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EXPONENTIAL STABILITY OF SAINT-VENANT EQUATIONS IN SUPERCRITICAL FLOW

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Abstract In this paper, the stability of a class of 2×2 hyperbolic systems with the same sign propagation speeds is investigated under proportional feedback control. Semigroup theory is used to verify the well-posedness of the system. The exponential stability of the closed-loop system is analyzed by constructing a strictly weighted Lyapunov function. In addition, the characteristic equation and eigenfunctions of the system are deduced. For the special case where γ is equal to 0, by using Schur-Cohn criterion, the sufficient and necessary stability conditions, which make the eigenvalues have the negative real part, is obtained.

Keywords Saint-Venant equations, supercritical flow, Lyapunov function, exponential stability.

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1. Introduction

The Saint-Venant equations, as the typical mathematical model of open channel flow dynamics, play an important role in revealing the fluid motion laws of natural and artificial waterway systems. These equations are not only applicable to depicting natural processes such as flood wave propagation and estuary tides, but also widely applied in human engineering scenarios such as irrigation canals and navigable rivers [5,15].

The properties of the steady-state flow in the open channel systems depend on the Froude's number (F_r). When $F_r < 1$, the flow state is a subcritical flow regime; when $F_r > 1$, the flow state is a supercritical flow regime. The boundary feedback control problems for hyperbolic systems with subcritical flow have been extensively studied [2,16]. In [4], the exponential stability of the linearized Saint-Venant equations under subcritical flow conditions is investigated, and extended to the cascade network systems with n pools. Furthermore, in [19], the spectral analysis and Riesz basis properties of the linear hyperbolic system are verified, and the exponential stability is established. For nonlinear Saint-Venant equations with non-uniform steady-states, the local exponential stability of the system in the H^2-norm is proved by constructing an appropriate quadratic Lyapunov function in [13]. Stability analysis of partial differential equations (PDE) systems can be conducted using various approaches, including the Lyapunov function method, spectral analysis, and frequency-domain techniques [6,10,17]. In [1,7], the controller of PDE systems is designed by backstepping method. In [3], the feedback stabilization problem of the open channel system under proportional-integral control was studied, and the stability was established for the linearized and nonlinear Saint-Venant equations. In [18], the stability regions of feedback parameters for hyperbolic conservation law systems are established via

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Walton-Marshall stability criterion and Schur-Cohn criterion. Through eigenvalue analysis, the delay-independent and delay-dependent stability results are derived, respectively. Reference [12] discusses the input-to-state stability (ISS) of cascade channels governed by the Saint-Venant equations under PI control. Specifically, exponential stability and ISS conditions are provided by constructing explicit quadratic Lyapunov functions.

There are few studies on analyzing hyperbolic systems with the same sign propagation speeds, i.e., the case of supercritical flow. In [8], a full-state feedback controller is designed using the backstepping method to ensure exponential stability of the desired setpoint for the Saint-Venant-Exner model. The controller's performance under supercritical flow regimes is verified through numerical simulation. As shown in Figure 10, for a supercritical flow regime ($F_r = 1.6$), the backstepping method significantly improves system stability compared to the traditional Lyapunov method. In [9], the controllability of single-channel supercritical flow is first investigated. This study established global controllability between two constant supercritical steady-states with the same flow direction within the same channel. By analyzing the controllability of supercritical steady-state flow in a single channel and extending this result to the network structure, it is strictly proved that the smooth solution of the tree-like open channel network under supercritical flow conditions has global controllability. In [11], a basic C^1 Lyapunov function is constructed to verify the exponential stability of a class of 2×2 hyperbolic systems with variable coefficients, and it is pointed out that when the propagation speeds of the system have the same sign, the system stability can be achieved by static boundary feedback control.

In this paper, we dedicated to give a sufficient condition for exponential stability of the linearised Saint-Venant equations with supercritical flow. It is found that the characteristic equation is a transcendental equation with multiple exponential terms. Hence, the characteristic root method and the frequency domain analysis can not be used to obtain the stability condition, and the asymptotic expression of its eigenvalues are difficult to carry out. If the system parameter $\gamma \neq 0$, by constructing a suitable strictly weighted Lyapunov function, we derive an sufficient condition for the feedback parameters and establish the exponential stability of the closed-loop system. For the case $\gamma = 0$, we establish sufficient and necessary stability conditions by using the Schur-Cohn criterion.

The structure of this paper is as follows: In Section 2, we linearize the Saint-Venant equations and rewrite the closed-loop system into the form of an abstract evolution equation by defining the linear operator. In Section 3, by applying the operator semigroup theory and Sobolev embedding theorem, the well-posedness of the closed-loop system is proved. Additionally, if $\gamma \neq 0$, by constructing a suitable strictly weighted Lyapunov function, we derive a sufficient condition for the feedback parameters and establish the exponential stability of the closed-loop system. In Section 4, we give the characteristic equation and eigenfunctions for the closed-loop system. Especially, if $\gamma = 0$, the sufficient and necessary conditions for stability is obtained. In Section 5, we verify the feasibility of the theoretical results through numerical simulations.

