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THE THRESHOLD OF A STOCHASTIC SIR EPIDEMIC MODEL WITH LOGISTIC GROWTH*

Zeyu Xu[1], Liang Wang[1,†] and Daqing Jiang[2]

Abstract In this paper, we propose and study a stochastic SIR epidemic model with logistic growth. The threshold dynamics of the stochastic model are governed by a parameter R_0^S. Precisely, the disease goes to extinction at an exponential rate and the distribution of susceptible individuals converges weakly to a unique invariant probability measure provided that $R_0^S < 1$; whereas if $R_0^S > 1$, we show that there is a unique ergodic stationary distribution of the positive solutions to the model by constructing a suitable stochastic Lyapunov function. Numerical simulations are introduced to illustrate our theoretical results. The aim of the work is to investigate the impact of environmental noise on the transmission of infectious diseases.

Keywords SIR epidemic model, logistic growth, stationary distribution, ergodicity, extinction, threshold.

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1. Introduction

Infectious diseases have persistently remained a significant latent threat to the development of human society. Throughout history, each major pandemic outbreak, from the medieval Black Death and the worldwide spread of smallpox to more modern instances of influenza and AIDS, and further to the 21st-century outbreaks of SARS and COVID-19, has exerted profound and severe impacts on the global economy and human health.

Recently, mathematical models have provided a deeper understanding of the transmission process of infectious diseases in the world. Starting with Kermack and McKendrick [21], a large number of mathematicians and ecologists have developed various epidemic models to realize and control the spread of transmissible diseases in the community [2, 19, 26, 29, 30, 36, 50]. The SIR model is a compartmental model that tracks the flow of individuals among three states: Susceptible (S), Infectious (I), and Recovered (R). Its core purpose is to mathematically describe how an infection spreads in a population. It is suitable for simulating infectious diseases where patients acquire long-lasting immunity after infection. For example, Liu et al. [29] proposed a

[†]The corresponding author.
[1]School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China
[2]College of Science, China University of Petroleum (East China), Qingdao 266580, China
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Email: xuzeyu@njust.edu.cn(Z. Xu), ntumacwangl@hotmail.com(L. Wang), daqingjiang2010@hotmail.com(D. Jiang)

deterministic SIR epidemic model with logistic growth which takes the following form

$$\begin{cases} \dfrac{dS(t)}{dt} = rS(t)\Big(1 - \dfrac{S(t)+I(t)}{K}\Big) - \beta S(t)I(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\gamma+\alpha)I(t), \end{cases} \tag{1.1}$$

where the parameters r, K, β, μ, γ and α are positive constants. As the dynamics of compartment R have no effect on the disease transmission dynamics, we omit it from their model. In system (1.1), $S(t)$ denotes the number of susceptible individuals and $I(t)$ denotes the number of infected individuals. The parameters r, K are the intrinsic birth rate and carrying capacity of susceptible host individuals, β denotes the disease transmission coefficient between compartments S and I, μ is the natural death rate of infected individuals, γ denotes the rate of recovery from infection and α is the disease-caused death rate of infected individuals. The model has a disease-free equilibrium point at $(K, 0)$, and an endemic equilibrium at $\left(\frac{\mu+\gamma+\alpha}{\beta}, \frac{r\left(1-\frac{\mu+\gamma+\alpha}{K\beta}\right)}{\beta+\frac{r}{K}}\right)$. Furthermore, the basic reproduction number $R_0 = \frac{\beta K}{\mu+\gamma+\alpha}$ is derived. Specifically, when $R_0 > 1$, the disease will persist in the population, becoming endemic. Conversely, when $R_0 < 1$, the disease will inevitably die out.

However, epidemic dynamics are always subject to environmental noise in an ecosystem. The environmental variations have an important effect on the development of an epidemic [37, 47]. For human diseases, the nature of epidemic growth and spread is inherently random due to the unpredictability of person-to-person contacts [44], and population is affected by a continuous spectrum of disturbances [5]. Thus, the variability and randomness of the environment are fed through to the state of the epidemic [47]. Keeping this in mind, the stochastic differential equations (SDEs) could be a more appropriate way of modeling disease spreading in many circumstances [16, 51, 52]. In recent years, various forms of stochastic epidemic models with logistic growth have been formulated and studied [7, 27, 28, 31, 49]. There exist three different possible approaches which lead to different influences on the population system to include random perturbations in the models. In [10, 16, 46], the situation of the parameter perturbation was investigated. The second approach is that white noise stochastic perturbations around the positive endemic equilibrium of epidemic models which was considered in [6, 8]. The last approach to include stochastic perturbations in a biological model was used by Imhof and Walcher in [18].

In this paper, our approach to include stochastic perturbation is similar to that of [18]. Here we assume that stochastic perturbations are of the white noise type which are directly proportional to $S(t)$ and $I(t)$. Then we obtain the following stochastic model

$$\begin{cases} dS(t) = \Big[rS(t)\Big(1 - \dfrac{S(t)+I(t)}{K}\Big) - \beta S(t)I(t)\Big]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\gamma+\alpha)I(t)]\,dt + \sigma_2 I(t)dB_2(t), \end{cases} \tag{1.2}$$

where $\dot{B}_i(t)$ are the white noise, which are formally regarded as the derivative of the Brownian motions $B_i(t)$, i.e., $\dot{B}_i(t) = dB_i(t)/dt$, $\sigma_i^2 > 0 (i = 1, 2)$ are the intensities of the white noise.

As far as we know, the dynamics of system (1.2) have not been studied yet. In this paper, we devote to studying the dynamics of system (1.2). We establish sufficient conditions for the disease to prevail and to disappear, and then we give the threshold of system (1.2). The organization of this paper is as follows: In Section 2, we show that system (1.2) has a unique global positive solution with any positive initial value. In Section 3, we establish the sufficient conditions for the disease to die out at an exponential rate and the distribution of susceptible individuals converges weakly to a unique invariant probability measure. In Section 4, we obtain the conditions for the existence of a unique ergodic stationary distribution of the positive solutions to system (1.2) by constructing a suitable stochastic Lyapunov function. Following the discussion in Sections 3 and 4, the threshold of system (1.2) is obtained. In Section 5, we provide some numerical simulations to show the effectiveness of our theoretical results. Finally, some concluding remarks and future directions are presented to end this paper.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), we also let $B_i(t)$ be defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $i = 1, 2$. Define

$$\mathbb{R}^d_+ = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}.$$

In general, consider the d-dimensional stochastic differential equation

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t) \text{ for } t \geq t_0, \tag{1.3}$$

with the initial value $X(0) = X_0 \in \mathbb{R}^d$. $B(t)$ denotes a d-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Denote by $C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$ the family of all nonnegative functions $V(X, t)$ defined on $\mathbb{R}^d \times [t_0, \infty)$ such that they are continuously twice differentiable in X and once in t. The differential operator L of Eq. (1.3) is defined by [34]

