

BÄCKLUND TRANSFORMATION AND SOME EXACT SOLITON SOLUTIONS FOR A (2+1)-DIMENSIONAL NONLINEAR EVOLUTION EQUATION

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Abstract This study presents a systematic derivation of exact solutions for a (2+1)-dimensional nonlinear evolution equation through Bäcklund transformation techniques. This equation can be reduced to the standard Korteweg-de Vries (KdV) equation. A Bäcklund transformation of the generalized (2+1)-dimensional KdV equation is constructed, and some exact soliton solutions are produced. In addition, the superposition formula is obtained.

Keywords Bäcklund transformation, nonlinear evolution equation, superposition formula, soliton solution.

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1. Introduction

Nonlinear evolution equations have long shaped modern physics through landmark methods like inverse scattering [5, 18] and Bäcklund transformations [7–10]. These tools decode intricate systems from vector field interactions, producing fundamental models like the dispersionless Kadomtsev-Petviashvili (dKP) equation [2–4, 6, 13, 15–17, 19, 20] and Korteweg-de Vries (KdV) equation [1, 11, 12, 14, 21, 22]. Recently, the Bäcklund transformation method has been extensively studied for investigating solutions of many nonlinear evolution equations with physical backgrounds and applications. The famous examples are the Korteweg-de Vries equation, the nonlinear Schrodinger equation, and so on. These (1+1)-dimensional integrable equations all have soliton solutions which can be obtained by the Bäcklund transformation method. However, the Bäcklund transformations of (2+1)-dimensional integrable evolution equations lacked a systematic study. Chen [6] once studied the Kadomtsev-Petviashvili (KP) equation, which describes disturbances in a weakly dispersive, weakly nonlinear medium. The Bäcklund transformation of the (2+1)-dimensional KP equation was obtained, which was used to investigate soliton solutions. In addition, the multi-soliton solutions of the KP equation were constructed by a superposition formula.

In [19], the following system of two nonlinear partial differential equations (PDEs) in (2+1) dimensions was proposed:

$$\begin{aligned}u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} &= 0\end{aligned}\tag{1.1}$$

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arising from the commutation of the two-dimensional vector fields

$$\hat{L}_1 = \partial_y + (\lambda + v_x)\partial_x - u_x\partial_\lambda, \quad (1.2)$$

$$\hat{L}_2 = \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda. \quad (1.3)$$

Strikingly, in the case where $v = 0$, the Eq. (1.1) is reduced to the dKP equation

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in R, \quad x, y, t \in R. \quad (1.4)$$

In the case where $u = 0$, the Eq. (1.1) is reduced to the Pavlov equation

$$v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{xy}, \quad v = v(x, y, t) \in R, \quad x, y, t \in R. \quad (1.5)$$

In [6], Hsing-Hen Chen conducted a study of the equation:

$$q_{xt} + 3b^2 q_{yy} + q_{xx} + (3q^2)_{xx} + q_{xxxx} = 0 \quad (1.6)$$

arising from the commutation $[\hat{L}, \hat{A}] = 0$ of the two-dimensional vector fields

$$\hat{L} = \partial_x^2 + q(x, y, t) - b\partial_y, \quad (1.7)$$

$$\hat{A} = 4i\partial_x^3 + 3i(q\partial_x + \partial_x q) + 3ib \int^x q_y dx + i\partial_x + i\partial_t. \quad (1.8)$$

By choosing $b = \pm \frac{i}{\sqrt{3}}$ in Eq. (1.6), we get

$$q_{xt} - q_{yy} + q_{xx} + (3q^2)_{xx} + q_{xxxx} = 0. \quad (1.9)$$

And by choosing $b = \pm \frac{1}{\sqrt{3}}$ in Eq. (1.6), we get

$$q_{xt} + q_{yy} + q_{xx} + (3q^2)_{xx} + q_{xxxx} = 0. \quad (1.10)$$

And the transformation $y \rightarrow iy$ to the Eq. (1.10) automatically carries to the Eq. (1.9).