2. Saint-Venant equations and model building

In this paper, we consider an open channel with a constant bottom slope and a rectangular cross section, whose flow dynamics can be characterized by nonlinear Saint-Venant equations

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \partial_x \begin{pmatrix} HV \\ \dfrac{V^2}{2} + gH \end{pmatrix} + \begin{pmatrix} 0 \\ g[S_f(H,V) - S_b] \end{pmatrix} = 0, \tag{2.1}$$

where $H(t,x)$ be the water depth and $V(t,x)$ the horizontal water velocity at time t and position x along the channel, $t \in [0,+\infty]$, $x \in [0,L]$, g is the constant gravity acceleration, L is the length of the channel, S_b is the constant bottom slope, $S_f(H,V) = C_f \frac{V^2}{H}$ is the friction slope and C_f is a constant friction coefficient.

2.1. Steady-state and linearization

Assume that the steady-state of the system is a constant state (H^*,V^*) and that it satisfies the relation $S_f(H^*,V^*) = S_b$. By linearizing the system given by (2.1), we obtain

$$\begin{cases} \partial_t h + V^* \partial_x h + H^* \partial_x v = 0, \\ \partial_t v + g\partial_x h - \left[\dfrac{g}{H^*} S_b\right] h + \left[\dfrac{2g}{V^*} S_b\right] v = 0, \end{cases} \tag{2.2}$$

where, $h(t,x) = H(t,x) - H^*$, $v(t,x) = V(t,x) - V^*$. By making a Riemannian coordinates transformation:

$$\begin{cases} \xi_1(t,x) = v(t,x) + h(t,x)\sqrt{\dfrac{g}{H^*}}, \\ \xi_2(t,x) = v(t,x) - h(t,x)\sqrt{\dfrac{g}{H^*}}. \end{cases} \tag{2.3}$$

Then the characteristic form of the linearized Saint-Venant equations (2.2) is

$$\begin{cases} \partial_t \xi_1(t,x) + \lambda_1 \partial_x \xi_1(t,x) + \gamma \xi_1(t,x) + \delta \xi_2(t,x) = 0, \\ \partial_t \xi_2(t,x) + \lambda_2 \partial_x \xi_2(t,x) + \gamma \xi_1(t,x) + \delta \xi_2(t,x) = 0, \end{cases} \tag{2.4}$$

with the characteristic velocities

$$\lambda_1 = V^* + \sqrt{gH^*}, \quad \lambda_2 = V^* - \sqrt{gH^*},$$

and the parameters

$$\gamma = gS_b\left(\frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}}\right), \quad \delta = gS_b\left(\frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right).$$

2.2. Supercritical flow

The steady-state flow is supercritical (or torrential) if the following condition holds

$$gH^* - (V^*)^2 < 0. \tag{2.5}$$

Under this condition, the system is strictly hyperbolic with

$$0 < \lambda_2 = V^* - \sqrt{gH^*} < \lambda_1 = V^* + \sqrt{gH^*}.$$

It is obvious that $\delta > 0$, and the parameter γ can be positive (if $V^* < 2\sqrt{gH^*}$), negative (if $V^* > 2\sqrt{gH^*}$), or $\gamma = 0$ (if $V^* = 2\sqrt{gH^*}$).

2.3. Model building

We assume that the initial conditions of the system (2.4) are

$$\xi_1(0,x)=\xi_1^0(x),\ \ \xi_2(0,x)=\xi_2^0(x), \tag{2.6}$$

and the proportional boundary feedback laws are

$$\xi_1(t,0)=k_1\xi_1(t,L),\ \ \xi_2(t,0)=k_2\xi_2(t,L), \tag{2.7}$$

where the boundary feedback parameters $k_1,k_2\in\mathbb{R}$. Hence, we are concerned with the stability analysis of the following closed-loop system:

$$\begin{cases}\partial_t\xi_1(t,x)+\lambda_1\partial_x\xi_1(t,x)+\gamma\xi_1(t,x)+\delta\xi_2(t,x)=0,\\ \partial_t\xi_2(t,x)+\lambda_2\partial_x\xi_2(t,x)+\gamma\xi_1(t,x)+\delta\xi_2(t,x)=0,\\ \xi_1(t,0)=k_1\xi_1(t,L),\\ \xi_2(t,0)=k_2\xi_2(t,L).\end{cases} \tag{2.8}$$

Let the Hilbert state space $\mathcal{H}=L^2(0,L)\times L^2(0,L)$ with the inner product defined by

$$\langle X_1,X_2\rangle=\int_0^L\left(f_1(x)\overline{f_2(x)}+g_1(x)\overline{g_2(x)}\right)\mathrm{d}x, \tag{2.9}$$

where $X_i=(f_i,g_i)\in\mathcal{H}(i=1,2)$, and $\overline{f}$ is the conjugate of f.

Define a linear operator $\mathcal{A}:\ D(\mathcal{A})\subseteq\mathcal{H}\to\mathcal{H}$ by

$$\mathcal{A}X=\begin{pmatrix}-\lambda_1\dfrac{\partial}{\partial x}-\gamma & -\delta\\ -\gamma & -\lambda_2\dfrac{\partial}{\partial x}-\delta\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}, \tag{2.10}$$

$$D(\mathcal{A})=\left\{(f,g)\in(H^1(0,L))^2\mid f(0)=k_1f(L),\ \ g(0)=k_2g(L)\right\}. \tag{2.11}$$

Then system (2.8) can be written as an abstract evolution equation in $\mathcal{H}$:

$$\begin{cases}\dot{X}(t)=\mathcal{A}X(t),\ t>0,\\ X(0)=X_0,\end{cases} \tag{2.12}$$

where $X(t)=(\xi_1(t,\cdot),\,\xi_2(t,\cdot))$, $X_0=(\xi_1^0(\cdot),\,\xi_2^0(\cdot))$.

