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EXPLORING THE DOUBLE ARA-KAMAL TRANSFORM FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract In this article, we explore the application of the double ARA-Kamal transformation (DARA-KT) to the nonlocal fractional Caputo derivative operator. Our investigation yields intriguing findings, particularly in solving certain classes of fractional partial differential equations (FPDEs). We delve into various physical scenarios, including Wave, Klein-Gordon, and Fokker-Planck equations, analyzing their implications. The incorporation of the DARA-KT technique proves to be both efficient and precise in deriving exact solutions for FPDEs. To demonstrate the practicality of our approach, we present some numerical examples and figures.

Keywords ARA transform, Kamal transform, double ARA-Kamal transform, fractional partial differential equations.

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1. Introduction and preliminaries

Fractional calculus extends the concepts of differentiation and integration to noninteger orders, offering a powerful framework for understanding various physical phenomena, engineering applications, and scientific disciplines. Its utility spans diverse fields such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population dynamics, optics, and signal processing. Notably, fractional calculus techniques have emerged as indispensable tools for addressing (long) memory effects inherent in many systems. Various definitions of fractional derivatives exist, including the Riemann-Liouville, Caputo, Caput-Fabrizio, Atangana-Baleanu, conformable, and generalized fractional derivatives. These definitions provide versatile mathematical tools that enable researchers to tackle intricate properties of systems with fractional dynamics, offering insights into complex behaviors and phenomena across different domains of applied sciences [1, 4, 5, 8, 11, 17, 29].

Fractional partial differential equations feature prominently in a wide array of scientific disciplines, including chemistry, physics, engineering, and mathematics. Consequently, researchers have developed numerous techniques for effectively solving these equations. These methods

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include the homotopy perturbation method, variation iteration method, Adomian decomposition method, finite difference method, among others. Through the application of these diverse methodologies, researchers can address complex problems arising in real-world scenarios and gain deeper insights into the dynamics of fractional systems across various scientific domains [10, 16, 18, 24, 33, 37–39, 41].

A recent development in this field involves the integration of established methods with integral transforms such as the Laplace transform, Sumudu transform, Elzaki transform, and ARA transform. These amalgamations have given rise to novel approaches, including the Laplace decomposition method, Laplace variation iteration method, Sumudu decomposition method, Sumudu homotopy perturbation method, Elzaki variation iteration method, Elzaki project differential transform method, Elzaki homotopy perturbation method, Elzaki decomposition method, ARA residual power series method, among others. These innovative techniques offer enhanced capabilities for solving both linear and nonlinear fractional partial differential equations (FPDEs), contributing significantly to the advancement of research in this area [6, 7, 12–15, 21, 22, 27, 30, 32, 36, 40].

Recent research has been actively exploring the method of double integral transforms, which revolves around applying a single transformation twice to functions of two variables or employing two distinct transformations on the same function. This innovative approach has garnered significant attention as a potent tool for solving partial differential equations (PDEs). Despite being a relatively recent area of study, the properties, theorems, and applications of double integral transformations have piqued the interest of numerous mathematicians, highlighting its promising potential in advancing the field of mathematical analysis.

Consequently, a multitude of researchers have delved into exploring innovative combinations, such as the double Laplace transform, double Sumudu transform, double Elzaki transform, double Laplace-Sumudu transform, and various others. These endeavors, documented in recent literature, signify a concerted effort to harness the potential of dual integral transformations for tackling complex mathematical problems and advancing the realm of applied mathematics [2, 3, 9, 19, 20, 23, 25, 28, 35].

In a recent publication [31], a novel fusion between the ARA transform and the Kamal transform was introduced. This innovative combination is presented as follows:

$$\mathcal{A}_x K_t[g(x,t)] = G(u,v) = u\int_0^\infty\int_0^\infty e^{-(ux+\frac{t}{v})}g(x,t)dxdt,$$

where $g(x,t)$ is a continuous function of two variables $t>0$ and $x>0$.

In this Study, we implement DARA-KT to solve families of FPDEs of the form:

$$A\ D_x^{\xi} g(x,t) + B\ D_t^{\zeta} g(x,t) + C\ Lg(x,t) = z(x,t), \quad x,t \geq 0, \tag{1.1}$$

$n-1<\xi\leq n, m-1<\zeta\leq m$ and $m,n\in\mathbb{N}$.

The initial conditions (ICs) are given by:

$$\frac{\partial^j g(x,0)}{\partial t^j} = f_j(x),\ \ j=0,1,2,....,m-1, \tag{1.2}$$

and the boundary conditions (BCs)

$$\frac{\partial^i g(0,t)}{\partial x^i} = h_i(t),\ \ i=0,1,2,....,n-1, \tag{1.3}$$

where A, B and C are real constants, D_x^{ξ}, and D_t^{ζ} are the fractional Caputo's derivatives with respect to x and t, respectively, L is a linear operator and $z(x,t)$ is the source function.

The primary objective of this study is to broaden the scope of applications for the DARA-KT method by employing it in the solution of fractional partial differential equations (FPDEs). We demonstrate the effectiveness of this approach by applying the DARA-KT technique to various intriguing scenarios, enabling the derivation of exact solutions and subsequent analysis of the results. The novelty of our work lies in the development of a straightforward formula for solving PDEs with fractional orders. This newly established formula offers simplicity and practicality, as demonstrated through its application in solving significant FPDEs within diverse contexts.

The structure of this article unfolds as follows: The subsequent two sections delve into fundamental definitions and theorems pertinent to our research. Section 4 introduces a novel algorithm tailored for solving families of FPDEs utilizing the DARA-KT method. To exemplify the efficacy of our proposed technique, Section 5 offers several illustrative examples. Lastly, Section 6 engages in a comprehensive discussion of our findings.

2. Kamal and ARA transformations

In this part, we present the definitions of the Kamal and ARA transforms, along with some of their key properties.

Definition 2.1. [26] Kamal transform of the function $g(x)$ is given by

$$K[g(x)] = \int_0^{\infty} e^{-(\frac{x}{v})} g(x)dx,\ x \geq 0, K_1 \leq v \leq K_2,$$

where K is Kamal transform operator.