Our innovation introduces a unified framework:

$$u_{xt} + u_{yy} + u_{xx} + (3u^2)_{xx} + u_{xxxx} + \lambda u_{xy} = 0, \quad (1.11)$$

where $\lambda \in C$ is the spectral parameter and $u = u(x, y, t) \in R, \quad x, y, t \in R$. In the case of setting the parameter λ to zero, the Eq. (1.11) becomes the Eq. (1.10), thus resulting in the promotion and inclusion of [6]. When the Eq. (1.11) is plotted on the xt plane, it is found to decrease to a KdV equation:

$$q_t + q_{xxx} + 6qq_x = 0.$$

Furthermore, when the equation is plotted on the xy plane, it is seen to reduce to a Boussinesq equation, which is expressed as follows:

$$\begin{aligned} q_{tt} - q_{xx} - (q^2)_{xx} - q_{xxxx} &= 0, \\ q_{tt} - q_{xx} + (q^2)_{xx} + q_{xxxx} &= 0. \end{aligned} \quad (1.12)$$

The Eq. (1.11) describes disturbances in a weakly dispersive, weakly nonlinear medium. At the same time, A novel Bäcklund transformation of the generalized (2+1)-dimensional KdV equation is constructed, and some exact soliton solutions are produced. Finally, we also get the superposition formula for the solutions. This work bridges dimensional hierarchies in integrable systems, offering new tools to probe wave phenomena in complex media.

This paper is organized as follows: In Sections 2 and 3, respectively, the Bäcklund transformation is constructed using the Lax paired transformation. In Section 4, a superposition formula is obtained via the Bäcklund transformation. In Section 5, a new soliton solution is obtained.

2. Generalized Lax criterion

In the context of two linear Hermitian operators: $\tilde{A}_1[\partial_x, u(x)]$ and $\tilde{A}_2[\partial_x, u(x)]$. The variables x and u represent the totality of $x_1, x_2, \dots, x_n$ and $u_1, u_2, \dots, u_m$. The operators commute with each other, i.e.,

$$[\tilde{A}_1, \tilde{A}_2] = 0. \quad (2.1)$$

It can be demonstrated that a simultaneous set of eigenfunctions $f(x)$ exists such that

$$\tilde{A}_1 f = \alpha f \text{ and } \tilde{A}_2 f = \beta f, \quad (2.2)$$

where α and β are eigenvalues independent of x . Eq. (2.1) will provide the evolution equations of interest, while Eq. (2.2) are fundamental. The Bäcklund transformation for the evolution equations obtained from Eq. (2.1) is derived from these fundamental equations. The subsequent example, adapted from the source text, will serve to elucidate the points above:

$$\tilde{A}_1 = \partial_x^2 + u(x, y, t) - m\partial_y, \quad (2.3)$$

$$\tilde{A}_2 = 4i\partial_x^3 + 3i(u\partial_x + \partial_x u) + 3im \int^x u_y dx + i\partial_x + i\partial_t + \lambda\partial_y. \quad (2.4)$$

In the event of the selection of $m = \pm \frac{1}{\sqrt{3}}$ from Eq. (2.3) and (2.4) being made, the ensuing equations are obtained:

$$u_{yy} + u_{xxx} + (3u^2)_{xx} + u_{xx} + u_{xt} + \lambda u_{xy} = 0. \quad (2.5)$$

In the event of the Eq. (2.5) being set to have a value of zero for the parameter λ , the resultant equation would be:

$$u_{yy} + u_{xxx} + (3u^2)_{xx} + u_{xx} + u_{xt} = 0 \quad (2.6)$$

which was the 2-dimensional KdV equation. The following discussion will concentrate on Eq. (2.5).