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(X, t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d}[g^T(X, t)g(X, t)]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, then

$$LV(X, t) = V_t(X, t) + V_X(X, t)f(X, t) + \frac{1}{2}\text{trace}[g^T(X, t)V_{XX}(X, t)g(X, t)],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_X = (\frac{\partial V}{\partial X_1}, \ldots, \frac{\partial V}{\partial X_d})$, $V_{XX} = (\frac{\partial^2 V}{\partial X_i \partial X_j})_{d\times d}$. In view of Itô's formula [34], if $X(t) \in \mathbb{R}^d$, then

$$dV(X(t), t) = LV(X(t), t)dt + V_X(X(t), t)g(X(t), t)dB(t).$$

2. Existence and uniqueness of the global positive solution

To study the dynamical behavior of an epidemic model, the first concerning thing is that whether the solution is global and positive. In what follows, we will prove that there is a unique global positive solution of system (1.2) via the stochastic Lyapunov function method.

Theorem 2.1. *For any given initial value $(S(0), I(0)) \in \mathbb{R}^2_+$, there exists a unique solution $(S(t), I(t))$ of system (1.2) on $t \geq 0$ and the solution will remain in $\mathbb{R}^2_+$ with probability one, namely, $(S(t), I(t)) \in \mathbb{R}^2_+$ for all $t \geq 0$ almost surely (a.s.).*

Proof. Since the coefficients of system (1.2) satisfy the local Lipschitz condition, then for any given initial value $(S(0), I(0)) \in \mathbb{R}^2_+$, there is a unique local solution $(S(t), I(t)) \in \mathbb{R}^2_+$ on $t \in [0, \tau_e)$ a.s., where τ_e denotes the explosion time [34]. To show this solution is global, we only need to prove that $\tau_e = \infty$ a.s. To this end, let $n_0 \geq 1$ be sufficiently large such that $S(0)$, $I(0)$ all lie within the interval $\left[\frac{1}{n_0}, n_0\right]$. For each integer $n \geq n_0$, define the stopping time as [34]

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), I(t)\} \geq n\right\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Obviously, τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and $(S(t), I(t)) \in \mathbb{R}^2_+$ a.s. for all $t \geq 0$. That is to say, to complete the proof we only need to prove $\tau_\infty = \infty$ a.s. If this assertion is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Thus, there is an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \epsilon, \ \forall n \geq n_1. \tag{2.1}$$

Define a C^2-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+ \cup \{0\}$ by

$$V(S, I) = \left(S - a - a \ln \frac{S}{a}\right) + (I - 1 - \ln I),$$

where a is a positive constant to be determined later. The nonnegativity of this function can be seen from

$$u - 1 - \ln u \geq 0 \text{ for any } u > 0.$$

Applying Itô's formula to V, we have

$$dV(S, I) = LV(S, I)dt + \sigma_1(S - a)dB_1(t) + \sigma_2(I - 1)dB_2(t),$$

where $LV : \mathbb{R}^2_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}
LV &= \left(1 - \frac{a}{S}\right)\left[rS\left(1 - \frac{S+I}{K}\right) - \beta SI\right] + \frac{a}{2}\sigma_1^2 + \left(1 - \frac{1}{I}\right)[\beta SI - (\mu + \gamma + \alpha)I] + \frac{\sigma_2^2}{2} \\
&= -\frac{r}{K}S^2 + \frac{(a+K)r}{K}S + \left[\left(\frac{r}{K} + \beta\right)a - (\mu + \gamma + \alpha)\right]I \\
&\quad -\frac{r}{K}SI - \beta S - ar + \mu + \gamma + \alpha + \frac{a}{2}\sigma_1^2 + \frac{\sigma_2^2}{2} \\
&\leq -\frac{r}{K}S^2 + \frac{(a+K)r}{K}S + \left[\left(\frac{r}{K} + \beta\right)a - (\mu + \gamma + \alpha)\right]I + \mu + \gamma + \alpha + \frac{a}{2}\sigma_1^2 + \frac{\sigma_2^2}{2} \\
&\leq \sup_{S \in \mathbb{R}_+}\left\{-\frac{r}{K}S^2 + \frac{(a+K)r}{K}S\right\} + \left[\left(\frac{r}{K} + \beta\right)a - (\mu + \gamma + \alpha)\right]I + \mu + \gamma + \alpha + \frac{a}{2}\sigma_1^2 + \frac{\sigma_2^2}{2}.
\end{aligned}$$

Choose $a = \frac{\mu+\gamma+\alpha}{\frac{r}{K}+\beta}$ such that $(\frac{r}{K} + \beta)a - (\mu + \gamma + \alpha) = 0$, then we obtain

$$LV \leq \sup_{S \in \mathbb{R}_+}\left\{-\frac{r}{K}S^2 + \frac{(a+K)r}{K}S\right\} + \mu + \gamma + \alpha + \frac{a}{2}\sigma_1^2 + \frac{\sigma_2^2}{2} := \tilde{K},$$

where $\tilde{K}$ is a positive constant. Thus

$$dV(S,I) \le \tilde{K}dt + \sigma_1(S-a)dB_1(t) + \sigma_2(I-1)dB_2(t). \tag{2.2}$$

Integrating (2.2) from 0 to $\tau_k \wedge T = \min\{\tau_k, T\}$ and then taking the expectation on both sides, we have

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T)) \le V(S(0), I(0)) + \tilde{K}\mathbb{E}(\tau_k \wedge T).$$

Hence

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T)) \le V(S(0), I(0)) + \tilde{K}T. \tag{2.3}$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and by (2.1), we have $\mathbb{P}(\Omega_k) \ge \epsilon$. Note that for every $\omega \in \Omega_k$, there is $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. So $V(S(\tau_k, \omega), I(\tau_k, \omega))$ is no less than either

$$\begin{aligned} & k - a - a\ln\frac{k}{a} \text{ or } \frac{1}{k} - a - a\ln\frac{1}{ak} = \frac{1}{k} - a + a\ln(ak) \\ & \text{or } k - 1 - \ln k \text{ or } \frac{1}{k} - 1 - \ln\frac{1}{k} = \frac{1}{k} - 1 + \ln k. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & V(S(\tau_k, \omega), I(\tau_k, \omega)) \\ & \ge \left(k - a - a\ln\frac{k}{a}\right) \wedge \left(\frac{1}{k} - a + a\ln(ak)\right) \wedge (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k\right). \end{aligned}$$