Definition 2.2. [34] ARA transform of order n of the function $g(x)$ is defined as

$$\mathcal{A}_n[g(x)](u) = G(n,u) = u\int_0^{t} t^{n-1} e^{-(ux)} g(x)dx,\ u > 0, n \in \mathbb{N},$$

and the ARA transform of the function $g(x)$ of order one is given by:

$$\mathcal{A}_1[g(x)] = u\int_0^{\infty} e^{-(ux)} g(x)dx,\ u > 0.$$

For the sake of simplicity, we will denote $\mathcal{A}_1[g(x)]$ by $\mathcal{A}[g(x)]$.

3. Basic definitions and theorems of DARA-KT

This section introduces the definition of the DARA-KT for functions of two variables, along with the conditions for its existence and some fundamental properties of this novel double transform.

Definition 3.1. [31] Let $g(x,t)$ be a continuous function of two variables $x > 0$ and $t > 0$ then the DARA-Kamal transform of the function $g(x,t)$ is given by

$$\mathcal{A}_x K_t[g(x,t)] = G(u,v) = u\int_0^{\infty}\int_0^{\infty} e^{-(ux+\frac{t}{v})} g(x,t)dxdt,$$

Table 1. Presents the fundamental properties of ARA and Kamal transforms.

$F(x)$	$\mathcal{A}[f(x) = F(u)]$	$K[f(x)] = F(v)$
1	1	v
x^a	$\dfrac{\Gamma(a+1)}{u^a}$	$\Gamma(a+1)v^{a+1}$
e^{ax}	$\dfrac{u}{u-a}$	$\dfrac{v}{1-av}$
$\sin(ax)$	$\dfrac{a\,u}{u^2+a^2}$	$\dfrac{a\,v^2}{1+a\,v^2}$
$\cos(ax)$	$\dfrac{u^2}{u^2+a^2}$	$\dfrac{v}{1+a\,v^2}$
$\sinh(ax)$	$\dfrac{a\,u}{u^2-a^2}$	$\dfrac{a\,v^2}{1-a\,v^2}$
$\cosh(ax)$	$\dfrac{u^2}{u^2-a^2}$	$\dfrac{v}{1-a\,v^2}$

provided the integral exists.

Clearly, the DARA-KT is linear, since

$$\mathcal{A}_x K_t[z_1 g_1(x,t) + z_2 g_2(x,t)] = z_1 G_1(u,v) + z_2 G_2(u,v),$$

where z_1 and z_2 are constants.

The inverse of the DARA-KT is expressed as:

$$\begin{aligned} G(x,t) =& \mathcal{A}_x^{-1} K_t^{-1}[g(x,t)] \\ =& \frac{1}{(2\pi i)^2} \int_{\xi - i\infty}^{\xi + i\infty} e^{-ux} \int_{\zeta - i\infty}^{\zeta + i\infty} e^{\left(\frac{-t}{v}\right)} G\left(u, \frac{1}{v}\right) dv du. \end{aligned}$$

Definition 3.2. The Caputo derivatives of orders ξ and ζ of the function $g(x,t)$ with respect to x and t, respectively, are expressed as follows:

$$D_x^{\xi} g(x,t) = \frac{\partial^{\xi} g(x,t)}{\partial x^{\xi}} = \begin{cases} \dfrac{1}{\Gamma(n-\xi)} \displaystyle\int_0^x (x-\phi)^{n-\xi-1} \frac{\partial^n g(\phi,t)}{\partial \phi^n} d\phi, & n-1 < \xi < n, n \in \mathbb{N}, \\ \dfrac{\partial^n g}{\partial x^n}, \quad n = \xi, \end{cases}$$

$$D_t^{\zeta} g(x,t) = \frac{\partial^{\zeta} g(x,t)}{\partial t^{\zeta}} = \begin{cases} \dfrac{1}{\Gamma(m-\zeta)} \displaystyle\int_0^t (t-\tau)^{m-\zeta-1} \frac{\partial^m g(x,\tau)}{\partial \tau^m} d\tau, & m-1 < \zeta < m, m \in \mathbb{N}, \\ \dfrac{\partial^m g}{\partial x^m}, \quad m = \zeta. \end{cases}$$

Definition 3.3. The Mittag-Leffler function is given by:

$$E_{\xi,\zeta}(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\xi r + \zeta)}, \quad (\zeta, \xi \in \mathbb{C}, \Re(\zeta), \Re(\xi) > 0).$$

The single ARA transform of $x^{\zeta-1}E_{\xi,\zeta}(\lambda x^{\xi})$ takes the value:

$$\mathcal{A}_x[x^{\zeta-1}E_{\xi,\zeta}(\lambda x^{\xi})] = \frac{u^{\xi-\zeta+1}}{u^{\xi}-\lambda}, \qquad |\lambda| < |u^{\xi}|.$$

The single Kamal transform of $t^{\zeta-1}E_{\xi,\zeta}(\lambda t^{\xi})$ takes the value:

$$K_t[t^{\zeta-1}E_{\xi,\zeta}(\lambda x^{\xi})] = \frac{v^{\zeta}}{1-\lambda v^{\xi}}, \qquad |\lambda| < |v^{\xi}|.$$

Table 2. Illustrates the values of DARA-KT for some basic functions.