3. Bäcklund transformation

From Eqs. (2.2), (2.3) and (2.4), it can be deduced that

$$\begin{aligned} \partial_x^2 f + u f \mp \frac{1}{\sqrt{3}} \partial_y f &= \alpha f, \\ 4i\partial_x^3 f + 3i(u\partial_x + \partial_x u)f \pm \sqrt{3}i \int^x u_y dx f + i\partial_x f + i\partial_t f + \lambda\partial_y f &= \beta f. \end{aligned} \quad (3.1)$$

This system of equations defines a transformation mapping a solution u of Eq. (2.5) to a function f . It will be shown that f corresponds to another solution, denoted $\acute{u}$, of the same equation. Thus, Eq. (3.1) constitutes a Bäcklund transformation between two distinct solutions u and $\acute{u}$ of Eq. (2.5).

We let $\Phi \equiv \ln f$ and $S_x \equiv u$, the Eq. (3.1) can be rewritten as follows:

$$\begin{aligned} \Phi_x^2 + \Phi_{xx} + S_x \mp \frac{1}{\sqrt{3}} \Phi_y &= \alpha, \\ 4i(\Phi_x^3 + 3\Phi_x \Phi_{xx} + \Phi_{xxx}) + 3i(2S_x \Phi_x + S_{xx}) \pm \sqrt{3}iS_y + i\Phi_x + i\Phi_t + \lambda i\Phi_y &= \beta. \end{aligned} \quad (3.2)$$

The elimination of S from Eq. (3.2) results in the subsequent nonlinear evolution equation for Φ :

$$\Phi_{xxx} + 6\alpha\Phi_x - 2\Phi_x^3 \pm 2\sqrt{3} \int^x \Phi_y \Phi_{xx} dx + \int^x \Phi_{yy} dx + \Phi_x + \Phi_t + \lambda\Phi_y = -i\beta. \quad (3.3)$$

It can be shown that for any solution Φ to Eq. (3.2), there corresponds a solution $S_x \equiv u$ to Eq. (2.5). Moreover, for any solution triple (Φ, β, λ) satisfying the equation, the triple $(-\Phi, -\beta, \lambda)$ also constitutes a valid solution, differing only by a sign change in the fourth term. For this new pair, denoted by $(-\Phi, -\beta, \lambda)$, there is a corresponding solution, denoted by $\dot{u} \equiv \dot{S}_x$ of Eq. (2.5) such that

$$\begin{aligned} -\Phi_{xx} + \Phi_x^2 + \dot{S}_x \mp \frac{1}{\sqrt{3}}\Phi_y &= \alpha, \\ 4i(-\Phi_{xxx} + 3\Phi_x\Phi_{xx} - \Phi_x^3) + 3i(-2\dot{S}_x\Phi_x + \dot{S}_{xx}) \\ &\mp \sqrt{3}i\dot{S}_y - i\Phi_x - i\Phi_t - \lambda i\Phi_y = -\beta. \end{aligned} \quad (3.4)$$

The difference between Eqs. (3.2) and (3.4) is calculated as follows:

$$\begin{aligned} 2\Phi_{xx} &= \dot{S}_x - S_x, \\ 8i\Phi_{xxx} + 8\Phi_x^3 i + 6i\Phi_x(\dot{S}_x + S_x) + 3i(S_{xx} - \dot{S}_{xx}) \\ &\mp \sqrt{3}i(\dot{S}_y + S_y) + 2i\Phi_x + 2i\Phi_t + 2\lambda i\Phi_y = 2\beta. \end{aligned} \quad (3.5)$$

Eq. (3.2) in conjunction with Eq. (3.4):

$$\begin{aligned} 2\Phi_x^2 + (S_x + \dot{S}_x) \mp \frac{2}{\sqrt{3}}\Phi_y &= 2\alpha, \\ 24i\Phi_x\Phi_{xx} + 6i\Phi_x(S_x - \dot{S}_x) + 3i(\dot{S}_{xx} + S_{xx}) \mp \sqrt{3}i(S_y - \dot{S}_y) &= 0. \end{aligned} \quad (3.6)$$

By (3.5), we choose

$$\Phi = \frac{1}{2} \int^x (\dot{S} - S) dx \pm \sqrt{3}\alpha y + (-i\beta \pm \sqrt{3}\lambda\alpha)t.$$

