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ANALYTICAL STUDY OF A FRACTIONAL PARTIAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract In this paper, we consider a fractional partial integro-differential equation (frPI-DE) associated with a quadratic integral equation of order $\gamma > 0$. By employing the properties of fractional integrals, the frPI-DE is shown to be equivalent to a nonlinear quadratic Volterra–Fredholm integral equation. We investigate the existence of solutions to such nonlinear functional integral equations in the Banach space $L_2[0,1] \times C[0,T]$. The existence results are established via a generalized form of Darbo's fixed-point theorem, which is based on the measure of noncompactness. To illustrate the applicability and effectiveness of the theoretical findings, two examples are provided. In addition, a numerical scheme is introduced to demonstrate the convergence of approximate solutions to the nonlinear quadratic Volterra–Fredholm integral problem. The uniqueness and stability of the error are also analyzed.

Keywords Fractional partial integro-differential equation, Caputo fractional derivative, quadratic Volterra–Fredholm integral equation, measure of noncompactness, Darbo's fixed point theorem, Riemann integral.

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1. Introduction

Nonlinear integral equations, which combine both integral operators and nonlinear functions, play a fundamental role in modeling complex phenomena across various scientific and engineering disciplines [7, 10, 12]. Among them, quadratic integral equations (QIEs) have attracted significant attention due to their ability to describe systems where interactions between variables are inherently nonlinear. These equations arise naturally in numerous applied contexts, including the kinetic theory of gases, radiative transfer, neutron transport, traffic flow, and queuing theory [24, 37]. Consequently, a growing body of research has focused on analyzing the existence and uniqueness of solutions to various forms of nonlinear QIEs [8, 9, 11, 19].

Fractional calculus and fractional-order differential equations have been used in many fields of science and engineering. Additionally, compared to integer-order derivatives, this technique provides more accurate representations of real-world problems. Because of their many applications in fluid flow, biological processes, and physics, see [14, 30, 33, 34]. Numerous researchers, such as Patil et al. [28], Petras et al. [29], Baitiche et al. [13], Shakeel et al. [31], Zhou et al. [38],

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Kai et al. [22], Wahash et al. [36], Noor et al. [27], and many others, have given the analysis of time-fractional ordinary and partial differential equations.

The intersection of these two domains, fractional and quadratic integro-differential equations, has spurred extensive research. In particular, fractional integro-differential equations provide a flexible framework for modeling dynamic systems with nonlocal and memory-dependent behavior, making them highly relevant in areas such as biology, electrochemistry, economics, electromagnetics, and control theory [18,32]. Consequently, numerous works have explored the analytical properties of these equations under various assumptions and settings.

Several authors have addressed the existence and uniqueness of solutions to abstract fractional integro-differential equations using a variety of methods. For instance, Aissaoui et al. [6] examined the solvability of nonlinear Volterra–Fredholm equations under suitable conditions. Abdo et al. [1] applied Banach's and Krasnoselskii's fixed-point theorems to fractional integro-differential equations with nonlocal boundary conditions, while Hamdy et al. [5] studied impulsive delay equations of Sobolev type. The semigroup approach has also been utilized effectively to establish well-posedness in Banach spaces, as demonstrated in Abdou et al. [2].

The fixed-point theory (FPT) remains a central analytical tool in this area. Originally developed within pure mathematics, FPT has found broad application across nonlinear analysis and applied sciences [15,21,23,26,35]. In particular, it serves as a fundamental technique for proving the existence of solutions to functional integral equations, an esteemed class of equations known for modeling a wide variety of real-world processes [3,17,25].

Motivated by these developments, this paper addresses a class of fractional partial integro-differential equations that emerge from nonlinear quadratic Volterra–Fredholm integral equations. Our objective is to establish existence results within the Banach space $L_2[0,1] \times C[0,T]$ using a generalized form of Darbo's fixed-point theorem, which is based on the measure of noncompactness. Additionally, we present illustrative examples to validate the theoretical findings and propose a numerical method to study the convergence, uniqueness, and stability of the solutions.

This article is structured in the following basic way: We introduced the formulation of the nonlinear quadratic Volterra-Fredholm integral equation and special cases in Section 2. In Section 3, some basic definitions and preliminary information about fractional calculus are reviewed. In Section 4, we provide sufficient conditions to prove the existence of at least one solution for the problem (2.3) by using the generalized Darbo fixed point theorem associated with the measure of noncompactness in the space $L_2[0,1] \times C[0,T]$. Also, we present two examples to illustrate our theorem. In Section 5, the convergence of the solutions is proved. Afterward, in Section 6, we elucidate the stability of the error. A conclusion is provided in Section 7. At last, in Section 8, future work is given.

2. Formulation of the nonlinear quadratic Volterra-Fredholm integral equation and special cases

In our work, we study the following fractional partial integro-differential equation (frPI-DE) with kind of the quadratic integral equation:

$$\frac{\partial^\gamma}{\partial t^\gamma}\left[\frac{\Psi(x,t)-f(x,t)}{\Psi(x,t)+\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau}\right] = \xi(t)\int_0^1 k(x,y)\nu(\Psi(y,t))\mathrm{d}y;\ \gamma>0, \tag{2.1}$$

with the initial condition

$$\Psi(x,0) = f(x,0), \tag{2.2}$$

where $t \in J = [0,T]$, with Caputo fractional derivatives $\frac{\partial^\gamma}{\partial t^\gamma}$ order γ, so that $\gamma > 0$, and $\Psi(x,t) \in L_2[0,1] \times C[0,T]$, where $L_2[0,1] \times C[0,T]$ is a Banach space. $f(x,t)$ is the known continuous function in $L_2[0,1] \times C[0,T]$. In addition, $p(t,\tau)$, $\xi(t)$ and $\nu(\Psi(y,t))$ are the known continuous functions. The kernel $k(x,y)$, in general, has a nonsingular term.

Integrating equation (2.1) and using equation (2.2), we get

$$\left[\frac{\Psi(x,t) - f(x,t)}{\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau}\right]_0^t = \frac{1}{\Gamma(\gamma)}\int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau,$$

we obtain

$$\frac{\Psi(x,t) - f(x,t)}{\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau} = \frac{1}{\Gamma(\gamma)}\int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau.$$

Then

$$\begin{aligned}\Psi(x,t) = f(x,t) + \frac{1}{\Gamma(\gamma)}[\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\ \times \int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{2.3}$$

The equation (2.3) is called the nonlinear quadratic Volterra-Fredholm integral equation (NQ-VFIE).

We thank the reviewer for this note. It is noticeable that the kernel in equation (2.1) is position-specific, describing the various properties of the matter being studied. Then, after performing fractional integration, we find that the new kernel in equation (2.3) became position- and time-specific. This means that the general kernel is now associated with changes in both position and time.