For the convenience of calculations and without losing generality, let $L=1$.

3. Well-posedness and Lyapunov stability

In this section, we first prove the well-posedness of the system (2.12). And then analyze the exponential stability of the system (2.8) by constructing a suitable Lyapunov function.

3.1. Well-posedness of system (2.12)

Theorem 3.1. *Let* $\mathcal{A}$ *be given by* (2.10) *and* (2.11)*. Then* $\mathcal{A}^{-1}$ *exists and is compact on* $\mathcal{H}$*, if the feedback parameters* k_1*,* k_2 *satisfy*

$$e^{c_1}\left(k_1k_2e^{c_1}-k_1-k_2\right)+\left(e^{c_1}-1\right)\left[\frac{k_1\delta}{\lambda_2c_1}(k_2e^{c_1}-1)+\frac{k_2\gamma}{\lambda_1c_1}(k_1e^{c_1}-1)\right]+1\neq 0,$$

where $c_1=-\frac{\gamma}{\lambda_1}-\frac{\delta}{\lambda_2}\neq 0$*. Therefore,* $\sigma(\mathcal{A})$*, the spectrum set of* $\mathcal{A}$ *consists of isolated eigenvalues of finite algebraic multiplicity only.*

Proof. For $\forall X_1=(f_1,g_1)\in\mathcal{H}$, solve $\mathcal{A}X=X_1$, where $X=(f,g)\in D(\mathcal{A})$, we have

$$\mathcal{A}\begin{pmatrix}f\\ g\end{pmatrix}=\begin{pmatrix}-\lambda_1f'-\gamma f-\delta g\\ -\lambda_2g'-\gamma f-\delta g\end{pmatrix}=\begin{pmatrix}f_1\\ g_1\end{pmatrix}, \tag{3.1}$$

i.e.,

$$\begin{cases}\lambda_1f'+\gamma f+\delta g+f_1=0,\\ \lambda_2g'+\gamma f+\delta g+g_1=0,\\ f(0)=k_1f(1),\\ g(0)=k_2g(1).\end{cases} \tag{3.2}$$

From the first two equations of (3.2), we have

$$\lambda_1f(x)-\lambda_2g(x)-\lambda_1f(0)+\lambda_2g(0)=\int_0^x\big(g_1(s)-f_1(s)\big)\mathrm{d}s, \tag{3.3}$$

then

$$g(x)=\frac{\lambda_1}{\lambda_2}f(x)-\frac{\lambda_1}{\lambda_2}f(0)+g(0)-\frac{1}{\lambda_2}\int_0^x\big(g_1(s)-f_1(s)\big)\mathrm{d}s. \tag{3.4}$$

Substitute (3.4) into the first equation of (3.2), we get

$$\begin{aligned}f'(x)=&\big(-\frac{\gamma}{\lambda_1}-\frac{\delta}{\lambda_2}\big)f(x)+\frac{\delta}{\lambda_2}f(0)-\frac{\delta}{\lambda_1}g(0)\\ &+\frac{\delta}{\lambda_1\lambda_2}\int_0^x\big(g_1(s)-f_1(s)\big)\mathrm{d}s-\frac{1}{\lambda_1}f_1(x).\end{aligned} \tag{3.5}$$

From the general solution formula of the first order linear ordinary differential equation

$$\begin{aligned}f(x)=&ce^{c_1x}-\int_0^xe^{c_1(x-v)}\Big[-\frac{\delta}{\lambda_2}f(0)+\frac{\delta}{\lambda_1}g(0)\\ &-\frac{\delta}{\lambda_1\lambda_2}\int_0^v\big(g_1(s)-f_1(s)\big)\mathrm{d}s+\frac{1}{\lambda_1}f_1(v)\Big]\mathrm{d}v,\end{aligned} \tag{3.6}$$

where $c=f(0)$, $c_1=-\frac{\gamma}{\lambda_1}-\frac{\delta}{\lambda_2}\neq 0$.

From (3.6), when $x=1$, we can have

$$\begin{aligned}f(1)=&f(0)e^{c_1}-\int_0^1e^{c_1(1-v)}\Big[-\frac{\delta}{\lambda_2}f(0)+\frac{\delta}{\lambda_1}g(0)\\ &-\frac{\delta}{\lambda_1\lambda_2}\int_0^v\big(g_1(s)-f_1(s)\big)\mathrm{d}s+\frac{1}{\lambda_1}f_1(v)\Big]\mathrm{d}v.\end{aligned} \tag{3.7}$$

By (3.7) and the boundary condition $f(0)=k_1f(1)$, we have

$$\begin{aligned}&\left[k_1e^{c_1}+\frac{k_1\delta}{\lambda_2c_1}(e^{c_1}-1)-1\right]f(0)-\frac{k_1\delta}{\lambda_1c_1}(e^{c_1}-1)g(0)\\=&\frac{k_1\delta}{\lambda_1\lambda_2c_1}\int_0^1\big(1-e^{c_1(1-s)}\big)\big(g_1(s)-f_1(s)\big)\mathrm{d}s+\frac{k_1}{\lambda_1}\int_0^1e^{c_1(1-v)}f_1(v)\mathrm{d}v.\end{aligned}\tag{3.8}$$