It is shown that this procedure eliminates the constants α and β in the Bäcklund transformation, indicating their non-essential role in solution construction. Substituting the relation between Φ and $\dot{S}$ into Eqs. (3.5) and (3.6) then yields the Bäcklund transformation in the form:

$$\begin{aligned} (\dot{S} - S)^2 + 2(\dot{S}_x + S_x) \mp \frac{2}{\sqrt{3}} \int^x (\dot{S}_y - S_y) dx &= 0, \\ 4(\dot{S} - S)_x + (\dot{S} - S)^3 + 3(\dot{S} - S)(\dot{S} + S)_x + 3(S - \dot{S})_{xx} \pm \sqrt{3}(S + \dot{S})_y \\ + (\dot{S} - S) + \int^x (\dot{S} - S)_t dx + \lambda \int^x (\dot{S} - S)_y dx &= 0. \end{aligned} \quad (3.7)$$

The double sign in the transformation equations indicates bidirectional wave propagation along the y -axis. Using Eq. (3.7), particular solutions to Eq. (2.5) can be constructed. Starting from the trivial solution $S = 0$, multiple new solutions $\dot{S}$ can be generated through integration of Eq. (3.7), or more directly by applying Eq. (3.1),

$$\begin{aligned} \partial_x^2 f \mp \frac{1}{\sqrt{3}} \partial_y f &= 0, \\ 4i\partial_x^3 f + i\partial_x f + i\partial_t f + \lambda i\partial_y f &= 0. \end{aligned} \quad (3.8)$$

If $f = B \exp(ikx + My + Nt)$ is a solution of Eq. (3.8), where B, k, M, N are undetermined constants, then

$$f_x = ikf, f_y = Mf, f_t = Nf, f_{xx} = -k^2 f, f_{xxx} = -k^2 f_x = -ik^3 f$$

substitute them into Eq. (3.8), we get

$$M = \mp \sqrt{3}k^2, N = (4k^3 - k)i \pm \lambda \sqrt{3}k^2.$$

Therefore, the most general solution for f can be expressed as follows:

$$f = \sum_k a_k \exp \left[ikx \mp \sqrt{3}k^2 y + ((4k^3 - k)i \pm \lambda \sqrt{3}k^2)t \right] \equiv \sum_k a_k \exp(\xi_k), \quad (3.9)$$

where

$$\xi_k = ikx \mp \sqrt{3}k^2 y + ((4k^3 - k)i \pm \lambda \sqrt{3}k^2)t. \quad (3.10)$$

The solution is applicable to all complex values of k and a_k is a spectral function. When $S = 0$ in Eq. (3.5), the relation

$$\begin{aligned} \dot{u} = \dot{S}_x = 2\Phi_{xx} &= 2\left(\frac{f_x}{f}\right)_x = 2\frac{f_{xx}f - f_x^2}{f^2} \\ &= \frac{-2\sum_k a_k k^2 \exp(\xi_k) \sum_k a_k \exp(\xi_k) + [\sum_k a_k k \exp(\xi_k)]^2}{[\sum_k a_k \exp(\xi_k)]^2} \end{aligned} \quad (3.11)$$

is a solution of Eq. (2.5).

A specific selection of a_k will yield particular solutions. To illustrate this point, consider the following example: Let $a_k = \delta_{k, -ik_1} + \delta_{k, -ik_2}$, where

$$\delta_{k, -ik_j} = \begin{cases} 1, & k = -ik_j, j = 1, 2, \\ 0, & k \neq -ik_j, j = 1, 2. \end{cases} \quad (3.12)$$