Special cases

(I) In the case where $\xi(t) = \lambda$, the nonlinear quadratic Volterra-Fredholm integral equation becomes

$$\frac{\Psi(x,t) - f(x,t)}{\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau} = \frac{\lambda}{\Gamma(\gamma)}\int_0^t\int_0^1 (t-\tau)^{\gamma-1}k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau;\ 0 < \gamma < 1. \tag{2.4}$$

From equation (2.4), when $\gamma \to 0$, we obtain

$$\frac{\Psi(x,t) - f(x,t)}{\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau} = \lambda\int_0^t\int_0^1 k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau;\ \gamma \to 0. \tag{2.5}$$

It is noticeable in this case that the time kernel of the Volterra-Fredholm integral is a constant in the case of ordinary time integration, meaning that time affects only the unknown function. In fractional differentiation, however, its effect appears in the Volterra-Fredholm integral as the time-dependent Abel function. This type of differentiation clearly demonstrates the effect of fractional differentiation or integration on the result of ordinary differentiation, which carries physical implications in describing and sequencing the unknown function over fractional time.

(II) Assuming that $k(x,y) \to 1$, we obtain

$$\frac{\partial^\gamma}{\partial t^\gamma}\left[\frac{\Psi(x,t)-f(x,t)}{\Psi(x,t)+\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau}\right] = \xi(t)M(t); \quad \int_0^1 \nu(\Psi(y,t))\mathrm{d}y = M(t). \tag{2.6}$$

A crucial case in studying a problem is the use of what is known as the pressure condition. This condition is that the integration of the unknown function of the position over the integration period gives the total pressure on the body. If the problem to be studied is at rest, the amount of pressure on the body is provided, and this is called the static pressure condition, to ensure a solution. If time is considered as a variable, the condition is called the dynamic pressure.

In this case, we obtain the following fractional differential equation:

$$\frac{\partial^\gamma}{\partial t^\gamma}\left[\frac{\Psi(x,t)-f(x,t)}{\Psi(x,t)+\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau}\right] = F(t). \tag{2.7}$$

In this case, it is possible to study the following special cases:

1. $(1-F(t))\Psi(x,t) = f(x,t) + F(t)\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau; \quad \gamma \to 0.$
2. $\Psi(x,t) = f(x,t) + \frac{1}{\Gamma(\gamma)}[\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau] \times \int_0^t (t-\tau)^{\gamma-1}F(\tau)\mathrm{d}\tau;\ 0<\gamma<1.$
3. $(1-V(t))\Psi(x,t) = f(x,t) + V(t)\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau; \quad \gamma \to 1,\ \int_0^t F(\tau)\mathrm{d}\tau = V(t).$

(III) If $p(t,\tau) = 0$, we have

$$\frac{\partial^\gamma}{\partial t^\gamma}\left[\frac{\Psi(x,t)-f(x,t)}{\Psi(x,t)}\right] = \xi(t)\int_0^1 k(x,y)\nu(\Psi(y,t))\mathrm{d}y. \tag{2.8}$$

This condition, in physical terms, means that the body to be studied is free of internal resistance over time (loss of immunity). In this case, the researcher can address the solutions to the following problems:

1. $\Psi(x,t) = f(x,t) + \xi(t)\Psi(x,t)\int_0^1 k(x,y)\nu(\Psi(y,t))\mathrm{d}y; \quad \gamma \to 0.$
2. $\Psi(x,t) = f(x,t) + \frac{\Psi(x,t)}{\Gamma(\gamma)}\int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau; \quad 0<\gamma<1.$
3. $\Psi(x,t) = f(x,t) + \Psi(x,t)\int_0^t\int_0^1 \xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau; \quad \gamma \to 1.$

(IV) If $f(x,t) = 0$ we have

$$\frac{\partial^\gamma}{\partial t^\gamma}\left[\frac{\Psi(x,t)}{\Psi(x,t)+\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau}\right] = \xi(t)\int_0^1 k(x,y)\nu(\Psi(y,t))\mathrm{d}y. \tag{2.9}$$

This equation and this type of problem are of interest to students of space physics and lasers. They are also of interest to researchers studying pulses, with the important note: When studying this type of problem, no initial conditions are imposed on the problem.

3. Definitions and fundamentals

This section presents a comprehensive overview of the fundamental concepts and distinctive characteristics underpinning the theory of fractional calculus. Key concepts discussed include the definitions of fractional derivatives and integrals, such as the Riemann–Liouville, Caputo function and their mathematical properties. Also, we also provide a definition of the measure of noncompactness, and among the most important theorems that we present here is the fixed-point theorem with Darbo type. Last, we define some quantities.

Definition 3.1. ([20]) Fractional integral of function based on the Riemann-Liouville formula $\Psi : (0, \infty) \to R$, consists of the definition that

$$I_x^\gamma \Psi(x) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^x \Psi(y)(x-y)^{\gamma-1} \mathrm{d}y; & 0 < \gamma < 1, \\ \Psi(x); & \gamma \to 0. \end{cases} \tag{3.1}$$

Definition 3.2. ([4]) The fractional derivative of the Caputo function $\Psi : (0, \infty) \to R$, consists of a description that

$$D_x^\gamma \Psi(x) = \frac{1}{\Gamma(k-\gamma)} \int_0^x \Psi^{(k)}(y)(x-y)^{k-\gamma-1} \mathrm{d}y, \tag{3.2}$$

where Γ is the gamma function and $\gamma \in (k-1, k);\ k \in \mathbb{N}$.

Lemma 3.1. (see [20]) *Let $k-1 < \gamma < k,\ k \in \mathbb{N},\ \gamma \in \mathbb{R}$, and $\Psi(x)$ be a function such that $D_x^\gamma(\Psi(x))$ exists. Then the following for the Caputo fractional derivative hold:*

$$\lim_{\gamma \to k} D_x^\gamma(\Psi(x)) = \Psi^{(k)}(x),$$
$$\lim_{\gamma \to k-1} D_x^\gamma(\Psi(x)) = \Psi^{(k-1)}(x) - \Psi^{(k-1)}(0).$$

Therefore, we have the following characteristics:

1. $I_x^\gamma I_x^\mu \Psi(x) = I_x^{\gamma+\mu} \Psi(x);\ \gamma, \mu > 0.$
2. $I_x^\gamma x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha};\ \gamma, x > 0,\ \alpha > -1.$
3. $I_x^\gamma D_x^\gamma \Psi(x) = \Psi(x) - \sum_{n=0}^{k-1} \Psi^n(0) \frac{x^n}{n!};\ x > 0,\ k-1 < \gamma < k.$
4. $D_x^\gamma I_x^\gamma \Psi(x) = \Psi(x);\ x > 0,\ k-1 < \gamma < k.$

Definition 3.3. (see [4]) The Riemann – Liouville fractional derivative of order $\gamma > 0$ is normally defined as

$$D_x^k I_x^{k-\gamma} \Psi(x) = D_x^\gamma \Psi(x);\ k-1 < \gamma < k.$$

In the following, we will introduce some notations and background information used in the analytical section of the work.

