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BOUNDEDNESS OF BILINEAR INTEGRAL OPERATORS RELATED TO GENERALIZED KERNELS AND THEIR COMMUTATORS ON PRODUCT OF GENERALIZED VARIABLE EXPONENTS MORREY SPACES*

Miaomiao Wang[1] and Guanghui Lu[1,†]

Abstract The purpose of this paper is to investigate the boundedness of a bilinear integral operator T and its commutator T_{b_1,b_2} formed by $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and the T on product of generalized variable Morrey spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$. Under assumption that Lebesgue measurable functions φ_i belong to the class $\mathbb{W}_{p_i(\cdot)}$, $i = 1, 2$, the authors prove that the T is bounded from product of spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into spaces $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$; furthermore, the authors show that the T_{b_1,b_2} is bounded from product of spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into spaces $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$, where $\varphi_1\varphi_2 = \varphi$, and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$ with $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. As corollaries, the boundedness of the T and T_{b_1,b_2} on product of variable exponent Morrey spaces $L^{p_1(\cdot),\kappa}(\mathbb{R}^n) \times L^{p_2(\cdot),\kappa}(\mathbb{R}^n)$ is established, respectively.

Keywords Bilinear singular integral operator related to generalized kernels, space $\mathrm{BMO}(\mathbb{R}^n)$, commutator, generalized variable exponent Morrey space.

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1. Introduction

The main theme of this article is the mapping properties of a bilinear singular integral operator associated with a class of generalized kernels and their related commutators on product of generalized variable exponents Morrey spaces. We obtain these results by using some known results of the multilinear singular integral operators and their corresponding commutators on product of variable exponents Lebesgue spaces established by Lin and Xiao [15] and some common inequalities of harmonic analysis.

The multilinear singular integral operators associated with a class of generalized kernels introduced in [15, 30] extend the classical multilinear Calderón-Zygmund operators firstly introduced by Coifman and Meyer in [2, 3] and the multilinear singular integral with the kernels of type ω studied by some authors in [21, 22]. One of the remarkable purpose of these operators is to establish their bounded properties on various function spaces. For example, in 2023, Yang et al. [29] obtained the definition of a kind of multilinear singular integral operators T associated with generalized kernels, whose kernel is much weaker than the kernel of certain Dini's type, and then they showed that the T, the m-linear commutator $T_{\vec{b}}$ generated

†The corresponding author.
[1]College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China
*The authors were supported by National Natural Science Foundation of China (No. 12201500).
Email: wmmlgh1990@126.com(M. Wang), lghwmm1989@126.com(G. Lu)

by $\vec{b}=(b_1,\cdots,b_m)\in \mathrm{BMO}^m_{\vec{\theta}}(\varphi)(\vec{\theta}=(\theta_1,\cdots,\theta_m)$ with $\theta_1,\cdots,\theta_m\ge 0)$ and the T, and the m-linear iterative commutator $T_{\prod\vec{b}}$ formed by $\vec{b}\in \mathrm{BMO}^m_{\vec{\theta}}(\varphi)$ are bounded from product of weighted Lebesgue spaces $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ into weighted Lebesgue spaces $L^p(\nu_{\vec{\omega}})$, where $\vec{\omega}=(\omega_1,\cdots,\omega_m)\in A^{\infty}_{\vec{p}/q}(\varphi)$, $\nu_{\vec{\omega}}=\prod_{j=1}^{m}\omega_j^{p/p_j}$ and $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ with $q'<p_j<\infty$, $j=1,\cdots,m$. Recently, Yang et al. [30] show that multilinear singular integral operators T related to a kind of generalized kernels is bounded from product of spaces $L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m)$ into spaces $L^p(\nu_{\vec{\omega}})$, where $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $\vec{P}=(p_1,\cdots,p_m)$ and $\vec{\omega}=(\omega_1,\cdots,\omega_m)$ satisfies $\vec{\omega}\in A_{\vec{P}/(q')}$ and $\nu_{\vec{\omega}}=\prod_{j=1}^{m}\omega_j^{\frac{p}{p_j}}$. More researchers about the bounded properties of bilinear singular integral operators on various function spaces can be seen in [6,16,17,27,28] and their references therein.

In this paper, we extend generalized Morrey spaces introduce in [7,8] to the setting of variable exponents, but the definition differs from the generalized variable exponents Morrey spaces $M^{p(\cdot),\varphi}$ introduced by Ekincioglu et al. [5]. The generalized variable exponents Morrey spaces of this paper extend the variable exponents Morrey spaces and Lebesgue spaces. The readers can see [13,14,18,19,26] for the recent development and applications of the variable exponents spaces. Moreover, as applications, we show that the bilinear singular integral operators T associated with generalized kernels is bounded from product of generalized variable exponents Morrey spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into spaces $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$, and it is also bounded from product of variable exponent Morrey spaces $L^{p_1(\cdot),\kappa}(\mathbb{R}^n)\times L^{p_2(\cdot),\kappa}(\mathbb{R}^n)$ into spaces $L^{p(\cdot),\kappa}(\mathbb{R}^n)$; furthermore, the boundedness of commutators T_{b_1,b_2} formed by $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$ and the T on product of spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ and product of spaces $L^{p_1(\cdot),\kappa}(\mathbb{R}^n)\times L^{p_2(\cdot),\kappa}(\mathbb{R}^n)$ is obtained.

Before formulating the organization of this article, we should recall some necessary definitions and notion. The following definition of a space BMO comes from [4].

Definition 1.1. A real-valued function $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the *spaces* $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}=\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B|b(x)-b_B|\mathrm{d}y, \tag{1.1}$$

where the number b_B represents the mean value of functions b over all balls B, i.e.,

$$b_B=\frac{1}{|B|}\int_B b(y)\mathrm{d}y.$$

The following notation of a bilinear singular integral operator T related to generalized kernels is from [15].

