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EXISTENCE OF POSITIVE SOLUTIONS FOR DAMPING ELASTIC SYSTEMS IN BANACH SPACES*

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Abstract The goal of this paper is to deal with the damped elastic systems with delay and nonlocal conditions in the framework of ordered Banach spaces. By combining a fixed point theorem for convex-power condensing operators with measures of noncompactness, we establish the existence of positive mild solutions for the aforementioned system. Our analysis assumes that the nonlinear function satisfies both measure conditions and order conditions. To demonstrate the practical applicability of our theoretical findings, we present a concrete example involving the vibration equation of a simply supported beam.

Keywords Damped elastic systems, nonlocal conditions, measure of noncompactness, positive mild solutions, convex-power condensing operator.

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1. Introduction

Second order systems arise in many fields, ranging from economics and biology to mathematical physics. The dynamic behaviors of certain natural phenomena can be characterized by these systems. A multitude of problems in elasticity theory, quantum mechanics, molecular dynamics, and mechanics can be formulated using time second order nonlinear partial differential equations. In mechanical or electronic devices, the function of a substance or component that curbs the growth of oscillation, vibration, or signal intensity is termed "damping". For example, sound proofing technologies serve to reduce sound wave fluctuations. Mathematically, damping can be represented as a force that acts opposite to, yet synchronously with, the motion of an object. Instances include the equations for a vibrating membrane or a vibrating nonlocal beam [23], the equations governing phase transitions in shape memory alloys [23], the viscous regularization of the Sine or Klein Gordon equation [15], and other equations in thermo viscoelasticity.

Damping denotes the energy dissipation that arises when a system is subjected to external forces or vibrations. Within a damped elastic system, energy loss mitigates vibration amplitudes and stabilizes the system. As a result, damping enjoys extensive applications in aerospace, construction, machinery, transportation, and electronics. For instance, implementing damping techniques at the nodes of a bridge's vertical support system can alleviate vibrations and cracking induced by natural disasters and wind, thereby bolstering the stability and safety of the support

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structure. The sine and cosine family theories were defined by Fattorini in 1969, and their properties have been explored in numerous research articles. Since damped elastic systems do not satisfy the assumptions of the cosine-family theory, we use the semigroup approach.

In recent years, the vibration equation of structural damping beams has emerged as an important mathematical model for spacecraft dynamics, attracting increasing attention from researchers. The theoretical foundation for such problems dates back to 1751, when Leonhard Euler and Daniel Bernoulli first proposed the renowned Euler-Bernoulli beam equation

$$\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\Big(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\Big)=0, \quad x\in(0,1),\ t>0,$$

where $\rho(x)$ denotes the mass density of the beam, E represents the elastic modulus, and $I(x)$ is the moment of inertia of the beam's cross-section. In the early decades, such equations were extensively studied. Typically, the beam vibration equation is transformed into an equivalent abstract elastic system, whose properties are then analyzed see [10, 11, 19, 22, 33–38, 48, 56]. In 1981, Chen and Russel [9] proposed the damped elastic system

$$\begin{cases}\ddot{x}(t)+B\dot{x}(t)+Ax(t)=0,\\ x(0)=x_0,\ \dot{x}(0)=y_0.\end{cases}$$

This class of problems has emerged as a prototypical research focus in the theory of second-order evolution equations. Recent years have witnessed growing scholarly interest in these issues, as evidenced by [9, 20, 21, 27, 28, 46, 50, 58] and the references therein. Notably, Fan et al. [11] investigated the existence, uniqueness, and asymptotic stability of solutions for semilinear structurally damped elastic systems described by

$$\begin{cases}\ddot{x}(t)+\rho A\dot{x}(t)+A^2x(t)=f(t,x(t)), \quad t\in[0,a],\\ x(0)=x_0,\ \dot{x}(0)=y_0,\end{cases}$$

employing operator semigroup theory and diverse nonlinear analysis techniques.

In [49], Luong and Tung analyzed structurally damped elastic systems

$$\begin{cases}x''(t)+\rho Ax'(t)+A^2x(t)=f(t,x(t)), \quad t>0,\\ x(0)+g(x)=x_0, \quad x'(0)+h(x)=y_0,\end{cases}$$

where $A: D(A)\subset E\to E$ is a closed linear operator, $\rho\geq 2$ is given constant, $x_0\in D(A), y_0\in E$. By using measure of noncompactness, they established the existence of mild solutions with explicit decay rate of exponential type.

And then, Luong et al. [50] considered nonlinear evolution equations of second order in Banach spaces

$$\begin{cases}x''(t)+\rho Ax'(t)+A^2x(t)=f(t,x(t),x_t), \quad t\in I=[0,T],\\ x(s)=\varphi(s), \quad s\leq 0,\\ x'(0)+h(x)=\psi,\end{cases}$$

where x is the unknown function in E, x_t is the history state defined by $x_t : (-\infty, 0] \to E, x_t(s) = x(t+s), t \in I$, by using the condensing maps, they proved the existence and exponential decay of mild solutions.

In [14], T. Diagana studied the well-posedness and existence of bounded solutions to the linear elastic systems with damping

$$\begin{cases} u''(t) + \rho B u'(t) + A u(t) = f(t), \quad t > 0, \\ u(0) = u_0 \in D(A), \quad u'(0) = u_1 \in E, \end{cases}$$

where $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ are densely defined closed (possibly unbounded) linear operators on a complex Banach space E and $f : \mathbb{R}^+ \to E$ is a continuous function.

The theory of nonlocal evolution equations constitutes a significant branch of mathematical analysis [41, 42, 44, 50]. The foundation for this field was laid in 1991 by Polish mathematician Byszewski [8], who first introduced the nonlocal Cauchy problem and investigated the semilinear parabolic evolution equation with nonlocal conditions:

$$x(t_0) + g(t_1, t_2, \cdots, t_k, x(\cdot)) = x_0,$$

where $0 \le t_0 < t_1 < \cdots < t_k \le t_0 + a$, with a a being a positive constant. Byszewski demonstrated that these abstract results could be applied to determine position changes of physical objects in kinematic and dynamic systems. Subsequent research has revealed that nonlocal problems often yield superior applicability compared to classical Cauchy problems. Consequently, differential equations with nonlocal conditions have attracted considerable attention, leading to fundamental advances in the theory of nonlocal problems [2,5–8,18,32,55,57]. For comprehensive reviews and additional references, we refer readers to the cited literature.

We employ nonlocal initial conditions in our analysis, motivated by their superior practicality compared to classical conditions when modeling physical phenomena. A key advantage of this approach lies in its enhanced measurement accuracy, for instance, the composite measurement:

$$u(x, 0) + \sum_{k=1}^{n} \beta_k(x) u(x, T_k)$$

provides a more reliable characterization of system states than the traditional single-point measurement $u(x, 0)$ alone. This methodology was originally introduced by Deng [13] in modeling gas diffusion through narrow tubes, where it proved particularly valuable for low-concentration scenarios. When the initial gas quantity is extremely small, the aggregated measurement approach yields significantly more reliable results than relying solely on the instantaneous measurement at $t = 0$. For comprehensive discussions of nonlocal condition applications, we refer readers [2, 5, 8, 32, 50, 54, 55, 59] and the references cited therein.

Motivated by these considerations, we investigate the existence of positive mild solutions for an abstract damped beam vibration model incorporating time-delay effects and nonlocal conditions within an ordered Banach space E

$$\begin{cases} u''(t) + \rho B u'(t) + A u(t) = F(t, u(t), u_t), \quad t \in [0, a], \\ u(t) = \varphi(t), \quad t \in [-r, 0], \\ u'(0) = g(u) + \phi, \end{cases} \tag{1.1}$$

where $u(\cdot) \in E$; $J = [0, a]$ with $a > 0$, u'' and u' denote the second and first order time derivatives of u respectively, where $\rho > 0$ represents the damping coefficient. The operators $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ are densely defined, closed (and potentially unbounded) linear operators on a Banach space E, $F : J \times K \times K_{\mathcal{B}} \to K$, $g : C(\mathbb{R}^+, E) \to E$, $\mathcal{B} := C([-r, 0], E)$. For $t \geq 0$, $u_t \in \mathcal{B}$ with $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, $\varphi \in \mathcal{B}$ and $\varphi(0) \in \mathcal{D}(A)$, $\phi \in E$, $r > 0$ is a constant.

Based on the idea of paper [39]. The use of this equation as follows: If a mechanical system possesses both mass and a restoring force, it will vibrate at one or more natural frequencies. Previous responses have addressed the role of damping in the free vibration of such systems. However, when the system is subjected to an external force at a frequency matching one of its natural frequencies—a condition known as resonance—the amplitude of vibration increases with each cycle. In the absence of damping, the amplitude would theoretically grow without bound. If damping (i.e., energy dissipation) is present, the amplitude will stabilize at a level where the work input per cycle is fully dissipated within the same cycle. Hence, the amplitude at resonance is primarily controlled by damping.

Elastic damping systems are widely employed in practical applications to reduce vibrations and absorb shock through elastic deformation. Shock absorbers are among the most common examples, used in vehicles, suspension bridges, tall buildings, and sensitive machinery or electronic equipment, where controlling unwanted oscillations is essential.

Shock absorbers are typically designed to be critically damped, allowing the system to return to equilibrium in the shortest possible time. Similarly, buildings—though they may appear rigid—are in fact elastic structures capable of bending. In many cases, their dynamic behavior can be compared to that of inverted pendulums, particularly under lateral loads.

Bridges and towers with elastic damping: A partial differential equation is commonly employed to model the dynamic behavior of such systems. This formulation incorporates deterministic terms, such as those representing elastic damping caused by wind. The equation of motion generally includes deterministic components related to damping, stiffness, and inertia.