In view of (2.3), we have

$$\begin{aligned} & V(S(0), I(0)) + \tilde{K}T \\ & \ge \mathbb{E}[\mathrm{I}_{\Omega_k(\omega)} V(S(\tau_k, \omega), I(\tau_k, \omega))] \\ & \ge \epsilon\left[\left(k - a - a\ln\frac{k}{a}\right) \wedge \left(\frac{1}{k} - a + a\ln(ak)\right) \wedge (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 + \ln k\right)\right], \end{aligned}$$

where I_{Ω_k} denotes the indicator function of Ω_k. Letting $k \to \infty$, then we obtain

$$\infty > V(S(0), I(0)) + \tilde{K}T = \infty,$$

which leads to the contradiction. So we must have $\tau_\infty = \infty$ a.s. This completes the proof. □

In the following analysis, we first define a parameter

$$R_0^S = \frac{\beta K(r - \frac{\sigma_1^2}{2})}{r(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2})}.$$

3. Dynamical behavior of system (1.2): Case $R_0^S < 1$

In this section, we will establish sufficient conditions for the disease to die out at an exponential rate and the distribution of susceptible individuals converges weakly to a unique invariant probability measure provided that $R_0^S < 1$. Firstly, we give a lemma as follows.

Lemma 3.1. ([13]) *Consider the following one-dimensional stochastic differential equation*

$$d\tilde{S}(t) = r\tilde{S}(t)\left(1 - \frac{\tilde{S}(t)}{K}\right)dt + \sigma_1\tilde{S}(t)dB_1(t),\ \tilde{S}(0) = x. \tag{3.1}$$

(a) If $r > \frac{\sigma_1^2}{2}$, the diffusion process $\tilde{S}(t)$ of system (3.1) admits a unique stationary distribution with density $\pi(x)$ and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{S}(u)du = \frac{K(r - \frac{\sigma_1^2}{2})}{r}\ a.s., \tag{3.2}$$

where

$$\pi(x) = \left(\frac{2r}{K\sigma_1^2}\right)^{\frac{2r}{\sigma_1^2}-1}\Gamma^{-1}\left(\frac{2r}{\sigma_1^2} - 1\right)x^{-2(1-\frac{r}{\sigma_1^2})}e^{-\frac{2r}{K\sigma_1^2}x}, x \in (0,\infty)$$

and $\Gamma(\cdot)$ is the Gamma function.
(b) If $r < \frac{\sigma_1^2}{2}$, the diffusion process $\tilde{S}(t)$ of system (3.1) goes to extinction a.s., i.e., $\lim_{t\to\infty}\tilde{S}(t) = 0$ a.s.

In view of the stochastic comparison theorem for Itô's processes [17], we obtain

$$\mathbb{P}(S(t) \le \tilde{S}(t), t \ge 0) = 1,\ S(0) = \tilde{S}(0) = x. \tag{3.3}$$

This together with Lemma 3.1 implies that $\lim_{t\to\infty} S(t) = 0$ a.s. provided that $r < \frac{\sigma_1^2}{2}$. Consequently, throughout this paper, we always suppose that $r > \frac{\sigma_1^2}{2}$.

Theorem 3.1. *Let $(S(t), I(t))$ be the solution of system (1.2) with any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$. If $R_0^S < 1$, then I converges to 0 a.s. at an exponential rate, i.e.,*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} = \left(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2}\right)(R_0^S - 1) < 0\ a.s.$$

Moreover, the distribution of $S(t)$ converges weakly to the unique invariant probability measure μ^* with density $\pi(x)$.

Proof. Making use of Itô's formula to $\ln I(t)$ leads to

$$\begin{aligned} d\ln I(t) &= \left[\frac{1}{I}(\beta SI - (\mu+\gamma+\alpha)I) - \frac{1}{2I^2}\sigma_2^2 I^2\right]dt + \sigma_2 dB_2(t) \\ &= \left(\beta S - \mu - \gamma - \alpha - \frac{\sigma_2^2}{2}\right)dt + \sigma_2 dB_2(t). \end{aligned} \tag{3.4}$$

Integrating (3.4) from 0 to t and then dividing by t on both sides, we have

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &= \frac{\beta}{t}\int_0^t S(u)du - \mu - \gamma - \alpha - \frac{\sigma_2^2}{2} + \frac{\sigma_2 B_2(t)}{t} \\ &= \frac{\beta}{t}\int_0^t S(u)du - \mu - \gamma - \alpha - \frac{\sigma_2^2}{2} + \frac{M(t)}{t}, \end{aligned} \tag{3.5}$$

where $M(t) = \sigma_2 B_2(t)$ is a continuous local martingale [34] vanishing at $t = 0$ and

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} = \sigma_2^2 < \infty.$$

By the strong law of large numbers [34], we get

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \text{ a.s.}$$

Taking the superior limit on both sides of (3.5) and combining with (3.2) and (3.3), we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{\ln I(t)}{t} &= \frac{\beta K(r - \frac{\sigma_1^2}{2})}{r} - \mu - \gamma - \alpha - \frac{\sigma_2^2}{2} \\ &= \left(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2}\right)(R_0^S - 1) \\ &:= \lambda \\ &< 0 \text{ a.s.,}\end{aligned} \tag{3.6}$$

where

$$R_0^S = \frac{\beta K(r - \frac{\sigma_1^2}{2})}{r(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2})} \text{ and } \lambda = \left(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2}\right)(R_0^S - 1),$$

which implies that $I(t)$ converges exponentially to 0 a.s.

In what follows, we focus on proving that the distribution of $S(t)$ converges weakly to a unique invariant probability measure. To proceed, we set

$$\Omega_\epsilon = \left\{\omega : \ln I(t) \le \frac{\lambda t}{2}\right\},$$

where $\epsilon > 0$ is an arbitrary number. According to (3.6), we can obtain that there exists a positive constant t_0 such that $\mathbb{P}(\Omega_\epsilon) > 1 - \epsilon$ holds for any $t \ge t_0$. Applying Itô's formula to $\ln \tilde{S}(t)$ and $\ln S(t)$, respectively, we get that

$$\begin{aligned} d\ln \tilde{S}(t) &= \left(r - \frac{\sigma_1^2}{2} - \frac{r}{K}\tilde{S}(t)\right)dt + \sigma_1 dB_1(t), \\ d\ln S(t) &= \left[r - \frac{\sigma_1^2}{2} - \frac{r}{K}S(t) - \left(\frac{r}{K} + \beta\right) I(t)\right] dt + \sigma_1 dB_1(t).\end{aligned}$$

Direct calculation leads to that

$$\begin{aligned} 0 &\le \ln \tilde{S}(t) - \ln S(t) \\ &= \frac{r}{K}\int_{t_0}^t (S(u) - \tilde{S}(u))du + \left(\frac{r}{K} + \beta\right)\int_{t_0}^t I(u)du \\ &\le \left(\frac{r}{K} + \beta\right)\int_{t_0}^t I(u)du \\ &\le \left(\frac{r}{K} + \beta\right)\int_{t_0}^t e^{\frac{\lambda}{2}u}du \\ &\le -\frac{2}{\lambda}\left(\frac{r}{K} + \beta\right)e^{\frac{\lambda}{2}t_0}. \end{aligned} \tag{3.7}$$