Definition 1.2. A function $K(\cdot,\cdot,\cdot)\in L^1_{\mathrm{loc}}\big((\mathbb{R}^n)^3\setminus\{(x,x,x):x\in\mathbb{R}^n\}\big)$ is called a bilinear generalized Calderón-Zygmund kernel if there exists some constant $C>0$ such that,

(i) for all $x,y_1,y_2\in\mathbb{R}^n$ with $x\ne y_i$, $i=1,2$,

$$|K(x,y_1,y_2)|\le C\big(|x-y_1|+|x-y_2|\big)^{-2n}; \tag{1.2}$$

(ii) there exist positive constant C_{k_i} only depending on $k_i\in\mathbb{N}_+$, $i=1,2$, such that, for all $x,x',y_1,y_2\in\mathbb{R}^n$,

$$\bigg(\int_{2^{k_2}|x-x'|\le|y_2-x|<2^{k_2+1}|x-x'|}\int_{2^{k_1}|x-x'|\le|y_1-x|<2^{k_1+1}|x-x'|}$$

$$|K(x,y_1,y_2)-K(x',y_1,y_2)|^q\mathrm{d}y_1\mathrm{d}y_2\Bigg)^{\frac{1}{q}}$$

$$\le C|x-x'|^{-\frac{2n}{q'}}\prod_{i=1}^{2}C_{k_i}2^{-\frac{n}{q'}k_i}; \tag{1.3}$$

(iii) there exist positive constant C_{k_i} only depending on $k_i\in\mathbb{N}_+$, $i=1,2$, such that, for all $x,y_1,y_1',y_2\in\mathbb{R}^n$,

$$\Bigg(\int_{2^{k_2}|y_1-y_1'|\le|y_2-y_1|<2^{k_2+1}|y_1-y_1'|}\int_{2^{k_1}|y_1-y_1'|\le|x-y_1|<2^{k_1+1}|y_1-y_1'|}$$

$$|K(x,y_1,y_2)-K(x,y_1',y_2)|^q\mathrm{d}x\mathrm{d}y_2\Bigg)^{\frac{1}{q}}$$

$$\le C|y_1-y_1'|^{-\frac{2n}{q'}}\prod_{i=1}^{2}C_{k_i}2^{-\frac{n}{q'}k_i}; \tag{1.4}$$

(iv) there exist positive constant C_{k_i} only depending on $k_i\in\mathbb{N}_+$, $i=1,2$, such that, for all $x,y_1,y_2,y_2'\in\mathbb{R}^n$,

$$\Bigg(\int_{2^{k_2}|y_2-y_2'|\le|y_1-y_2|<2^{k_2+1}|y_2-y_2'|}\int_{2^{k_1}|y_2-y_2'|\le|x-y_2|<2^{k_1+1}|y_2-y_2'|}$$

$$|K(x,y_1,y_2)-K(x,y_1,y_2')|^q\mathrm{d}x\mathrm{d}y_1\Bigg)^{\frac{1}{q}}$$

$$\le C|y_2-y_2'|^{-\frac{2n}{q'}}\prod_{i=1}^{2}C_{k_i}2^{-\frac{n}{q'}k_i}. \tag{1.5}$$

Remark 1.1. Comparing to the smooth conditions of m-linear Calderón-Zygmund kernels (for example, see [6]), Lin and Xiao [15] showed that the condition (1.3) is much weaker. Furthermore, Lin and Xiao pointed out that (1.3) is weaker than the Calderón-Zygmund kernels of Dini's type via putting $C_{k_i}=[\omega(2^{-k_i})]^{\frac{1}{2}}$.

Let $L_b^\infty(\mathbb{R}^n)$ be the spaces consist of all $L^\infty(\mathbb{R}^n)$ functions with bounded support. A bilinear operator T is claimed a bilinear singular integral operator T related to generalized kernels K meeting (1.2), (1.3), (1.4) and (1.5) if, for all $f_1,f_2\in L_b^\infty(\mathbb{R}^n)$ and $x\in\mathbb{R}^n\backslash\big(\mathrm{supp}(f_1)\bigcap\mathrm{supp}(f_2)\big)$,

$$T(f_1,f_2)(x)=\int_{(\mathbb{R}^n)^2}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2. \tag{1.6}$$

Given $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$, the commutator T_{b_1,b_2} formed by b_1,b_2 and the T is defined by

$$\begin{aligned}T_{b_1,b_2}(f_1,f_2)(x)&=b_1(x)b_2(x)T(f_1,f_2)(x)-b_1(x)T\big(f_1,b_2(\cdot)f_2\big)(x)\\&\quad-b_2(x)T\big(b_1(\cdot)f_1,f_2\big)(x)+T\big(b_1(\cdot)f_1,b_2(\cdot)f_2\big)(x).\end{aligned} \tag{1.7}$$

Equivalently, we can write the $T_{b_1,b_2}(f_1,f_2)(x)$ as

$$T_{b_1,b_2}(f_1,f_2)(x)=\int_{(\mathbb{R}^n)^2}K(x,y_1,y_2)\big(b_1(x)-b_1(y_1)\big)\big(b_2(x)-b_2(y_2)\big)$$

$$\times f_1(y_1)f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

Also, the commutators $T_{b_1}(f_1, f_2)(x)$ and $T_{b_2}(f_1, f_2)(x)$, respectively, are defined by

$$T_{b_1}(f_1, f_2)(x) = b_1(x)T(f_1, f_2)(x) - T\big(b_1(\cdot)f_1, f_2\big)(x)$$

and

$$T_{b_2}(f_1, f_2)(x) = b_2(x)T(f_1, f_2)(x) - T\big(f_1, b_2(\cdot)f_2\big)(x).$$

To state the definition of a generalized variable exponent space, we need to recall some notions about the variable exponent and variable Lebesgue spaces introduced in [25]. A measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ is said to be a variable exponent; moreover, for any variable exponent $p(\cdot)$, set

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) = p_- > 0, \qquad \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) = p_- < \infty.$$

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ such that $1 < p_- \leq p(x) \leq p_+ < \infty$. Denote by $\mathcal{P}_0(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that $0 < p_- \leq p(x) \leq p_+ < \infty$. Let $\mathcal{P}_1$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 \leq p_- \leq p(x) \leq p_+ < \infty$.

For any $p(\cdot) \in \mathcal{P}_1(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ represents the set of real-valued measurable functions defined on $\mathbb{R}^n$ such that for some $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} \big(\varepsilon|f(x)|\big)^{p(x)}\mathrm{d}x < \infty.$$

This become a Banach function space respect to the Luxemburg-Nakano norm,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}\mathrm{d}x \leq 1\right\}.$$

Next, we recall some classes of variable exponent functions (see [27]). Given a real-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is respectively defined by

$$M(f)(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. The set $\mathcal{B}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) \in \mathcal{P}_1(\mathbb{R}^n)$ satisfying that the Hardy-Littlewood maximal operator M is bounded on spaces $L^{p(\cdot)}(\mathbb{R}^n)$. An important subset of $\mathcal{B}(\mathbb{R}^n)$ is the class of globally log-Hölder continuous functions $p(\cdot) \in LH(\mathbb{R}^n)$, with $p(\cdot) \in \mathcal{P}_1(\mathbb{R}^n)$. Recall that $p(\cdot) \in LH(\mathbb{R}^n)$, if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x - y| \leq \frac{1}{2}$$

and

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n,$$

where $p_\infty = \lim_{x\to+\infty} p(x)$.

We now state a generalized variable exponent Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$ as follows.