A mathematical model incorporating elastic damping is developed for bridges and towers, taking into account uncertainties in loads—such as those from wind, traffic, and seismic events—as well as in material properties. Such probabilistic modeling is crucial for real-world structures like bridges and towers, where both environmental forces and structural responses are highly unpredictable.

Consider a beam-like structure, such as a bridge or tower, of length L. Its deformation is governed by a displacement field $u(x, t)$, where $x \in [0, L]$ denotes the position along the structure and t represents time. The system incorporates elastic damping and is subjected to environmental forces arising from wind, traffic, or seismic activity.

The transverse displacement $u(x, t)$ of the system, incorporating elastic damping, is governed by the following general partial differential equation:

$$\rho A \frac{\partial^2 u(x,t)}{\partial t^2} + c \frac{\partial^2 u(x,t)}{\partial x^2} + EI \frac{\partial^4 u(x,t)}{\partial x^4} = f(t, u(x,t)),$$

ρ represents the mass density of the material, A is the cross-sectional area, c denotes the damping coefficient (accounting for elastic damping), EI corresponds to the flexural rigidity—determined by Young's modulus E and the moment of inertia I, and $f(t, u(x, t))$ models the deterministic external load, such as that induced by traffic.

The term $c \frac{\partial^2 u(x,t)}{\partial x^2}$ represents elastic damping, which models energy dissipation caused by elastic deformation within the material. This mechanism captures how structures such as bridges or towers attenuate vibrational energy through internal damping. The damping coefficient c quantitatively characterizes the rate of this energy dissipation.

The key contributions of our work compared to existing literature are summarized as follows:

(1) We establish existence results requiring only measure conditions and order conditions on the nonlinear term F, eliminating the need for any Lipschitz-type assumptions that are commonly imposed in previous studies.

(2) Our methodology employs a novel estimation technique combining convex-power condensing operators with Hausdorff measures of noncompactness. This innovative approach allows us to prove the existence of positive solutions without requiring uniform continuity of the nonlinearity.

This paper is organized as follows. Section 2 provides the necessary preliminary material. In Section 3, we establish the existence of positive mild solutions through the application of convex-power condensing operators and a novel estimation technique involving Hausdorff measures of noncompactness. Finally, Section 4 presents a concrete example demonstrating the applicability of our theoretical results.

2. Preliminaries

Throughout this paper, let $(E, \|\cdot\|)$ be an ordered Banach space with partial order "$\leq$". The positive cone $K = \{u \in E | u \geq \theta\}$ (θ is the zero element of E) is normal with normal constant N. Let $\mathcal{L}(E)$ denote the Banach space of all bounded linear operators on E, endowed with the operator norm $\|\cdot\|_{\mathcal{L}(E)}$. We write $C(J, E)$ for the Banach space of all continuous E-valued functions defined on J, equipped with the supremum norm $\|u\|_C = \sup\limits_{t\in J} \|u(t)\|$. Furthermore, let $\mathcal{B} := C([-r, 0], E)$ represent the Banach space of continuous functions from $[-r, 0]$ to E, with norm $\|\phi\|_{\mathcal{B}} = \sup\limits_{s\in[-r,0]} \|\phi(s)\|$, where $r > 0$.

We define the positive cone K_C by

$$K_C = \{u \in C(J, E) | u(t) \in K, \quad t \in J\},$$

which is normal with normality constant N. Consequently, $C(J, E)$ becomes an ordered Banach space when equipped with the cone K_C. Similarly, $\mathcal{B}$ is an ordered Banach space endowed with the positive cone

$$K_{\mathcal{B}} = \{\phi \in \mathcal{B} | \phi(s) \in K, \ s \in [-a, 0]\}.$$

Let $A : D(A) \subset E \to E$ be a closed linear operator such that $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ on E. By the exponential boundedness property of C_0-semigroups, there exist constants $M \geq 1$ and $\nu \in \mathbb{R}$ satisfying

$$\|T(t)\| \leq Me^{\nu t}, \quad t \geq 0. \tag{2.1}$$

When $\nu = 0$, the semigroup is called uniformly bounded, meaning $\|T(t)\| \leq M$ for all $t \geq 0$.

Definition 2.1. ([17, 51]) Let $T(t)(t \geq 0)$ be a C_0-semigroup on E. We define the growth exponent ν_0 as the infimum of all $\nu \in \mathbb{R}$ for which there exists $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\nu t} \text{ for all } t \geq 0, \tag{2.2}$$

that is,

$$\nu_0 = \inf\{\nu \in \mathbb{R} | \exists\ M \ \geq 1 \text{ such that } \|T(t)\| \leq Me^{\nu t}, \quad t \geq 0\}.$$

When $\nu_0 < 0$, the C_0-semigroup $T(t)(t \geq 0)$ is called exponentially stable.

Definition 2.2. ([17,51]) A C_0-semigroup $T(t)(t \geq 0)$ on E is called positive if $T(t)x \geq \theta$ for all $x \in E$ with $x \geq \theta$ and all $t \geq 0$.

Definition 2.3. ([25,26]) A C_0-semigroup $T(t)(t \geq 0)$ on E is called compact if $T(t)$ is a compact operator on E for every $t > 0$.

Definition 2.4. ([21,30]) A C_0-semigroup $T(t)(t \geq 0)$ on E is called equicontinuous if the mapping $t \to T(t)$ is norm-continuous for every$t > 0$.

By Definition 2.1, any exponentially stable C_0-semigroup $T(t)(t \geq 0)$ is uniformly bounded. When $T(t)(t \geq 0)$ is continuous in the uniform operator topology for each $t > 0$, the growth exponent ν_0 can be characterized in terms of the spectral bound of its generator A, i.e.,

$$\nu_0 = -\inf\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}. \tag{2.3}$$

In fact, it is known from [51,53] that for any compact C_0-semigroup $T(t)(t \geq 0)$, the mapping $t \mapsto T(t)$ is norm-continuous for all $t \geq 0$.

In order to investigate the well-posedness of problem (1.1), we will extensively utilize the aforementioned linear operator that connects both operators A and B: $C(\rho) = \rho^2 B^2 - 4A = L^2(\rho)$ with $D(C(\rho)) = D(B^2) \cap D(A)$. In this paper, we will focus on the cases: $C(\rho) = L^2(\rho) > 0$, for densely closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$. Clearly, the case of $C(\rho) = 0$ corresponds to the scenario analyzed in [22] and [20]. When $C(\rho) = L^2(\rho) < 0$, further details can be found in [27,28].

In the following, we recall some basic facts on the second order linear evolution equation, which are needed to prove our main results.

Let $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$ be densely defined, closed (possibly unbounded) linear operators on a Banach space E, and let $h \in C(J, E)$. Assume there exists a densely defined, closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$ such that $\varphi(0) \in D(L(\rho)) \cap D(B)$ and

(F1) $C(\rho) = \rho^2 B^2 - 4A = L^2(\rho) > 0$;

(F2) $BL(\rho) = L(\rho)B$;

(F3) $-E_1(\rho)$ and $-E_2(\rho)$ are respectively the infinitesimal generators of compact, analytic and exponentially stable C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ on X with growth exponent $\nu_i < 0$ satisfy $\nu_2 < \nu_1 < 0$ and

$$E_1(\rho) = \frac{1}{2}(\rho B - L(\rho)), \quad E_2(\rho) = \frac{1}{2}(\rho B + L(\rho)). \tag{2.4}$$

For $u'(t) \in D(E_1(\rho)) \cap D(E_2(\rho))$ and $u(t) \in D(E_2(\rho)), E_1(\rho)u(t) \in D(E_2(\rho))$, the equation

$$u''(t) + \rho Bu'(t) + Au(t) = h(t), \quad t > 0, \tag{2.5}$$

can be decomposed as

$$\left(\frac{d}{dt} + E_1(\rho)\right)\left(\frac{d}{dt} + E_2(\rho)\right) u = h(t), \quad t > 0.$$

This leads to the form

$$\frac{d^2u}{dt^2} + (E_1(\rho) + E_2(\rho))\frac{du}{dt} + E_1(\rho)E_2(\rho)u = h(t). \tag{2.6}$$

By (2.5) and (2.6), we derive

$$E_1(\rho) + E_2(\rho) = \rho B, \quad E_1(\rho)E_2(\rho) = A. \tag{2.7}$$

Let $C(\rho) = \rho^2B^2 - 4A = L^2(\rho) > 0$, then:

$$E_1(\rho) = \frac{\rho B - \sqrt{\rho^2B^2 - 4A}}{2} = \frac{\rho B - L(\rho)}{2},$$
$$E_2(\rho) = \frac{\rho B + \sqrt{\rho^2B^2 - 4A}}{2} = \frac{\rho B + L(\rho)}{2}.$$

Letting $u'(t) + E_2(\rho)u = v(t)$, by defining

$$v_0 := v(0) = g(u) + \phi + E_2(\rho)\varphi(0). \tag{2.8}$$

This reduces the system (2.5) to Cauchy problems

$$\begin{cases} v'(t) + E_1(\rho)v = h(t), \\ v(0) = v_0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} u'(t) + E_2(\rho)u = v(t), \\ u(0) = \varphi(0). \end{cases} \tag{2.10}$$

By operator semigroup theory [51], if $h \in C(J, E)$, then the problem (2.9) has a mild solution

$$v(t) = T_1(t)v_0 + \int_0^t T_1(t-s)h(s)ds. \tag{2.11}$$

Similarly, for any $v \in C(J, E)$, the mild solution to problem (2.10) can be expressed as

$$u(t) = T_2(t)\varphi(0) + \int_0^t T_2(t-s)v(s)ds. \tag{2.12}$$

Inserting (2.11) into (2.12) yields

$$\begin{aligned} u(t) =& T_2(t)\varphi(0) + \int_0^t T_2(t-s)T_1(s)(g(u) + \phi + E_2(\rho)\varphi(0))ds \\ &+ \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)h(\tau)d\tau ds. \end{aligned}$$

Definition 2.5. Let $F \in C(\mathbb{R}^+ \times E \times \mathcal{B}, E)$, $-E_1(\rho)$ and $-E_2(\rho)$ are the infinitesimal generators of C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ respectively. A function $u \in C([-r, +\infty), E)$ is called a mild solution of the nonlocal problem (1.1) if it satisfies $u(t) = \varphi(t)$ for $t \in [-r, 0]$ and

$$u(t) = T_2(t)\varphi(0) + \int_0^t T_2(t-s)T_1(s)(g(u) + \phi + E_2(\rho)\varphi(0))ds$$

$$+\int_0^t\int_0^s T_2(t-s)T_1(s-\tau)F(\tau,u(\tau),u_\tau)d\tau ds,\quad \text{for } t\geq 0. \tag{2.13}$$

Moreover, when $u(t)\geq\theta$ for all $t\in[-r,a)$, the solution is called a positive mild solution of the nonlocal problem (1.1).