Definition 3.4. (see [16]) A function $\mu : N_W \to R^+ = [0, +\infty)$ is said to be the measure of noncompactness in W, if it satisfies the following conditions:

(1*) The family $\ker \mu = \{\Theta \in N_W : \mu(\Theta) = 0\}$ is nonempty and $\ker \mu \subset M_W$;

(2*) $\Theta \subset \Phi \Rightarrow \mu(\Theta) \leq \mu(\Phi)$;

(3*) $\mu(\bar{\Theta}) = \mu(\mathrm{Conv}\ \Theta) = \mu(\Theta)$;

(4*) $\mu(\lambda\Theta + (1-\lambda)\Phi) \leq \lambda\mu(\Theta) + (1-\lambda)\mu(\Phi)$ for $0 \leq \lambda \leq 1$;

(5*) if $\{\Theta_j\}$ is a sequence of closed sets from N_W such that $\Theta_{j+1} \subset \Theta_j$ for $j = 1, 2, ...$, and if $\lim_{j\to\infty} \mu\{\Theta_j\} = 0$, then the set $\Theta_\infty = \bigcap_{j=1}^\infty \Theta_j$ is nonempty.

For our purposes, we will only need the following fixed–point theorem of Darbo type. Let us suppose that N is a nonempty subset of a Banach space W and the operator $V : N \to W$ is continuous and transforms bounded sets onto bounded ones. We say that V satisfies the Darbo condition (with constant $\beta \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset Θ of N, we have

$$\mu(V\Theta) \leq \beta\mu(\Theta).$$

If V satisfies the Darbo condition with $\beta < 1$, then it is called a contraction with respect to μ.

Theorem 3.1. *Let S be a nonempty, bounded, closed and convex subset of the Banach space W and μ a measure of noncompactness in W. Let $V : S \to S$ be a contraction with respect to μ, i.e., there exists a constant $\beta \in [0, 1)$ such that*

$$\mu(V\Theta) \leq \beta\mu(\Theta),$$

for any nonempty subset Θ of S. Then V has at least one fixed point in the set S.

Let us fix a nonempty and bounded subset Θ of $C(I)$. For $\Psi \in \Theta$ and $\epsilon \geq 0$ denoted by $\omega(\Psi, \epsilon)$, the modulus of continuity of Ψ defined by

$$\omega(\Psi, \epsilon) = \sup\{|\Psi(x, t) - \Psi(x, s)| : x, t, s \in [0, 1],\ |t - s| \leq \epsilon\}.$$

Further, let us put

$$\begin{aligned}\omega(\Theta, \epsilon) &= \sup\{\omega(\Psi, \epsilon) : \Psi \in \Theta\},\\ \omega_0(\Theta) &= \sup_{\epsilon\to 0}\{\omega(\Theta, \epsilon)\}.\end{aligned}$$

Moreover, let us define the following quantities,

$$\begin{aligned}d(\Psi) &= \sup\{|\Psi(x, s) - \Psi(x, t)| - [\Psi(x, s) - \Psi(x, t)] : x, t, s \in [0, 1],\ t \leq s\},\\ d(\Theta) &= \sup\{d(\Psi) : \Psi \in \Theta\}.\end{aligned}$$

The quantity $d(\Theta)$ measures the degree of decrease of functions from the set Θ.

Finally, let us define the function μ on the family $N_{L_2[0,1]\times C[0,T]}$ by the formula

$$\mu(\Theta) = \omega_0(\Theta) + d(\Theta).$$

The function μ is a measure of noncompactness in the space $L_2[0, 1] \times C[0, T]$.

4. Existence of the solution

Equation (2.3) can be expressed using the integral operator form as follows:

$$(V\Psi)(x, t) = f(x, t) + \frac{1}{\Gamma(\gamma)}(E_1\Psi)(x, t) \times (E_2\Psi)(x, t), \tag{4.1}$$

where

$$\begin{aligned}E_1\Psi(x, t) &= \Psi(x, t) + \int_0^t p(t, \tau)\Psi(x, \tau)\mathrm{d}\tau,\\ E_2\Psi(x, t) &= \int_0^t\int_0^1 (t - \tau)^{\gamma-1}\xi(\tau)k(x, y)\nu(\Psi(y, \tau))\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

In order to discuss the existence of at least one solution of equation (2.3), we assume the following assumptions:

(i) $f:[0,1]\to R$ is a continuous, nondecreasing and nonnegative function on $[0,1]$.

(ii) $k:[0,1]\times[0,1]\to R$ is continuous and the functions $x\to k(x,y)$ and $y\to k(x,y)$ are continuous on $[0,1]$ for fixed $y\in[0,1]$ and $x\in[0,1]$, respectively, such that $|k(x,y)|\le A$, where A is a positive constant.

(iii) The functions of time $p(t,\tau),\ \xi(\tau)$ are continuous in the space $C[0,T]$ and satisfies $|p(t,\tau)|\le Q,\quad |\xi(\tau)|\le D$, where Q,D are positive constants.

(iv) The known function $\nu(\Psi(x,\tau))$, for the constants $B>B_1$ and $B>B_2$, satisfies:

$$\begin{aligned}&(iv-a)\quad |\nu(\Psi(x,\tau))|\le B_1\Psi(x,\tau),\\ &(iv-b)\quad |\nu(\Psi_1(x,t))-\nu(\Psi_2(x,t))|\le B_2|\Psi_1(x,t)-\Psi_2(x,t)|.\end{aligned}$$

(v) There exists $r_0>0$ such that,

$$\|f\|+\frac{T^\gamma}{\Gamma(\gamma+1)}DAB(1+QT)r_0^2\le r_0;\ T^\gamma DABr_0\le\Gamma(\gamma+1).$$

Now we can formulate the main existence theorem.

Theorem 4.1. *Let the assumptions $(i)-(v)$ be satisfied, the equation (2.2) has at least one solution $\Psi=\Psi(x,t)$ which is nondecreasing on the interval $[0,1]$.*

Proof. Let us consider the operator V defined on the space $L_2[0,1]\times C[0,T]$ by the formula (4.2).

Taking into account assumptions $(i)-(iv)$ and the properties of the superposition operator, we infer that the function $V\Psi$ is continuous on $[0,1]$ for any function $\Psi\in L_2[0,1]\times C[0,T]$, i.e., the operator V transforms the space $L_2[0,1]\times C[0,T]$ into itself. Further, applying our assumptions, we derive the following estimate:

$$\begin{aligned}(V\Psi)(x,t)=&f(x,t)+\frac{1}{\Gamma(\gamma)}[\Psi(x,t)+\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\ &\times\int_0^t\int_0^1(t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Then from assumptions $(i),(iv),\ V(\Psi)\in L_2[0,1]\times C[0,T]$ and V is will defined

$$\begin{aligned}|(V\Psi)(x,t)|\le&|f(x,t)|+\frac{1}{\Gamma(\gamma)}[|\Psi(x,t)|+\int_0^t|p(t,\tau)||\Psi(x,\tau)|\mathrm{d}\tau]\\ &\times\int_0^t\int_0^1|(t-\tau)^{\gamma-1}||\xi(\tau)||k(x,y)||\nu(\Psi(y,\tau))|\mathrm{d}y\mathrm{d}\tau\\ \le&|f(x,t)|+\frac{1}{\Gamma(\gamma)}[|\Psi|+QT|\Psi|]\times\frac{T^\gamma}{\gamma}DAB|\Psi|.\end{aligned}$$

Hence, we obtain

$$\|(V\Psi)(x,t)\|\le\|f(x,t)\|+\frac{1}{\Gamma(\gamma+1)}\|T^\gamma DAB(1+QT)\Psi^2\|.$$

For $r_0>0$ such that $\|\Psi\|\le r_0$, by assumption (v), $\|(V\Psi)(x,t)\|\le r_0$, i.e., the operator V transforms the ball B_{r_0} into itself.