$f(x,t)$	$\mathcal{A}_x k_t[f(x,t)] = F(u,v)$
1	v
$x^n y^m$	$u^{-n}\Gamma(n+1)v^{m+1}\Gamma(m+1)$
$e^{\alpha x+\beta y}$	$\frac{uv}{(u-\alpha)(1-\beta v)}$
$\sin(nx+my)$	$\frac{u(uv^2m+nv)}{(u^2+n^2)(1+m^2v^2)}$
$\cos(nx+my)$	$\frac{u(uv-nmu^2)}{(u^2+n^2)(1+m^2v^2)}v$
$\sinh(nx+my)$	$\frac{u(uv^2m+nv)}{(u^2-n^2)(m^2v^2-1)}$
$\cosh(nx+my)$	$\frac{u(uv-nmu^2)}{(u^2-n^2)(m^2v^2-1)}$

Theorem 3.1. *Let $g(x,t)$ be a continuous function in every finite intervals $(0,M)$ and $(0,N)$ and of exponential order $e^{\lambda x+\eta t}$, then the double ARA-Kamal transform of function $f(x,t)$ exists for all $u>\lambda$ and $u>\eta$ we have*

$$|g(x,t)| \leq Pe^{(\lambda x+\eta y)}, \ \forall x > M, y < N.$$

Theorem 3.2. (Derivatives properties) [31] *If $\mathcal{A}_x K_t[g(x,t)] = G(u,v)$, then:*

(i)

$$\mathcal{A}_x K_t\left[\frac{\partial g(x,t)}{\partial x}\right] = uG(u,v) - uk[g(0,t)];$$

(ii)

$$\mathcal{A}_x K_t\left[\frac{\partial g(x,t)}{\partial t}\right] = \frac{1}{u}G(u,v) - A[g(x,0)];$$

(iii)

$$\mathcal{A}_x K_t\left[\frac{\partial^2 g(x,t)}{\partial x^2}\right] = -uK\left[g_x(0,t)\right] - u^2 K\left[g(0,t)\right] + v^2 G(u,v);$$

(iv)

$$\mathcal{A}_x K_t\left[\frac{\partial^2 g(x,t)}{\partial t^2}\right] = -\mathcal{A}_x[g_t(x,0)] - \frac{1}{v}\mathcal{A}_x[g(x,0)] + \frac{1}{v^2}G(u,v).$$

Theorem 3.3. *Let $G(u,v) = \mathcal{A}_x K_t\big[g(x,t)\big]$. Then,*

$$\mathcal{A}_x K_t[g(x-\delta,t-\epsilon)H(x-\delta,t-\epsilon)] = e^{-u\delta-\frac{\epsilon}{v}}G(u,v).$$

Where $H(x,t)$ represents the Heaviside unit step function, defined by:

$$H(x-\delta,t-\epsilon) = \begin{cases} 1, & x>\delta, t>\epsilon, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} \mathcal{A}_x K_t[g(x-\delta,t-\epsilon)H(x-\delta,t-\epsilon)] &= u\int_0^\infty\int_0^\infty e^{-ux-\frac{t}{v}}\big(g(x-\delta,t-\epsilon)H(x-\delta,t-\epsilon)\big)dxdt \\ &= u\int_0^\infty\int_0^\infty e^{-ux-\frac{y}{v}}g(x-\delta,t-\epsilon)dxdt. \end{aligned} \tag{3.1}$$

By letting $x-\delta = p$ and $t-\epsilon = q$ in equation (3.1), we obtain

$$\mathcal{A}_x K_t[g(x-\delta,t-\epsilon)H(x-\delta,t-\epsilon)] = u\int_0^\infty\int_0^\infty e^{-u(\delta+p)-\frac{\epsilon+q}{v}}g(p+q)dpdq. \tag{3.2}$$

Equation (3.2) can be simplified into

$$\begin{aligned} \mathcal{A}_x K_t[g(x-\delta,t-\epsilon)H(x-\delta,t-\epsilon)] &= e^{-u\delta-\frac{\epsilon}{v}}\left(u\int_0^\infty\int_0^\infty e^{-up-\frac{q}{v})}g(p+q)dpdq\right) \\ &= e^{-u\delta-\frac{\epsilon}{v}}G(u,v). \end{aligned}$$

□

Theorem 3.4. (Convolution theorem) *If $\mathcal{A}_x K_t[g(x,t)] = F(u,v)$ and $\mathcal{A}_x K_t[g(x,t)] = G(u,v)$ then:*

$$\mathcal{A}_x K_t[(f*g)(x,t)] = \frac{1}{u}F(u,v)\cdot G(u,v),$$

where

$$\mathcal{A}_x K_t[f*g(x,t)] = \int_0^x\int_0^y f(x-\delta,t-\epsilon)g(\delta,\epsilon)d\delta d\epsilon.$$

Proof.

$$\begin{aligned} \mathcal{A}_x K_t[f*g(x,t)] &= u\int_0^\infty\int_0^\infty e^{-ux-\frac{t}{v}}\cdot(f*g)(x,t)dxdt \\ &= u\int_0^\infty\int_0^\infty e^{-ux-\frac{t}{v}}\left(\int_0^x\int_0^t f(x-\delta,t-\epsilon)g(\delta,\epsilon)d\delta d\epsilon\right)dxdt. \end{aligned} \tag{3.3}$$

Equation (3.3) can be rewritten using the Heaviside unit step function as:

$$\mathcal{A}_x K_t[f*g(x,t)] = u\int_0^\infty\int_0^\infty e^{-ux-\frac{t}{v}}\left(\int_0^x\int_0^t f(x-\delta,t-\epsilon)g(x-\delta,t-\epsilon)d\delta d\epsilon\right)dxdt.$$

Thus

$$\begin{aligned}\mathcal{A}_x K_t[f * g(x,t)] &= \int_0^\infty \int_0^\infty g(\delta,\epsilon) d\delta d\epsilon \left(u \int_0^\infty \int_0^\infty e^{-ux-\frac{t}{v}} f(x-\delta, t-\epsilon) g(x-\delta, t-\epsilon) dx dt \right) \\ &= \int_0^\infty \int_0^\infty g(\delta,\epsilon) d\delta d\epsilon \left(e^{-ux-\frac{t}{v}} \cdot F(u,v) \right) \\ &= F(u,v) \int_0^\infty \int_0^\infty g(\delta,\epsilon) e^{-ux-\frac{t}{v}} d\delta d\epsilon \\ &= \frac{1}{u} F(u,v) \cdot G(u,v).\end{aligned}$$

□

4. Algorithm of DARA-KT method

In this section, we introduce the technique of using DARA-KT to solve families of fractional partial differential equations (FPDEs). To accomplish this, we first calculate the DARA-KT for the nonlocal Caputo fractional derivative, as stated in the following lemma.

