. In addition, when the deterministic term $f(x)$ is continuous and the variance sequence σ_n^2 in the random term follows an exponential decay of the form $n^{-\frac{1}{2}}$, the quasi-boundary regularized solution approximates the exact solution with high accuracy.

Example 5.2. Fix $\alpha = 0.3$, and consider the following functions

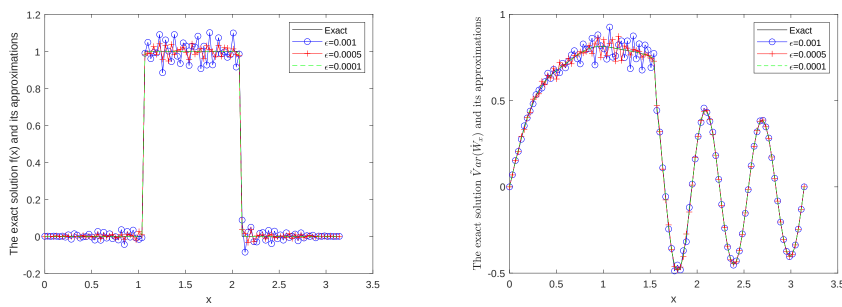
$$f(x) = \begin{cases} 0, & x \in [0, \frac{\pi}{4}), \\ 4\left(x - \frac{\pi}{4}\right), & x \in [\frac{\pi}{4}, \frac{\pi}{2}), \\ -4\left(x - \frac{3\pi}{4}\right), & x \in [\frac{\pi}{2}, \frac{3\pi}{4}), \\ 0, & x \in [\frac{3\pi}{4}, \pi], \end{cases} \quad \tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_x) = \sum_{n=1}^{10} \cos(-n) X_n(x).$$

Figure 3 displays the corresponding results for Example 5.2. Here, $f(x)$ is a non-smooth function, the variance sequence σ_n^2 is a trigonometric function of the form $\cos(-n)$. As can be seen, the numerical results are in good agreement with the exact solutions.



(a) $f(x)$ (b) $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)$

Figure 3. The comparison of the exact solutions and the regularized solutions with $\varepsilon = 0.001, 0.0005, 0.0001$ for Example 5.2.



(a) $f(x)$ (b) $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)$

Figure 4. The comparison of the exact solutions and the regularized solutions with $\varepsilon = 0.001, 0.0005, 0.0001$ for Example 5.3.

Example 5.3. Fix $\alpha = 0.3$, and consider the following functions

$$f(x) = \begin{cases} 0, & x \in [0, \dfrac{\pi}{3}), \\ 1, & x \in [\dfrac{\pi}{3}, \dfrac{2\pi}{3}), \\ 0, & x \in [\dfrac{2\pi}{3}, \pi], \end{cases} \quad \tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x) = \begin{cases} \displaystyle\sum_{n=1}^{10} n^{-2} X_n(x), & x \in [0, \dfrac{\pi}{2}), \\ \displaystyle\sum_{n=1}^{10} \cos(-\frac{1}{n}) X_n(x), & x \in [\dfrac{\pi}{2}, \pi]. \end{cases}$$

Figure 4 displays the corresponding results for Example 5.3. It can be observed that when $f(x)$ is a discontinuous function, the variance sequence σ_n^2 forms a sequence of discontinuous functions, the approximation of the exact solutions by the quasi-boundary regularized solutions is relatively poor.

Table 1 shows that, with fixed time-fractional order α, the relative errors of both the exact and regularized solutions decrease as the noise level ε decreases from 0.001 to 0.0001.

From the examples above and Table 1, it can be observed that smaller noise levels ε and larger time-fractional order α both lead to better approximation results. Moreover, the quasi-boundary regularization method performs better when the recovered function is continuous, compared to the discontinuous case. Figures 1 to 4 show that the regularized solutions closely approximate the exact ones, indicating the high effectiveness of the quasi-boundary regulariza-