By Definition 2.1, we have

$$\|T_i(t)\|\leq M_i e^{\nu_i t},\quad t\geq 0. \tag{2.14}$$

Definition 2.6. ([12]) The Hausdorff measure of noncompactness $\alpha(\cdot)$ for a bounded subset D of a Banach space E is defined as

$$\alpha(D):=\inf\{\varepsilon>0: D\subset\cup_{k=1}^n B(\xi_k,d_k)\ ,\xi_k\in E, d_k<\varepsilon(k=1,\cdots,n), n\in\mathbb{N}\},$$

where $B(\xi_k,d_k)$ denotes the open ball centered at ξ_k with radius d_k.

Lemma 2.1. ([3,4,12]) *Let C,D be bounded subsets of a Banach space E and let $\delta\in\mathbb{R}$. Then the Hausdorff measure of noncompactness $\alpha(\cdot)$ satisfies the following properties:*

(1) $\alpha(D)=0$ if and only if D is relatively compact;

(2) (Monotonicity) If $C\subseteq D$, then $\alpha(C)\leq\alpha(D)$;

(3) $\alpha(\overline{D})=\alpha(D)$; where $\overline{D}$ denotes the closure of D;

(4) (Union property) $\alpha(C\cup D)=\max\{\alpha(C),\alpha(D)\}$;

(5) (Homogeneity) $\alpha(\lambda C)=|\lambda|\alpha(C)$, where $\lambda S=\{x=\lambda z: z\in C\}$;

(6) (Subadditivity) $\alpha(C+D)\leq\alpha(C)+\alpha(D)$, where $S+T=\{x=y+z: y\in C, z\in D\}$;

(7) (Convex hull invariance) $\alpha(co(D))=\alpha(D)$;

(8) (Lipschitz mapping) For any Lipschitz continuous map $Q: D(Q)\subseteq E\to Z$ with constant k between Banach spaces, $\alpha_z(Q\Omega)\leq k\alpha(\Omega)$ holds for all bounded $\Omega\subseteq D(\Omega)$.

In this article, we denote by $\alpha(\cdot)$ and $\alpha_C(\cdot)$ the Hausdorff measure of noncompactness on the bounded set of E and $C(J,E)$ respectively. For any $D\subset C([c,d],E)$ and $t\in J$, we define $D(t)=\{x(t)|x\in D\}$, which is clearly a subset of E. Furthermore, if D is bounded in $C([c,d],E)$, then $D(t)$ is bounded in E and satisfies the inequality $\alpha(D(t))\leq\alpha_C(D)$.

Lemma 2.2. ([31]) *Let E be a Banach space, $B=\{u_n\}_{n=1}^\infty\subset C([0,a],E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integrable on $[0,a]$, and*

$$\alpha\Big(\Big\{\int_0^a u_n(t)dt|n\in\mathbb{N}\Big\}\Big)\leq 2\int_0^a\alpha(B(t))dt.$$

Lemma 2.3. ([30]) *Let E be a Banach space, $B\subset C([0,a],E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on $[0,a]$, and $\alpha(B)=\max_{t\in[0,a]}\alpha(B(t))$.*

Lemma 2.4. ([47]) *Let $B\subset C(J,E)$ be bounded and equicontinuous, then $\overline{co}B\subset C(J,E)$ is also bounded and equicontinuous.*

Lemma 2.5. ([43]) *Let E be a Banach space, and let $D\subset E$ be bounded. Then there exists a countable set $D_0\subset D$, such that $\alpha(D)\leq 2\alpha(D_0)$.*

The following fixed-point theorem extends Sadovskii's fixed-point theorem. Originally introduced by Sun and Zhang [52] in the context of the Kuratowski measure of noncompactness, this result can also be formulated in terms of the Hausdorff measure of noncompactness. Indeed, owing to the relationship between these two noncompactness measures (see Theorem 5.13 in [4]), an adaptation of the proof technique employed in Lemma 2.4 of [47] yields the analogous conclusion for the Hausdorff measure setting.

Definition 2.7. Let E be a real Banach space. An operator $\mathcal{O}: E \to E$ is called convex-power condensing (with respect to $\exists\ u_0 \in E$ and $n_0 > 0$) if the following conditions hold:

(1) $\mathcal{O}$ is continuous and bounded.

(2) For every bounded subset $W \subset E$ that is not precompact,

$$\alpha(\mathcal{O}^{(n_0,u_0)}(W)) < \alpha(W),$$

where the iterated operator $\mathcal{O}^{(n_0,u_0)}$ is defined recursively by

$$\mathcal{O}^{(1,u_0)}(W) \equiv \mathcal{O}(W), \mathcal{O}^{(n,u_0)}(W) = \mathcal{O}(\overline{co}\mathcal{O}^{(n_1,u_0)}(W), u_0), n = 2, 3, \cdots.$$

Lemma 2.6. ([52]) *Let E be a real Banach space and $D \subset E$ a bounded, closed, convex subset of E If If $\exists\ u_0 \in D$ and integer $n_0 > 0$ such that $\mathcal{O}: D \to D$ is a convex-power condensing operator (with respect to u_0 and n_0), then $\mathcal{O}$ admits at least one fixed point in D.*

Lemma 2.7. ([26]) *Assuming E is a Banach space, and $D \subset E$ is a bounded closed convex sets, $\mathcal{O}: D \to D$ is a condensing operator. Then $\mathcal{O}$ has at least a fixed point in D.*

Lemma 2.8. ([40]) *Let $T > 0$ and consider functions $a, m \in C([0,T], \mathbb{R}^+)$. For any nonnegative function $y \in C([0,T], \mathbb{R}^+)$ satisfying the integral inequality*

$$y(t) \le m(t) + \int_0^t a(s)y(s)ds,\ \forall\ t \in [0,T],$$

the following estimate holds:

$$y(t) \le m(t) + \int_0^t a(s)m(s)e^{\int_s^t a(\tau)dg(\tau)}ds.$$

3. Main result

We define the function $u_\varphi(t): [-r, a] \to K$ by the piecewise formula:

$$u_\varphi(t) = \begin{cases} u(t), & t \in J, \\ \varphi(t), & t \in [-r, 0], \end{cases} \tag{3.1}$$

where $\varphi \in K_{\mathcal{B}}$, $u \in C(J,K)$.

We define the closed subspace $C_\varphi(K)$ of $C(J,K)$ by

$$C_\varphi(K) := \{u \in C(J,K) \mid u(0) = \varphi(0)\} \tag{3.2}$$

equipped with the norm $\|\cdot\|_\varphi$.

In what follows, we demonstrate the central result of this section.

Theorem 3.1. *Let E be an ordered Banach space with a normal positive cone $K \subset E$. Consider two densely defined, closed (possibly unbounded) linear operators $A : D(A) \subset E \to E$ and $B : D(B) \subset E \to E$. Suppose there exists another densely defined, closed linear operator $L(\rho) : D(L(\rho)) \subset E \to E$ such that $\varphi(0) \in D(L(\rho)) \cap D(B)$ and conditions (F1)-(F3) are satisfied. If $\varphi \in K_{\mathcal{B}}, \varphi(0) \in K \cap \mathcal{D}(A)$, $\phi \in K$ and the following conditions hold:*

$(H1)$ *The operator A generates a positive, bounded, strongly continuous semigroup $\{T(t)\}_{t\geq 0}$.*

$(H2)$ *The function $F(\cdot, x, \phi)$ is measurable for all $\forall\ u \in G(J, E)$ $\phi \in \mathcal{B}$, while $F(t, \cdot, \cdot)$ is continuous for a.e., $t \in J$.*

$(H3)$ *The nonlinear functions $F : J \times E \times \mathcal{B} \to E$ and $g : C([-r, 0], E) \to E$ are continuous. $\exists\ L_1, L_2, L_g > 0$, for $\forall\ D \in K, \mathcal{D} \in K_{\mathcal{B}}$, $t \in J$,*

$$\alpha(F(t, D, \mathcal{D})) \leq L_1\alpha(D) + L_2 \sup_{\tau\in[-r,0]} \alpha(\mathcal{D}(\tau)),$$

$$\alpha(\{g(D)\}) \leq L_g\alpha(D).$$

$(H4)$ *$\exists\ c_1, c_2 > 0$, for $t \in J$, $x \in K, \phi \in K_{\mathcal{B}}$,*

$$F(t, x, \phi) \leq c_1x + c_2\phi(\cdot).$$

$(H5)$ *$g : G(J, K) \to K$ is continuous mapping, and $\exists\ c_0, d_0 > 0$,*

$$\|g(u)\| \leq c_0\|u\| + d_0.$$

$(H6)$ *The condition*

$$\Theta = aM_1M_2[c_0 + a(c_1 + c_2)] \in (0, 1).$$

The problem (1.1) admits at least one positive mild solution $u_\varphi \in C(J, K)$.