Similarly,

$$\begin{aligned}&\left[k_2e^{c_1}+\frac{k_2\gamma}{\lambda_1c_1}(e^{c_1}-1)-1\right]g(0)-\frac{k_2\gamma}{\lambda_2c_1}(e^{c_1}-1)f(0)\\=&-\frac{k_2\gamma}{\lambda_1\lambda_2c_1}\int_0^1\big(1-e^{c_1(1-s)}\big)\big(g_1(s)-f_1(s)\big)\mathrm{d}s+\frac{k_2}{\lambda_2}\int_0^1e^{c_1(1-v)}g_1(v)\mathrm{d}v.\end{aligned}\tag{3.9}$$

According to (3.8), (3.9), we can obtain

$$f(0)=\frac{M_1}{M_0},\ \ g(0)=\frac{M_2}{M_0},\tag{3.10}$$

where $M_0\neq 0$, and

$$\begin{cases}M_0=&e^{c_1}\left(k_1k_2e^{c_1}-k_1-k_2\right)+(e^{c_1}-1)\left[\frac{k_1\delta}{\lambda_2c_1}(k_2e^{c_1}-1)+\frac{k_2\gamma}{\lambda_1c_1}(k_1e^{c_1}-1)\right]+1,\\M_1=&\frac{k_1k_2\delta e^{c_1}-k_1\delta}{\lambda_1\lambda_2c_1}\int_0^1\big(1-e^{c_1(1-s)}\big)\big(g_1(s)-f_1(s)\big)\mathrm{d}s\\&+\frac{k_1k_2\lambda_1c_1e^{c_1}+k_1k_2\gamma(e^{c_1}-1)-k_1\lambda_1c_1}{\lambda_1^2c_1}\int_0^1e^{c_1(1-v)}f_1(v)\mathrm{d}v\\&+\frac{k_1k_2\delta(e^{c_1}-1)}{\lambda_1\lambda_2c_1}\int_0^1e^{c_1(1-v)}g_1(v)\mathrm{d}v,\\M_2=&\frac{k_2\gamma-k_1k_2\gamma e^{c_1}}{\lambda_1\lambda_2c_1}\int_0^1\big(1-e^{c_1(1-s)}\big)\big(g_1(s)-f_1(s)\big)\mathrm{d}s\\&+\frac{k_1k_2\lambda_2c_1e^{c_1}+k_1k_2\delta(e^{c_1}-1)-k_2\lambda_2c_1}{\lambda_2^2c_1}\int_0^1e^{c_1(1-v)}g_1(v)\mathrm{d}v\\&+\frac{k_1k_2\gamma(e^{c_1}-1)}{\lambda_1\lambda_2c_1}\int_0^1e^{c_1(1-v)}f_1(v)\mathrm{d}v.\end{cases}\tag{3.11}$$

Moreover, based on (3.4), (3.6) and (3.10), we can get expressions for $f(x)$ and $g(x)$:

$$\begin{cases}f(x)=&\frac{M_1}{M_0}e^{c_1x}-\int_0^xe^{c_1(x-v)}\Big[-\frac{\delta}{\lambda_2}\frac{M_1}{M_0}+\frac{\delta}{\lambda_1}\frac{M_2}{M_0}\\&-\frac{\delta}{\lambda_1\lambda_2}\int_0^v\big(g_1(s)-f_1(s)\big)\mathrm{d}s+\frac{1}{\lambda_1}f_1(v)\Big]\mathrm{d}v,\\g(x)=&\frac{\lambda_1}{\lambda_2}f(x)-\frac{\lambda_1}{\lambda_2}\frac{M_1}{M_0}+\frac{M_2}{M_0}-\frac{1}{\lambda_2}\int_0^x\big(g_1(s)-f_1(s)\big)\mathrm{d}s.\end{cases}\tag{3.12}$$

Hence, $\mathcal{A}^{-1}$ exists and is compact by the Sobolev embedding theorem. Therefore, $\sigma(\mathcal{A})$ consists of isolated eigenvalues of finite algebraic multiplicity only. $\square$

3.2. Lyapunov stability of system (2.8)

The following is an analysis of the system stability in the case where $\gamma \neq 0$.

Definition 3.1. The closed-loop system (2.8) is exponentially stable (in L^2-norm) if there exist $v > 0$ and $C > 0$ such that, for every initial condition $(\xi_1^0(x), \xi_2^0(x)) \in L^2((0,1);\mathbb{R}^2)$ the solution to the Cauchy problem (2.8) satisfies

$$\|\xi_1(t,\cdot), \xi_2(t,\cdot)\|_{L^2((0,1);\mathbb{R}^2)} \leq Ce^{-vt}\|\xi_1^0, \xi_2^0\|_{L^2((0,1);\mathbb{R}^2)}.$$

Theorem 3.2. *The closed-loop system* (2.8) *is exponentially stable, if the boundary feedback parameters* $k_i (i = 1, 2)$ *satisfy*

$$\max\{k_1^2, k_2^2\} < e^{-\nu}, \tag{3.13}$$

where $\nu > 0$.