Finally, we can choose t_0 sufficiently large such that

$$t_0 > \frac{2}{\lambda} \ln \left(-\frac{\lambda \epsilon}{2\left(\frac{r}{K} + \beta\right)} \right).$$

It follows that for all $t \geq t_0$,

$$\mathbb{P}\{|\ln \tilde{S}(t) - \ln S(t)| > \epsilon\} \leq 1 - \mathbb{P}(\Omega_\epsilon) < \epsilon. \tag{3.8}$$

Denote by $X(t)$ and $Y(t)$ two random variables, let ν^* be the distribution of $\ln X(t)$ provided that $X(t)$ admits μ^* as its distribution. In view of the Portmanteau theorem [4], if

$$\mathbb{E}g(\ln Y(t)) \to \bar{g} := \int_{\mathbb{R}} g(x)\nu^*(dx) = \int_0^\infty g(\ln x)\mu^*(dx)$$

holds for every

$$g(\cdot) : \mathbb{R} \mapsto \mathbb{R} : |g(x) - g(y)| \leq |x - y|, \ |g(x)| < 1, \ \forall x, y \in \mathbb{R},$$

then the distribution of $Y(t)$ converges weakly to the measure μ^*. Thus, to prove that the distribution of $S(t)$ converges weakly to the unique invariant probability measure μ^*, it only needs to prove that

$$\mathbb{E}g(\ln S(t)) \to \bar{g} := \int_{\mathbb{R}} g(x)\nu^*(dx) = \int_0^\infty g(\ln x)\mu^*(dx). \tag{3.9}$$

Since the distribution of $\tilde{S}(t)$ converges to μ^* as $t \to \infty$, we have

$$\lim_{t\to\infty} \mathbb{E}g(\ln \tilde{S}(t)) = \bar{g}. \tag{3.10}$$

By a simple calculation, we obtain

$$\begin{aligned}
&|\mathbb{E}g(\ln S(t)) - \bar{g}| \\
\leq &|\mathbb{E}g(\ln S(t)) - \mathbb{E}g(\ln \tilde{S}(t))| + |\mathbb{E}g(\ln \tilde{S}(t)) - \bar{g}| \\
\leq &\epsilon\mathbb{P}\{|\ln S(t) - \ln \tilde{S}(t)| \leq \epsilon\} + |\mathbb{E}g(\ln \tilde{S}(t)) - \bar{g}| + 2\mathbb{P}\{|\ln S(t) - \ln \tilde{S}(t)| > \epsilon\}.
\end{aligned} \tag{3.11}$$

As $t \to \infty$, taking the superior limit on both sides of (3.11) and using (3.8) and (3.10) yields

$$\limsup_{t\to\infty} |\mathbb{E}g(\ln S(t)) - \bar{g}| \leq \epsilon \cdot 1 + \epsilon + 2 \cdot \epsilon = 4\epsilon.$$

This, together with the arbitrariness of ϵ, proves (3.9). This completes the proof. □

4. Dynamical behavior of system (1.2): $R_0^S > 1$

In this section, we will pay our main attention to the dynamical behavior of system (1.2) provided that $R_0^S > 1$. More specifically, we will establish sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the positive solutions to system (1.2), implying that the stochastic system (1.2) will reach a state of weak stability.

Definition 4.1. ([14]) The transition probability function $P(s,x,t,A)$ is said to be time-homogeneous (and the corresponding Markov process is called time-homogeneous) if the function $P(s,x,t+s,A)$ is independent of s, where $0 \le s \le t$, $x \in \mathbb{R}^d$ and $A \in \mathcal{B}$ and $\mathcal{B}$ denotes the σ-algebra of Borel sets in $\mathbb{R}^d$.

Let $X(t)$ be a regular time-homogeneous Markov proccess in $\mathbb{R}^d$ described by the following stochastic differential equation

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t).$$

The diffusion matrix is defined as follows

$$A(x) = (a_{ij}(x)), a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 4.1. ([14]) *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded domain $D \subset \mathbb{R}^d$ with regular boundary Γ, having the following properties:*

A_1: There is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge M|\xi|^2,\ x \in D,\ \xi \in \mathbb{R}^d$.

A_2: There exists a nonnegative C^2-function V such that LV is negative for any $\mathbb{R}^d \setminus D$.

Theorem 4.1. *Assume that $R_0^S > 1$, then for any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$, the solution $(S(t), I(t))$ of system (1.2) is positive recurrent and has a unique ergodic stationary distribution $\pi(\cdot)$.*

Proof. In order to prove Theorem 4.1, we only need to validate conditions A_1 and A_2 in Lemma 4.1. We first prove the condition A_1. The diffusion matrix of system (1.2) is given by

$$A = \begin{pmatrix} \sigma_1^2 S^2 & 0 \\ 0 & \sigma_2^2 I^2 \end{pmatrix}.$$

Obviously, the matrix A is positive definite for any compact subset of $\mathbb{R}_+^2$, so the condition A_1 in Lemma 4.1 holds.

Now we prove the condition A_2. To this end, we define a C^2-function $\tilde{V} : \mathbb{R}_+^2 \to \mathbb{R}$ as follows

$$\tilde{V}(S,I) = M\left(-\ln I - \frac{\beta K}{r}\ln S + \frac{\beta(\beta K + r)}{r(\mu+\gamma+\alpha)}I\right) + \frac{1}{\theta+1}(S+I)^{\theta+1}.$$

Note that $\tilde{V}(S,I)$ is not only continuous, but also tends to ∞ as (S,I) approaches the boundary of $\mathbb{R}_+^2$ and as $||(S,I)|| \to \infty$, where $||\cdot||$ denotes the Euclidean norm of a point in $\mathbb{R}_+^2$. Thus, it must be lower bounded and achieve this lower bound at a point (S_0, I_0) in the interior of $\mathbb{R}_+^2$. Then we define a C^2-function $\bar{V} : \mathbb{R}_+^2 \to \mathbb{R}_+ \cup \{0\}$ in the following form

$$\begin{aligned}\bar{V}(S,I) &= M\left(-\ln I - \frac{\beta K}{r}\ln S + \frac{\beta(\beta K + r)}{r(\mu+\gamma+\alpha)}I\right) + \frac{1}{\theta+1}(S+I)^{\theta+1} - \tilde{V}(S_0, I_0) \\ &:= MV_1 + V_2,\end{aligned}$$

where $(S,I) \in (\frac{1}{n}, n) \times (\frac{1}{n}, n)$ and $n > 1$ is a sufficiently large integer,

$$V_1 = -\ln I - \frac{\beta K}{r}\ln S + \frac{\beta(\beta K + r)}{r(\mu+\gamma+\alpha)}I,\ V_2 = \frac{1}{\theta+1}(S+I)^{\theta+1} - \tilde{V}(S_0, I_0),$$