Proof. Consider the operator $\mathcal{O}$ acting on $C_\varphi(K)$ given by

$$(\mathcal{O}u)(t) = \begin{cases} T_2(t)\varphi(0) + \displaystyle\int_0^t T_2(t-s)T_1(s)(g(u) + \phi + E_2(\rho)\varphi(0))ds \\ + \displaystyle\int_0^t \int_0^s T_2(t-s)T_1(s-\tau)F(\tau, u(\tau), u_\tau)d\tau ds, \quad t \in J, \quad \varphi(t), \quad t \in [-a, 0]. \end{cases} \tag{3.3}$$

By assumption (H2), the iterated integral $\int_0^t \int_0^s T_2(t-s)T_1(s-\tau)F(\tau, u(\tau), u_\tau)d\tau ds$ is well defined. Consequently, the operator $\mathcal{O} : C_\varphi(K) \to C_\varphi(K)$ is well defined. Moreover, according to Definition 2.5, any fixed point $u \in C_\varphi(K)$ of $\mathcal{O}$ yields a mild solution to problem (1.1).

In what follows, we establish the existence of at least one fixed point of $\mathcal{O}$ by applying Lemma 2.6.

Define the closed ball

$$\Omega_R = \{u \in C_\varphi(K) \mid \|u\|_C \leq R\}, \tag{3.4}$$

centered at θ with radius R in $C_\varphi(K)$. For any $\forall\ u \in \Omega_R$ and $t \in J$, we have $\|u_\varphi(t)\| \leq R$.

We establish the result through four steps.

Step 1. To check that $\exists\ R_0 > 0$, $\mathcal{O}(\Omega_{R_0}) \subset \Omega_{R_0}$.

Indeed, were this not the case, it would follow that for every $\forall\ R > 0$, $\exists\ u \in \Omega_R$ with $\|\mathcal{O}u\|_C > R$. For $\forall\ u \in \Omega_{R_0}$, $t \in J$, by the definition of $\|\cdot\|_{\mathcal{B}}$ and $\mathcal{O}$, we have

$$\|u_t\|_{\mathcal{B}} = \sup_{-a\leq s\leq 0} \|u(t+s)\| \leq \sup_{-a\leq s\leq a} \|u(s)\| = \|u\|_C \leq R_0,$$

thus, we have

$$\begin{aligned}
&\|(\mathcal{O}u)(t)\| \\
&\le \|T(t)\varphi(0)\| + \Big\| \int_0^t T_2(t-s)T_1(s)(g(u)+\phi+E_2(\rho)\varphi(0))ds\Big\| \\
&\quad + \Big\| \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)F(\tau,u(\tau),u_\varphi(\tau))d\tau\Big\| \\
&\le M_2\|\varphi\|_{\mathcal{B}} + aM_1M_2\|(\|\phi\| + c_0\|u\| + d_0 + \|E_2(\rho)\varphi(0)\|) \\
&\quad + M_1M_2\| \int_0^t \int_0^s \|F(\tau,u(\tau),u_\varphi(\tau))\|d\tau \\
&\le M\|\varphi\|_{\mathcal{B}} + aM_1M_2(\|\phi\| + c_0R + d_0 + \|E_2(\rho)\varphi(0)\|) \\
&\quad + a^2M_1M_2(c_1+c_2)R. \qquad (3.5)
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
R < &\Big(M_2\|\varphi\|_{\mathcal{B}} + aM_1M_2(\|\phi\| + d_0 + \|E_2(\rho)\varphi(0)\|)\Big) \\
&+ a^2M_1M_2(c_1+c_2)R + aM_1M_2c_0R.
\end{aligned}$$

Taking the inferior limit as $R \to \infty$, we obtain the inequality

$$aM_1M_2[c_0 + a(c_1+c_2)] \ge 1.$$

However, this contradicts our assumption that $\Theta < 1$. Consequently, $\exists\ R_0 > 0$ such that $\mathcal{O}(\Omega_{R_0}) \subset \Omega_{R_0}$.

Step 2. We prove that $\mathcal{O} : \Omega_{R_0} \to \Omega_{R_0}$ is equicontinuous.

For $\forall\ u \in \Omega_{R_0}$ and $0 \le t_1 \le t_2 \le a$, by (H4) and (H5), we obtain

$$\begin{aligned}
&\|(\mathcal{O}u)(t_2) - (\mathcal{O}u)(t_1)\| \\
&\le \|(T(t_2) - T(t_1))\varphi(0)\| \\
&\quad + \| \int_0^t T_2(t_2-s)T_1(s)[g(u)+\phi+E_2(\rho)\varphi(0)]ds \\
&\quad - \int_0^t T_1(t_1-s)T_1(s)[g(u)+\phi+E_2(\rho)\varphi(0)]ds\| \\
&\quad + \| \int_0^{t_2}\int_0^s T_2(t_2-s)T_1(s-\tau)F(\tau,u(\tau),u_\varphi(\tau))d\tau ds \\
&\quad - \int_0^{t_1}\int_0^s T_2(t_1-s)T_1(s-\tau)F(\tau,u(\tau),u_\varphi(\tau))d\tau ds\| \\
&\le \|T_2(t_2) - T_2(t_1)\|_{\mathcal{L}(\mathbb{E})} \cdot \|\varphi(0)\| \\
&\quad + \| \int_0^{t_1} (T_2(t_2-s) - T_2(t_1-s))T_1(s)(g(u)+\varphi+E_2(\rho)\varphi(0))\| \\
&\quad + \| \int_{t_1}^{t_2} T_2(t_2-s)T_1(s)(g(u)+\varphi+E_2(\rho)\varphi(0))ds\| \\
&\quad + \| \int_0^{t_1}\int_0^s (T_2(t_2-s) - T_2(t_1-s))T_1(s-\tau)F(\tau,u(\tau),u_\varphi(\tau))d\tau ds\|
\end{aligned}$$

$$
\begin{aligned}
&+\|\int_{t_1}^{t_2}\int_0^s T_2(t_2-s)T_1(s-\tau)F(\tau,u(\tau),u_\varphi(\tau))d\tau ds\| \\
&:=I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t).
\end{aligned}
\tag{3.6}
$$

Since $T_2(t)$ for $t>0$ is continuous, then $I_1(t)\to 0$ as $t_2-t_1\to 0$.

For I_2, when $t_1=0$, we have $I_2=0$, when $0<t_1<t_2\le a$, we have

$$
\begin{aligned}
I_2(t) &\le \int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|\cdot\|T_1(s)\|\cdot\|g(u)+\phi+E_2(\rho)\varphi(0)\|ds \\
&\le M_1(c_0R_0+d_0+\|\phi\|+\|E_2(\rho)\varphi(0)\|)\cdot\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
&\to 0,\quad t_2-t_1\to 0.
\end{aligned}
$$

For $I_3(t)$, we have

$$
\begin{aligned}
I_3(t) &\le \int_{t_1}^{t_2}\|T_2(t_2-s)\|\cdot\|T_1(s)\|\cdot\|g(u)+\phi+E_2(\rho)\varphi(0)\|ds \\
&\le M_1M_2(c_0R_0+d_0+\|\phi\|+\|E_2(\rho)\varphi(0)\|)\cdot|t_2-t_1| \\
&\to 0,\quad t_2-t_1\to 0.
\end{aligned}
$$

For $I_4(t)$, when $t_1=0, 0<t_2\le a$, we have $I_2(t)=0$, when $0<t_1<t_2\le a$, take a sufficiently small constant $\epsilon\to 0^+$, we have

$$
\begin{aligned}
&\int_0^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
\le&\int_0^{t_1-\epsilon}\|T_2(t_2-s)-T_2(t_1-s)\|ds+\int_{t_1-\epsilon}^{t_1}\|T_2(t_2-s)-T_2(t_1-s)\|ds \\
&+\|T_2(t_2-t_1+\epsilon)-T_2(\epsilon)\|\cdot\int_0^{t_1-\epsilon}\|T_2(t_1-\epsilon-s)\|ds \\
&+\int_{t_1-\epsilon}^{t_1}(\|T_2(t_2-s)\|+\|T_2(t_1-s)\|)ds \\
\le& M_2|a-\epsilon|\cdot\|T_2(t_2-t_1+\epsilon)-T_2(\epsilon)\|+2M_2\epsilon \\
\to& 0,\quad t_2-t_1\to 0.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
I_4(t) &\le M_1((c_1+c_2)\|u\|_C)\int_0^{t_1}\int_0^s\|T_2(t_2-s)-T_2(t_1-s)\|d\tau ds \\
&\le aM_1((c_1+c_2)\|u\|_C)\int_0^{t_1}\|T_2(t_2-t_1+s)-T_2(s)\|ds \\
&\to 0,\quad t_2-t_1\to 0.
\end{aligned}
$$

For $I_5(t)$, we have

$$
I_5(t)\le\int_{t_1}^{t_2}\int_0^s\|T_2(t_2-s)T_1(s-\tau)F(\tau,u(\tau),u_\tau)\|d\tau ds
$$

$$\leq \frac{1}{2}M_1M_2((c_1+c_2)\|u\|_C)(t_2^2-t_1^2)$$
$$\to 0, \quad t_2-t_1\to 0.$$

Hence, when $0\leq t_1<t_2\leq a$ and $t_2-t_1\to 0$, $\|(\mathcal{O}u)(t_2)-(\mathcal{O}u)(t_1)\|\to 0$ does not depend on $u\in\Omega_{R_0}$. And for $-a\leq t_1<t_2\leq 0$, by the definition $\mathcal{O}$ and the continuity of φ, we have

$$\|(\mathcal{O}u)(t_2)-(\mathcal{O}u)(t_1)\|=\|\varphi(t_2-\varphi(t_1))\|\to 0, \quad t_2-t_1\to 0.$$

Hence, $\mathcal{O}(\Omega_{R_0})$ is equicontinuous.