In what follows, we will consider the operator V on the subset $B_{r_0}^+$ of the ball B_{r_0} defined by:

$$B_{r_0}^+ = \{\Psi \in B_{r_0} : \Psi(x,t) \geq 0, \quad \text{for} \quad (x,t) \in [0,1] \times [0,1]\}.$$

Notice that the set $B_{r_0}^+$ is nonempty, bounded, closed, and convex. Hence and in view of assumptions $(i)-(v)$, we deduce easily that V transforms the set $B_{r_0}^+$ into itself.

Now, we show that V is continuous on the set $B_{r_0}^+$. To do this let us fix $\epsilon > 0$ and choose $\varrho > 0$ according to the continuity of V. Further, take arbitrarily $\Psi, \Phi \in B_{r_0}^+$ such that $\|\Psi - \Phi\| \leq \varrho$. Then, for $(x,t) \in [0,1] \times [0,1]$, we derive the following estimates:

$$\begin{aligned}
(V\Psi)(x,t) - (V\Phi)(x,t) = {} & \frac{1}{\Gamma(\gamma)}[\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\
& \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\
& - \frac{1}{\Gamma(\gamma)}[\Phi(x,t) + \int_0^t p(t,\tau)\Phi(x,\tau)\mathrm{d}\tau] \\
& \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\
& + \frac{1}{\Gamma(\gamma)}[\Phi(x,t) + \int_0^t p(t,\tau)\Phi(x,\tau)\mathrm{d}\tau] \\
& \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\
& - \frac{1}{\Gamma(\gamma)}[\Phi(x,t) + \int_0^t p(t,\tau)\Phi(x,\tau)\mathrm{d}\tau] \\
& \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Phi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\
= {} & \frac{1}{\Gamma(\gamma)}[(\Psi(x,t) - \Phi(x,t)) + \int_0^t p(t,\tau)(\Psi(x,\tau) - \Phi(x,\tau))\mathrm{d}\tau] \\
& \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\
& + \frac{1}{\Gamma(\gamma)}[\Phi(x,t) + \int_0^t p(t,\tau)\Phi(x,\tau)\mathrm{d}\tau] \\
& \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi(y,\tau)) - \nu(\Phi(y,\tau))\}\mathrm{d}y\mathrm{d}\tau.
\end{aligned}$$

Using the properties of the normal, we get

$$\begin{aligned}
\|(V\Psi)(x,t) - (V\Phi)(x,t)\| \leq {} & \left\| \frac{1}{\Gamma(\gamma)}[(\Psi(x,t) - \Phi(x,t)) + \int_0^t p(t,\tau)(\Psi(x,\tau) - \Phi(x,\tau))\mathrm{d}\tau] \right\| \\
& \times \left\| \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \right\| \\
& + \left\| \frac{1}{\Gamma(\gamma)}[\Phi(x,t) + \int_0^t p(t,\tau)\Phi(x,\tau)\mathrm{d}\tau] \right\| \\
& \times \left\| \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi(y,\tau)) - \nu(\Phi(y,\tau))\}\mathrm{d}y\mathrm{d}\tau \right\|.
\end{aligned}$$

Applying assumptions $(i)-(iv)$, we obtain

$$\begin{aligned}\|(V\Psi)(x,t)-(V\Phi)(x,t)\| \leq & \frac{\|\Psi(x,t)-\Phi(x,t)\|}{\Gamma(\gamma+1)}[1+QT]\times T^{\gamma}DAB\|\Psi(x,t)\| \\ & +\frac{\|\Phi(x,t)\|}{\Gamma(\gamma+1)}[1+QT]\times T^{\gamma}DAB\|\Psi(x,t)-\Phi(x,t)\| \\ \leq & \frac{2r_0}{\Gamma(\gamma+1)}T^{\gamma}DAB[1+QT]\varrho.\end{aligned}$$

The above estimate allows us to deduce that the operator V is continuous on the set $B^{+}_{r_0}$.

Now, let us take a nonempty set Θ, such that $\Theta \in B^{+}_{r_0}$. Further, fix arbitrarily $\epsilon > 0$ and choose $\Psi \in \Theta$ and $t_1, t_2 \in [0,1]$ such that $|t_2 - t_1| \leq \epsilon$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned} & (V\Psi)(x,t_2)-(V\Psi)(x,t_1) \\ = & (f(x,t_2)-f(x,t_1))+\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\ & \times\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ & -\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\ & \times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ & +\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\ & \times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ & -\frac{1}{\Gamma(\gamma)}[\Psi(x,t_1)+\int_0^{t_1}p(t_1,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\ & \times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ = & (f(x,t_2)-f(x,t_1))+\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\ & \times\Big(\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ & -\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\Big) \\ & +\frac{1}{\Gamma(\gamma)}\Big[(\Psi(x,t_2)-\Psi(x,t_1))+(\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau-\int_0^{t_1}p(t_1,\tau)\Psi(x,\tau)\mathrm{d}\tau)\Big] \\ & \times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Using the properties of the normal, we obtain

$$\|(V\Psi)(x,t_2)-(V\Psi)(x,t_1)\|$$

$$
\begin{aligned}
&\le \|f(x,t_2)-f(x,t_1)\| + \frac{\|\Psi(x,t_2)\|}{\Gamma(\gamma)}\left\|1+\int_0^{t_2} p(t_2,\tau)\mathrm{d}\tau\right\| \\
&\quad \times \left\|\int_0^{t_2}\int_0^1 (t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)|\nu(\Psi(y,\tau))|\mathrm{d}y\mathrm{d}\tau\right. \\
&\quad \left. - \int_0^{t_1}\int_0^1 (t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)|\nu(\Psi(y,\tau))|\mathrm{d}y\mathrm{d}\tau\right\| \\
&\quad + \frac{1}{\Gamma(\gamma)}\left\|\|\Psi(x,t_2)-\Psi(x,t_1)\| + \|\Psi(x,t_2)\|\|\int_0^{t_2} p(t_2,\tau)\mathrm{d}\tau - \int_0^{t_1} p(t_1,\tau)\mathrm{d}\tau)\|\right\| \\
&\quad \times \left\|\int_0^{t_1}\int_0^1 (t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)|\nu(\Psi(y,\tau))|\mathrm{d}y\mathrm{d}\tau\right\|.
\end{aligned}
$$

From condition (iv), we get

$$
\begin{aligned}
&\|(V\Psi)(x,t_2)-(V\Psi)(x,t_1)\| \\
\le &\|f(x,t_2)-f(x,t_1)\| + \frac{B\|\Psi(x,t_2)\|^2}{\Gamma(\gamma)}\left\|1+\int_0^{t_2} p(t_2,\tau)\mathrm{d}\tau\right\| \\
&\times \left\|\int_0^{t_2}\int_0^1 (t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau - \int_0^{t_1}\int_0^1 (t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau\right\| \\
&+ \frac{B\|\Psi(x,t_2)\|}{\Gamma(\gamma)}\left\|\|\Psi(x,t_2)-\Psi(x,t_1)\| + \|\Psi(x,t_2)\|\|\int_0^{t_2} p(t_2,\tau)\mathrm{d}\tau - \int_0^{t_1} p(t_1,\tau)\mathrm{d}\tau)\|\right\| \\
&\times \left\|\int_0^{t_1}\int_0^1 (t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau\right\|.
\end{aligned}
$$