Proof. The following candidate Lyapunov function is considered:

$$V(t) = \int_0^1 \left(|\gamma| e^{-\nu x}\xi_1^2 + \delta e^{-\nu x}\xi_2^2\right) \mathrm{d}x, \tag{3.14}$$

where, $|\gamma| \neq 0$ and δ is positive which are denote the system parameters, $\nu > 0$ is to be determined.

The derivation of the Lyapunov function along the system (2.8) is

$$\dot{V} = \dot{V}_1 + \dot{V}_2, \tag{3.15}$$

with

$$\dot{V}_1 = -\left[\lambda_1|\gamma|\xi_1^2 e^{-\nu x} + \lambda_2\delta\xi_2^2 e^{-\nu x}\right]_0^1, \tag{3.16}$$

$$\dot{V}_2 = -\int_0^1 \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} \begin{pmatrix} (\lambda_1\nu + 2|\gamma|)|\gamma|e^{-\nu x} & 2|\gamma|\delta e^{-\nu x} \\ 2|\gamma|\delta e^{-\nu x} & (\lambda_2\nu + 2\delta)\delta e^{-\nu x} \end{pmatrix} \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} \mathrm{d}x. \tag{3.17}$$

It is obvious that $\dot{V}_2 < 0$. For $\dot{V}_1$, substituting the boundary conditions $\xi_1(t,0) = k_1\xi_1(t,1)$, $\xi_2(t,0) = k_2\xi_2(t,1)$ into (3.16), we have

$$\dot{V}_1 = (\lambda_1|\gamma|k_1^2 - \lambda_1|\gamma|e^{-\nu})\xi_1^2(1) + (\lambda_2\delta k_2^2 - \lambda_2\delta e^{-\nu})\xi_2^2(1). \tag{3.18}$$

If the system parameters satisfy condition (3.13), then $\dot{V}_1 < 0$.

Therefore, there must exist a positive constant number ε such that $\dot{V}(t) < -\varepsilon V(t)$. □

Remark 3.1. When $\nu (> 0)$ is sufficiently small, it is easy to find that $\max\{k_1^2, k_2^2\} < e^{-\nu}$ is equivalent to $k_1^2 < 1, k_2^2 < 1$.

Remark 3.2. The stability condition (3.13) is sufficient, but not necessary. In fact, the integral term $\dot{V}_2(t)$ in $\frac{dV(t)}{dt}$ is always less than 0. However, if $k_1^2 \geq 1$ or $k_2^2 \geq 1$, the sign of $\dot{V}_1(t)$ is uncertain.

4. The eigenvalue problems of the system operator

In this section, we first give the characteristic equation and eigenfunctions of linear operator $\mathcal{A}$. Then, for the case $\gamma = 0$, by using Schur-Cohn criterion(see Proposition 5.3 of [14, p27]), we establish necessary and sufficient conditions on the system parameters that guarantee all eigenvalues have negative real parts.

4.1. The characteristic equation

Considering the eigenvalue problem of $\mathcal{A}$:

$$\mathcal{A}X = \mu X, \quad X = (f, g) \in D(\mathcal{A}),$$

if and only if f, g satisfy

$$\begin{cases} -\lambda_1 f' - \gamma f - \delta g = \mu f, \\ -\lambda_2 g' - \gamma f - \delta g = \mu g, \\ f(0) = k_1 f(1), \\ g(0) = k_2 g(1). \end{cases} \tag{4.1}$$

Theorem 4.1. *Let* $\mathcal{A}$ *be given by* (2.10) *and* (2.11), *and let*

$$\begin{aligned} \Delta(\mu) = & (\gamma + \mu - \frac{1}{2} a\lambda_1) \left(k_1 e^{-\frac{1}{2}a} - k_2 e^{-\frac{1}{2}a} \right) \sinh \omega \\ & + \lambda_1 \omega \left(k_1 k_2 e^{-a} - k_1 e^{-\frac{1}{2}a} \cosh \omega - k_2 e^{-\frac{1}{2}a} \cosh \omega + 1 \right). \end{aligned} \tag{4.2}$$

Then

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\mu \in \mathbb{C} | \Delta(\mu) = 0\}. \tag{4.3}$$

For each $\mu \in \sigma(\mathcal{A})$ *is geometrically simple and its eigenfunctions* $\phi_\mu = (f_\mu, g_\mu)$ *have the following form:*

$$\begin{cases} f_\mu(x) = e^{-\frac{1}{2}ax} \left[\sinh \omega x - \dfrac{k_1 e^{-\frac{1}{2}a} \sinh \omega}{k_1 e^{-\frac{1}{2}a} \cos \omega - 1} \cosh \omega x \right], \\ g_\mu(x) = -\dfrac{1}{\delta} e^{-\frac{1}{2}ax} \left(\gamma + \mu - \dfrac{1}{2} a\lambda_1 - \dfrac{\lambda_1 \omega k_1 e^{-\frac{1}{2}a} \sinh \omega}{k_1 e^{-\frac{1}{2}a} \cosh \omega - 1} \right) \sinh \omega x \\ \qquad\qquad - \dfrac{1}{\delta} e^{-\frac{1}{2}ax} \left(\lambda_1 \omega - \dfrac{(\gamma + \mu - \frac{1}{2} a\lambda_1) k_1 e^{-\frac{1}{2}a} \sinh \omega}{k_1 e^{-\frac{1}{2}a} \cosh \omega - 1} \right) \cosh \omega x, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} & a \doteq a(\mu) = \frac{(\lambda_1 + \lambda_2)\mu + \lambda_1 \delta + \lambda_2 \gamma}{\lambda_1 \lambda_2}, \ b \doteq b(\mu) = \frac{\mu^2 + (\gamma + \delta)\mu}{\lambda_1 \lambda_2}, \\ & \omega \doteq \omega(\mu) = \sqrt{\frac{1}{4} a^2 - b}. \end{aligned} \tag{4.5}$$