Definition 1.3. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\varphi(\cdot,\cdot)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. Then the generalized variable exponent Morrey $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is defined by

$$\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\begin{aligned}\|f\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))} \\ &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{1.8}$$

Remark 1.2. (1) If we take $\varphi(x,r) = r^{\frac{\lambda}{p(x)}}$ for $0 < \lambda < n$ and $p(x) \in \mathcal{P}(\mathbb{R}^n)$, then the space $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the variable exponent Morrey space introduced in [1].

(2) If we take $\varphi(x,r) \equiv 1$, then $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ is just the variable Lebesgue space.

(3) If we take $\varphi(x,r) = |B(x,r)|^{\frac{\kappa}{p(x)}}$ for $\kappa \in (0,1)$, then $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n) = L^{p(\cdot),\kappa}(\mathbb{R}^n)$ is just the variable exponent Morrey space.

(3) If we take $p(\cdot) \equiv const$, then the generalized variable exponent Morrey space $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the generalized Morrey space $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ introduced in [7, 24].

(4) If we take $p(x) \equiv const$ and $\varphi(x,r) = r^{\frac{\lambda}{p(x)}}$ with $0 \le \lambda \le n$ in Definition 1.3, then the space $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$ is just the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ introduced in [23].

We now define a class of functions as follows.

Definition 1.4. Let $p(\cdot) : \mathbb{R}^n \to (1,\infty)$. A Lebesgue measurable function $\varphi(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is said to belong to the class $\mathbb{W}_{p(\cdot)}$ if there exist positive constants C_1, C_2 and C_3 such that for all $x \in \mathbb{R}^n$ and $r > 0$, $u(x,r)$ fulfills

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{\varphi(x,2^{k+1}r)}{\varphi(x,r)} \le C_1, \quad \text{for any } r > 0, \\ &C_2 \le \varphi(x,r), \quad \text{for any } r \ge 1, \\ &\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_3\varphi(x,r), \quad \text{for any } 0 < r < 1.\end{aligned} \tag{1.9}$$

Remark 1.3. We notice that the above definition is meaningful. In fact, we set that $\kappa \in (0,1)$ and $u(x,r) = \|\chi_{B(x,r)}\|^{\kappa}_{L^{p(\cdot)}(\mathbb{R}^n)}$ in (1.9), we deduce that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{\varphi(x,2^{k+1}r)}{\varphi(x,r)} &= \sum_{k=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{\|\chi_{B(x,2^{k+1}r)}\|^{\kappa}_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|^{\kappa}_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &= \sum_{k=0}^{\infty} \frac{\|\chi_{B(x,r)}\|^{1-\kappa}_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{k+1}r)}\|^{1-\kappa}_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ &\le C \sum_{k=0}^{\infty} \left(\frac{|B(x,r)|}{|B(x,2^{k+1}r)|} \right)^{\frac{1-\kappa}{p}} \\ &\le C_1,\end{aligned}$$

where we use the following result being modified from [9]

$$\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|B(x,r)|}{|B(x,2^{k+1}r)|}\right)^{\frac{1}{p}}, \quad \text{for any } p > p_+ \ \text{ and } r > 0.$$

Hence, we have $u(x,r) \in \mathbb{W}_{p(\cdot)}$. In addition, with a slightly modified method that used in [31], it is not difficult to show that $\chi_B \in \mathbb{W}_{p(\cdot)}$.

The organization of this paper is stated as follows. In Section 2, the authors prove that the T is bounded from product of generalized variable exponents spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ into spaces $\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)$, and it is also bounded from product of variable exponents Morrey spaces $L^{p_1(\cdot),\kappa}(\mathbb{R}^n) \times L^{p_2(\cdot),\kappa}(\mathbb{R}^n)$ into spaces $L^{p(\cdot),\kappa}(\mathbb{R}^n)$, where $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\kappa \in (0,1)$, and Lebesgue measurable functions $\varphi_i (i = 1,2)$ belong to $\mathbb{W}_{p_i(\cdot)}$ with satisfying $\varphi_1\varphi_2 = \varphi$. By using the boundedness of the commutator T_{b_1,b_2} associated with BMO functions on product of variable exponents Lebesgue spaces $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ and some inequalities of harmonic analysis, the authors show that the T_{b_1,b_2} is bounded on product of spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$ and product of spaces $L^{p_1(\cdot),\kappa}(\mathbb{R}^n) \times L^{p_2(\cdot),\kappa}(\mathbb{R}^n)$, respectively.

Finally, we make some conventions on notation. Throughout this paper, we use the letter C to represent a positive constant which is independent of main parameters, but it varies from line to line and C_α stands for a positive constant only relying on α. Furthermore, subscript $C_i (i = 1,2,3)$ denote a positive constant. For any given $p(\cdot) \in (1,\infty)$, if $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, then we call $p'(\cdot)$ the conjugate index of $p(\cdot)$. For any subset $E \subset \mathcal{X}$, χ_E represents its characteristic function.

2. Estimates for T on product of spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n) \times \mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$

The main theorem of this section is stated as follows.

Theorem 2.1. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, T be a bilinear singular integral operator defined by* (1.6) *whose kernels K meet the conditions* (1.2), (1.3), (1.4) *and* (1.5) *with $1 < q' \le \min\{q_0^1, q_0^2\}$ and $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$, and $\varphi_1\varphi_2 = \varphi$ for $\varphi_i \in \mathbb{W}_{p_i(\cdot)}$, $i = 1,2$. Suppose that T is bounded from product of Lebesgue spaces $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ into spaces $L^{r,\infty}(\mathbb{R}^n)$ for fixed $1 \le r_1, r_2 \le q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then there exists a positive constant C such that, for all $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1,2$,*

$$\|T(f_1,f_2)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

As a corollary of Theorem 2.1, it is not difficult to obtain the following consequences about the T on product of variable exponents Morrey spaces $L^{p(\cdot),\kappa}(\mathbb{R}^n)$.

Corollary 2.1. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\kappa \in (0,1)$ and T be a bilinear singular integral operator defined by* (1.6) *whose kernels K meet the conditions* (1.2), (1.3), (1.4) *and* (1.5) *with $1 < q' \le \min\{q_0^1, q_0^2\}$ and $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$. Suppose for fixed $1 \le r_1, r_2 \le q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, T is bounded from product of Lebesgue spaces $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ into spaces $L^{r,\infty}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for all $f_i \in L^{p_i(\cdot),\kappa}(\mathbb{R}^n)$, $i = 1,2$,*

$$\|T(f_1,f_2)\|_{L^{p(\cdot),\kappa}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot),\kappa}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot),\kappa}(\mathbb{R}^n)}.$$

To prove the above theorem, we need to recall and establish some necessary results.