4.1. DARA-KT of fractional derivatives

Lemma 4.1. *The DARA-KT for Caputo fractional derivatives can expressed as:*

(i) $\mathcal{A}_x K_t[D_x^\xi f(x,t)] = u^\xi F(u,v) - \sum_{i=0}^{n-1} u^{\xi-i} K_t\left[\frac{\partial^i f(0,t)}{\partial x^i}\right], \quad n-1 < \xi < n.$

(ii) $\mathcal{A}_x K_t[D_t^\zeta f(x,t)] = v^{-\zeta} F(u,v) - \sum_{j=0}^{m-1} v^{j-\xi-1} \mathcal{A}_x\left[\frac{\partial^j f(x,0)}{\partial t^j}\right], \quad m-1 < \zeta < m.$

Proof. Proof of Lemma 1. i. Applying DARA-KT on $D_x^\xi f(x,t)$, we get

$$\mathcal{A}_x K_t[D_x^\xi f(x,t)] = \mathcal{A}_x K_t\left[\frac{1}{\Gamma(n-\xi)} \int_0^x (x-\phi)^{n-\xi-1} \frac{\partial^n f(\phi,t)}{\partial \phi^n} d\phi\right].$$

From the definition of convolution, we obtain the following:

$$\begin{aligned}\mathcal{A}_x K_t[D_x^\xi f(x,t)] &= \mathcal{A}_x K_t\left[\frac{1}{\Gamma(n-\xi)}\left(x^{n-\xi-1} * \frac{\partial^n f(x,t)}{\partial x^n}\right)\right] \\ &= K_t\left[\frac{1}{\Gamma(n-\xi)} \mathcal{A}_x\left(x^{n-\xi-1} * \frac{\partial^n f(x,t)}{\partial x^n}\right)\right].\end{aligned}$$

By utilizing the convolution property of the ARA transform, we obtain:

$$\mathcal{A} K_t[D_x^\xi f(x,t)] = K_t\left[\frac{1}{\Gamma(n-\xi)}\left(\frac{1}{u}\mathcal{A}\left[x^{n-\xi-1}\right]\mathcal{A}\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]\right)\right].$$

By applying the derivative property of the ARA transform, we obtain:

$$\mathcal{A}_x K_t[D_x^\xi f(x,t)] = \frac{1}{\Gamma(n-\xi)} K_t\left[\frac{\Gamma(n-\xi)}{u^{n-\xi}}\left(u^n \mathcal{A}_x\left[f(x,t)\right] - u^n f(0,t)\right.\right.$$

$$- \dots - u\frac{\partial^{n-1}f(0,t)}{\partial x^{n-1}})\Big].$$

After performing some straightforward calculations, we obtain:

$$\begin{aligned}\mathcal{A}_xK_t[D_x^{\xi}f(x,t)] &= u^{\xi}F(u,v) - u^{\xi}K_t\big[f(0,t)\big] - \dots - u^{\xi-n+1}K_t\left[\frac{\partial^{n-1}f(0,t)}{\partial x^{n-1}}\right] \\ &= u^{\xi}F(u,v) - \sum_{i=0}^{n-1} u^{\xi-i}K_t\left[\frac{\partial^{i}f(0,t)}{\partial x^{i}}\right].\end{aligned}$$

ii. Applying DARA-KT on $D_t^{\zeta}f(x,t)$, we obtain

$$\mathcal{A}_xK_t[D_t^{\zeta}f(x,t)] = \mathcal{A}_xK_t\left[\frac{1}{\Gamma(m-\zeta)}\int_0^t (t-\tau)^{m-\zeta-1}\frac{\partial^m f(x,\tau)}{\partial\tau^m}d\tau\right].$$

From the definition of convolution, we obtain the following:

$$\begin{aligned}\mathcal{A}_xK_t[D_t^{\zeta}f(x,t)] &= \mathcal{A}_xK_t\left[\frac{1}{\Gamma(m-\zeta)}\left(t^{m-\zeta-1} * \frac{\partial^m f(x,t)}{\partial t^m}\right)\right] \\ &= \mathcal{A}_x\left[\frac{1}{\Gamma(m-\zeta)}K_t\left(t^{m-\zeta-1} * \frac{\partial^m f(x,t)}{\partial t^m}\right)\right].\end{aligned}$$

By utilizing the convolution property of the Kamal transform, we obtain

$$\mathcal{A}_xK_t[D_t^{\zeta}f(x,t)] = \mathcal{A}_x\left[\frac{1}{\Gamma(m-\zeta)}\left(\frac{1}{v}K_t\left[t^{m-\zeta-1}\right]K_t\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]\right)\right].$$

Applying the derivative property of Kamal transform, we obtain

$$\begin{aligned}\mathcal{A}_xK_t[D_t^{\zeta}f(x,t)] =& \frac{1}{\Gamma(m-\zeta)}\mathcal{A}_x\Big[\frac{\Gamma(m-\zeta)}{v^{\zeta-m}}\big(v^{-m}K_t\big[f(x,t)\big] - v^{-m}f(x,0) \\ & - \dots - \frac{\partial^{m-1}f(x,0)}{\partial t^{m-1}})\Big].\end{aligned}$$

After performing some straightforward calculations, we obtain:

$$\begin{aligned}\mathcal{A}_xK_t[D_t^{\zeta}f(x,t)] &= v^{-\zeta}F(u,v) - v^{\zeta}\mathcal{A}_x\big[f(x,0)\big] - \dots - v^{m-\zeta-1}\mathcal{A}_x\left[\frac{\partial^{m-1}f(x,0)}{\partial t^{m-1}}\right] \\ &= v^{-\zeta}F(u,v) - \sum_{j=0}^{m-1} v^{j-\xi-1}\mathcal{A}_x\left[\frac{\partial^{j}f(x,0)}{\partial t^{j}}\right].\end{aligned}$$