Proof. From the first equation of (4.1), we obtain

$$g = -\frac{1}{\delta} [\lambda_1 f' + (\gamma + \mu) f], \tag{4.6}$$

and then

$$g' = -\frac{1}{\delta} [\lambda_1 f'' + (\gamma + \mu) f']. \tag{4.7}$$

Substituting (4.6) and (4.7) into the second equation of equation (4.1), we get

$$f'' + af' + bf = 0, \tag{4.8}$$

where a, b are given in (4.5). Now we assume

$$f(x) = m(x) e^{-\frac{1}{2}ax}, \tag{4.9}$$

and

$$m(x) = C_1 \sinh \sqrt{\frac{1}{4}a^2 - b}x + C_2 \cosh \sqrt{\frac{1}{4}a^2 - b}x. \tag{4.10}$$

By (4.6) and (4.9), we have

$$g(x) = -\frac{1}{\delta} e^{-\frac{1}{2}ax} \left[\lambda_1 m'(x) + (\gamma + \mu - \frac{1}{2}a\lambda_1) m(x) \right]. \tag{4.11}$$

Combing with (4.9), (4.11) and the boundary conditions of (4.1), we have

$$\begin{cases} k_1 e^{-\frac{1}{2}a} \sinh \omega C_1 + \left(k_1 e^{-\frac{1}{2}a} \cosh \omega - 1 \right) C_2 = 0, \\ \left[\lambda_1 \omega - k_2 e^{-\frac{1}{2}a} \left(\lambda_1 \omega \cosh \omega + (\gamma + \mu - \frac{1}{2}a\lambda_1) \sinh \omega \right) \right] C_1 \\ + \left[(\gamma + \mu - \frac{1}{2}a\lambda_1) - k_2 e^{-\frac{1}{2}a} \left(\lambda_1 \omega \sinh \omega + (\gamma + \mu - \frac{1}{2}a\lambda_1) \cosh \omega \right) \right] C_2 = 0. \end{cases} \tag{4.12}$$

Hence, the characteristic polynomial $\Delta(\mu)$ can be written as

$$\begin{aligned} &\Delta(\mu) \\ &= k_1 e^{-\frac{1}{2}a} \sinh \omega \left[(\gamma + \mu - \frac{1}{2}a\lambda_1) - k_2 e^{-\frac{1}{2}a} \lambda_1 \omega \sinh \omega - k_2 e^{-\frac{1}{2}a} (\gamma + \mu - \frac{1}{2}a\lambda_1) \cosh \omega \right] \\ &\quad - \left[k_1 e^{-\frac{1}{2}a} \cosh \omega - 1 \right] \left[\lambda_1 \omega - k_2 e^{-\frac{1}{2}a} \lambda_1 \omega \cosh \omega - k_2 e^{-\frac{1}{2}a} (\gamma + \mu - \frac{1}{2}a\lambda_1) \sinh \omega \right] \\ &= (\gamma + \mu - \frac{1}{2}a\lambda_1) \left(k_1 e^{-\frac{1}{2}a} - k_2 e^{-\frac{1}{2}a} \right) \sinh \omega \\ &\quad + \lambda_1 \omega \left(k_1 k_2 e^{-a} - k_1 e^{-\frac{1}{2}a} \cosh \omega - k_2 e^{-\frac{1}{2}a} \cosh \omega + 1 \right). \end{aligned} \tag{4.13}$$

Let $C_1 = 0$, combining the first equation of (4.12), (4.6) and (4.9), we can obtain the eigenfunctions as (4.4). ☐

4.2. The stability for $\gamma = 0$

If $\gamma \neq 0$, the stability condition is presented in Section 3. Now, we dedicated to the case if $\gamma = 0$, i.e, $\lambda_1 = 3\lambda_2$. According to (4.2), the characteristic equation can be written as follows:

$$c e^{\frac{5\mu}{3\lambda_2}} - \hat{k}_1 e^{\frac{4\mu}{3\lambda_2}} - \hat{k}_2 e^{\frac{2\mu}{3\lambda_2}} + d e^{\frac{\mu}{3\lambda_2}} = 0, \tag{4.14}$$

where,

$$c = e^{\frac{3\delta}{2\lambda_2}},\ \hat{k}_1 = k_1 e^{\frac{3\delta}{2\lambda_2}},\ \hat{k}_2 = k_2 e^{\frac{\delta}{2\lambda_2}},\ d = k_1 k_2 e^{\frac{\delta}{2\lambda_2}}.$$

Denote

$$e^{\frac{\mu}{3\lambda_2}} = z, \tag{4.15}$$

and then, the equation (4.14) can be written as follows:

$$F(z) \doteq c z^5 - \hat{k}_1 z^4 - \hat{k}_2 z^2 + d z = 0, \tag{4.16}$$

which is a real 5-order polynomial in z. Hence, Re$\mu < 0$ in (4.14) if and only if $|z| < 1$ in (4.16).