Lemma 2.1 (Theorem 4.2, [15]). *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Let q_0^j be given (see Lemma 4.4 in [15]) for $p_j(\cdot)$, $j = 1, 2$, and T be a bilinear singular integral operator defined by* (1.6) *whose kernel K satisfies the conditions* (1.2), (1.3), (1.4) *and* (1.5) *with $1 < q' \le \min\{q_0^1, q_0^2\}$ and $\sum_{k_i=1}^{\infty} C_{k_i} < \infty$, $i = 1, 2$. Suppose for fixed $1 \le r_1, r_2 \le q'$ with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, T is bounded from product of spaces $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ into spaces $L^{r,\infty}(\mathbb{R}^n)$. Then there exists some positive constant C such that, for all $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1, 2$,*

$$\|T(f_1, f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

We now recall the following lemmas.

Lemma 2.2 ([25]). *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a positive constant C such that, for all $B \subset \mathbb{R}^n$,*

$$C^{-1}|B| \le \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C|B|.$$

Lemma 2.3 ([10]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \tag{2.1}$$

Lemma 2.4 ([11]). *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, then there exists some positive constant C such that, for all $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$,*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \tag{2.2}$$

Lemma 2.5 ([12]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, there exist constants $\delta, \tau, C > 0$ such that, for all balls $B \subset \mathbb{R}^n$ and measurable set $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \ \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta} \text{ and } \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\tau}.$$

Also, we should establish the following lemma on the class $\mathbb{W}_{p(\cdot)}$.

Lemma 2.6. *Let $\varphi \in \mathbb{W}_{p(\cdot)}$ with $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant $\widetilde{C}_n$ only depending on n such that, for any $B = B(x, r) \subset \mathbb{R}^n$, the following inequality*

$$\varphi(x, 2r) \le \widetilde{C}_n \varphi(x, r) \tag{2.3}$$

holds. In other word, the $\varphi(x, r)$ is a doubling function on the second variable $r > 0$.

Proof. By using (1.9) and Lemma 2.2, we deduce

$$\begin{aligned}\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\frac{\varphi(x,2r)}{\varphi(x,r)} \le C_1 \Leftrightarrow \varphi(x,2r) &\le C_1\frac{\|\chi_{B(x,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\varphi(x,r) \\ \Rightarrow \varphi(x,2r) &\le C_1C\frac{|B(x,2r)|}{|B(x,r)|}\varphi(x,r)\end{aligned}$$

$$\Rightarrow \varphi(x,2r) \le \widetilde{C}_n \varphi(x,r).$$

Hence, the proof of Lemma 2.6 is finished. □

It is now position to state the proof of Theorem 2.1 as follows.

Proof. Let $B = B(x,r)$ be a fixed ball centered at $x \in \mathbb{R}^n$ and its radius $r > 0$. And represent functions $f_i (i = 1, 2)$ as

$$f_i = f_i^1 + f_i^\infty = f_i\chi_{2B} + f_i\chi_{\mathbb{R}^n\setminus(2B)}. \tag{2.4}$$

Then, by the Minkowski's inequality, write

$$\begin{aligned}\|T(f_1,f_2)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} &\le \|T(f_1^1,f_2^1)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T(f_1^1,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\ &\quad + \|T(f_1^\infty,f_2^1)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T(f_1^\infty,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\ &= \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4.\end{aligned}$$

From (1.8), Lemma 2.1, $\varphi_1\varphi_2 = \varphi$, Lemma 2.6 and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, it then follows that

$$\begin{aligned}\mathrm{D}_1 &\le \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|T(f_1^1,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le C \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\le C \sup_{x\in\mathbb{R}^n, r>0} \frac{\varphi(x,2r)}{\varphi(x,r)} \frac{1}{\varphi_1(x,2r)} \|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{1}{\varphi_2(x,2r)} \|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

To estimate D_2, we first consider the $|T(f_1^1,f_2^\infty)(y)|$ with $y \in B$. By using (1.2), (1.8), (1.9), Lemma 2.2 and (2.1), we have

$$\begin{aligned}&|T(f_1^1,f_2^\infty)(y)| \\ &\le C \int_{2B} |f_1(z_1)|\mathrm{d}z_1 \int_{\mathbb{R}^n\setminus(2B)} \frac{1}{|x-z_2|^{2n}} |f_2(z_2)|\mathrm{d}z_2 \\ &\le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \left(\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} \frac{|f_2(z_2)|}{|x-z_2|^{2n}}\mathrm{d}z_2\right) \\ &\le C\frac{1}{\varphi(x,2r)} \|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \varphi_1(x,2r) \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \left(\sum_{k=1}^\infty \frac{1}{|2^kB|^2} \int_{2^{k+1}B} |f_2(z_2)|\mathrm{d}z_2\right) \\ &\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \varphi_1(x,2r) \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\ &\quad \times \left(\sum_{k=1}^\infty \frac{1}{|2^kB|^2} \varphi_2(x,2^{k+1}r) \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right) \\ &\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \varphi_1(x,2r) \frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \\ &\quad \times \left(\sum_{k=1}^\infty \frac{1}{|2^kB|^2} \varphi_2(x,2^{k+1}r) \frac{\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\frac{1}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{|2B|}{|2^kB|}\varphi_2(x,2^{k+1}r)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\frac{1}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\varphi_2(x,2^{k+1}r)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right),
\end{aligned}
$$

from this, (1.8), (1.9), (2.2), Lemma 2.2, $\varphi_1\varphi_2=\varphi$ and Lemma 2.6, it then follows that

$$
\begin{aligned}
\mathrm{D}_2 &\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\varphi_1(x,2r)\\
&\quad\times\frac{1}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\varphi_2(x,2^{k+1}r)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,2r)\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\varphi_2(x,2^{k+1}r)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,2r)\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\right)\\
&\quad\times\left(\sum_{k=1}^{\infty}\varphi_2(x,2^{k+1}r)\frac{\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\varphi_1(x,2r)\varphi_2(x,2r)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)},
\end{aligned}
$$

where we use the following inequality (see [20])

$$
\sum_{k=1}^{\infty}a_kb_k\le\left(\sum_{k=1}^{\infty}a_k^r\right)^{\frac{1}{r}}\left(\sum_{k=1}^{\infty}b_k^{r'}\right)^{\frac{1}{r'}}\le\left(\sum_{k=1}^{\infty}a_k\right)\left(\sum_{k=1}^{\infty}b_k\right),\tag{2.5}
$$

for all $a_k,b_k>0$ and $r>1$.