Applying assumptions $(i)-(v)$, we obtain

$$
\begin{aligned}
\|(V\Psi)(x,t_2)-(V\Psi)(x,t_1)\| \le &\omega(f,\epsilon) + \frac{BDAr_0^2}{\Gamma(\gamma+1)}(1+Qt_2)\times(t_2^\gamma - t_1^\gamma) \\
&+ \frac{T^\gamma BDAr_0}{\Gamma(\gamma+1)}(\omega(\Psi,\epsilon)+r_0Q(t_2-t_1).
\end{aligned}
$$

We are enabled to deduce that $\omega(f,\epsilon)\to 0, (t_2^\gamma - t_1^\gamma)\to 0$, and $(t_2-t_1)\to 0$, when $\epsilon\to 0$. Thus, we can find

$$
\omega_0(V\Theta) \le \frac{T^\gamma BDAr_0}{\Gamma(\gamma+1)}\omega_0(\Theta). \tag{4.2}
$$

In what follows, fix arbitrary $\Psi\in\Theta$ and $t_1,t_2\in[0,1]$ with $t_2\ge t_1$. Then, taking into account our assumptions, we have

$$
\begin{aligned}
&|(V\Psi)(x,t_2)-(V\Psi)(x,t_1)| - [(V\Psi)(x,t_2)-(V\Psi)(x,t_1)] \\
= &|f(x,t_2)-f(x,t_1)| - [f(x,t_2)-f(x,t_1)] + \left|\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2} p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\right. \\
&\times \int_0^{t_2}\int_0^1 (t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\
&- \frac{1}{\Gamma(\gamma)}[\Psi(x,t_1)+\int_0^{t_1} p(t_1,\tau)\Psi(x,\tau)\mathrm{d}\tau] \\
&\left.\times \int_0^{t_1}\int_0^1 (t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\right|
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigg[\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&-\frac{1}{\Gamma(\gamma)}[\Psi(x,t_1)+\int_0^{t_1}p(t_1,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\Bigg]\\
=&\,|f(x,t_2)-f(x,t_1)|-[f(x,t_2)-f(x,t_1)]+\Bigg|\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&-\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&-\frac{1}{\Gamma(\gamma)}[\Psi(x,t_1)+\int_0^{t_1}p(t_1,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\Bigg|\\
&+\Bigg[\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&-\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\gamma)}[\Psi(x,t_2)+\int_0^{t_2}p(t_2,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\\
&-\frac{1}{\Gamma(\gamma)}[\Psi(x,t_1)+\int_0^{t_1}p(t_1,\tau)\Psi(x,\tau)\mathrm{d}\tau]\\
&\times\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\Bigg].
\end{aligned}
$$

Using the properties of the normal and condition (iv), we obtain

$$\begin{aligned}
&|(V\Psi)(x,t_2)-(V\Psi)(x,t_1)|-[(V\Psi)(x,t_2)-(V\Psi)(x,t_1)]\\
&\le |f(x,t_2)-f(x,t_1)|-[f(x,t_2)-f(x,t_1)]+\frac{B|\Psi(x,t_2)|^2}{\Gamma(\gamma)}\left|1+\int_0^{t_2}p(t_2,\tau)\mathrm{d}\tau\right|\\
&\times\left|\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau-\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau\right|\\
&+\frac{B|\Psi(x,t_2)|}{\Gamma(\gamma)}\left||\Psi(x,t_2)-\Psi(x,t_1)|+|\Psi(x,t_2)||\int_0^{t_2}p(t_2,\tau)\mathrm{d}\tau-\int_0^{t_1}p(t_1,\tau)\mathrm{d}\tau)|\right|\\
&\times\left|\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau\right|+\frac{B[\Psi(x,t_2)]^2}{\Gamma(\gamma)}\left[1+\int_0^{t_2}p(t_2,\tau)\mathrm{d}\tau\right]\\
&\times\left[\int_0^{t_2}\int_0^1(t_2-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau-\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau\right]\\
&+\frac{B[\Psi(x,t_2)]}{\Gamma(\gamma)}\left[[\Psi(x,t_2)-\Psi(x,t_1)]+[\Psi(x,t_2)][\int_0^{t_2}p(t_2,\tau)\mathrm{d}\tau-\int_0^{t_1}p(t_1,\tau)\mathrm{d}\tau)]\right]\\
&\times\left[\int_0^{t_1}\int_0^1(t_1-\tau)^{\gamma-1}\xi(\tau)k(x,y)\mathrm{d}y\mathrm{d}\tau\right].
\end{aligned}$$

Using conditions $(i)-(v)$, we get

$$\begin{aligned}
&|(V\Psi)(x,t_2)-(V\Psi)(x,t_1)|-[(V\Psi)(x,t_2)-(V\Psi)(x,t_1)]\\
&\le |f(x,t_2)-f(x,t_1)|-[f(x,t_2)-f(x,t_1)]+\frac{BDAr_0^2}{\Gamma(\gamma+1)}(1+Qt_2)\times(t_2^\gamma-t_1^\gamma)\\
&+\frac{BDAr_0T^\gamma}{\Gamma(\gamma+1)}|\Psi(x,t_2)-\Psi(x,t_1)|+r_0\epsilon Q-\frac{BDAr_0^2}{\Gamma(\gamma+1)}(1+Qt_2)\times(t_2^\gamma-t_1^\gamma)\\
&-\frac{BDAr_0T^\gamma}{\Gamma(\gamma+1)}[\Psi(x,t_2)-\Psi(x,t_1)]+r_0\epsilon Q\\
&\le\frac{BDAr_0T^\gamma}{\Gamma(\gamma+1)}\{|\Psi(x,t_2)-\Psi(x,t_1)|-[\Psi(x,t_2)-\Psi(x,t_1)]\}.
\end{aligned}$$

Now, we are enabled to write

$$d(V\Psi)\le\frac{T^\gamma BDAr_0}{\Gamma(\gamma+1)}d(\Psi).$$

Thus, we can find

$$d(V\Theta)\le\frac{T^\gamma BDAr_0}{\Gamma(\gamma+1)}d(\Theta). \tag{4.3}$$

Finally, from equations (4.2) and (4.3) give us that

$$\omega_0(V\Theta)+d(V\Theta)\le\frac{T^\gamma BDAr_0}{\Gamma(\gamma+1)}(\omega_0(\Theta)+d(\Theta)).$$

Following in mind the concepts of the measure of noncompactness μ in Section 3 , we get

$$\mu(V\Theta)\le\frac{T^\gamma BDAr_0}{\Gamma(\gamma+1)}\mu(\Theta).$$

Now, keeping in mind the above inequality and the fact that $T^{\gamma}BDAr_0 < \Gamma(\gamma+1)$, in view of Theorem 4.1, then equation (2.3) has at least one solution $\Psi \in L_2[0,1] \times C[0,T]$. This completes the proof. □

Further, we give two examples with a verification of all the five assumptions of our main theorem.