We have

$$\begin{aligned} u_1 &= \frac{-2[(-ik_1)^2 \exp \xi_{-ik_1} + (-ik_2)^2 \exp \xi_{-ik_2}](\exp \xi_{-ik_1} + \exp \xi_{-ik_2})}{(\exp \xi_{-ik_1} + \exp \xi_{-ik_2})^2} \\ &\quad + \frac{[(-ik_1) \exp \xi_{-ik_1} + (-ik_2) \exp \xi_{-ik_2}]^2}{(\exp \xi_{-ik_1} + \exp \xi_{-ik_2})^2} \\ &= \frac{2[(k_1)^2 \exp \xi_{-ik_1} + (k_2)^2 \exp \xi_{-ik_2}](\exp \xi_{-ik_1} + \exp \xi_{-ik_2})}{(\exp \xi_{-ik_1} + \exp \xi_{-ik_2})^2} \\ &\quad + \frac{-k_1^2 \exp 2\xi_{-ik_1} - k_2^2 \exp 2\xi_{-ik_2} - 2k_1 k_2 \exp(\xi_{-ik_1} + \xi_{-ik_2})}{(\exp \xi_{-ik_1} + \exp \xi_{-ik_2})^2} \\ &= \frac{2(k_1 - k_2)^2 \exp(\xi_{-ik_1} + \xi_{-ik_2})}{(\exp \xi_{-ik_1} + \exp \xi_{-ik_2})^2} \\ &= \frac{2(k_1 - k_2)^2}{(\exp \xi_{-ik_1} + \exp \xi_{-ik_2})^2} \\ &= \frac{2(k_1 - k_2)^2 \exp \left(\frac{-\xi_{-ik_1} - \xi_{-ik_2}}{2} \right)^2}{(\exp \frac{\xi_{-ik_1} - \xi_{-ik_2}}{2} + \exp \frac{-\xi_{-ik_1} - \xi_{-ik_2}}{2})^2} \end{aligned}$$

$$= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \frac{\xi_{-ik_1} - \xi_{-ik_2}}{2}, \quad (3.13)$$

here

$$\begin{aligned} \xi_{-ik_1} &= i(-ik_1)x \mp \sqrt{3}(-ik_1)^2 y + [4(-ik_1)^3 i - (-ik_1)i \pm \lambda\sqrt{3}(-ik_1)^2]t \\ &= k_1 x \pm \sqrt{3}k_1^2 y - (4k_1^3 + k_1 \pm \lambda\sqrt{3}k_1^2)t, \\ \xi_{-ik_2} &= k_2 x \pm \sqrt{3}k_2^2 y - (4k_2^3 + k_2 \pm \lambda\sqrt{3}k_2^2)t, \end{aligned} \quad (3.14)$$

therefore

$$\xi_{-ik_1} - \xi_{-ik_2} = (k_1 - k_2)x \pm \sqrt{3}(k_1^2 - k_2^2)y - (4k_1^3 - 4k_2^3 + k_1 - k_2 \pm \lambda\sqrt{3}(k_1^2 - k_2^2))t.$$

We get

$$\begin{aligned} u_1 &= \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \left\{ \frac{1}{2}(k_1 - k_2)x \pm \frac{\sqrt{3}}{2}(k_1^2 - k_2^2)y \right. \\ &\quad \left. - \frac{1}{2}[4k_1^3 - 4k_2^3 + k_1 - k_2 \pm \lambda\sqrt{3}(k_1^2 - k_2^2)]t \right\}. \end{aligned} \quad (3.15)$$

It is a two-dimensional soliton with amplitude $\frac{1}{2}(k_1 - k_2)^2$ and velocity

$$\begin{aligned} v_x &= \frac{-\frac{1}{2}[4k_1^3 - 4k_2^3 + k_1 - k_2 \pm \lambda\sqrt{3}(k_1^2 - k_2^2)]}{\frac{1}{2}(k_1 - k_2)} = -4[k_1^2 + k_1 k_2 + k_2^2 + 1 \pm \lambda\sqrt{3}(k_1 + k_2)], \\ v_y &= \frac{-\frac{1}{2}[4k_1^3 - 4k_2^3 + k_1 - k_2 \pm \lambda\sqrt{3}(k_1^2 - k_2^2)]}{\pm \frac{\sqrt{3}}{2}(k_1^2 - k_2^2)} = \frac{\pm v_x}{\sqrt{3}(k_1 + k_2)}. \end{aligned}$$