With an argument similar to that used in the estimation for D_2, it is easy to get

$$
\mathrm{D}_3\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
$$

We turn D_4. For any $y\in B$, by applying (1.2), (1.8), (1.9), Lemma 2.2 and (2.1), we get

$$
\begin{aligned}
&|T(f_1^{\infty},f_2^{\infty})(y)|\\
&\le C\int_{\mathbb{R}^n\setminus(2B)}\frac{|f_1(z_1)|}{|x-z_1|^n}\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|f_2(z_2)|}{|x-z_2|^n}\mathrm{d}z_2
\end{aligned}
$$

$$
\begin{aligned}
&\le C\left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{2^{j+1}B}|f_1(z_1)|\mathrm{d}z_1\right)\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|}\int_{2^{k+1}B}|f_2(z_2)|\mathrm{d}z_2\right)\\
&\le C\left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\|f_1\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)\\
&\quad\times\left(\sum_{j=1}^{\infty}\frac{1}{|2^kB|}\|f_1\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\
&\quad\times\left(\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\frac{\|\chi_{2^{j+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\\
&\quad\times\left(\sum_{j=1}^{\infty}\frac{1}{|2^kB|}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\left(\sum_{j=1}^{\infty}\frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\\
&\quad\times\left(\sum_{j=1}^{\infty}\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right),
\end{aligned}
$$

by this, (1.8), (1.9), (2.2), Lemma 2.6, $\varphi_1\varphi_2=\varphi$ and $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, we have

$$
\begin{aligned}
\mathrm{D}_4 &\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\left(\sum_{j=1}^{\infty}\frac{\varphi_1(x,2^{j+1}r)}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\right)\left(\sum_{j=1}^{\infty}\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\quad\times\left(\sum_{j=1}^{\infty}\frac{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\left(\sum_{j=1}^{\infty}\frac{\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Which, combining the estimates for $\mathrm{D}_1,\mathrm{D}_2$ and D_3, yields our desired result. Hence, the proof of Theorem 2.1 is completed. □

3. Estimates for T_{b_1,b_2} on product of spaces $\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)\times\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)$

The main theorem of this section is as follows.

Theorem 3.1. *Let* $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *meet* $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, *$T$ be a bilinear singular integral operator defined by* (1.6) *whose kernel K meets the conditions* (1.2), (1.3), (1.4) *and* (1.5) *with* $1 < q' \le \min\{q_0^1, q_0^2\}$ *and* $\sum\limits_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i = 1, 2$, *and the Lebesgue measurable functions φ_i defined on* $\mathbb{R}^n \times (0, \infty)$ *meet the following inequality*

$$\sum_{k=0}^{\infty}(k+1)\frac{\|\chi_{B(x,r)}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{k+1}r)}\|_{L^{p_i(\cdot)}(\mathbb{R}^n)}}\frac{\varphi_i(x,2^{k+1}r)}{\varphi_i(x,r)} \le C \tag{3.1}$$

and $\varphi_1\varphi_2 = \varphi$. *Suppose for fixed* $1 \le r_1, r_2 \le q'$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, *$T$ is bounded from product of Lebesgue spaces* $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ *into spaces* $L^{r,\infty}(\mathbb{R}^n)$. *Then there exists some positive constant C such that, for all* $f_i \in \mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)$, $i = 1, 2$,

$$\|T_{b_1,b_2}(f_1, f_2)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{\mathcal{M}^{p_i(\cdot),\varphi_i}(\mathbb{R}^n)}.$$

As an corollary of Theorem 3.1, it is easy to get the following result about the T_{b_1,b_2} on variable exponent Morrey spaces $L^{p(\cdot),\kappa}(\mathbb{R}^n)$.

Corollary 3.1. *Let* $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *meet* $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\kappa \in (0, 1)$ *and T be a bilinear singular integral operator defined by* (1.6) *whose kernel K meets the conditions* (1.2), (1.3), (1.4) *and* (1.5) *with* $1 < q' \le \min\{q_0^1, q_0^2\}$ *and* $\sum\limits_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i = 1, 2$. *Suppose for fixed* $1 \le r_1, r_2 \le q'$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, *$T$ is bounded from product of Lebesgue spaces* $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ *into spaces* $L^{r,\infty}(\mathbb{R}^n)$. *Then there exists a positive constant C such that, for all* $f_i \in L^{p_i(\cdot),\kappa}(\mathbb{R}^n)$, $i = 1, 2$,

$$\|T_{b_1,b_2}(f_1, f_2)\|_{L^{p(\cdot),\kappa}(\mathbb{R}^n)} \le C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_i\|_{L^{p_i(\cdot),\kappa}(\mathbb{R}^n)}.$$

To prove the above results, we need to recall and establish some necessary results.

Lemma 3.1 (Theorem 6.4, [15])**.** *Let* $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *satisfy* $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. *Let q_0^j be given (see Lemma 4.4 in [15]) for $p_j(\cdot)$, $j = 1, 2$, and T be a bilinear singular integral operator defined by* (1.6) *whose kernel K meets the conditions* (1.2), (1.3), (1.4) *and* (1.5) *with* $1 < q' \le \min\{q_0^1, q_0^2\}$ *and* $\sum\limits_{k_i=1}^{\infty} k_i C_{k_i} < \infty$, $i = 1, 2$. *Suppose for fixed* $1 \le r_1, r_2 \le q'$ *with* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, *$T$ is bounded from product of Lebsgue spaces* $L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n)$ *into spaces* $L^{r,\infty}(\mathbb{R}^n)$. *Then there exists some positive constant C such that, for all* $f_i \in L^{p_i(\cdot)}(\mathbb{R}^n)$, $i = 1, 2$,

$$\|T_{b_1,b_2}(f_1, f_2)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Also, we need the following characterizations of spaces $\mathrm{BMO}(\mathbb{R}^n)$.

Lemma 3.2 ([14])**.** *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then there exists a positive constant C such that, for all* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $i, j \in \mathbb{N}$ *with* $j > i$,

$$C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le \sup_{B\subset\mathbb{R}^n}\|\chi_B\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}\big\|\big(b(\cdot) - b_B\big)\chi_B\big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \tag{3.2}$$

and

$$\|\big(b(\cdot)-b_{B_i}\big)\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(j-i)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{3.3}$$

where B_i represents a ball with the same center to the B and radius 2^i times of B.

It is now position to state the proof of Theorem 3.1 as follows.