□

4.2. Solving FPDEs by DARA-KT

In this section, we apply DARA-KT to derive solutions for certain fractional partial differential equations (FPDEs). We focus on the initial boundary value problems described in equations

(1.1)-(1.3). To find the solution using this new approach, we apply DARA-KT to both sides of equation (1.1), resulting in:

$$\mathcal{A}_x K_t\Big[A\big[D_x^{\xi} f(x,t)\big] + \mathcal{A}_x K_t\big[BD_t^{\zeta} f(x,t)\big]\Big] + \mathcal{A}_x K_t\big[CL\big[f(x,t)\big]\big] = \mathcal{A}_x K_t\big[Z(x,t)\big],$$

which implies

$$A\Big(u^{\xi}F(u,v) - \sum_{i=0}^{n-1} u^{\xi-i} K_t\Big[\frac{\partial^i f(0,t)}{\partial x^i}\Big]\Big) + B\Big(v^{-\zeta}F(u,v) - \sum_{j=0}^{m-1} v^{j-\xi+1}\mathcal{A}_x\Big[\frac{\partial^j f(x,0)}{\partial t^j}\Big]\Big)$$
$$+ C\mathcal{A}_x K_t\big[L\big[f(x,t)\big]\big] = Z(u,v). \tag{4.1}$$

Additionally, we apply the single ARA transform to the initial conditions (1.3) and the single Kamal transform to the boundary conditions (1.2), resulting in:

$$\mathcal{A}_x\Big[\frac{\partial^j f(x,0)}{\partial t^j}\Big] = G\big[g_j(x)\big] = G_j(u),\ \forall j = 1,2,3,....,m-1, \tag{4.2}$$

$$K_t\Big[\frac{\partial^i f(0,t)}{\partial x^i}\Big] = K\big[h_i(t)\big] = H_i(v),\ \forall i = 1,2,3,....,n-1. \tag{4.3}$$

By simplifying equation (4.1) and substituting the values from equations (4.2) and (4.3), we obtain:

$$F(u,v) = \frac{1}{Au^{\xi} + Bv^{-\zeta}}\big(A\sum_{i=0}^{n-1} u^{\xi-i}H_i(v) + B\sum_{j=0}^{m-1} v^{-\zeta+j+1}G_j(u)$$
$$- C\mathcal{A}_x K_t\big[L\big[f(x,t)\big]\big] + Z(u,v)\big). \tag{4.4}$$

Applying the inverse DARA-KT,$\mathcal{A}_x^{-1}K_t^{-1}$ to both sides of equation (4.4), we obtain:

$$f(x,t) = \mathcal{A}_x^{-1}K_t^{-1}\big[\frac{1}{Au^{\xi} + Bv^{-\zeta}}\big(A\sum_{i=0}^{n-1} u^{\xi-i}H_i(v) + B\sum_{j=0}^{m-1} v^{-\zeta+j+1}G_j(u)$$
$$- C\mathcal{A}_x K_t\big[L\big[f(x,t)\big]\big] + Z(u,v)\big)\big], \tag{4.5}$$

which provides the solution to the target problem.

5. Illustrative examples

In this section, we present several well-known partial differential equations (PDEs) from mathematical physics, including the reaction–diffusion equation, advection–diffusion equation, wave equation, Klein–Gordon equation, and Fokker–Planck equation. The primary aim is to demonstrate the applicability and efficiency of the new double transform method. We apply this approach to obtain solutions for these equations and implement the formula derived in equation (4.5) to solve fractional PDEs (FPDEs), showcasing its simplicity and versatility.

5.1. Fractional reaction-diffusion equation

Consider the fractional reaction-diffusion equation:

$$A\ D_x^{\xi} f(x,t) + B\ D_t^{\zeta} f(x,t) + C\ f(x,t) = 0, \quad 1 < \xi \leq 2,\ 0 < \zeta \leq 1, \tag{5.1}$$

with the initial condition (IC):

$$f(x,0) = g_0(x), \tag{5.2}$$

and the boundary conditions (BCs):

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{5.3}$$

Applying the single ARA transform to $g_0(x)$ in equation (5.2), we get:

$$G_0(u) = \mathcal{A}_x[g_0(x)].$$

Applying the single Kamal transform to $h_0(t)$ and $h_1(t)$ in equation (5.3), we get:

$$H_0(v) = K_t[h_0(t)], \quad H_1(v) = K_t[h_1(t)].$$

Substituting $B = -1, L(f(x,t)) = f(x,t), Z(x,t) = 0, n = 2, m = 1$, and the functions $G_0(u), H_0(v), H_1(v)$ into the general formula in equation (4.5), and performing the computations, we obtain:

$$f(x,t) = \mathcal{A}_x^{-1} K_t^{-1} \left[\frac{1}{Au^{\xi} + v^{-\zeta} + C} \big(Au^{\xi} H_0(v) + Au^{\xi-1} H_1(v) - u^{-\zeta+1} G_0(u) \big) \right]. \tag{5.4}$$

Example 5.1. Heat Diffusion Equation.
Consider the heat diffusion equation:

$$f_{xx}(x,t) - D_t^{\zeta} f(x,t) = 0, \quad 0 < \zeta \leq 1, \tag{5.5}$$

with the IC:

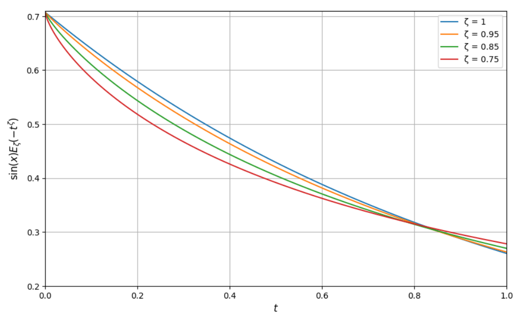
$$f(x,0) = \sin x, \tag{5.6}$$

and the BCs:

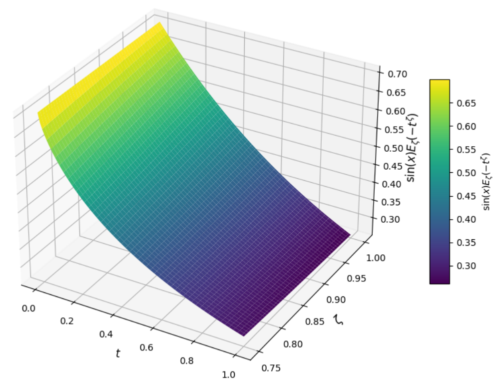
$$f(0,t) = 0, \quad f_x(0,t) = E_{\zeta}(-t^{\zeta}). \tag{5.7}$$

Solution: Substituting $A = 1, C = 0, \zeta = 2, G_0(u) = \frac{u}{u^2+1}, H_0(v) = 0, H_1(v) = \frac{v}{1+v^{\zeta}}$ into equation (5.4), we find the solution:

$$f(x,t) = \mathcal{A}_x^{-1} K_t^{-1} \left[\frac{uv}{(u^2+1)(1+v^{\zeta})} \right] = \sin x\, E_{\zeta}(-t^{\zeta}). \tag{5.8}$$



(A) Plot of $\sin x\, E_{\zeta}(-t^{\zeta})$ vs t.



(B) 3D plot of $\sin x\, E_{\zeta}(-t^{\zeta})$.

Figure 1. A) The plots display the exact solution for $\zeta = 1$ as a reference and compare it with solutions for $\zeta = 0.95$, $\zeta = 0.85$, and $\zeta = 0.75$. This shows how changes in ζ affect the system's behavior. B) The surface graph of the solution $f(x,t)$ for the heat diffusion equation at $\zeta = 1$ for the problem in Example 5.1.

5.2. Fractional advection-diffusion equation

Consider the fractional advection-diffusion equation:

$$A\ D_x^{\xi} f(x,t) - D_t^{\zeta} f(x,t) + C\ f(x,t) = 0, \quad 1 < \xi \leq 2,\ 0 < \zeta \leq 1, \tag{5.9}$$

with the initial condition (IC):

$$f(x,0) = g_0(x), \tag{5.10}$$
$$f(x,0) = g_0(x), \tag{5.11}$$

and the BCs

$$f(0,t) = h_0(t), f_x(0,t) = h_1(t). \tag{5.12}$$

Applying the single ARA transform on $g_0(x)$ in equation (5.10), we obtain

$$G_0(u) = \mathcal{A}_x[g_0(x)].$$

Applying the single Kamal transform on $h_0(t)$ and $h_1(t)$ in equation (5.3), we obtain

$$H_0(v) = K_t[h_0(t)],$$
$$H_1(v) = K_t[h_1(t)].$$

Substituting $B = -1, L(f(x,t)) = f_x(x,t), Z(x,t) = 0, n = 2, m = 1$, and the functions $G_0(u), H_0(v), H_1(v)$ in the general formula in equation (4.5), after simple computations, we obtain

$$f(x,t) = \mathcal{A}_x^{-1} K_t^{-1}\left[\frac{1}{Au^{\xi} + v^{-\zeta} + Cu}\left(Au^{\xi} H_0(v) + Au^{\xi-1} H_1(v) - u^{-\zeta+1} G_0(u) + CuH_0(v)\right)\right]. \tag{5.13}$$

Example 5.2. Consider the fractional advection-diffusion equation

$$f_{xx}(x,t) - D_t^{\zeta} f(x,t) - f_x(x,t) = 0, \quad 0 < \zeta \leq 1, \tag{5.14}$$

with the IC

$$f(x,0) = e^{-x}, \tag{5.15}$$

and the BCs

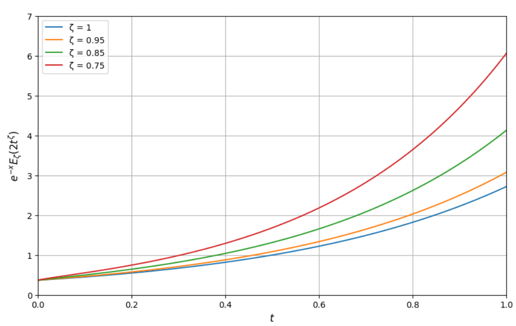
$$f(0,t) = E_{\zeta}(2t^{\zeta}), f_x(0,t) = -E_{\zeta}(2t^{\zeta}). \tag{5.16}$$

Solution: Putting $A = 1, C = -1, \xi = 2, G_0(u) = \frac{u}{u+1}, H_0(v) = \frac{v}{1-2v^{\zeta}}, H_1(v) = \frac{-v}{1-2v^{\zeta}}$, in equation (5.12), we obtain the solution of (5.13) as follows:

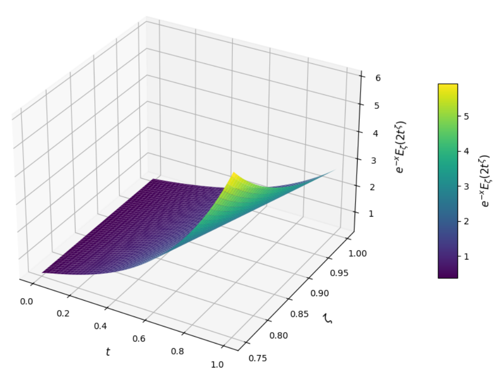
$$f(x,t) = \mathcal{A}_x^{-1} K_t^{-1}\left[\frac{vu}{u^2 - v^{\zeta} - u}\left(\frac{u}{1-2v^{\zeta}} - \frac{1}{1-2v^{\zeta}} - \frac{v^{-\zeta}}{u+1} - \frac{1}{1-2v^{\zeta}}\right)\right] \tag{5.17}$$

$$= \mathcal{A}_x^{-1} K_t^{-1}\left[\frac{uv}{(u+1)(1-2v^{\zeta})}\right]$$

$$= e^{-x} E_{\zeta}(2t^{\zeta}). \tag{5.18}$$



(A) Plot of $e^{-x}E_{\zeta}(2t^{\zeta})$ vs t.