Example 4.1. Consider the following frPI-DE with kind of the quadratic integral equation:

$$\frac{\partial^{\frac{1}{5}}}{\partial t^{\frac{1}{5}}}\left[\frac{\Psi(x,t)-x^2(0.1+t^2)}{\Psi(x,t)+\int_0^t t^2\tau^2\Psi(x,\tau)\mathrm{d}\tau}\right] = (t+0.003)\int_0^1 \frac{x^2y^2}{9}\Psi^2(y,t)\mathrm{d}y;\ \Psi(x,0)=0.1x^2. \tag{4.4}$$

Integrating equation (4.4), we obtain

$$\begin{aligned}\Psi(x,t) =\,& x^2(0.1+t^2) + \frac{1}{\Gamma(\frac{1}{5})}[\Psi(x,t)+\int_0^t t^2\tau^2\Psi(x,\tau)\mathrm{d}\tau] \\ &\times \int_0^t\int_0^1 (t-\tau)^{\frac{1}{5}-1}(\tau+0.003)\frac{x^2y^2}{9}\Psi^2(y,\tau)\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{4.5}$$

In this example, comparing with equation (2.3) and assumptions (i)–(v) in Section 4, we have

- **(i)** $f:[0,1]\to R$ is a continuous, such that $\|f(x,t)\| = 0.193793$.
- **(ii)** $k:[0,1]\times[0,1]\to R$ is continuous and the functions $x\to k(x,y)$ and $y\to k(x,y)$ are nondecreasing on $[0,1]$ for fixed $y\in[0,1]$ and $x\in[0,1]$, respectively, such that $|k(x,y)| \le \frac{1}{9}$, where $A=\frac{1}{9}$.
- **(iii)** The functions of time $p(t,\tau),\ \xi(\tau)$ are continuous in the space $C[0,T]$ and satisfies $|p(t,\tau)| \le 1,\ \ |\xi(\tau)| \le 1.003$, where $Q=1$ and $D=1.003$.
- **(iv)** The known function $\nu(\Psi(x,\tau)) = \Psi^2(x,\tau)$, for the constants $2.5 > B_2 > B_1$.

Further, let us consider the inequality

$$0.193793 + \frac{2}{9\Gamma(\frac{1}{5}+1)}(2.5)(1.003)r_0^2 \le r_0.$$

Now, we can observe that the last inequality admits a positive solution for $r_0 = 1.4$. Hence, all assumptions (i)–(v) of Theorem 4.1 are fulfilled. Thus, we are enabled to conclude that the equation (4.5) admits at least one solution Ψ in the space $L_2[0,1]\times C[0,T]$.

Example 4.2. Consider the following frPI-DE with kind of the quadratic integral equation:

$$\begin{aligned}&\frac{\partial^{0.9}}{\partial t^{0.9}}\left[\frac{\Psi(x,t)-(x^2+0.005)e^{-t}}{\Psi(x,t)+\int_0^t \tau\cos(t)\Psi(x,\tau)\mathrm{d}\tau}\right] = \frac{t}{4(1+t^2)}\int_0^1 e^{x^2y^2}[ty+\ln(1+\Psi(y,t))]\mathrm{d}y,\\ &\Psi(x,0)=x^2+0.005.\end{aligned} \tag{4.6}$$

Integrating equation (4.6), we obtain

$$\begin{aligned}\Psi(x,t) =\,& (x^2+0.005)e^{-t} + \frac{1}{\Gamma(0.9)}[\Psi(x,t)+\int_0^t \tau\cos(t)\Psi(x,\tau)\mathrm{d}\tau] \\ &\times \int_0^t\int_0^1 (t-\tau)^{0.9-1}\frac{\tau}{4(1+\tau^2)}e^{x^2y^2}[\tau y+\ln(1+\Psi(y,\tau))]\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{4.7}$$

Comparing with equation (2.3) and assumptions (i)–(v), we obtain

(i) $f : [0,1] \to R$ is a continuous, such that $\|f(x,t)\| = 0.285069$.

(ii) $k : [0,1] \times [0,1] \to R$ is continuous and the functions $x \to k(x,y)$ and $y \to k(x,y)$ are nondecreasing on $[0,1]$ for fixed $y \in [0,1]$ and $x \in [0,1]$, respectively, such that $|k(x,y)| \leq 2.71828$, where $A = 2.71828$.

(iii) The functions of time $p(t,\tau)$, $\xi(\tau)$ are continuous in the space $C[0,T]$ and satisfies $|p(t,\tau)| \leq 1$, $|\xi(\tau)| \leq 0.125$, where $Q = 1$ and $D = 0.125$.

(iv) The known function $\nu(\Psi(x,\tau)) = \tau x + \ln(1 + \Psi(x,\tau))]$, for the constants $0.99875025 > B_2 > B_1$.

Further, let us consider the inequality

$$0.285069 + \frac{0.125}{\Gamma(0.9+1)}(2.71828)(0.99875025)(2)r_0^2 \leq r_0.$$

Now, we can observe that the last inequality admits a positive solution for $r_0 = 1.0129$. Hence, all assumptions (i)–(v) of Theorem 4.1 are fulfilled. Thus, we are enabled to conclude that the equation (4.7) admits at least one solution Ψ in the space $L_2[0,1] \times C[0,T]$.

5. Convergence of the solutions

Convergence of solutions to integral equations pertains to the stability and reliability of approximation methods, as well as the theoretical assurance that iterative or analytical solutions tend toward a true solution under certain conditions. In many practical scenarios, exact solutions are either difficult or impossible to obtain; hence, understanding the convergence properties of approximate solutions is critical for ensuring the validity of computational results. This is especially vital in nonlinear, singular, or ill-posed problems where small perturbations in data can lead to significant deviations in outcomes.

For the convergence of the solution of equation (2.3), we construct the sequence of functions $\{\Psi_0(x,t), \Psi_1(x,t), \ldots, \Psi_{k-1}(x,t), \Psi_k(x,t), \ldots\} = \{\Psi_m\}_{m=0}^{m=\infty}$, where the two functions$\{\Psi_{k-1}(x, t), \Psi_k(x,t)\}$ are used to satisfy

$$\begin{aligned}
\Psi_k(x,t) &= f(x,t) + \frac{1}{\Gamma(\gamma)}[\Psi_{k-1}(x,t) + \int_0^t p(t,\tau)\Psi_{k-1}(x,\tau)\mathrm{d}\tau] \\
&\quad \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_{k-1}(y,\tau))\mathrm{d}y\mathrm{d}\tau, \\
\Psi_{k-1}(x,t) &= f(x,t) + \frac{1}{\Gamma(\gamma)}[\Psi_{k-2}(x,t) + \int_0^t p(t,\tau)\Psi_{k-2}(x,\tau)\mathrm{d}\tau] \\
&\quad \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_{k-2}(y,\tau))\mathrm{d}y\mathrm{d}\tau, \\
\Psi_0(x,t) &= f(x,t).
\end{aligned}$$