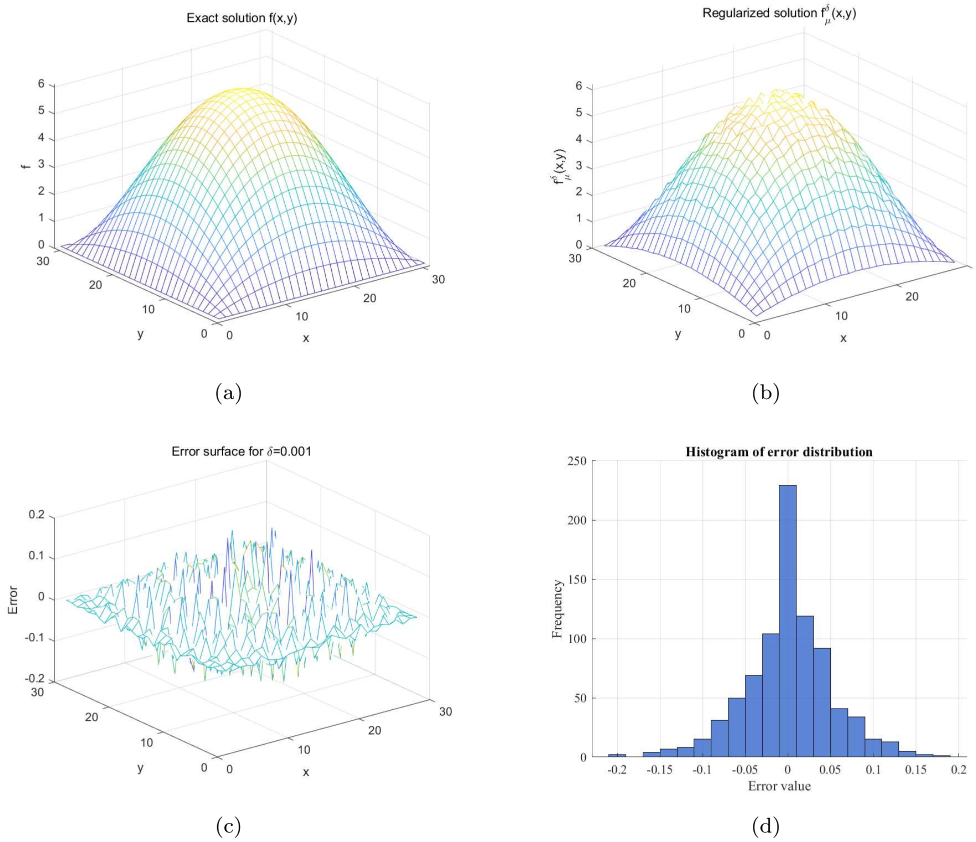


Figure 5. Exact solution, regularized solution, error surface, and error distribution histogram of $f(x, y)$.

Table 1. Relative errors for Examples 5.1 to 5.3 at different values of ε with fixed $\alpha = 0.3$.

	ε	0.001	0.0005	0.0001
Example 5.1	e_f	0.0738	0.0358	0.0009
	$e_{\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)}$	0.0573	0.0301	0.0007
Example 5.2	e_f	0.0705	0.0377	0.0006
	$e_{\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)}$	0.0285	0.0139	0.0005
Example 5.3	e_f	0.0633	0.0353	0.0053
	$e_{\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_x)}$	0.0656	0.0367	0.0013

tion method in identifying the source term in the stochastic Caputo-Hadamard time-fractional diffusion equation.

- **2D Numerical Example**

In 2D case, we suppose $l_1 = l_2 = \pi$, $a = 1$, $T = 5$, $\varphi(x, y) = x\cos(y)$, $M_1 = M_2 = 30$, $N = 30$, $\tau_1 = \tau_2 = 1.01$. Select $K = 5$ from the random term in $\tilde{\mathbb{V}}\mathrm{ar}(\dot{\mathcal{W}}_{xy}) = \sum_{n,m=1}^{K} \sigma_{n,m}^2 X_{n,m}(x, y)$.