Step 3. $\mathcal{O}:\Omega_{R_0}\to\Omega_{R_0}$ is continuous.

Consider a sequence $\{u^{(n)}\}_{n=1}^{\infty}\subset\Omega_{R_0}$ such that $u^{(n)}\to u(t)$ in E as $n\to\infty$, then, $u^{(n)}(t)\to u(t)\in E$ for each $t\in J$ as $n\to\infty$. Moreover, by (3.1), we have $u_t^{(n)}\to u_t\in\mathcal{B}$ as $n\to\infty$.

For $t\in J$, by the Continuity of nonlinear function f and g, we have

$$\lim_{n\to\infty}F(t,u^{(n)}(t),u_\varphi^{(n)}(t))=F(t,u(t),u_\varphi(t)), \quad \lim_{n\to\infty}g(u^{(n)})=g(u).$$

Thus,

$$\|F(t,u^{(n)}(t),u_\varphi^{(n)}(t))-F(t,u(t),u_\varphi(t))\|\leq 2N(2R_0+\|g\|_C). \tag{3.7}$$

By (3.7), we get that

$$\begin{aligned}
&\|\mathcal{O}(u^n)(t)-\mathcal{O}(u)(t)\|\\
\leq&\|\int_0^t T_2(t-s)T_1(s)(g(u^{(n)})-g(u))ds\|\\
&+\|\int_0^t\int_0^s T_2(t-s)T_1(s-\tau)(F(s,u^{(n)}(s),u_\varphi^{(n)}(s))-F(s,u(s),u_\varphi(s)))\|\\
\leq&\int_0^t\|T_2(t-s)\|_{\mathcal{L}(E)}\cdot\|T_1(s)\|_{\mathcal{L}(E)}\cdot\|g(u^{(n)})-g(u)\|ds\\
&+\int_0^t\int_0^s(t-s)\|T_2(t-s)\|_{\mathcal{L}(E)}\cdot\|T_1(s-\tau)\|_{\mathcal{L}(E)}\\
&\times\|F(\tau,u^{(n)}(\tau),u_\varphi^{(n)}(\tau))-F(\tau,u(\tau),u_\varphi(\tau))\|d\tau ds,\\
\leq& aM_1M_2(\|g(u^{(n)})-g(u)\|+a\|F(\tau,u^{(n)}(\tau),u_\varphi^{(n)}(\tau))-F(\tau,u(\tau),u_\varphi(\tau))\|)\\
\to&0, \quad n\to\infty.
\end{aligned} \tag{3.8}$$

For $t\in[-a,0]$, by the definition $\mathcal{O}$, we have

$$\|\mathcal{O}(u^n)(t)-\mathcal{O}(u)(t)\|=\|\varphi(t)-\varphi(t)\|=0.$$

Thus, we have

$$\|\mathcal{O}(u^{(n)})-\mathcal{O}(u)\|_C\to 0,\ n\to\infty.$$

Consequently, the operator $\mathcal{O}:\Omega_{R_0}\to\Omega_{R_0}$ is continuous.

Step 4. The operator $\mathcal{O}:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ is convex and condensing.

From the preceding arguments, we can directly deduce that $\mathcal{O}:\overline{\Omega}_{R_0}\to\overline{\Omega}_{R_0}$ equicontinuity, let $\Theta=\overline{co}\mathcal{O}(\overline{\Omega}_{R_0})$, from the foregoing analysis, it follows directly that$\mathcal{O}$ maps Θ onto itself, thus,

$\Theta \subset \overline{\Omega}_{R_0}$ is uniformly continuous. For any $B \subset \Theta, u_0 \in \Theta$, define the operator $\mathcal{O}^{(m,u_0)}(B)(m = 1, 2, \cdots)$ as follows:

$$\mathcal{O}^{(1,u_0)}(B) = \mathcal{O}(B),$$
$$\mathcal{O}^{(m,u_0)}(B) = \mathcal{O}(\overline{co}\{Q^{(m-1,u_0)}(B), u_0\}), \qquad m = 2, 3, \cdots .$$

Let $u_0 \in \Theta$, next, we show that $\exists\ m_0 > 0$, for non-relatively compact bounded set $B \subset \Theta$,

$$\alpha_C(\mathcal{O}^{(m_0,u_0)}(B)) < \alpha_C(B). \tag{3.9}$$

From the definition of $\mathcal{O}^{(m,u_0)}(B)$ and the equicontinuity of Θ, it follows that $\mathcal{O}^{(m,u_0)}(B) \subset \Omega_{R_0}(m = 1, 2, \cdots)$ is uniformly continous. Therefore, by Lemma 2.3, it can be concluded that

$$\alpha_C(\mathcal{O}^{(m,u_0)}(B)) = \max_{t\in[-r,a]} \alpha(\mathcal{O}^{(m,u_0)}(B)(t)), \quad m = 1, 2, \cdots . \tag{3.10}$$

In view of the definition of $\mathcal{O}$ and $\mathcal{O}^{(m,u_0)}(B)$, for $t \in [-r, 0]$, we have

$$\alpha(\mathcal{O}^{(m,u_0)}(B)(t)) \le \alpha(\mathcal{O}(B)(t)) = \alpha(\{\varphi(t)\}) = 0,\ m = 1, 2, \cdots .$$

Thus by virtue of the properties of noncompact measures, it can be concluded that for $\forall\ t \in [-r, 0]$,

$$\alpha(\mathcal{O}^{(m,u_0)}(B)(t)) \equiv 0, m = 1, 2, \cdots . \tag{3.11}$$

Based on the properties of the (3.10), (3.11) and noncompactness measures, it can be concluded that

$$\alpha_C(\mathcal{O}^{(m,u_0)}(B)) = \max_{t\in[0,a]} \alpha(\mathcal{O}^{(m,u_0)}(B)(t)), m = 1, 2, \cdots . \tag{3.12}$$

By Lemma 2.5, $\exists\ B^1 = \{u_1^{(n)} : n \in \mathbb{N}\} \subset B$,

$$\alpha_C(\mathcal{O}(B)) \le 2\alpha_C(\mathcal{O}(B^1)).$$

For $\forall\ t \in [0, a]$, let

$$B^1(t) = \{u_1^{(n)}(t) : u_1^{(n)} \in B^1\}, B_t^1 = \{(u_1^{(n)})_t : u_1^{(n)} \in B^1\} \subset \overline{\Omega}_{R_0},$$
$$\alpha_{\mathcal{B}}(\{u_1^{(n)}\}) = \sup_{\tau\in[-a,0]} \alpha(\{u_1^{(n)}(t+\tau)\}) \le \alpha(\{u_1^{(n)}(t)\}_\rho). \tag{3.13}$$

Then, according to (3.13), we get that $\alpha_{\mathcal{B}}(B_t^1) \le \alpha_C(B^1)$. Thus, for any $t \in [0, a]$, y combining the definition of operator $\mathcal{O}$ with condition (H3) and Lemmas 2.3–2.5, we rigorously establish that

$$\begin{aligned}
&\alpha(\mathcal{O}^{(1,u_0)}(B)(t))\\
&= \alpha(\mathcal{O}(B)(t))\\
&\le \alpha(\mathcal{O}(B^1)(t))\\
&\le \alpha(\{\int_0^t T_2(t-s)T_1(s)g(u_1^{(n)})ds : n \in \mathbb{N}\})\\
&\quad + \alpha(\{\int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, u_1^{(n)}(\tau), (u_1^{(n)})_\tau)d\tau ds : n \in \mathbb{N}\})
\end{aligned}$$

$$
\begin{aligned}
&\le 2\int_0^t \|T_2(t-s)\|_{\mathcal{L}(E)}\cdot\|T_1(s)\|_{\mathcal{L}(E)}\cdot\alpha(\{g(u_1^{(n)}):n\in\mathbb{N}\})d\tau ds\\
&\quad+4\int_0^t\int_0^s \|T_2(t-s)\|_{\mathcal{L}(E)}\cdot\|T_1(s-\tau)\|_{\mathcal{L}(E)}\cdot\alpha(\{f(\tau,u_1^{(n)}(\tau),(u_1^{(n)})_\tau):n\in\mathbb{N}\})d\tau ds\\
&\le 2M_1M_2\int_0^t (L_g\alpha_C(B^1))ds+4M_1M_2\int_0^t\int_0^s(L_1\alpha(B^1(\tau))+L_2\alpha_{\mathcal{B}}(B^1_\tau))d\tau ds\\
&\le 2M_1M_2L_g\cdot\alpha_C(B^1)\cdot t+4aM_1M_2(L_1+L_2)\cdot\alpha_C(B^1)\cdot t\\
&\le 2M_1M_2(L_g+2a(L_1+L_2))\cdot\alpha_C(B)\cdot t.
\end{aligned}
$$

By Lemma 2.5, $\exists\ B^2=\{u_2^{(n)}:n\in\mathbb{N}\}\subset\overline{co}\{\mathcal{O}^{(1,u_0)}(B),u_0\}$, such that

$$
\alpha_C(\mathcal{O}(\overline{co}\{\mathcal{O}^{(1,u_0)}(B),u_0\}))=\alpha_C(\mathcal{O}(B^2)).
$$