Since all of the functions $\Psi_k(x,t)$ are continuous, $\Psi_k(x,t)$ can be expressed as the following:

$$\Psi_k(x,t) = \Psi_0(x,t) + \sum_{m=1}^{k}(\Psi_m(x,t) - \Psi_{m-1}(x,t)).$$

The solution will be,

$$\Psi(x,t) = \lim_{k\to\infty} \Psi_k(x,t).$$

For this, we write

$$\begin{aligned}\Psi_k(x,t) - \Psi_{k-1}(x,t) =& \frac{1}{\Gamma(\gamma)}[\Psi_{k-1}(x,t) + \int_0^t p(t,\tau)\Psi_{k-1}(x,\tau)\mathrm{d}\tau] \\ & \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_{k-1}(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ & - \frac{1}{\Gamma(\gamma)}[\Psi_{k-2}(x,t) + \int_0^t p(t,\tau)\Psi_{k-2}(x,\tau)\mathrm{d}\tau] \\ & \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_{k-2}(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ =& \frac{1}{\Gamma(\gamma)}[(\Psi_{k-1}(x,t) - \Psi_{k-2}(x,t)) + \int_0^t p(t,\tau)(\Psi_{k-1}(x,\tau) - \Psi_{k-2}(x,\tau))\mathrm{d}\tau] \\ & \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_{k-1}(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ & + \frac{1}{\Gamma(\gamma)}[\Psi_{k-2}(x,t) + \int_0^t p(t,\tau)\Psi_{k-2}(x,\tau)\mathrm{d}\tau] \\ & \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi_{k-1}(y,\tau)) - \nu(\Psi_{k-2}(y,\tau))\}\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\Psi_k(x,t) - \Psi_{k-1}(x,t)\| \\ \le & \left\| \frac{1}{\Gamma(\gamma)}[(\Psi_{k-1}(x,t) - \Psi_{k-2}(x,t)) + \int_0^t p(t,\tau)(\Psi_{k-1}(x,\tau) - \Psi_{k-2}(x,\tau))\mathrm{d}\tau] \right\| \\ & \times \left\| \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_{k-1}(y,\tau))\mathrm{d}y\mathrm{d}\tau \right\| \\ & + \left\| \frac{1}{\Gamma(\gamma)}[\Psi_{k-2}(x,t) + \int_0^t p(t,\tau)\Psi_{k-2}(x,\tau)\mathrm{d}\tau] \right\| \\ & \times \left\| \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi_{k-1}(y,\tau)) - \nu(\Psi_{k-2}(y,\tau))\}\mathrm{d}y\mathrm{d}\tau \right\|.\end{aligned}$$

From conditions $(i)-(iv)$, we obtain

$$\begin{aligned}\|\Psi_k(x,t) - \Psi_{k-1}(x,t)\| \le & \frac{\|\Psi_{k-1}(x,t) - \Psi_{k-2}(x,t)\|}{\Gamma(\gamma+1)}[1+QT] \times T^\gamma DAB\|\Psi_{k-1}(x,t)\| \\ & + \frac{\|\Psi_{k-2}(x,t)\|}{\Gamma(\gamma+1)}[1+QT] \times T^\gamma DAB\|\Psi_{k-1}(x,t) - \Psi_{k-2}(x,t)\|.\end{aligned} \tag{5.1}$$

Since

$$\|\Psi(x,t)\| \le r \Rightarrow \|\Psi_{k-1}(x,t)\| \le r, \quad \text{also} \quad \|\Psi_{k-2}(x,t)\| \le r, \quad r > 0. \tag{5.2}$$

Using equation (5.2) in equation (5.1), we obtain

$$\|\Psi_k(x,t) - \Psi_{k-1}(x,t)\| \le \frac{[1+QT]}{\Gamma(\gamma+1)} \times T^\gamma DABr\|\Psi_{k-1}(x,t) - \Psi_{k-2}(x,t)\|$$

$$+\frac{[1+QT]}{\Gamma(\gamma+1)} \times T^{\gamma} DABr\|\Psi_{k-1}(x,t)-\Psi_{k-2}(x,t)\|.$$

This last formula can be modified to get the following:

$$\|\Psi_k(x,t)-\Psi_{k-1}(x,t)\| \leq \frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr\|\Psi_{k-1}(x,t)-\Psi_{k-2}(x,t)\|.$$

Letting $k=1$, we have

$$\begin{aligned}\|\Psi_1(x,t)-\Psi_0(x,t)\| &\leq \frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr\|\Psi_0(x,t)\|; \quad \Psi_0(x,t)=f(x,t)\\ &\leq \frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr\|f(x,t)\|\\ &\leq \beta\|f(x,t)\|; \quad \beta=\frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr.\end{aligned}$$

Let $k=2$,

$$\|\Psi_2(x,t)-\Psi_1(x,t)\| \leq \frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr\|\Psi_1(x,t)-\Psi_0(x,t)\| \leq \beta^2\|f(x,t)\|.$$

Let $k=3$,

$$\|\Psi_3(x,t)-\Psi_2(x,t)\| \leq \frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr\|\Psi_2(x,t)-\Psi_1(x,t)\| \leq \beta^3\|f(x,t)\|.$$

By induction, we obtain

$$\|\Psi_k(x,t)-\Psi_{k-1}(x,t)\| \leq \beta^k\|f(x,t)\|; \quad \beta=\frac{2[1+QT]}{\Gamma(\gamma+1)} T^{\gamma} DABr;\ k=1,2,3,\ldots.$$

Since $2[1+QT]T^{\gamma}DABr<\Gamma(\gamma+1)$, then the uniform convergence of

$$\sum_{k=1}^{\infty}\left(\Psi_k(x,t)-\Psi_{k-1}(x,t)\right),$$

is proved and so the sequence $\{\Psi_k(x,t)\}$ is uniformly convergent.

$$\begin{aligned}\Psi(x,t)=&\lim_{k\to\infty}\Bigg(f(x,t)+\frac{1}{\Gamma(\gamma)}[\Psi_k(x,t)+\int_0^t p(t,\tau)\Psi_k(x,\tau)\mathrm{d}\tau]\\ &\times\int_0^t\int_0^1(t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_k(y,\tau))\mathrm{d}y\mathrm{d}\tau\Bigg)\\ =&f(x,t)+\frac{1}{\Gamma(\gamma)}[\Psi(x,t)+\int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau]\int_0^t\int_0^1(t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

6. Stability of the error

In numerical analysis, the stability of an algorithm plays a pivotal role in ensuring the reliability and accuracy of computational results. As numerical methods are increasingly applied to solve complex real-world problems—ranging from fluid dynamics to financial modeling—the behavior of error propagation within these methods becomes a critical consideration. Stability of the error, in particular, refers to the algorithm's capacity to control the growth of rounding errors, discretization errors, and perturbations introduced at any stage of computation.