θ is a sufficiently small positive constant and $M > 0$ is a sufficiently large number satisfying the following condition

$$-M\left(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}\right)(R_0^S-1)+B \leq -2 \tag{4.1}$$

and

$$B = \sup_{(S,I)\in\mathbb{R}_+^2}\left\{-\frac{r}{2K}S^{\theta+2}+\frac{2^\theta\theta}{2}(\sigma_1^2\vee\sigma_2^2)S^{\theta+1}+rS(S+I)^\theta-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}\right\}<\infty.$$

Applying Itô's formula to V_1, we have

$$\begin{aligned}
LV_1 &= -\beta S+\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}-\frac{\beta K}{r}\left(r-\frac{\sigma_1^2}{2}\right)+\frac{\beta K}{r}\frac{r}{K}S+\frac{\beta K}{r}\left(\frac{r}{K}+\beta\right)I\\
&\quad+\frac{\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI-\frac{\beta(\beta K+r)}{r}I\\
&=\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}-\frac{\beta K}{r}\left(r-\frac{\sigma_1^2}{2}\right)+\frac{\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI\\
&=-\left(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}\right)(R_0^S-1)+\frac{\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI,
\end{aligned} \tag{4.2}$$

where

$$R_0^S=\frac{\beta K(r-\frac{\sigma_1^2}{2})}{r(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2})}.$$

Similarly

$$\begin{aligned}
LV_2 &= (S+I)^\theta\left[rS\left(1-\frac{S+I}{K}\right)-(\mu+\gamma+\alpha)I\right]+\frac{\theta}{2}(S+I)^{\theta-1}(\sigma_1^2S^2+\sigma_2^2I^2)\\
&\leq (S+I)^\theta\left[rS\left(1-\frac{S}{K}\right)-(\mu+\gamma+\alpha)I\right]+\frac{\theta}{2}(S+I)^{\theta+1}(\sigma_1^2\vee\sigma_2^2)\\
&\leq rS(S+I)^\theta-\frac{r}{K}S^{\theta+2}-(\mu+\gamma+\alpha)I^{\theta+1}+\frac{\theta}{2}(S+I)^{\theta+1}(\sigma_1^2\vee\sigma_2^2)\\
&\leq -\frac{r}{2K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{2}I^{\theta+1}-\frac{r}{2K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{2}I^{\theta+1}+rS(S+I)^\theta\\
&\quad+\frac{2^\theta\theta}{2}(S^{\theta+1}+I^{\theta+1})(\sigma_1^2\vee\sigma_2^2)\\
&=-\frac{r}{2K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{2}I^{\theta+1}-\frac{r}{2K}S^{\theta+2}+\frac{2^\theta\theta}{2}(\sigma_1^2\vee\sigma_2^2)S^{\theta+1}+rS(S+I)^\theta\\
&\quad-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}-\left(\frac{\mu+\gamma+\alpha}{4}-\frac{2^\theta\theta}{2}(\sigma_1^2\vee\sigma_2^2)\right)I^{\theta+1}\\
&\leq -\frac{r}{2K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{2}I^{\theta+1}-\frac{r}{2K}S^{\theta+2}+\frac{2^\theta\theta}{2}(\sigma_1^2\vee\sigma_2^2)S^{\theta+1}+rS(S+I)^\theta\\
&\quad-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}
\end{aligned}$$

$$\leq -\frac{r}{2K}S^{\theta+2} - \frac{\mu+\gamma+\alpha}{2}I^{\theta+1} + B, \tag{4.3}$$

where

$$B = \sup_{(S,I)\in\mathbb{R}^2_+}\left\{ -\frac{r}{2K}S^{\theta+2} + \frac{2^\theta\theta}{2}(\sigma_1^2\vee\sigma_2^2)S^{\theta+1} + rS(S+I)^\theta - \frac{\mu+\gamma+\alpha}{4}I^{\theta+1}\right\}$$

and in the third inequality we have used the inequality $|\sum_{i=1}^k v_i|^p \leq k^{p-1}\sum_{i=1}^k |v_i|^p$ for $\forall p\geq 1$.

By (4.2) and (4.3), we obtain

$$L\bar{V} \leq -M\left(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}\right)(R_0^S-1) + \frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI - \frac{r}{2K}S^{\theta+2} - \frac{\mu+\gamma+\alpha}{2}I^{\theta+1} + B.$$

Now we are in the position to construct a compact subset D_ϵ such that the condition A_2 in Lemma 4.1 holds. Define the following bounded closed set

$$D_\epsilon = \left\{(S,I)\in\mathbb{R}^2_+ : \epsilon\leq S\leq\frac{1}{\epsilon}, \epsilon\leq I\leq\frac{1}{\epsilon}\right\},$$

where $0<\epsilon<1$ is a sufficiently small constant. In the set $\mathbb{R}^2_+\setminus D_\epsilon$, we can choose ϵ sufficiently small such that the following conditions hold

$$\epsilon < \frac{r(\mu+\gamma+\alpha)(\theta+1)}{M\beta^2(\beta K+r)\theta}, \tag{4.4}$$

$$\epsilon < \frac{r(\mu+\gamma+\alpha)^2(\theta+1)}{2M\beta^2(\beta K+r)}, \tag{4.5}$$

$$\epsilon < \frac{r(\mu+\gamma+\alpha)(\theta+2)}{M\beta^2(\beta K+r)(\theta+1)}, \tag{4.6}$$

$$\epsilon < \frac{r^2(\mu+\gamma+\alpha)(\theta+2)}{2KM\beta^2(\beta K+r)}, \tag{4.7}$$

$$-\frac{r}{4K\epsilon^{\theta+2}} + C \leq -1, \tag{4.8}$$

$$-\frac{\mu+\gamma+\alpha}{4\epsilon^{\theta+1}} + C \leq -1, \tag{4.9}$$

where C is a positive constant which will be given explicitly in expression (4.13). For convenience we can divide $D_\epsilon^c = \mathbb{R}^2_+\setminus D_\epsilon$ into four domains,

$$D_\epsilon^1 = \{(S,I)\in\mathbb{R}^2_+ : S<\epsilon\}, D_\epsilon^2 = \{(S,I)\in\mathbb{R}^2_+ : I<\epsilon\},$$
$$D_\epsilon^3 = \{(S,I)\in\mathbb{R}^2_+ : S>\frac{1}{\epsilon}\}, D_\epsilon^4 = \{(S,I)\in\mathbb{R}^2_+ : I>\frac{1}{\epsilon}\}.$$

It is easy to see that $D_\epsilon^c = D_\epsilon^1\cup D_\epsilon^2\cup D_\epsilon^3\cup D_\epsilon^4$. Next, we will prove that $L\bar{V}(S,I)\leq -1$ for any $(S,I)\in D_\epsilon^c$, which is equivalent to prove it on the above four domains, respectively.