Similarly, for $\forall\ t\in[0,a]$, let

$$
B^2(t)=\{u_2^{(n)}(t):u_2^{(n)}\in B^2\},B^2_t=\{(u_2^{(n)})_t:u_2^{(n)}\in B^2\}\subset\overline{\Omega}_{R_0},
$$

$$
\alpha_{\mathcal{B}}(\{u_2^{(n)}(t)\})=\sup_{t\in[-a,0]}\alpha(\{u_2^{(n)}(t+\tau)_\rho\})\le\alpha(\{u_2^{(n)}(t)\}). \tag{3.14}
$$

And by (3.14) and (H3), $\alpha_{\mathcal{B}}(B^2_t)\le\alpha_C(B^2)$. Thus, for $\forall\ t\in[0,a]$, we get that

$$
\begin{aligned}
&\quad\alpha(\mathcal{O}^{(2,u_0)}(B)(t))\\
&=\alpha(\mathcal{O}(\overline{co}\{\mathcal{O}^{(1,u_0)}(B),u_0\})(t))\\
&=\alpha(\mathcal{O}(B^2)(t))\\
&\le\alpha(\{\int_0^t T_2(t-s)T_1(s)g(u_2^{(n)})ds:n\in\mathbb{N}\})\\
&\quad+\alpha(\{\int_0^t\int_0^s T_2(t-s)T_1(s-\tau)f(\tau,u_2^{(n)}(\tau),(u_2^{(n)})_\tau)d\tau ds:n\in\mathbb{N}\})\\
&\le 2\int_0^t \|T_2(t-s)\|_{\mathcal{L}(E)}\cdot\|T_1(s)\|_{\mathcal{L}(E)}\cdot\alpha(\{g(u_2^{(n)}):n\in\mathbb{N}\})d\tau ds\\
&\quad+4\int_0^t\int_0^s \|T_2(t-s)\|_{\mathcal{L}(E)}\cdot\|T_1(s-\tau)\|_{\mathcal{L}(E)}\cdot\alpha(\{f(\tau,u_2^{(n)}(\tau),(u_2^{(n)})_\tau):n\in\mathbb{N}\})d\tau ds\\
&\le 2M_1M_2\int_0^t (L_g\alpha_C(B^2))ds+4M_1M_2\int_0^t\int_0^s(L_1\alpha(B^2(\tau))+L_2\alpha_{\mathcal{B}}(B^2_\tau))d\tau ds\\
&\le 2M_1M_2(L_g+2a(L_1+L_2))\cdot\int_0^t\alpha_C(B^2)ds\\
&\le 2M_1M_2(L_g+2a(L_1+L_2))\cdot\int_0^t\alpha_C(\overline{co}\{Q^{(1,u_0)}(B),u_0\})ds\\
&\le(2M_1M_2(L_g+2a(L_1+L_2)))^2\cdot\alpha_C(B)\cdot\frac{t^2}{2!}.
\end{aligned}
$$

Thus,

$$
\alpha(\mathcal{O}^{(k,u_0)}(B)(t))\le(2aM_1M_2(L_g+2a(L_1+L_2)))^k\cdot\alpha_C(B)\cdot\frac{t^k}{k!}.
$$

By Lemma 2.5, $\exists\ B^{k+1} = \{u_{k+1}^{(n)} : n \in \mathbb{N}\} \subset \overline{co}\{\mathcal{O}^{(k,u_0)}(B), u_0\}$, such that

$$\alpha_C(\mathcal{O}(\overline{co}\{\mathcal{O}^{(k,u_0)}(B), u_0\})) = \alpha_C(\mathcal{O}(B^{k+1})).$$

Similarly, for $\forall\ t \in [0, a]$, let

$$B^{k+1}(t) = \{u_{k+1}^{(n)}(t) : u_{k+1}^{(n)} \in B^{k+1}\}, B_t^{k+1} = \{(u_{k+1}^{(n)})_t : u_{k+1}^{(n)} \in B^{k+1}\} \subset \overline{\Omega}_{R_0},$$

$$\alpha_{\mathcal{B}}(\{u_{k+1}^{(n)}(t)\}) = \sup_{t\in[-a,0]} \alpha(\{u_{k+1}^{(n)}(t+\tau)\}) \leq \alpha(\{u_{k+1}^{(n)}(t)\}). \tag{3.15}$$

Thus, by (3.15), we have $\alpha_{\mathcal{B}}(B_t^{k+1}) \leq \alpha_C(B^{k+1})$. For $\forall\ t \in [0, a]$, such that

$$\begin{aligned}
&\alpha(\mathcal{O}^{(k+1,u_0)}(B)(t)) \\
&= \alpha(\mathcal{O}(\overline{co}\{Q^{(k,u_0)}(B), u_0\})(t)) \\
&= \alpha(\mathcal{O}(B^{k+1})(t)) \\
&\leq \alpha(\{\int_0^t T_2(t-s)T_1(s)g(u_{k+1}^{(n)})ds : n \in \mathbb{N}\}) \\
&\quad + \alpha(\{\int_0^t \int_0^s T_2(t-s)T_1(s-\tau)f(\tau, u_{k+1}^{(n)}(\tau), (u_{k+1}^{(n)})_\tau)d\tau ds : n \in \mathbb{N}\}) \\
&\leq 2\int_0^t \|T_2(t-s)\|_{\mathcal{L}(E)} \cdot \|T_1(s)\|_{\mathcal{L}(E)} \cdot \alpha(\{g(u_{k+1}^{(n)}) : n \in \mathbb{N}\})d\tau ds \\
&\quad + 4\int_0^t \int_0^s \|T_2(t-s)\|_{\mathcal{L}(E)} \cdot \|T_1(s-\tau)\|_{\mathcal{L}(E)} \cdot \alpha(\{f(\tau, u_{k+1}^{(n)}(\tau), (u_{k+1}^{(n)})_\tau) : n \in \mathbb{N}\})d\tau ds \\
&\leq 2M_1M_2 \int_0^t (L_g\alpha_C(B^{k+1}))ds + 4M_1M_2 \int_0^t \int_0^s (L_1\alpha(B^{k+1}(\tau)) + L_2\alpha_{\mathcal{B}}(B_\tau^{k+1}))d\tau ds \\
&\leq 2M_1M_2(L_g + 2a(L_1 + L_2)) \cdot \int_0^t \alpha_C(B^{k+1})ds \\
&\leq 2M_1M_2(L_g + 2a(L_1 + L_2)) \cdot \int_0^t \alpha_C(\overline{co}\{Q^{(k,u_0)}(B), u_0\})ds \\
&\leq (2M_1M_2(L_g + 2a(L_1 + L_2)))^{k+1} \cdot \alpha_C(B) \cdot \frac{t^{k+1}}{(k+1)!}.
\end{aligned}$$

Hence, proceeding by mathematical induction, we establish that for all $\forall\ m > 0$, $t \in J$, we have

$$\alpha(\mathcal{O}^{(m,u_0)}(B)(t)) \leq (2aM_1M_2(L_g + 2a(L_1 + L_2)))^m \cdot \frac{t^m}{m!} \cdot \alpha_C(B),$$

from which it necessarily follows that

$$\alpha_C(\mathcal{O}^{(m,u_0)}(B)) \leq (2aM_1M_2(L_g + 2a(L_1 + L_2)))^m \cdot \frac{t^m}{m!} \cdot \alpha_C(B). \tag{3.16}$$

Thus, we get that $m \to \infty$, then

$$(2aM_1M_2(L_g + 2a(L_1 + L_2)))^m \cdot \frac{t^m}{m!} \to 0.$$

Thus, $\exists\ m_0$, which is large enough, such that

$$(2aM_1M_2(L_g + 2a(L_1 + L_2)))^{m_0} \cdot \frac{t^{m_0}}{m_0!} < 1.$$

By (3.16), we have

$$\alpha_C(\mathcal{O}^{(m_0,u_0)}(B)) < \alpha_C(B).$$

Hence, by Lemma 2.6, we get that the operator $\mathcal{O}$ defined by (3.3) has at least one fixed point $u \in \Theta \subset C_\varphi(K)$, which is a positive mild solution to (1.1). □

Theorem 3.2. *Under the conditions of Theorem 3.1, then the system (1.1) admits at least one positive mild solution $u_\varphi \in C(J, K)$ possessing the property that*

$$2aM_1M_2(L_g + 2a(L_1 + L_2)) < 1. \tag{3.17}$$

Proof. Define the operator $\mathcal{O} : C_\varphi(K) \to C_\varphi(K)$. By Theorem 3.1, we get that $\mathcal{O} : \Omega_R \to \Omega_R$ is a continuous and mild solution to (1.1) is equivalent to the fixed point of $\mathcal{O}$.