Similarly, if we let a_k equal $\delta_{k,-ik_1} - \delta_{k,-ik_2}$, we may similarly arrive at

$$\begin{aligned} u_2 &= -\frac{1}{2}(k_1 - k_2)^2 \operatorname{csch}^2 \left\{ \frac{1}{2}(k_1 - k_2)x \pm \frac{\sqrt{3}}{2}(k_1^2 - k_2^2)y \right. \\ &\quad \left. - \frac{1}{2}[4k_1^3 - 4k_2^3 + k_1 - k_2 \pm \lambda\sqrt{3}(k_1^2 - k_2^2)]t \right\}. \end{aligned} \quad (3.16)$$

4. Superposition formula

To obtain the superposition formula, let S_1 be a solution generated via the Bäcklund transformation. Starting from a known solution S_0 with spectral function $a_{1,k}$, a second solution S_2 is constructed from w_0 using spectral function $a_{2,k}$. A third solution S_3 is then derived from S_1 with spectral function $a_{2,k}$. In accordance with the established definition and the application of the Bäcklund transformation Eq. (3.7), we have

$$(S_1 - S_0)^2 + 2(S_1 + S_0)_x \mp \frac{2}{\sqrt{3}} \int^x (S_1 - S_0)_y dx = 0, \quad (4.1)$$

$$(S_2 - S_0)^2 + 2(S_2 + S_0)_x \mp \frac{2}{\sqrt{3}} \int^x (S_2 - S_0)_y dx = 0, \quad (4.2)$$

$$(S_3 - S_1)^2 + 2(S_3 + S_1)_x \mp \frac{2}{\sqrt{3}} \int^x (S_3 - S_1)_y dx = 0, \quad (4.3)$$

$$(S_3 - S_2)^2 + 2(S_3 + S_2)_x \mp \frac{2}{\sqrt{3}} \int^x (S_3 - S_2)_y dx = 0. \quad (4.4)$$

Adding (3.15) to (4.2) yields:

$$(S_3 - S_2)^2 + (S_1 - S_0)^2 + 2(S_1 + S_0 + S_3 + S_2)_x \mp \frac{2}{\sqrt{3}} \int^x (S_1 - S_0 + S_3 - S_2)_y dx = 0. \quad (4.5)$$

Adding (3.16) to (4.1) yields:

$$(S_2 - S_0)^2 + (S_3 - S_1)^2 + 2(S_1 + S_0 + S_3 + S_2)_x \mp \frac{2}{\sqrt{3}} \int^x (S_2 - S_0 + S_3 - S_1)_y dx = 0. \quad (4.6)$$

Subtracting (4.4) from (4.3), we obtain:

$$(S_3 - S_2)^2 + (S_1 - S_0)^2 - (S_2 - S_0)^2 - (S_3 - S_1)^2 \mp \frac{4}{\sqrt{3}} \int^x (S_1 - S_2)_y dx = 0,$$

i.e.,

$$S_0 S_2 - S_3 S_2 + S_3 S_1 - S_0 S_1 \mp \frac{2}{\sqrt{3}} \int^x (S_1 - S_2)_y dx = 0. \quad (4.7)$$

Taking the difference of (3.15) and (3.16), we get

$$S_1^2 - 2S_1 S_0 - S_2^2 + 2S_2 S_0 + 2(S_1 - S_2)_x \mp \frac{2}{\sqrt{3}} \int^x (S_1 - S_2)_y dx = 0. \quad (4.8)$$

Finally, taking the difference of (4.5) and (4.6), we get

$$S_0 S_2 - S_3 S_2 + S_3 S_1 - S_0 S_1 = S_1^2 - S_1 S_0 - S_2^2 + S_2 S_0 + 2(S_1 - S_2)_x,$$

i.e.,

$$S_0 + S_3 = S_1 + S_2 + 2 \frac{(S_1 - S_2)_x}{S_1 - S_2} \quad (4.9)$$

is therefore the superposition formula we are searching for.