Case 1. If $(S,I)\in D_\epsilon^1$, due to $SI<\epsilon I\leq\epsilon\frac{\theta+I^{\theta+1}}{\theta+1} = \frac{\theta}{\theta+1}\epsilon + \frac{\epsilon}{\theta+1}I^{\theta+1}$, we obtain

$$L\bar{V} \leq -M\left(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}\right)(R_0^S-1) + \frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}\frac{\theta}{\theta+1}\epsilon$$

$$+\left(\frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}\frac{\epsilon}{\theta+1}-\frac{\mu+\gamma+\alpha}{2}\right)I^{\theta+1}+B$$
$$\le -2+1$$
$$=-1, \tag{4.10}$$

which follows from (4.1), (4.4) and (4.5). Thus

$$L\bar{V}\le -1 \text{ for any } (S,I)\in D_\epsilon^1.$$

Case 2. If $(S,I)\in D_\epsilon^2$, due to $SI<\epsilon S\le \epsilon\frac{\theta+1+S^{\theta+2}}{\theta+2}=\frac{\theta+1}{\theta+2}\epsilon+\frac{\epsilon}{\theta+2}S^{\theta+2}$, we get

$$\begin{aligned}L\bar{V}&\le -M\left(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2}\right)(R_0^S-1)+\frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}\frac{\theta+1}{\theta+2}\epsilon\\&\quad+\left(\frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}\frac{\epsilon}{\theta+2}-\frac{r}{2K}\right)S^{\theta+2}+B\\&\le -2+1\\&=-1,\end{aligned}\tag{4.11}$$

which follows from (4.1), (4.6) and (4.7). Therefore

$$L\bar{V}\le -1 \text{ for any } (S,I)\in D_\epsilon^2.$$

Case 3. If $(S,I)\in D_\epsilon^3$, we have

$$\begin{aligned}L\bar{V}&\le -\frac{r}{4K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}-\frac{r}{4K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}+\frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI+B\\&\le -\frac{r}{4K}S^{\theta+2}+C\\&\le -\frac{r}{4K\epsilon^{\theta+2}}+C\\&\le -1,\end{aligned}\tag{4.12}$$

which follows from (4.8) and

$$C=\sup_{(S,I)\in\mathbb{R}_+^2}\left\{-\frac{r}{4K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}+\frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI+B\right\}.\tag{4.13}$$

So

$$L\bar{V}\le -1 \text{ for any } (S,I)\in D_\epsilon^3.$$

Case 4. If $(S,I)\in D_\epsilon^4$, we get

$$\begin{aligned}L\bar{V}&\le -\frac{r}{4K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}-\frac{r}{4K}S^{\theta+2}-\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}+\frac{M\beta^2(\beta K+r)}{r(\mu+\gamma+\alpha)}SI+B\\&\le -\frac{\mu+\gamma+\alpha}{4}I^{\theta+1}+C\\&\le -\frac{\mu+\gamma+\alpha}{4\epsilon^{\theta+1}}+C\end{aligned}$$

$$\leq -1, \tag{4.14}$$

which follows from (4.9). Hence

$$L\bar{V} \leq -1 \text{ for any } (S, I) \in D_\epsilon^4.$$

Obviously, from (4.10), (4.11), (4.12) and (4.14), we can obtain that for a sufficiently small ϵ,

$$L\bar{V}(S, I) \leq -1 \text{ for any } (S, I) \in \mathbb{R}_+^2 \setminus D_\epsilon.$$

So the condition A_2 in Lemma 4.1 holds. In view of Lemma 4.1, we can obtain that system (1.2) admits a unique ergodic stationary distribution $\pi(\cdot)$. This completes the proof. □

Remark 4.1. According to Theorems 3.1 and 4.1, we can find that the value of R_0^S mainly determines the extinction and persistence of the disease. If $R_0^S > 1$, there is a unique ergodic stationary distribution $\pi(\cdot)$ of system (1.2) which implies that the disease will be stochastic persistent a.s., while if $R_0^S < 1$, the disease will be extinct a.s. So we consider R_0^S as the threshold of system (1.2).

Remark 4.2. From the expression for R_0^S and the discussions in Sections 3 and 4, it can be seen that the disease is more prone to die out when the transmission coefficient β and carrying capacity of susceptible host individuals K are smaller, the death rates (both disease-induced α and natural death rate μ) and recovery rate γ are larger, and the intensities of stochastic perturbations σ_1^2, σ_2^2 are greater. Conversly, the disease is more likely to reach a weakly stable state and persist.

Remark 4.3. From the expression for R_0^S, it is evident that, unlike the basic SIR model, the carrying capacity K in the model with logistic growth exerts a strong influence on disease extinction or persistence. An insufficient carrying capacity can also lead to disease extinction.

5. Examples and numerical simulations

In this section, we introduce some numerical simulations to illustrate our theoretical results. We numerically simulate the solution of system (1.2) with the initial value $(S(0), I(0)) = (0.6, 0.8)$. For the numerical simulations, we use Milstein's Higher Order Method mentioned in [15] to obtain the discretization equations of system (1.2):

$$\begin{cases} S_{j+1} = S_j + \left[rS_j \left(1 - \dfrac{S_j + I_j}{K} \right) - \beta S_j I_j \right] \Delta t + \sigma_1 S_j \sqrt{\Delta t} \nu_{1,j} + \dfrac{\sigma_1^2}{2} S_j (\nu_{1,j}^2 - 1) \Delta t, \\ I_{j+1} = I_j + [\beta S_j I_j - (\mu + \gamma + \alpha) I_j] \Delta t + \sigma_2 I_j \sqrt{\Delta t} \nu_{2,j} + \dfrac{\sigma_2^2}{2} I_j (\nu_{2,j}^2 - 1) \Delta t, \end{cases}$$

where the time increment $\Delta t > 0$, $\sigma_i^2 > 0$ $(i = 1, 2)$ denote the intensities of the white noise, $\nu_{i,j}$ $(i = 1, 2)$ denote independent Gaussian random variables which follow the distribution $N(0, 1)$ for $j = 1, 2, \cdots, n$.