Finally, we shall apply Schur-Cohn criterion to check that the roots of the polynomial $F(z)$ are inside the unit circle.

Theorem 4.2. *All the roots of $F(z)$ in (4.16) lie inside the unit circle, if and only if:*

$$c+d>|\hat{k}_1+\hat{k}_2|,\ c>|d|,\ c^3-c\hat{k}_1\hat{k}_2+d(\hat{k}_1^2-d)>\left|dc^2-c\hat{k}_2^2\right|. \tag{4.17}$$

Proof. By Schur-Cohn criterion, we have

$$F(1)>0,\ \ (-1)^5F(-1)>0, \tag{4.18}$$

i.e.,

$$c+d>|\hat{k}_1+\hat{k}_2|. \tag{4.19}$$

Noting that

$$\begin{aligned}\Delta_4^{\pm}&=\begin{pmatrix} c & 0 & 0 & 0\\ -\hat{k}_1 & c & 0 & 0\\ 0 & -\hat{k}_1 & c & 0\\ -\hat{k}_2 & 0 & -\hat{k}_1 & c\end{pmatrix}\pm\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & d\\ 0 & 0 & d & -\hat{k}_2\\ 0 & d & -\hat{k}_2 & 0\end{pmatrix}\\ &=\begin{pmatrix} c & 0 & 0 & 0\\ -\hat{k}_1 & c & 0 & d\\ 0 & -\hat{k}_1 & c\pm d & \mp\hat{k}_2\\ -\hat{k}_2 & \pm d & -\hat{k}_1\mp\hat{k}_2 & c\end{pmatrix},\end{aligned} \tag{4.20}$$

when the determinants of all inner submatrixes of $\Delta_4^{\pm}$ are greater than 0, it yields the last two inequalities in (4.17). □

5. Numerical simulations

In this part, we verify the feasibility of the parameter condition given by Theorem 3.2 and Theorem 4.2 through some numerical simulations.

Example 5.1. Consider a single prismatic channel model described by (2.1) with the following parameters:

- Channel length: $L=2000\ m$, width: $W=80\ m$.
- The bottom slope: $S_b=0.245$, the friction coefficient: $C_f=0.009\ \sec^2/m$.
- The steady-state flow rate: $Q^*=90.144\ m^3/s$.

By $S_bH^*=C_fV^{*^2}$ and $Q^*=WH^*V^*$. We can calculate the steady-state water depth and flow velocity, which are $H^*=0.36\ m$ and $V^*=3.13\ m/\sec$. Then we have

$$\lambda_1=5.008\ m/\sec,\ \lambda_2=1.252\ m/\sec,\ \underline{\gamma=0.1279\ \sec^{-1}},\ \delta=1.4062\ \sec^{-1}.$$

According to Theorem 3.2, we choose $k_1=0.6$, $k_2=0.8$, and then, Figure 1 show the stability of the system (2.8) with initial condition $\xi_1(0,x)=4\cos(2\pi x)$ and $\xi_2(0,x)=2\cos(2\pi x)$.

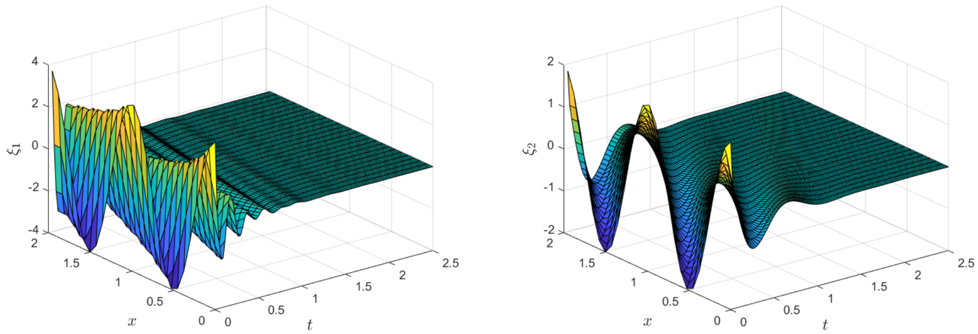

Figure 1. The convergence of state $\xi_1(t,x)$ and $\xi_2(t,x)$ of the system (2.8).

Example 5.2. The system parameters are chosen as follows:

$$L = 1000\ m, W = 20\ m, S_b = 0.0392, C_f = 0.001\ \sec^2/m, Q^* = 89.6\ m^3/s,$$

then we have

$$H^* = 0.8\ m,\ V^* = 5.6\ m/\sec, \lambda_1 = 8.4\ m/\sec,$$
$$\lambda_2 = 2.8\ m/\sec, \underline{\gamma = 0},\ \delta = 0.1372\ \sec^{-1}.$$

According to the conditions in Theorem 4.2, the stability of the system can be guaranteed if and only if the feedback parameters $k_i (i = 1, 2)$ satisfy:

$$\begin{aligned} & e^{0.0735} + k_1k_2e^{0.0245} > |k_1e^{0.0735} + k_2e^{0.0245}|, \quad e^{0.0735} > |k_1k_2e^{0.0245}|, \\ & e^{0.2205} - k_1k_2e^{0.1715} + k_1k_2(k_1^2e^{0.1715} - k_1k_2e^{0.049}) > |k_1k_2e^{0.1715} - k_2^2e^{0.1225}|. \end{aligned} \tag{5.1}$$

Now, we choose $k_1 = 0.5$, $k_2 = 0.6$, and it is easy to verify that they satisfy the condition (5.1). Figure 2 show the stability of the system (2.8) under initial condition $\xi_1(0,x) = 4\cos(2\pi x)$ and $\xi_2(0,x) = 2\cos(2\pi x)$.