Proof. Let $B = B(x,r)$ be the fixed ball centered at $x\in\mathbb{R}^n$ and $r>0$, and decompose functions f_i using the same forms in (2.4), i.e.,

$$f_i = f_i^1 + f_i^\infty,\ i=1,2,$$

where $f_i^1 = f\chi_{2B}$ and $f_i^\infty = f_i\chi_{\mathbb{R}^n\setminus(2B)}$. Then, write

$$\begin{aligned}\|T_{b_1,b_2}(f_1,f_2)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} &\le \|T_{b_1,b_2}(f_1^1,f_2^1)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T_{b_1,b_2}(f_1^1,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &\quad + \|T_{b_1,b_2}(f_1^\infty,f_2^1)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} + \|T_{b_1,b_2}(f_1^\infty,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &= \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4.\end{aligned}$$

From (1.8), Lemma 2.2, $\varphi_1\varphi_2=\varphi$, Lemma 2.6 and $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$, it then follows that

$$\begin{aligned}\mathrm{E}_1 &\le \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\|T_{b_1,b_2}(f_1^1,f_2^1)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\ &\quad\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)\varphi_2(x,2r)}{\varphi(x,r)}\\ &\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.\end{aligned}$$

For $y\in B(x,r)$, since

$$\begin{aligned}|T_{b_1,b_2}(f_1^1,f_2^\infty)(y)| &\le |b_1(y)-(b_1)_{2B}||b_2(y)-(b_2)_{2B}||T(f_1^1,f_2^\infty)(y)|\\ &\quad+|b_1(y)-(b_1)_{2B}||T\big(f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)|\\ &\quad+|b_2(y)-(b_2)_{2B}||T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,f_2^\infty\big)(y)|\\ &\quad+|T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)|,\end{aligned}$$

then, by the Minkowski's inequality, write

$$\begin{aligned}\|T_{b_1,b_2}(f_1^1,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} &\le \|(b_1(\cdot)-(b_1)_{2B})(b_2(\cdot)-(b_2)_{2B})T(f_1^1,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &\quad+\|(b_1(\cdot)-(b_1)_{2B})T\big(f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &\quad+\|(b_2(\cdot)-(b_2)_{2B})T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,f_2^\infty\big)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &\quad+\|T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}\\ &=\mathrm{E}_2^1+\mathrm{E}_2^2+\mathrm{E}_2^3+\mathrm{E}_2^4.\end{aligned}$$

From (1.8), the estimation of $|T(f_1^1,f_2^\infty)(y)|$ in D_2, (2.2), (2.3), (2.5), $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\frac{1}{p_2(\cdot)}$ and $\varphi_1\varphi_2=\varphi$, it follows that

$$\|(b_1(\cdot)-(b_1)_{2B})(b_2(\cdot)-(b_2)_{2B})T(f_1^1,f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)}$$

$$
\begin{aligned}
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\frac{1}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\|(b_1(\cdot)-(b_1)_{2B})(b_2(\cdot)-(b_2)_{2B})\|_{L^{p(\cdot)}(B(x,r))}\\
&\quad\times\bigg(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\varphi_2(x,2^{k+1}r)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\bigg)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\
&\quad\times\frac{\|(b_1(\cdot)-(b_1)_{2B})\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\|(b_2(\cdot)-(b_2)_{2B})\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\bigg(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\varphi_2(x,2^{k+1}r)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\bigg)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}\\
&\quad\times\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\bigg(\sum_{k=1}^{\infty}\varphi_2(x,2^{k+1}r)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\bigg)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}
$$

To estimate E_2^2, we first consider $|T\big(f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^{\infty}\big)(y)|$ for $y\in B$. By applying (1.2), (1.8), (2.2), Lemmas 2.2 and 2.6, (3.1) and Lemma 3.2, we obtain

$$
\begin{aligned}
&|T\big(f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^{\infty}\big)(y)|\\
&\le C\int_{2B}|f_1(z_1)|\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}|}{|x-z_2|^{2n}}|f_2(z_2)|\mathrm{d}z_2\\
&\le C\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\bigg(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|\mathrm{d}z_2\bigg)\\
&\le C\frac{1}{\varphi_1(x,2r)}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\varphi_1(x,2r)\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\bigg(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}|(b_2)_{2B}-(b_2)_{2^{k+1}B}|\int_{2^{k+1}B}|f_2(z_2)|\mathrm{d}z_2\\
&\quad+\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|\mathrm{d}z_2\bigg)\\
&\le C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\varphi_1(x,2r)\frac{|2B|}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\bigg(\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{k=1}^{\infty}k\frac{1}{|2^kB|^2}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\\
&\quad+\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\big(b_2(\cdot)-(b_2)_{2^{k+1}B}\big)\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\bigg)
\end{aligned}
$$

$$\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\frac{|2B|}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}$$
$$\times\left(\sum_{k=1}^{\infty}(k+1)\frac{1}{|2^kB|^2}\varphi_2(x,2^{k+1}r)\frac{|2^{k+1}B|}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)$$
$$\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\varphi_1(x,2r)\frac{|2B|}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}$$
$$\times\left(\sum_{k=1}^{\infty}(k+1)\frac{1}{|2^kB|}\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right),$$

by this, (1.8), (2.2), (2.3), (2.5), (3.1) and $\varphi_1\varphi_2=\varphi$, we deduce

$$\mathrm{E}_2^2\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}$$
$$\times\frac{|2B|}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\|(b_1(\cdot)-(b_1)_{2B})\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\times\left(\sum_{k=1}^{\infty}(k+1)\frac{1}{|2^kB|}\frac{\varphi_2(x,2^{k+1}r)}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\right)$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}$$
$$\times\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\left(\sum_{k=1}^{\infty}(k+1)\frac{|2B|}{|2^kB|}\frac{\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,2r)}{\varphi(x,r)}$$
$$\times\left(\sum_{k=1}^{\infty}(k+1)\frac{\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

To estimate E_2^3, we first consider the equation $|T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,f_2^\infty\big)(y)|$ with $y\in B$. By (1.2), (1.8), Lemma 2.2, (2.1), (2.2) and (2.5), we get

$$|T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,f_2^\infty\big)(y)|$$
$$\leq C\int_{2B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|f_2(z_2)|}{|y-z_2|^{2n}}\mathrm{d}z_2$$
$$\leq C\|\big(b_1(\cdot)-(b_1)_{2B}\big)\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$$
$$\times\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|f_2(z_2)|\mathrm{d}z_2\right)$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\varphi_1(x,2r)$$
$$\times\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right)$$

$$
\begin{aligned}
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\frac{\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\varphi_1(x,2r)\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\frac{\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\frac{|2B|}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2r)\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\frac{|2^{k+1}B|}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\right)\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right),
\end{aligned}
$$

from this, (1.8), (2.2), (2.3), (3.1) and $\varphi_1\varphi_2=\varphi$, it follows that

$$
\begin{aligned}
\mathrm{E}_2^3&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\frac{\varphi_1(x,2r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\quad\times\|\big(b_2(\cdot)-(b_2)_{2B}\big)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=1}^{\infty}\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,r)}{\varphi(x,r)}\\
&\quad\times\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\left(\sum_{k=1}^{\infty}\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}
$$