Example 5.1. In order to verify the extinction of the disease, we choose the values of the system parameters as follows: $r = 0.6$, $K = 0.9$, $\beta = 0.9$, $\mu = 0.2$, $\gamma = 0.2$, $\alpha = 0.1$, $\sigma_1^2 = 0.35$ and $\sigma_2^2 = 0.4$. By a simple calculation, we obtain $r = 0.6 > 0.175 = \frac{\sigma_1^2}{2}$ and $R_0^S = \frac{\beta K (r - \frac{\sigma_1^2}{2})}{r(\mu + \gamma + \alpha + \frac{\sigma_2^2}{2})} =$

$0.8196 < 1$. In other words, the condition of Theorem 3.1 is satisfied. In view of Theorem 3.1, we obtain the result that the disease dies out exponentially with probability one. The simulation results presented in Figure 1 serve to validate our theoretical findings.

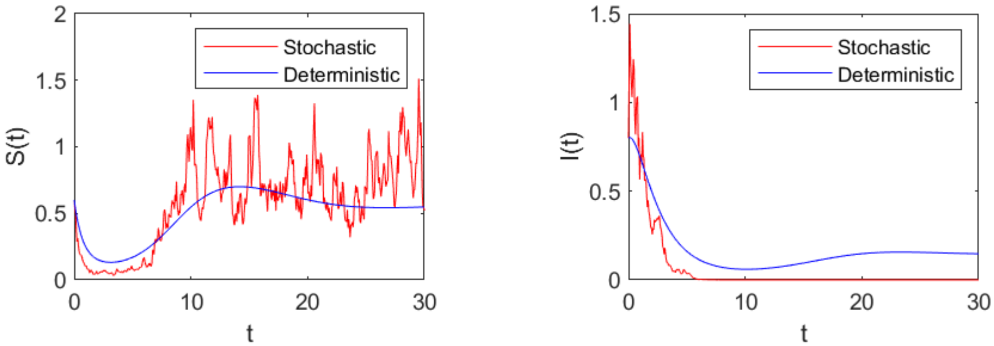

Figure 1. The left and right figures represent the paths of $S(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), I(0)) = (0.6, 0.8)$ under the noise intensities $\sigma_1^2 = 0.35$ and $\sigma_2^2 = 0.4$.

Example 5.2. To confirm the extinction of the disease, we use the same parameter values as in Example 5.1, but with $\sigma_1^2 = 0.6$ and $\sigma_2^2 = 0.6$. A straightforward calculation shows that $r = 0.6 > 0.3 = \frac{\sigma_1^2}{2}$ and $R_0^S = \frac{\beta K(r - \frac{\sigma_1^2}{2})}{r(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2})} = 0.5062 < 1$. That is to say, the condition of Theorem 3.1 is still satisfied. Then we derive the conclusion that the disease dies out exponentially with probability one. The simulation results presented in Figure 2 support our findings.

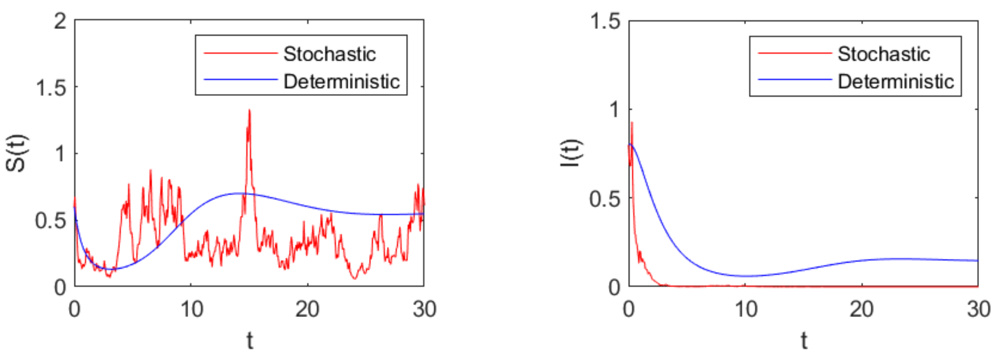

Figure 2. The left and right figures represent the paths of $S(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), I(0)) = (0.6, 0.8)$ under the noise intensities $\sigma_1^2 = 0.6$ and $\sigma_2^2 = 0.6$.

Remark 5.1. These two simulation results indicate that, given the same other parameters, a larger environmental noise leads to a smaller R_0^S and a faster extinction of the infectious disease.

Example 5.3. In order to check the existence of an ergodic stationary distribution, we choose the same parameter values as those in Example 5.1, except with $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.3$. Direct calculation leads to $r = 0.6 > 0.05 = \frac{\sigma_1^2}{2}$ and $R_0^S = \frac{\beta K(r - \frac{\sigma_1^2}{2})}{r(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2})} = 1.1423 > 1$. In other words, the condition of Theorem 4.1 holds. Therefore we can get that there is an ergodic stationary distribution $\pi(\cdot)$ of system (1.2). We provide simulation results in Figure 3.

Table 1. Parameter values for the examples.

Symbol	Explanation	Value1	Value2	Value3	Value4
r	Intrinsic birth rate	0.6	0.6	0.6	0.6
K	Carrying capacity	0.9	0.9	0.9	0.9
β	Disease transmission coefficient	0.9	0.9	0.9	0.9
μ	Natural death rate	0.2	0.2	0.2	0.2
γ	Rate of recovery	0.2	0.2	0.2	0.2
α	Disease-caused death rate	0.1	0.1	0.1	0.1
σ_1^2	Noise intensity of $S(t)$	0.35	0.6	0.1	0.05
σ_2^2	Noise intensity of $I(t)$	0.4	0.6	0.3	0.05
R_0^S	Threshold of the stochastic system	0.8196	0.5062	1.1423	1.4786
result		extinction	extinction	persistence	persistence