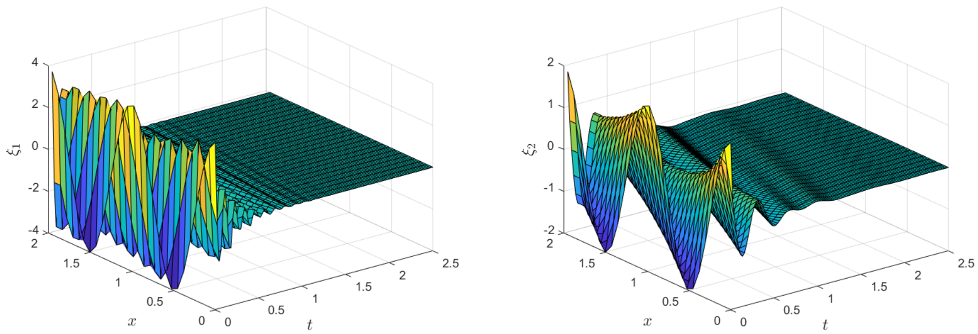

Figure 2. The convergence of state $\xi_1(t,x)$ and $\xi_2(t,x)$ for the case $\gamma = 0$.

References

[1] G. A. de Andrade, R. Vazquez, I. Karafyllis and M. Krstic, *Backstepping control of a hyperbolic PDE system with zero characteristic speed states*, IEEE Trans. Autom. Control, 2024, 69(10), 6988–6995.

[2] G. Bastin and J.-M. Coron, *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, Birkhäuser Cham, Switzerland, 2016.

[3] G. Bastin and J.-M. Coron, *Exponential stability of PI control for Saint-Venant equations with a friction term*, Methods Appl. Anal., 2019, 26(2), 101–112.

[4] G. Bastin, J.-M. Coron and B. d'Andréa Novel, *On lyapunov stability of linearised Saint-Venant equations for a sloping*, Netw. Heterog. Media, 2009, 4(2), 177–187.

[5] G. Bastin, J.-M. Coron and A. Hayat, *Feedforward boundary control of* 2×2 *nonlinear hyperbolic systems with application to Saint-Venant equations*, Eur. J. Control., 2021, 57, 41–53.

[6] R. Benzaid and A. Benaissa. *Indirect boundary stabilization for weakly coupled degenerate wave equations under fractional damping*, J. Appl. Anal. Comput., 2024, 14(3), 1735–1770.

[7] J. Deutscher, N. Gehring and N. Jung, *Backstepping control of coupled general hyperbolic-parabolic PDE-PED systems*, IEEE Trans. Autom. Control, 2023, 69(5), 3465–3472.

[8] A. Diagne, M. Diagne, S. Tang and M. Krstic, *Backstepping stabilization of the linearized Saint-Venant–Exner model*, Automatica, 2017, 76, 345–354.

[9] M. Gugat, G. Leugering and E. J. P. G. Schmidt, *Global controllability between steady supercritical flows in channel networks*, Math. Methods Appl. Sci., 2004, 27(7), 781–802.

[10] B. Guo, Y. Xiao and Y. Ban, *Orbital stability of solitary waves for the nonlinear Schrödinger-KdV system*, J. Appl. Anal. Comput., 2022, 12(1), 245–255.

[11] A. Hayat, *On boundary stability of inhomogeneous* 2×2 *1-D hyperbolic systems for the* C^1 *norm*, ESAIM: Control Optim. Calc. Var., 2019, 25(82).

[12] A. Hayat, Y. Hu and P. Shang, *PI control for the cascade channels modeled by general Saint-Venant equations*, IEEE Trans. Autom. Control, 2023, 69(8), 4974–4987.

[13] A. Hayat and P. Shang, *A quadratic lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope*, Automatica, 2019, 100, 52–60.

[14] J. P. LaSalle, *The Stability and Control of Discrete Processes*, Springer-Verlag, New York, 1986.

[15] E. Somathilake and M. Diagne, *Output feedback control of suspended sediment load entrainment in water canals and reservoirs*, IFAC-PapersOnLine, 2024, 58(28), 983–988.

[16] H. Yu and M. Krstic, *Traffic Congestion Control by PDE Backstepping*, Birkhäuser Cham, Switzerland, 2022.

[17] B. Zhang, F. Zheng and Y. He, *Uniformly exponentially stable approximation for the transmission line with variable coefficients and its application*, J. Appl. Anal. Comput., 2024, 14(4), 2228–2256.

[18] D. Zhao and F. Bao, *The stability analysis for a* 2×2 *conservation law with PI controller and PDP controller*, J. Math., 2023, 2023, 4382517.

[19] D. Zhao, D. Fan and Y. Guo, *The spectral analysis and exponential stability of a 1-d* 2×2 *hyperbolic system with proportional feedback control*, Int. J. Control, Autom. Syst., 2022, 20(8), 2633–2640.

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