Assume the approximate solution of equation (2.3) becomes

$$\begin{aligned}\Psi_k(x,t) = f_k(x,t) + \frac{1}{\Gamma(\gamma)}[\Psi_k(x,t) + \int_0^t p(t,\tau)\Psi_k(x,\tau)\mathrm{d}\tau] \\ \times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi_k(y,\tau))\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{6.1}$$

Hence, from (2.3) and (6.1) we have

$$\begin{aligned}&[\Psi(x,t) - \Psi_k(x,t)] \\ =&[f(x,t) - f_k(x,t)] + \frac{1}{\Gamma(\gamma)}[(\Psi(x,t) - \Psi_k(x,t)) + \int_0^t p(t,\tau)(\Psi(x,\tau) - \Psi_k(x,\tau))\mathrm{d}\tau] \\ &\times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau + \frac{1}{\Gamma(\gamma)}[\Psi_k(x,t) + \int_0^t p(t,\tau)\Psi_k(x,\tau)\mathrm{d}\tau] \\ &\times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi(y,\tau)) - \nu(\Psi_k(y,\tau))\}\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

Therefore, we get

$$\begin{aligned}R_k(x,t) =& F_k(x,t) + \frac{1}{\Gamma(\gamma)}[R_k(x,t) + \int_0^t p(t,\tau)R_k(x,\tau)\mathrm{d}\tau] \\ &\times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau + \frac{1}{\Gamma(\gamma)}[\Psi_k(x,t) + \int_0^t p(t,\tau)\Psi_k(x,\tau)\mathrm{d}\tau] \\ &\times \int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi(y,\tau)) - \nu(\Psi_k(y,\tau))\}\mathrm{d}y\mathrm{d}\tau,\end{aligned} \tag{6.2}$$

where

$$R_k(x,t) = [\Psi(x,t) - \Psi_k(x,t)],\ F_k(x,t) = [f(x,t) - f_k(x,t)].$$

Theorem 6.1. *Under the conditions $(i)-(iv)$, the equation of error (6.2) is stable in the space.*

Proof. Since

$$\begin{aligned}\|R_k(x,t)\| \leq& \|F_k(x,t)\| + \frac{1}{\Gamma(\gamma)}\left\|R_k(x,t) + \int_0^t p(t,\tau)R_k(x,\tau)\mathrm{d}\tau\right\| \\ &\times \left\|\int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau\right\| \\ &+ \frac{1}{\Gamma(\gamma)}\left\|\Psi_k(x,t) + \int_0^t p(t,\tau)\Psi_k(x,\tau)\mathrm{d}\tau\right\| \\ &\times \left\|\int_0^t \int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi(y,\tau)) - \nu(\Psi_k(y,\tau))\}\mathrm{d}y\mathrm{d}\tau\right\|.\end{aligned}$$

Applying assumptions $(i)-(iv)$, we obtain

$$\|R_k(x,t)\| \leq \|F_k(x,t)\| + \frac{\|R_k(x,t)\|}{\Gamma(\gamma+1)}[1+QT]\times T^{\gamma}DAB\|\Psi(x,t)\| + \frac{\|\Psi_k(x,t)\|}{\Gamma(\gamma+1)}[1+QT]\times T^{\gamma}DAB\|R_k(x,t)\|.$$

Assuming that $\|\Psi(x,t)\|\leq r \Rightarrow \|\Psi_k(x,t)\|\leq r$, hence we have

$$\|R_k(x,t)\| \leq \|F_k(x,t)\| + \frac{2[1+QT]}{\Gamma(\gamma+1)}T^{\gamma}DABr\|R_k(x,t)\|. \tag{6.3}$$

Hence, the error is stable under the condition

$$2[1+QT]T^{\gamma}DABr < \Gamma(\gamma+1).$$

□

Lemma 6.1. *As $k\to\infty$, the error $R_\infty(x,t)\to 0$.*

Proof. From inequality (6.3), since $F(x,t)=F_\infty(x,t)$, then the error $R_k(x,t)\to 0$ as $k\to\infty$. This led us to conclude that the error $R_k(x,t)$ inequality (6.3) has a unique representation. □

Theorem 6.2. *The representation of the modified error (6.2) is unique.*

Proof. Assume that there are two different forms to describe the modified error

$$\begin{aligned} R_k(x,t)-R_n(x,t) =& [F_k(x,t)-F_n(x,t)] \\ &+ \frac{1}{\Gamma(\gamma)}[(R_k(x,t)-R_n(x,t)) + \int_0^t p(t,\tau)(R_k(x,\tau)-R_n(x,\tau))\mathrm{d}\tau] \\ &\times \int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\nu(\Psi(y,\tau))\mathrm{d}y\mathrm{d}\tau \\ &+ \frac{1}{\Gamma(\gamma)}[(\Psi_k(x,t)-\Psi_n(x,t)) + \int_0^t p(t,\tau)(\Psi_k(x,\tau)-\Psi_n(x,\tau))\mathrm{d}\tau] \\ &\times \int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi(y,\tau))-\nu(\Psi_n(y,\tau))\}\mathrm{d}y\mathrm{d}\tau \\ &+ \frac{1}{\Gamma(\gamma)}[\Psi_k(x,t) + \int_0^t p(t,\tau)\Psi_k(x,\tau)\mathrm{d}\tau] \\ &\times \int_0^t\int_0^1 (t-\tau)^{\gamma-1}\xi(\tau)k(x,y)\{\nu(\Psi_n(y,\tau))-\nu(\Psi_k(y,\tau))\}\mathrm{d}y\mathrm{d}\tau. \end{aligned}$$

Then, we have

$$\begin{aligned} \|R_k(x,t)-R_n(x,t)\| &\leq \|F_k(x,t)-F_n(x,t)\| + \frac{2[1+QT]}{\Gamma(\gamma+1)}T^{\gamma}DABr\|\Psi_k(x,t)-\Psi_n(x,t)\| \\ &\leq \|F_k(x,t)-F_n(x,t)\| + \beta\|\Psi_k(x,t)-\Psi_n(x,t)\|; \quad \beta<1. \end{aligned}$$

In the above inequality, if $k\to n$, then $\{(F_k(x,t)-F_n(x,t)),\,(\Psi_k(x,t)-\Psi_n(x,t))\}\to 0 \Leftrightarrow \{R_k(x,t)-R_n(x,t)\}\to 0$. □

7. General conclusion

From the above results and discussion in this work, the following may be concluded: Using the Riemann-Liouville fractional integral of order $\gamma > 0$, the fractional partial integro-differential equation was converted to a nonlinear quadratic Volterra–Fredholm integral equation. The equation (2.3) has at least one solution $\Psi(x,t)$ in the space $L_2[0,1] \times C[0,T]$, under some conditions. The result is obtained by the application of the generalized Darbo fixed point theorem associated with the measure of noncompactness in the Banach space. After that, the convergence of the solutions of the nonlinear quadratic Volterra–Fredholm integral equation has been proved by using the Picard method. Moreover, an error analysis is presented. Our results are demonstrated with the two examples.

8. Future work

The authors aim to obtain a solution to the equation studied here analytically, but with developments in the equation as follows:

$$\frac{\partial^\gamma}{\partial t^\gamma}\left[\frac{\Psi(x,t) - f(x,t)}{\Psi(x,t) + \int_0^t p(t,\tau)\Psi(x,\tau)\mathrm{d}\tau}\right] = \xi(t)\int_0^1 k(|x-y|)\nu(\Psi(y,t))\mathrm{d}y;\ \gamma > 0,$$

with the initial condition

$$\Psi(x,0) = U(x).$$

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