We now turn E_2^4. For any $y\in B$, by applying (1.2), (1.8), Lemma 2.2, (2.1) and Lemma 3.2, we have

$$
\begin{aligned}
&|T\big((b_1(\cdot)-(b_1)_{2B})f_1^1,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)|\\
&\le C\int_{2B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}z_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}|}{|x-z_2|^{2n}}|f_2(z_2)|\mathrm{d}z_2\\
&\le C\|\big(b_1(\cdot)-(b_1)_{2B}\big)\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{|2^kB|^2}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|\mathrm{d}z_2\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\varphi_1(x,2r)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\sum_{k=1}^{\infty} \frac{1}{|2^k B|^2} |(b_2)_{2B} - (b_2)_{2^{k+1}B}| \int_{2^{k+1}B} |f_2(z_2)| \mathrm{d}z_2 \right.\\
&\left. + \sum_{k=1}^{\infty} \frac{1}{|2^k B|^2} \int_{2^{k+1}B} |b_2(z_2) - (b_2)_{2^{k+1}B}||f_2(z_2)| \mathrm{d}z_2 \right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \varphi_1(x,2r)\\
&\times \left(\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \sum_{k=1}^{\infty} k \frac{1}{|2^k B|^2} \|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \right.\\
&\left. + \sum_{k=1}^{\infty} \frac{1}{|2^k B|^2} \|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \big\| \big(b_2(\cdot) - (b_2)_{2^{k+1}B}\big) \chi_{2^{k+1}B} \big\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|\chi_{2B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \varphi_1(x,2r)\\
&\times \left[\sum_{k=1}^{\infty} (k+1) \frac{1}{|2^k B|^2} \|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \right]\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \frac{|2B|}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\times \varphi_1(x,2r) \left[\sum_{k=1}^{\infty} (k+1) \frac{1}{|2^k B|^2} \frac{|2^{k+1}B|}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \varphi_1(x, 2^{k+1}r) \right]\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \frac{\varphi_1(x,r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\times \left[\sum_{k=1}^{\infty} (k+1) \frac{|2B|}{|2^k B|} \frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \varphi_1(x, 2^{k+1}r) \right]\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \frac{\varphi_1(x,r)}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\
&\times \left[\sum_{k=1}^{\infty} (k+1) \frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \varphi_1(x, 2^{k+1}r) \right],
\end{aligned}
$$

from this, (1.8), (2.2), 3.1 and $\varphi_1\varphi_2 = \varphi$, it follows that

$$
\begin{aligned}
\mathrm{E}_2^4 &\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{\varphi_1(x,r)}{\varphi(x,r)}\\
&\times \frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \left[\sum_{k=1}^{\infty} (k+1) \frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \varphi_1(x, 2^{k+1}r) \right]\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{\varphi_1(x,r)}{\varphi(x,r)}\\
&\times \frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \left[\sum_{k=1}^{\infty} (k+1) \frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \varphi_1(x, 2^{k+1}r) \right]\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)} \|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}
\end{aligned}
$$

$$\times \sup_{x\in\mathbb{R}^n,r>0} \frac{\varphi_1(x,r)\varphi_2(x,r)}{\varphi(x,r)}$$
$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

To estimate E_4, we first consider the $|T_{b_1,b_2}(f_1^\infty, f_2^\infty)(y)|$ for $y \in B$. Because of

$$\begin{aligned}
&|T_{b_1,b_2}(f_1^\infty, f_2^\infty)(y)| \\
&\leq |b_1(y)-(b_1)_{2B}||b_2(y)-(b_2)_{2B}||T(f_1^\infty, f_2^\infty)(y)| \\
&\quad + |b_1(y)-(b_1)_{2B}||T\big(f_1^\infty, (b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)| \\
&\quad + |b_2(y)-(b_2)_{2B}||T\big((b_1(\cdot)-(b_1)_{2B})f_1^\infty, f_2^\infty\big)(y)| \\
&\quad + |T\big((b_1(\cdot)-(b_1)_{2B})f_1^\infty, (b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)|,
\end{aligned}$$

then, by the Minkowski's inequality, write

$$\begin{aligned}
&\|T_{b_1,b_2}(f_1^\infty, f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\
&\leq \|\big(b_1(\cdot)-(b_1)_{2B}\big)\big(b_2(\cdot)-(b_2)_{2B}\big)T(f_1^\infty, f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\
&\quad + \|\big(b_1(\cdot)-(b_1)_{2B}\big)T\big(f_1^\infty, (b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\
&\quad + \|\big(b_2(\cdot)-(b_2)_{2B}\big)T\big((b_1(\cdot)-(b_1)_{2B})f_1^\infty, f_2^\infty\big)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\
&\quad + \|T\big((b_1(\cdot)-(b_1)_{2B})f_1^\infty, (b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\
&= \mathrm{E}_4^1 + \mathrm{E}_4^2 + \mathrm{E}_4^3 + \mathrm{E}_4^4.
\end{aligned}$$

Using the estimate of $|T(f_1^\infty, f_2^\infty)(y)|$ in D_4, (1.8), (2.2), (2.3) and $\varphi_1\varphi_2 = \varphi$, we obtain

$$\begin{aligned}
&\|\big(b_1(\cdot)-(b_1)_{2B}\big)\big(b_2(\cdot)-(b_2)_{2B}\big)T(f_1^\infty, f_2^\infty)\|_{\mathcal{M}^{p(\cdot),\varphi}(\mathbb{R}^n)} \\
&\leq C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0} \frac{1}{\varphi(x,r)} \\
&\quad \times \|\big(b_1(\cdot)-(b_1)_{2B}\big)\big(b_2(\cdot)-(b_2)_{2B}\big)\|_{L^{p(\cdot)}(B(x,r))} \\
&\quad \times \left(\sum_{j=1}^\infty \frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\left(\sum_{j=1}^\infty \frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right) \\
&\leq C\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0} \|\big(b_1(\cdot)-(b_1)_{2B}\big)\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\quad \times \|\big(b_2(\cdot)-(b_1)_{2B}\big)\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\left(\sum_{j=1}^\infty \frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right) \\
&\quad \times \frac{1}{\varphi(x,r)}\left(\sum_{j=1}^\infty \frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right) \\
&\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0} \frac{1}{\varphi(x,r)} \\
&\quad \times \left(\sum_{j=1}^\infty \frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right) \\
&\quad \times \left(\sum_{j=1}^\infty \frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right)
\end{aligned}$$