Similar to Step 3 in Theorem 3.1, we verify $\{\mathcal{O}u : u \in \Omega_R\}$ is equicontinuous in $C_\varphi(J, K)$, for $\forall\ B \subset \overline{co}\mathcal{O}(\Omega_R)$, $\mathcal{O}(B) \subset \Omega_R$ is equicontinuous. In view of Lemma 2.3, we get that

$$\alpha_C(\mathcal{O}(B)) = \sup_{t\in J} \alpha(\mathcal{O}(B)(t)). \tag{3.18}$$

By Lemma 2.3, $\exists\ B_0 = u_m \subset B$,

$$\alpha_G(\mathcal{O}(B)) \le \alpha_G(\mathcal{O}(B_0)). \tag{3.19}$$

For $t \in J$,

$$\alpha_{\mathcal{B}}(\{u_m(t)_\rho\}) = \sup_{\tau\in[-a,0]} \alpha(\{u_m(t+\tau)_\rho\}) \le \alpha(\{u_m(t)\}). \tag{3.20}$$

In view of Lemma 2.12 and (H3), (3.3), (3.20),

$$\begin{aligned}
&\alpha\Big(\mathcal{O}^{(1,u_0)}(B)(t)\Big)\\
&= \alpha\Big(\mathcal{O}(B)(t)\Big)\\
&\le 2\alpha\Big(\mathcal{O}(B_1)(t)\Big)\\
&\le 2\alpha(\{T_2(t)\varphi(0)\}) + 2\alpha\Big(\Big\{\int_0^t T_2(t-s)T_1(s)[g(u_1^{(n)}) + \phi + E_2(\rho)\varphi(0)]ds : n \in \mathbb{N}\Big\}\Big)\\
&\quad + 2\alpha\Big(\Big\{\int_0^t\int_0^s T_2(t-s)T_1(s-\tau)F(\tau, u_1^{(n)}(\tau), u_1^{(n)}(\tau)_\rho)d\tau ds : n \in \mathbb{N}\Big\}\Big)\\
&\le 2\int_0^t \|T_2(t-s)\|_{\mathcal{L}(E)} \cdot \|T_1(s)\|_{\mathcal{L}(E)}\alpha\Big(\Big\{g(u_1^{(n)}) : n \in \mathbb{N}\Big\}\Big)ds\\
&\quad + 4\int_0^t\int_0^s \|T_2(t-s)\|_{\mathcal{L}(E)} \cdot \|T_1(s-\tau)\|_{\mathcal{L}(E)}\alpha\Big(\Big\{F(\tau, u_1^{(n)}(\tau), u_1^{(n)}(\tau)_\rho) : n \in \mathbb{N}\Big\}\Big)d\tau ds\\
&\le 2M_1M_2\int_0^t L_g\alpha_C(B_1)ds + 4M_1M_2\int_0^t\int_0^s (L_1\alpha(B_1(s)) + L_2\alpha_{\mathcal{B}}(B_1(s)_\tau))d\tau ds\\
&\le 2aM_1M_2(L_g + 2a(L_1 + L_2))\alpha_C(B_1)
\end{aligned}$$

$$\le 2aM_1M_2(L_g + 2a(L_1 + L_2))\alpha_C(B).$$

And by (3.17), (3.18) and Lemma 2.7, we have

$$\alpha_C(\mathcal{O}(B)) \le 2aM_1M_2(L_g + 2a(L_1 + L_2))\alpha_C(B) \le \alpha_C(B). \tag{3.21}$$

Thus, $\alpha_C(\mathcal{O}(B)) \le \alpha_C(B)$, i.e., $\mathcal{O} : B \to B$ is a condensing operator. By Lemma 2.14, has at least one fixed point $u \in B \subset \overline{co}\mathcal{O}(\Omega_R)$. Consequently, the function $u_\varphi \in C_\varphi(J, K)$ constitutes a positive mild solution to (1.1). □

Theorem 3.3. *Under the conditions of Theorem 3.1, if $\varphi \in K_{\mathcal{B}}$ with $\varphi(0) \in K \cap \mathcal{D}(A)$ and $\phi \in K$, then under hypotheses (H1)–(H3), (H5), and (H7) For $\forall\ t \ge 0$, let $x_1, x_2 \in K$ satisfy*

$$\theta \le x_1 \le x_2,\ \ \|x_i\| \le R \quad (i = 1, 2),$$

and let $\phi_1, \phi_2 \in K_{\mathcal{B}}$ satisfy

$$\theta \le \phi_1 \le \phi_2,\ \ \|\phi_i\|_{\mathcal{B}} \le R \quad (i = 1, 2),$$

the mapping F satisfies the condtion

$$F(t, x_2, \phi_2) \ge F(t, x_1, \phi_1) \ge \theta.$$

(H8) $\forall\ t \ge 0,\ x \in E,\ \phi \in \mathcal{B},\ \exists\ p_i(\cdot) \in L^1(\mathbb{R}^+, \mathbb{R}^+),\ \exists\ \overline{K} \ge 0$

$$\|F(t, x, \phi)\| \le p_1(t)\|x\| + p_2(t)\|\phi\|_{\mathcal{B}} + \overline{K},$$

the equation (1.1) admits a minimal positive mild solution $u \in C(J, K)$ satisfying

$$aM_1M_2(\|p_1\|_{L^1} + c_0) < 1, \quad L_g < \frac{1}{4aM_1M_2}. \tag{3.22}$$

Proof. We establish the continuity of the operator $\mathcal{O} : C_\varphi(K) \to C_\varphi(K)$ defined in (3.3), by invoking Theorem 3.1. From Definition 2.10, it follows that any fixed point u of $\mathcal{O}$ corresponds to a mild solution of problem (1.1). Combining hypothesis (H3) with the definition of $\mathcal{O}$ in (3.3), we derive, for all $\forall\ u \in C_\varphi(K)$, the inequality

$$\begin{aligned}\|(\mathcal{O}u)(t)\| &\le M_2\|\varphi\|_{\mathcal{B}} + M_1M_2a(\|\phi\| + c_0\|u\| + d_0 + \|E_2(\rho)\varphi(0)\|) \\ &\quad + \int_0^t \int_0^s \|T_2(t-s)T_1(s-\tau)\| \cdot \|F(\tau, u(\tau), u_\rho(\tau))\| d\tau ds \\ &\le M_2\|\varphi\|_{\mathcal{B}} + aM_1M_2(\|\phi\| + c_0\|u\| + d_0 + \|E_2(\rho)\varphi(0)\|) \\ &\quad + aM_1M_2 \int_0^t (p_1\|u(s)\| + p_2(s) \sup_{t\in[-a,0]} \|u(s+\tau)_\rho\| + \overline{K})ds \\ &\le M_2\|\varphi\|_{\mathcal{B}} + aM_1M_2(\|\phi\| + c_0\|u\|_C + d_0 + \|E_2(\rho)\varphi(0)\|) \\ &\quad + aM_1M_2\Big(\|u\|_C \cdot \|p_1\|_{L^1} + \|\varphi\|_{\mathcal{B}} \cdot \|p_2\|_{L^1} + \overline{K}),\end{aligned} \tag{3.23}$$

where

$$L = M_2(1 + aM_1\|p_2\|_{L^1})\|\varphi\|_{\mathcal{B}} + aM_1M_2(\|\phi\| + d_0 + \|E_2(\rho)\varphi(0)\| + \overline{K}),$$

$$\eta = aM_1M_2(\|p_1\|_{L^1} + c_0) < 1.$$

Next, we establish the existence of positive mild solutions.

Step 1. For all $\forall\, u, v \in C_\varphi(K)$, satisfying $u \le v$ and $t \in J$, the evaluations $u_\varphi(t), v_\varphi(t)$ belong to $K_{\mathcal{B}}$ and inherit the order relation $u_\varphi(t) \le v_\varphi(t)$. By hypothesis (H7), the operator definition (3.3), and the conditions $\varphi \in K_{\mathcal{B}}$ and $\phi \in K$, it follows that for all $t \in J$,

$$\begin{aligned}
\theta &\le \mathcal{O}u(t) \\
&\le T_2(t)\varphi(0) + \int_0^t T_2(t-s)T_1(s)[g(v) + \phi + E_2(\rho)\varphi(0)] \\
&\quad + \int_0^t \int_0^s T_2(t-s)T_1(s-\tau)F(\tau, v(\tau), v_\rho(\tau))d\tau ds \\
&= \mathcal{O}v(t),
\end{aligned}$$

then, for $u \le v$, we get $\theta \le \mathcal{O}u \le \mathcal{O}v$.

Let $v_0(t) \equiv \varphi(0)$. Thus, $v_0 \in C_\varphi(J, K)$, $v_0(t)_\varphi \in K_{\mathcal{B}}$.

Define a sequence $\{v_m\}$ satisfying

$$v_m = \mathcal{O}v_{m-1},\ \ m = 1, 2, \cdots. \tag{3.24}$$

By monotonicity of $\mathcal{O}$, we have $\{v_m\} \subset C_\varphi(J, K)$ and

$$v_0 \le v_1 \le \cdots \le v_m \le \cdots, \|v_m\|_C \le L + \eta\|v_{m-1}\|_C. \tag{3.25}$$

Since $\|v_0\| \le \|\varphi\|_{\mathcal{B}}$, $0 < \beta < 1$, we obtain

$$\begin{aligned}
\|v_m\|_C &\le L + L\eta + \cdots + L\eta^{m-1} + \eta^m\|v_0\|_C \\
&\le \frac{L(1-\eta^m)}{1-\eta} + \eta^m\|\varphi\|_{\mathcal{B}} \\
&\le \frac{L}{1-\eta} + \|\varphi\|_{\mathcal{B}},
\end{aligned}$$

i.e., the sequence $\{v_m\}$ is uniformly bounded.

Step 2. We demonstrate the convergence of the sequence $\{v_m\}$ in $C_\varphi(J, K)$. Define the sets $V = \{v_m\}$ and $V_0 = V \cup \{v_0\}$, hence $\mathcal{O}(V_0) = V$. By Step 3 of Theorem 3.1, $V \subset C_\varphi(K)$ is equicontinuous. Combining hypothesis (H3) with Lemmas 2.4–2.5 and the operator definition (3.3), we derive for all $t \in J$,

$$\begin{aligned}
\alpha\Big(V(t)\Big) &= \alpha\Big(\mathcal{O}V_0(t)\Big) \\
&\le 2\alpha(\{T_2(t)\varphi(0)\}) + 2\alpha\Big(\Big\{\int_0^t T_2(t-s)T_1(s)[g(v_{m-1}) + \phi + E_2(\rho)\varphi(0)]ds : m \in \mathbb{N}\Big\}\Big) \\
&\quad + 2\alpha\Big(\Big\{\int_0^t \int_0^s T_2(t-s)T_1(s-\tau)F(\tau, v_{m-1}(\tau), v_{m-1}(\tau)_\rho)d\tau ds : m \in \mathbb{N}\Big\}\Big) \\
&\le 4aM_1M_2L_g\alpha(V(t)) + 4aM_1M_2(L_1 + L_2)\int_0^t \alpha(V(s))ds.
\end{aligned}$$

Since $4aM_1M_2L_g < 1$, we get

$$\alpha\Big(V(t)\Big) \leq \frac{4aM_1M_2(L_1+L_2)}{1-4aM_1M_2L_g}\int_0^t \alpha(V(s))dh(s).$$

By Lemma 2.8, we conclude that the measure of noncompactness satisfies $\alpha(\{v_m(t)\}) \equiv 0$ for all $t \in J$. Furthermore, since the sequence $\{v_m\}$ s uniformly bounded and equicontinuous, it is relatively compact in the space $C_\varphi(J,K)$. Consequently, $\{v_m\}$ onverges to a limit $u^* \in C_\varphi(J,K)$. i.e., $\lim_{m\to\infty} v_m = u^*$. This u^* is a fixed point of the operator $\mathcal{O}$ ($u^* = \mathcal{O}u^*$) and constitutes a positive mild solution to problem (1.1).