(B) 3D plot of $e^{-x}E_{\zeta}(2t^{\zeta})$.

Figure 2. A) Plots show the exact solution for $\zeta = 1$ as a reference and compare it with solutions for $\zeta = 0.95, 0.85$, and 0.75 from Example 5.2, highlighting how changes in ζ affect the results. B) The surface graph of the solution $f(x,t)$ for the fractional advection-diffusion equation at $\zeta = 1$ for the problem in Example 5.2.

5.3. Fractional wave equation

Consider the fractional wave equation

$$A\ D_x^{\xi} f(x,t) - D_t^{\zeta} f(x,t) = 0, \quad 1 < \xi, \zeta \leq 2, \tag{5.19}$$

with the ICs

$$f(x,0) = g_0(x), \quad f_t(x,0) = g_1(x), \tag{5.20}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{5.21}$$

Applying the single ARA transform on $g_0(x)$, and $g_1(x)$ in equation (5.19), we obtain

$$G_0(u) = \mathcal{A}_x[g_0(x)],$$
$$G_1(u) = \mathcal{A}_x[g_1(x)].$$

Applying the single Kamal transform on $h_0(t)$, and $h_1(t)$ in equation (5.20), we obtain

$$H_0(v) = K_t[h_0(t)],$$
$$H_1(v) = K_t[h_1(t)].$$

Substituting $B = -1, C = 0, Z(x,t) = 0, n = m = 2$ and the functions $G_0(u), G_1(u), H_0(v)$, $H_1(v)$ in the general formula in equation (4.5), after simple computations, we obtain

$$f(x,t) = \mathcal{A}_x^{-1}K_t^{-1}\Bigg[\frac{1}{Au^{\xi} - v^{-\zeta}}\Bigg(Au^{\xi}H_0(v) + Au^{\xi-1}H_1(v) - v^{-\zeta}G_0(u) - v^{-\zeta}G_0(u) - v^{-\zeta+1}G_0(u) - v^{-\zeta+2}G_1(u)\Bigg)\Bigg]. \tag{5.22}$$

5.4. Fractional Klein-Gordon equation

Consider the fractional Klein-Gordon equation

$$D_x^{\xi}f(x,t) - D_t^{\zeta}f(x,t) + Cf(x,t) = z(x,t), \quad 1 < \xi, \zeta \leq 2, \tag{5.23}$$

with the ICs

$$f(x,0) = g_0(x), \quad f_t(x,0) = g_1(x) \tag{5.24}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{5.25}$$

Applying the single ARA transform on $g_0(x)$ and $g_1(x)$ in equation (5.23), we obtain

$$G_0(u) = \mathcal{A}_x[g_0(x)],$$
$$G_1(u) = \mathcal{A}_x[g_1(x)].$$

Applying the single Kamal transform on $h_0(t)$ and $h_1(t)$ in equation (5.24), we obtain

$$H_0(v) = K_t[h_0(t)],$$
$$H_1(v) = K_t[h_1(t)].$$

Substituting $A = -1, B = -1, L\big[f(x,t)\big] = f(x,t), n = m = 2$ and the functions $G_0(u), G_1(u)$, $H_0(v), H_1(v)$ in the general formula in equation (4.5), after simple computations, we obtain

$$f(x,t) = \mathcal{A}_x^{-1}K_t^{-1}\Bigg[\frac{1}{u^{\xi} + v^{-\zeta} + C}\Bigg(u^{\xi}H_0(v) + u^{\xi-1}H_1(v) - v^{-\zeta+1}G_0(u) - v^{-\zeta+2}G_1(u) + Z(u,v)\Bigg)\Bigg]. \tag{5.26}$$

Example 5.3. Consider the fractional Klein-Gordon equation

$$f_{xx}(x,t) - D_t^{\zeta} f(x,t) + f(x,t) = 0, \quad 1 < \zeta \leq 2, \tag{5.27}$$

with the ICs

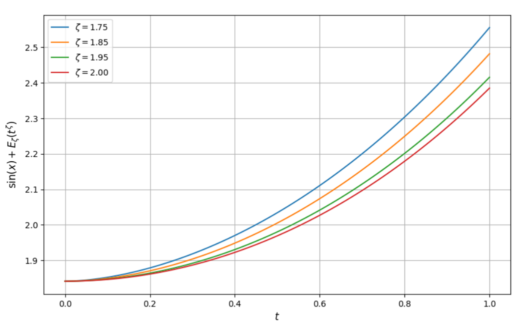
$$f(x,0) = \sin x + 1, \quad f_t(x,0) = 0, \tag{5.28}$$

and the BCs

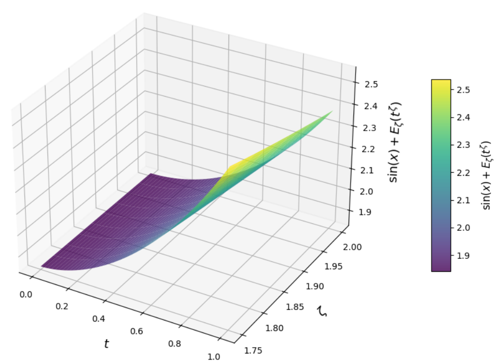
$$f(0,t) = E_{\zeta}(t^{\zeta}), \quad f_x(0,t) = 1. \tag{5.29}$$

Solution: Putting $C = 1, \xi = 2, z(x,t) = 0, G_0(u) = \frac{u}{u^2+1} + 1, G_1(u) = 0, H_0(v) = \frac{v}{1-v^{\zeta}}$, $H_1(v) = 1$, in equation (5.25), we obtain:

$$\begin{aligned} f(x,t) &= \mathcal{A}_x^{-1} K_t^{-1}\left[\frac{1}{u^2 + v^{-\zeta} + 1}\left(\frac{u^{\xi} v}{1 - v^{\zeta}} + u - \frac{v^{-\zeta+1} u}{u^2 + 1} - v^{-\zeta+1}\right)\right] \\ &= \mathcal{A}_x^{-1} K_t^{-1}\left[\frac{u}{u^2+1} + \frac{v}{1 - v^{\zeta}}\right] \\ &= \sin x + E_{\zeta}(t^{\zeta}). \end{aligned} \tag{5.30}$$



(A) Plot of $\sin x + E_{\zeta}(t^{\zeta})$ vs t.