Starting from the initial state $S_0 = 0$, the solutions S_1 and S_2 are derived using Eq. (3.11) with spectral functions $a_{1,k}$ and $a_{2,k}$, respectively, forming single-spectrum solutions. Substituting these into (4.7) yields a new solution S_3 , which incorporates both spectral functions $a_{1,k}$ and $a_{2,k}$, and is thus designated as a two-spectrum solution. In particular, when $a_{1,k}$ and $a_{2,k}$ take the specific forms that generate 2D solitary waves as in Eqs. (3.15) and (3.16), the resulting S_3 corresponds to a two-soliton solution with two straight-line wavefronts in two dimensions.

5. A new soliton solution

Eq. (3.8) admits not only the soliton solutions given in (3.13) and (3.16), but also the following solution:

$$f = \sum_k \frac{a_k e^{\theta_k}}{1 + b e^{\theta_k}}, \quad (5.1)$$

where b is a constant. When $a_k = \delta_{k,k_1} + \delta_{k,k_2}$, it is easy to find

$$\begin{aligned} f &= T + Q, \\ f_x &= k_1 b T^2 - k_1 T + k_2 b Q^2 - k_2 Q, \\ f_{xx} &= 2k_1^2 b^2 T^3 - 3k_1^2 b T^2 + k_1^2 T + 2k_2^2 b^2 Q^3 - 3k_2^2 b Q^2 + k_2^2 Q, \end{aligned} \quad (5.2)$$

where

$$T = \frac{e^{\theta_{k_1}}}{1 + b e^{\theta_{k_1}}}, \quad Q = \frac{e^{\theta_{k_2}}}{1 + b e^{\theta_{k_2}}}.$$

Therefore, a new solution to Eq. (3.8) is given by

$$\begin{aligned} u_3 = \dot{S}_x &= 2\Phi_{xx} = 2\left(\frac{f_x}{f}\right)_x = 2\frac{f_{xx}f - f_x^2}{f^2} \\ &= \frac{2(2k_1^2 b^2 T^3 - 3k_1^2 b T^2 + k_1^2 T + 2k_2^2 b^2 Q^3 - 3k_2^2 b Q^2 + k_2^2 Q)}{T + Q} \\ &\quad - 2\left(\frac{k_1 b T^2 - k_1 T + k_2 b Q^2 - k_2 Q}{T + Q}\right)^2. \end{aligned} \quad (5.3)$$

Similarly, when $a_k = \delta_{k,k_1} - \delta_{k,k_2}$, it is easy to find $f = T - Q$, another new solution (shown in Figure 1) to Eq. (3.8) is given by

$$\begin{aligned} u_4 &= \frac{2(2k_1^2 b^2 T^3 - 3k_1^2 b T^2 + k_1^2 T - 2k_2^2 b^2 Q^3 + 3k_2^2 b Q^2 - k_2^2 Q)}{T + Q} \\ &\quad - 2\left(\frac{k_1 b T^2 - k_1 T - k_2 b Q^2 + k_2 Q}{T + Q}\right)^2. \end{aligned} \quad (5.4)$$

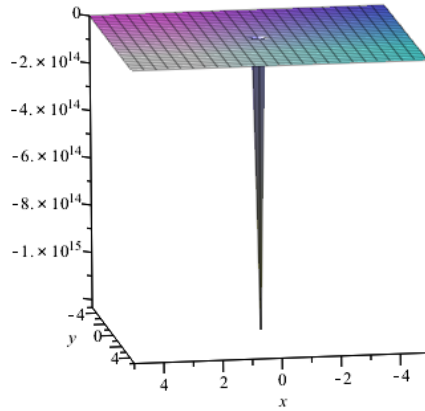


Figure 1. New solution (5.4). The parameters are selected as $k_1 = 1, k_2 = 2, b = 1, \lambda = 1, t = 0$.

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