To establish the minimality of u^*, let $\widetilde{u} \in C(J,E)$ be an arbitrary positive mild solution of (1.1). By definition $\widetilde{u} \in C_\varphi(K)$ with $\widetilde{u} = \mathcal{O}\widetilde{u}$. For all $t \in J$, the inequality $\widetilde{u}(t) \geq v_0(t) = \varphi(0)$ implies

$$\widetilde{u}(t) = \mathcal{O}\widetilde{u}(t) \geq \mathcal{O}v_0(t) = v_1(t), \quad t \in J,$$

and by induction $\widetilde{u} \geq v_1$ for all $m \in \mathbb{N}$. Taking the limit as $m \to \infty$, we obtain $\widetilde{u} \geq u^*$. Thus, u^* is the minimal positive mild solution of (1.1). □

4. Example

Example 4.1. Consider the following damping elastic system

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} - 2\gamma_1\Delta\dfrac{\partial u(t,x)}{\partial t} + \Delta^2 u(t,x) = \dfrac{\sin(u(t+s,x))}{1+e^{2t}}, \quad (t,x) \in [0,a]\times[0,\pi], \\ s \in [-r,0], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,a], \\ u(t,x) = \varphi(t,x), \quad (t,x) \in [-r,0]\times[0,\pi], \\ \dfrac{\partial u(t,x)}{\partial t}|_{t=0} = \dfrac{|u(t,x)|}{6+|u(t,x)|} + \phi(x), \quad x \in [0,\pi], \end{cases} \tag{4.1}$$

where $\gamma_1 = \rho > 0$ is constant, Δ stands for the Laplace operator in the space variable x, Assume that $f : [0,a]\times[0,\pi]\times K \times K_{\mathcal{B}} \to E$ be measurable function, $\varphi(\cdot,\cdot) \in C([-r,0]\times[0,\pi]), \phi(\cdot) \geq 0$, $a, r > 0$ are constants. Here the variable $u(t,x)$ represents the temperature of the point x at time t.

Let $p = 2$, and consider the Banach space $E = L^2([0,\pi]\times[0,a])$ equipped with the L^2-norm $\|\cdot\|_2$ and partial-order "$\leq$". The positive cone $K \subset E$ is defined as $K = \{u \in L^2([0,\pi]\times[0,a]) : u(x) \geq 0,\ a.e.\ x \in [0,\pi]\}$ which is a normal cone with normality constant $N = 1$. Define the space $\mathcal{B} := C([-r,0]\times[0,\pi], E)$, and let its associated positive cone be $K_{\mathcal{B}} = \{u \in \mathcal{B} : u(t,x) \in K, t \in [-r,0],\ a.e.\ x \in [0,\pi]\}$.

Defined the linear operators A and B in E by

$$\begin{aligned} &Au = \Delta^2 u, \quad u \in D(A) = D(\Delta^2) = \{u \in X : u', u'' \in X, u(0) = u(\pi) = 0\}, \\ &Bu = -2\Delta u, \quad u \in D(B) = H_0^1([0,\pi]) \cap H_0^2([0,\pi]). \end{aligned}$$

Obviously, $C(\rho) = \rho^2B^2 - 4A = 4\Delta^2(\rho^2-1) = L^2$, where $Lu = 2\Delta(\rho^2-1)^{\frac{1}{2}}u$ for all $u \in D(L) = H_0^1([0,\pi]) \cap H_0^2([0,\pi])$. It is evident that the linear operators $BL = LB$.Furthermore,

$$E_1(\rho) = -(\rho + (\rho^2-1)^{\frac{1}{2}})\Delta = -\sigma_1\Delta, \qquad E_2(\rho) = -(\rho - (\rho^2-1)^{\frac{1}{2}})\Delta = -\sigma_2\Delta, \tag{4.2}$$

where $\sigma_1 = (\rho + (\rho^2-1)^{\frac{1}{2}}), \sigma_2 = (\rho - (\rho^2-1)^{\frac{1}{2}})$, $E_1(\rho)$ and $E_2(\rho)$ are invertible bounded linear operator on $L^2(\Omega)$ for all $\rho > 0$.

Since Δ is a sectorial operator, it generates a compact, exponentially stable semigroup $T(t)(t \geq 0)$ for all $t \geq 0$. By the properties of positive semigroups (see [45]), there exists some sufficiently large $\lambda_0 > -\inf\{\text{Re}\lambda : \lambda \in \sigma(\Delta)\}$, such that the operator $\lambda_0 I + \Delta$ possesses a bounded, invertible, and positive inverse $(\lambda_0 I + \Delta)^{-1}$. Since $\sigma(A) \neq \emptyset$ the spectral radius

$$r\big((\lambda_0 I + \Delta)^{-1}\big) = \frac{1}{\text{dist}(-\lambda_0, \sigma(\Delta))} > 0.$$

Then, according to the famous Krein-Rutman theorem (see [51]), the operator Δ has the first eigenvalue $\lambda_1 > 0$, with corresponding positive eigenfunction ϕ_1, and

$$\lambda_1 = \inf\{\text{Re}\lambda : \lambda \in \sigma(\Delta)\}.$$

Hence, from equation (2.4), it directly follows that $T(t)$ satisfying $\|T(t)\| \leq e^{-\lambda_1 t}$ for $t \geq 0$, that is, $M = 1, \nu_0 = -\lambda_1$.

Furthermore, for any $\rho > 0$, (4.2) implies the existence of $\sigma_1 > 0$ and $\sigma_2 > 0$ with $\sigma_1 > \sigma_2$. By operator semigroup theory [17], if the semigroup $T(t)(t \geq)0$ is exponentially stable, we can deduce that $-E_1(\rho) = \sigma_1\Delta$ and $-E_2(\rho) = \sigma_2\Delta$ generate analytic and compact C_0-semigroups $T_1(t)(t \geq 0)$ and $T_2(t)(t \geq 0)$ on E, respectively. Consequently, we have

$$T_1(t) = T(\sigma_1 t), \quad T_2(t) = T(\sigma_2 t), \quad t \geq 0,$$

which is exponential stable, i.e.,

$$\|T_1(t)\| \leq e^{-\lambda_1\sigma_1 t} = e^{\nu_1 t}, \;\; \|T_2(t)\| \leq e^{-\lambda_1\sigma_2 t} = e^{\nu_2 t},$$

where λ_1 is the first eigenvalue of Δ, and since $\sigma_1, \sigma_2 > 0$, from Definition 2.1, it follows that ν_1, ν_2 are the growth exponents of $T_i(t)(t \geq 0)(i = 1, 2)$. As $\nu_0 < 0$, we have $\nu_1 < \nu_2 < 0$.

We define

$$\begin{aligned} f(t, x, u(t, x), u(t+s, x)) &= \frac{\sin(u(t+s))}{1+e^{2t}}, \quad t \in [0, a], \; s \in [-r, 0], \\ g(u(t, x)) &= \frac{|u(t, x)|}{6 + |u(t, x)|}. \end{aligned}$$

For $u \in L^2([0, \pi], [0, b])$, let $\varphi(t) = \varphi(t, \cdot)$, $\psi = \psi(\cdot)$, $u(t) = u(t, \cdot)$, $u_t(s) = u(t+s, \cdot)$

$$F(t, u(t), u_t) = f(t, \cdot, u(t, \cdot), u(t+s, \cdot)), \quad g(u) = \frac{|u|}{6+|u|}.$$

Consequently, equation (4.1) can be recast as equation (1.1) within the function space $L^2([0, \pi], [0, a])$.

From the definitions of the functions f and m,

$$\|F(t, u(t), u_t)\| \leq \frac{1}{2}\|u\|, \;\; \|g(u(t, x))\| \leq \frac{1}{6}\|u\|.$$

We conclude that condition (H5) is satisfied with parameters $c_0 = \frac{1}{6}$ and $d_0 = 0$. Furthermore, condition (H8) holds with $p_1(t) = \frac{1}{2}, p_2(t) = 0$ and $\overline{K} = 0$.

Theorem 4.1. *If assumptions (H1) and (H3), along with the following condition*

(A1) For $\forall \xi \in [0, \pi], t \in [0, a]$, $\theta \leq x_1 \leq x_2, \theta \leq \phi_1 \leq \phi_2$,

$$f(\xi, t, x_2, \phi_2) - f(\xi, t, x_1, \phi_1) \geq \theta,$$

hold, then equation (4.1) admits at least one positive mild solution.

Proof. Under the assumption that $f : [0, a] \times [0, \pi] \times K \times K_{\mathcal{B}} \to E$ is a continuous and measurable function, hypothesis (H2) is satisfied. By invoking condition (A1), we further deduce that hypothesis (H7) holds. Consequently, applying Theorem 3.3, we conclude that problem (4.1) admits at least one positive mild solution. □

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