(B) 3D plot of $\sin x + E_{\zeta}(t^{\zeta})$.

Figure 3. A) plot shows how the exact solution behaves when $\zeta = 2$, compared to how it changes for different values of $\zeta = 1.95, 1.85$, and 1.75. Each line represents the solution for one of these values of ζ, illustrating how the solution shifts as ζ decreases from 2. B) The surface graph of the solution $f(x,t)$ for the Klein-Gordon equation at $\zeta = 1$ for the problem in Example 5.3.

5.5. Fractional Fokker-Planck equation

Consider the fractional Fokker-Planck equation

$$D_x^{\xi} f(x,t) - D_t^{\zeta} f(x,t) + f_x(x,t) = 0, \quad 1 < \xi \leq 2, 0 < \zeta \leq 1, \tag{5.31}$$

with the IC

$$f(x,0) = g_0(x), \tag{5.32}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{5.33}$$

Applying the single ARA transform on $g_0(x)$ in equation (5.31), we obtain

$$G_0(u) = \mathcal{A}_x[g_0(x)].$$

Applying the single Kamal transform on $h_0(t)$ and $h_1(t)$ in Equation (5.32), we obtain

$$H_0(v) = K_t[h_0(t)],$$
$$H_1(v) = K_t[h_1(t)].$$

Substituting $A = 1, B = -1, C = 1, L\big[f(x,t)\big] = f_x(x,t), z(x,t) = 0, n = 2, m = 1$ and the functions $G_0(u), H_0(v), H_1(v)$ in the general formula in equation (4.5), after simple computations, we obtain

$$f(x,t) = \mathcal{A}_x^{-1}K_t^{-1}\left[\frac{1}{u^{\xi} - v^{-\zeta} + u}\bigg(u^{\xi}H_0(v) + u^{\xi-1}H_1(v) - v^{-\zeta+1}G_0(u) + uH_0(v)\bigg)\right]. \tag{5.34}$$

Example 5.4. Consider the fractional Fokker-Planck equation

$$f_{xx}(x,t) - D_t^{\zeta} f(x,t) + f_x(x,t) = 0, \quad 0 < \zeta \le 1, \tag{5.35}$$

with the IC

$$f(x,0) = x, \tag{5.36}$$

and the BCs

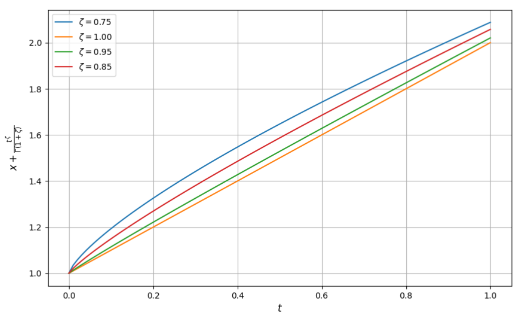
$$f(0,t) = \frac{t^{\zeta}}{\Gamma(1+\zeta)}, \quad f_x(0,t) = 1. \tag{5.37}$$

Solution: Putting $\xi = 2, G_0(u) = \frac{1}{u}, H_0(v) = v^{\zeta+1}, H_1(v) = v$, in equation (5.33), we obtain the solution of (5.34) as follows:

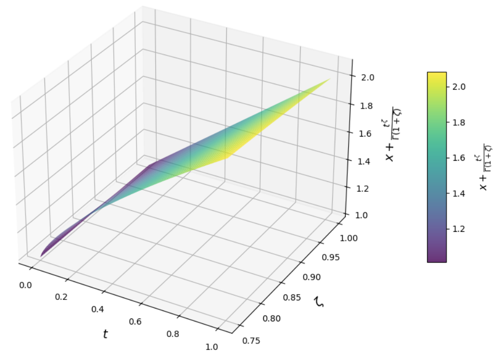
$$\begin{aligned} f(x,t) &= \mathcal{A}_x^{-1}K_t^{-1}\left[\frac{1}{u^2 - v^{-\zeta} + u}\bigg(u^2v^{\zeta+1} + uv - \frac{v^{-\xi+1}}{u} + uv^{\zeta+1}\bigg)\right] \\ &= \mathcal{A}_x^{-1}K_t^{-1}\left[\frac{1}{u} + v^{\zeta+1}\right] \\ &= x + \frac{t^{\zeta}}{\Gamma(1+\zeta)}. \end{aligned} \tag{5.38}$$

6. Conclusions

In this research, the DARA-KT method was applied to the Caputo fractional derivative, resulting in the derivation of a new and interesting formula, which was implemented to solve families of fractional partial differential equations (FPDEs). We introduced a novel method to obtain exact solutions for these equations and demonstrated the reliability and efficiency of the proposed approach through several compelling physical applications. For future work, we aim to combine the DARA-KT method with iterative approaches to solve nonlinear FPDEs, such as the nonlinear wave equation, the nonlinear Klein–Gordon equation, and the nonlinear Fokker–Planck equation. Furthermore, researchers can utilize new definitions in fractional calculus, such as the generalized fractional derivative, to explore and achieve new results in transformation theory.



(A) Plot of $x + \frac{t^\zeta}{\Gamma(1+\zeta)}$ vs t.



(B) 3D plot of $x + \frac{t^\zeta}{\Gamma(1+\zeta)}$.

Figure 4. A) The plot shows the exact solution of when $\zeta = 1$ and compares it to the solutions for different values of $\zeta = 0.95$, 0.85 and 0.75.from Example 5.4. B) The surface graph of the solution $f(x, t)$ for fractional Fokker-Planck equation at $\zeta = 1$ for the problem in Example 5.4.

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