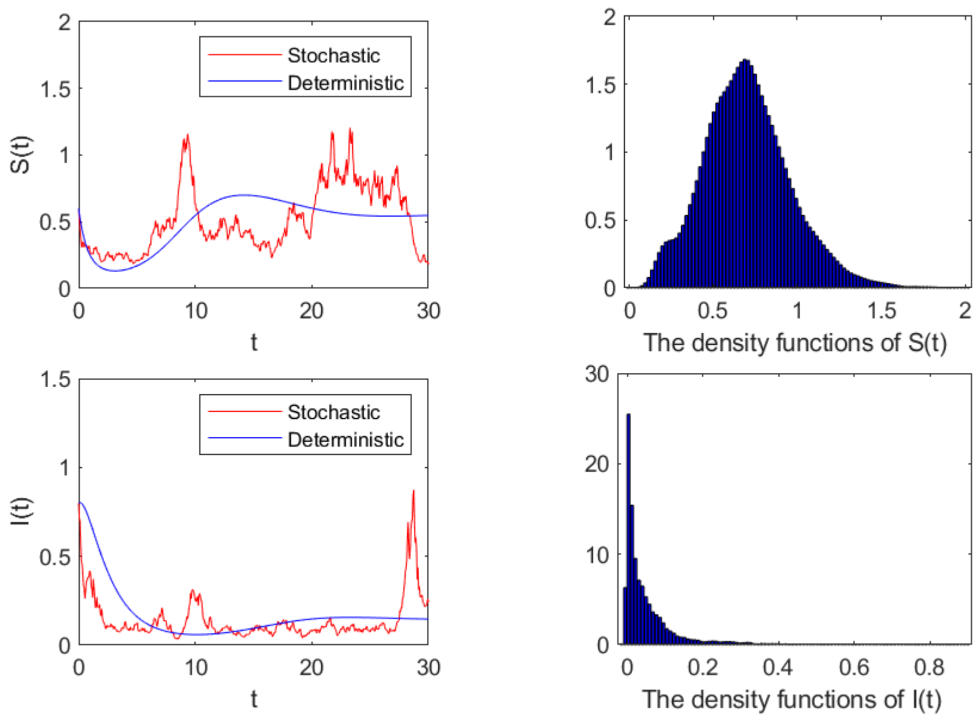

Figure 3. The left column shows the paths of $S(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), I(0)) = (0.6, 0.8)$ under the noise intensities $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.3$. The right column displays the histogram of the probability density functions of $S(t)$ and $I(t)$ with values of $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.3$.

Example 5.4. To check the existence of an ergodic stationary distribution, we select the same parameter values as those in Example 5.1, but with both σ_1^2 and σ_2^2 equal to 0.05. Direct calculation leads to $r = 0.6 > 0.025 = \frac{\sigma_1^2}{2}$ and $R_0^S = \frac{\beta K(r-\frac{\sigma_1^2}{2})}{r(\mu+\gamma+\alpha+\frac{\sigma_2^2}{2})} = 1.4786 > 1$. By Theorem 4.1, we can get that there is an ergodic stationary distribution $\pi(\cdot)$ of system (1.2). We give the simulations displayed in Figure 4 to support our results.

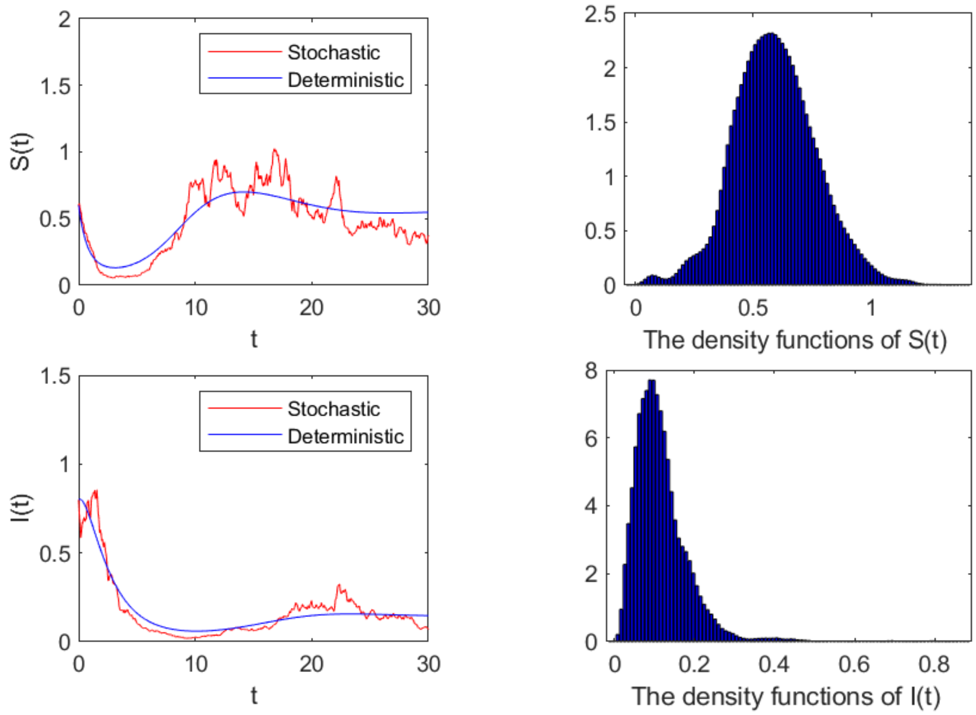

Figure 4. The left column displays the paths of $S(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), I(0)) = (0.6, 0.8)$ under the noise intensities $\sigma_1^2 = 0.05$ and $\sigma_2^2 = 0.05$. The right column shows the histogram of the probability density functions of $S(t)$ and $I(t)$ with values of $\sigma_1^2 = 0.05$ and $\sigma_2^2 = 0.05$.

Remark 5.2. Example 5.3 and Example 5.4 show that when $R_0^S > 1$, smaller environmental noise as well as a larger R_0^S drive the stochastic paths of $S(t)$ and $I(t)$ closer to the paths of the deterministic model (1.1).

6. Concluding remarks and future directions

In this paper, we considered a stochastic SIR epidemic model with logistic growth. We first discussed the existence of the global positive solution to model (1.2). The main result of this paper was to obtain sufficient conditions for the threshold dynamics of the stochastic model (1.2) which were governed by a parameter R_0^S. Precisely, the disease died out exponentially and the distribution of susceptible individuals converged weakly to a unique invariant probability measure provided that $R_0^S < 1$; whereas if $R_0^S > 1$, we proved that there was a unique ergodic stationary distribution of the positive solutions to model (1.2) by constructing a suitable stochastic Lyapunov function. Our analysis shows that the basic reproduction number R_0^S for the stochastic model is smaller than its deterministic counterpart R_0 from system (1.1). This reduction makes the disease less likely to reach an endemic state. Furthermore, we can also see that an increase in the noise intensity may lead to the extinction of the disease. The results of this study show that environmental noise suppresses the spread of the disease.

Some interesting topics deserve further investigation. On the one hand, one can further consider the optimal control of the epidemic model, see e.g. [12, 35]. For example, in [12], by introducing two control variables, the author identified the optimal control strategy and demonstrated that this strategy is highly effective in reducing the scale of infection and is cost-efficient. On the other hand, one can extend the results in this work by considering a second order stochastic perturbations [32], or some discontinuous perturbations such as Lévy noise [38]. In addition, the method used in this paper can be also applicable to investigate other biological models, such as HIV-1 infection model with logistic growth, predator-prey model with logistic growth, etc. Furthermore, it is noteworthy that in recent years, fractional systems, by virtue of their intrinsic memory and hereditary properties, have exerted a profound impact and garnered increasingly extensive application across numerous domains, notably in the modeling of biological systems [22, 24] and environmental and climate change [20, 23]. In addition, there is also extensive theoretical research on fractional calculus and its applications in other fields [1,9,11,25,33,40–43,45,48]. Fractional versions of infectious disease models can also be developed, see e.g. [3, 39]. We leave these problems for our future work.

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