$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

For any $y\in B$, by (1.2), (1.8), Lemma 2.2 and (2.2), we have

$$\begin{aligned}
&|T\big(f_1^\infty,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)|\\
&\leq C\int_{(\mathbb{R}^n)^2}\frac{|b_2(z_2)-(b_2)_{2B}|}{(|y-z_1|+|y-z_2|)^{2n}}|f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2\\
&\leq C\left(\sum_{j=1}^\infty\frac{1}{|2^jB|}\int_{2^{j+1}B}|f_1(z_1)|\mathrm{d}z_1\right)\left(\sum_{k=1}^\infty\frac{1}{|2^kB|}|(b_2)_{2B}-(b_2)_{2^{k+1}B}|\right.\\
&\quad\left.\times\int_{2^{k+1}B}|f_2(z_2)|\mathrm{d}z_2+\sum_{k=1}^\infty\frac{1}{|2^kB|}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|\mathrm{d}z_2\right)\\
&\leq C\left(\sum_{j=1}^\infty\frac{1}{|2^jB|}\|f_1\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)\\
&\quad\times\left(\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{k=1}^\infty k\frac{1}{|2^kB|}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right.\\
&\quad\left.+\sum_{k=1}^\infty\frac{1}{|2^kB|}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\big(b_2(\cdot)-(b_2)_{2^{k+1}B}\big)\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right)\\
&\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\
&\quad\times\left(\sum_{j=1}^\infty\frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\\
&\quad\times\left[\sum_{k=1}^\infty(k+1)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right],
\end{aligned}$$

by this, (1.8), (2.2), (2.3), (3.1) and $\varphi_1\varphi_2=\varphi$, we deduce

$$\begin{aligned}
\mathrm{E}_4^2&\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\quad\times\|\big(b_1(\cdot)-(b_1)_{2B}\big)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left(\sum_{j=1}^\infty\frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\\
&\quad\times\left[\sum_{k=1}^\infty(k+1)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right]\\
&\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\\
&\quad\times\left(\sum_{j=1}^\infty\frac{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right)\\
&\quad\times\left[\sum_{k=1}^\infty(k+1)\frac{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right]\\
&\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}$$

By employing an argument similar to that used in the estimation of E_4^2, it is easy to obtain

$$\mathrm{E}_4^3 \le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.$$

We now turn E_4^4. For any $y \in B$, by (1.2), (1.8), Lemma 2.2, (2.1) and Lemma 3.2, we get

$$\begin{aligned}
&|T\big((b_1(\cdot)-(b_1)_{2B})f_1^\infty,(b_2(\cdot)-(b_2)_{2B})f_2^\infty\big)(y)| \\
&\le C\int_{(\mathbb{R}^n)^2}\frac{|b_1(z_1)-(b_1)_{2B}||b_2(z_2)-(b_2)_{2B}|}{(|y-z_1|+|y-z_2|^{2n})}|f_1^\infty(z_1)||f_2^\infty(z_2)|\mathrm{d}z_1\mathrm{d}z_2 \\
&\le C\bigg(\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_1(z_1)-(b_1)_{2B}|}{|x-z_1|^n}|f_1(z_1)|\mathrm{d}z_1\bigg) \\
&\quad\times\bigg(\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(z_2)-(b_2)_{2B}|}{|x-z_2|^n}|f_2(z_2)|\mathrm{d}z_2\bigg) \\
&\le C\bigg(\sum_{j=1}^\infty\frac{1}{|2^jB|}\int_{2^{j+1}B}|b_1(z_1)-(b_1)_{2B}||f_1(z_1)|\mathrm{d}z_1\bigg) \\
&\quad\times\bigg(\sum_{k=1}^\infty\frac{1}{|2^kB|}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2B}||f_2(z_2)|\mathrm{d}z_2\bigg) \\
&\le C\bigg(\sum_{j=1}^\infty|(b_1)_{2B}-(b_1)_{2^{j+1}B}|\frac{1}{|2^jB|}\int_{2^{j+1}B}|f_1(z_1)|\mathrm{d}z_1 \\
&\quad+\sum_{j=1}^\infty\frac{1}{|2^jB|}\int_{2^{j+1}B}|b_1(z_1)-(b_1)_{2^{j+1}B}||f_1(z_1)|\mathrm{d}z_1\bigg) \\
&\quad\times\bigg(\sum_{k=1}^\infty|(b_2)_{2B}-(b_2)_{2^{k+1}B}|\frac{1}{|2^kB|}\int_{2^{k+1}B}|f_2(z_2)|\mathrm{d}z_2 \\
&\quad+\sum_{k=1}^\infty\frac{1}{|2^kB|}\int_{2^{k+1}B}|b_2(z_2)-(b_2)_{2^{k+1}B}||f_2(z_2)|\mathrm{d}z_2\bigg) \\
&\le C\bigg(\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{j=1}^\infty j\frac{1}{|2^jB|}\|f_1\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{j+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
&\quad+\sum_{j=1}^\infty\frac{1}{|2^jB|}\|\big(b_1(z_1)-(b_1)_{2^{j+1}B}\big)\chi_{2^{j+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\bigg) \\
&\quad\times\bigg(\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{k=1}^\infty k\frac{1}{|2^kB|}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)} \\
&\quad+\sum_{k=1}^\infty\frac{1}{|2^kB|}\|\big(b_2(z_2)-(b_2)_{2^{j+1}B}\big)\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\bigg) \\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\bigg[\sum_{j=1}^\infty(j+1)\frac{1}{|2^jB|}\|f_1\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&\quad\times\|\chi_{2^{j+1}B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\bigg]
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\sum_{k=1}^{\infty}(k+1)\frac{1}{|2^kB|}\|f_2\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\right]\\
\le{}& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\
&\times\left[\sum_{j=1}^{\infty}(j+1)\frac{1}{|2^jB|}\frac{|2^{j+1}B|}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right]\\
&\times\left[\sum_{k=1}^{\infty}(k+1)\frac{1}{|2^kB|}\frac{|2^{k+1}B|}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right]\\
\le{}& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\
&\times\left[\sum_{j=1}^{\infty}(j+1)\frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right]\\
&\times\left[\sum_{k=1}^{\infty}(k+1)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right],
\end{aligned}
$$

from this, (1.8), (2.2), (3.1) and $\varphi_1\varphi_2=\varphi$, it is easy to get

$$
\begin{aligned}
\mathrm{E}_4^4\le{}& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\
&\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\varphi(x,r)}\\
&\times\left[\sum_{j=1}^{\infty}(j+1)\frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\varphi_1(x,2^{j+1}r)\right]\\
&\times\left[\sum_{k=1}^{\infty}(k+1)\frac{1}{\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}\varphi_2(x,2^{k+1}r)\right]\\
\le{}& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}\\
&\times\sup_{x\in\mathbb{R}^n,r>0}\frac{\varphi_1(x,r)\varphi_2(x,r)}{\varphi(x,r)}\\
\le{}& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{\mathcal{M}^{p_1(\cdot),\varphi_1}(\mathbb{R}^n)}\|f_2\|_{\mathcal{M}^{p_2(\cdot),\varphi_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Which, combining the estimates for $\mathrm{E}_4^1,\mathrm{E}_4^2,\mathrm{E}_4^3$, E_1, E_2 and E_3, yields the desired result. Therefore, the proof of Theorem 3.1 is completed. □

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