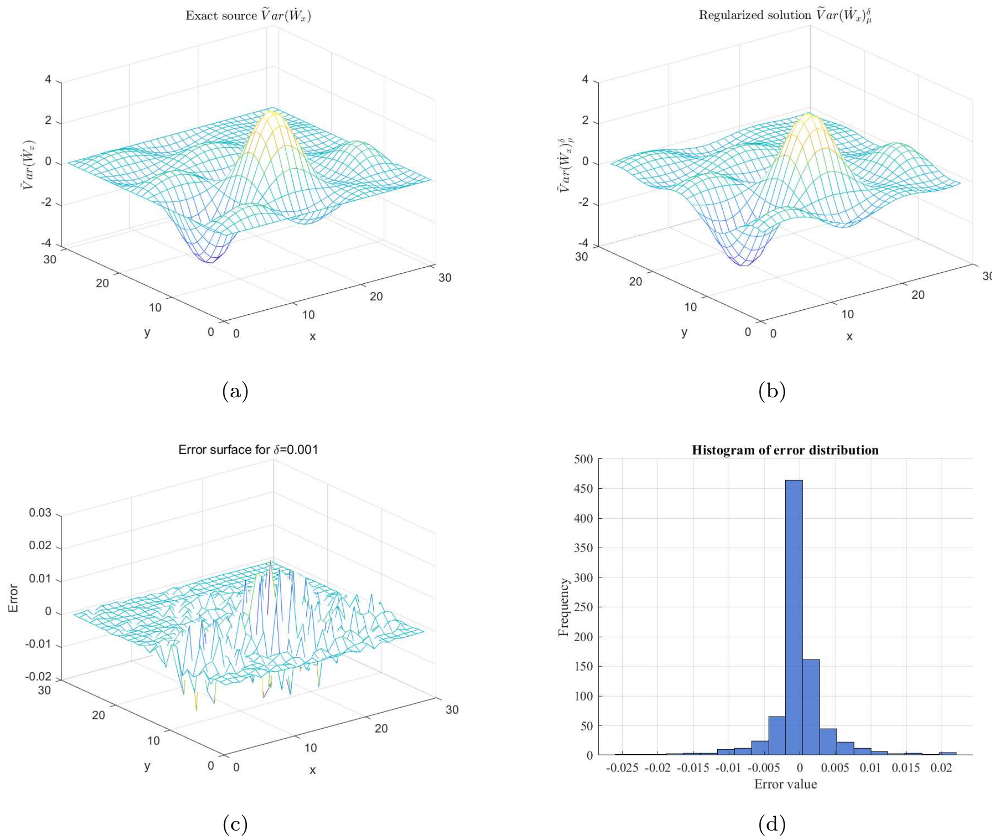


Figure 6. Exact solution, regularized solution, error surface, and error distribution histogram of $\tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_{xy})$.

The eigenvalues and eigenfunctions of the operator $\mathcal{L}$ are

$$\lambda_{n,m} = n^2 + m^2,\ X_{n,m}(x,y) = \sqrt{\frac{4}{\pi^2}} \sin(nx)\sin(my),\ n,m = 1,2,\cdots.$$

Example 5.4. Fix $\alpha = 0.3$ and $\varepsilon = 0.001$, we consider the following functions

$$f(x,y) = xy(x-\pi)(y-\pi),\ \tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_{xy}) = \sum_{n,m=1}^{5} \sin(n)\cos(m) X_{n,m}(x,y).$$

Figures 5 and 6 illustrate the reconstruction results of $f(x,y)$ and $\tilde{\mathbb{V}}\text{ar}(\dot{\mathcal{W}}_{xy})$, respectively, under fixed parameters $\alpha = 0.3$ and $\varepsilon = 0.001$. In both figures, subfigure (a) shows the exact solution, while (b) presents the corresponding regularized solution. Subfigure (c) displays the error surface between the exact and regularized solutions for $\delta = 0.001$, and (d) shows the histogram of the error distribution. It can be observed that most error values are concentrated near zero in both cases. In Figure 5(d), the errors are primarily within the range of -0.1 to 0.1, with only a few exceeding ± 0.15. In contrast, Figure 6(d) shows a narrower error range, mostly between -0.01 and 0.01, with only a small number exceeding ± 0.015. These results indicate that the regularized solutions exhibit relatively small systematic bias. These visualizations show that the quasi-boundary regularization method can produce accurate and stable results when the noise level is small.

6. Conclusions

In this paper, we investigate an inverse problem aimed at determining the random source term of the stochastic Caputo-Hadamard time-fractional diffusion equation. The solutions to this stochastic inverse problem are derived by utilizing the expectation and variance of the final value data $u(x,T)$. To achieve regularized solutions, we employ the quasi-boundary regularization method. By separately handling the deterministic and random terms, error estimates are derived using both a priori and a posteriori rules for selecting the regularization parameter. Numerical experiments in both one-dimensional and two-dimensional cases are conducted to validate the effectiveness of the proposed method. The results show that smaller noise level ε and larger time-fractional order α lead to better approximation performance. Moreover, the quasi-boundary regularization method yields more accurate and stable results when the recovered function is continuous, compared to the discontinuous case. However, these experiments are limited to idealized examples. In future work, we plan to extend the framework to practical applications involving complex physical processes by incorporating physical modeling and experimental data.

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