

# GROUND STATE ROTATING PERIODIC SOLUTIONS FOR SECOND-ORDER HAMILTONIAN SYSTEMS WITH “PINCHED” CONDITION

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**Abstract** This paper aims to investigate the existence of ground state rotating periodic solutions for second-order Hamiltonian systems. The rotating periodic solution may be periodic, anti-periodic, subharmonic or quasi periodic based on the forms of the orthogonal matrix. By using Nehari manifold and perturbation method, we obtain existence results under the nondecreasing monotone assumption and “pinched” condition. Our results extend some existing relevant work.

**Keywords** Hamiltonian systems, ground state rotating periodic solutions, Nehari manifold, perturbation method.

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## 1. Introduction

In this paper, we study the existence of ground state rotating periodic solutions for the following second-order Hamiltonian system:

$$u''(t) + \nabla V(t, u) = 0, \quad (1.1)$$

where  $V(t, x) = -K(t, x) + H(t, x) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ ,  $K(t+T, x) = K(t, Q^{-1}x)$  and  $H(t+T, x) = H(t, Q^{-1}x)$  with  $K(t, 0) = 0 = H(t, 0)$  for all  $t \in \mathbb{R}$  and  $x \in \mathbb{R}^N$ ,  $Q \in O(N)$ . Here,  $O(N)$  denotes the orthogonal group on  $\mathbb{R}^N$ .

We say  $u(t)$  is a rotating periodic solution to (1.1), if  $u(t)$  satisfies (1.1) and  $u(t+T) = Qu(t)$ ,  $\forall t \in \mathbb{R}$ . In fact, the rotating periodic solution contains several types of solutions based on the forms of the orthogonal matrix  $Q$ , that is, it may be periodic solution if  $Q = I$ , anti-periodic solution if  $Q = -I$ , subharmonic solution if  $Q^k = I$ , quasi-periodic solution if  $Q^k \neq I$ , where  $I$  is the unit matrix and  $k \in \mathbb{N}$  with  $k \geq 2$ . It is important to note that the rotating periodic problem not only encompasses the periodic problem, but also exhibits significant differences from it in certain cases. Hence, the rotating periodic problems are more general.

To use the variational method, we show that the rotating periodic solutions to the system (1.1) are equivalent to the solutions to the following rotating periodic boundary value problem:

$$\begin{cases} u''(t) - \nabla K(t, u) + \nabla H(t, u) = 0, \\ u(T) = Qu(0), u'(T) = Qu'(0). \end{cases} \quad (1.2)$$

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In fact, if  $u(t)$  is a solution to (1.2), then

$$\begin{aligned} (Q^{-1}u(t+T))'' &= Q^{-1}u''(t+T) \\ &= Q^{-1}(\nabla K(t+T, u(t+T)) - \nabla H(t+T, u(t+T))) \\ &= \nabla K(t, Q^{-1}u(t+T)) - \nabla H(t, Q^{-1}u(t+T)), \end{aligned}$$

which yields  $Q^{-1}u(t+T)$  is also a solution to (1.1). Combined with the boundary value conditions  $u(T) = Qu(0), u'(T) = Qu'(0)$ , we have  $Q^{-1}u(t+T) = u(t)$ (i.e.,  $u(t+T) = Qu(t)$ ) by the uniqueness of solution.

As a natural extension of periodic problem, the rotating periodic problems have attracted the interest of many scholars in recent years, and obtained many good results. In [1], Chang and Li established the existence of rotating periodic solutions for a class of second-order dissipative dynamical systems using the coincidence degree theory. Subsequently, they explored the existence of rotating periodic solutions for singular second-order dissipative dynamical systems (see [2]). In [5], Liu, Li, and Yang employed Morse theory to obtain the multiplicity of rotating periodic solutions for a class of asymptotically linear second-order Hamiltonian systems. In [14], using the fountain theorem, Shen and Liu obtained infinitely many rotating periodic solutions for super-linear second-order impulsive Hamiltonian systems. For more results on rotating periodic solutions, please refer to [6–8, 11, 17, 18] and references therein. However, there is little research on ground state rotating periodic solutions to second-order Hamiltonian systems.

In order to present our main conclusions, we provide the following conditions:

( $K_1$ ) There exist constants  $a_1, a_2 > 0$  such that

$$a_1|x|^2 \leq K(t, x) \leq a_2|x|^2,$$

for all  $(t, x) \in [0, T] \times \mathbb{R}^N$ .

( $K_2$ ) For any given  $x \in \mathbb{R}^N \setminus \{0\}$ , the mapping

$$s \rightarrow \frac{\nabla K(t, sx) \cdot x}{s}$$

is non-increasing on  $(0, +\infty)$ .

( $H_1$ ) For any given  $x \in \mathbb{R}^N \setminus \{0\}$ , the mapping

$$s \rightarrow \frac{\nabla H(t, sx) \cdot x}{s}$$

is strictly increasing on  $(0, +\infty)$ .

( $H_2$ ) As  $|x| \rightarrow 0$ ,  $\nabla H(t, x) = o(|x|)$  uniformly for  $t \in [0, T]$ .

( $H_3$ ) There are  $a_3 \geq 0, r_1 > 0, \mu > 2$  and  $0 \leq \gamma \leq 2$  such that

$$\nabla H(t, x) \cdot x - \mu H(t, x) \geq -a_3|x|^\gamma,$$

uniformly for  $|x| \geq r_1$  and  $t \in [0, T]$ .

( $H_4$ )  $\lim_{|x| \rightarrow +\infty} \frac{H(t, x)}{|x|^2} = +\infty, t \in [0, T]$ .

( $H_1$ )' For any given  $x \in \mathbb{R}^N \setminus \{0\}$ , the mapping

$$s \rightarrow \frac{\nabla H(t, sx) \cdot x}{s}$$

is non-decreasing on  $(0, +\infty)$ .

**Remark 1.1.** (i) The condition  $(K_1)$  is called “pinched” condition, as seen in [4, 9, 10, 19].

(ii) Note that the conditions  $(H_3)$  and  $(H_4)$  are weaker than the classical Ambrosetti-Rabinowitz (AR) condition [13]: There is a constant  $\mu > 0$  such that

$$0 < \mu H(t, x) \leq \nabla H(t, x) \cdot x,$$

uniformly for  $t \in [0, T]$  and  $x \in \mathbb{R}^N \setminus \{0\}$ . The (AR) condition implies  $H(t, x) \geq c|x|^\mu > 0$ , which yields condition  $(H_4)$ .

(iii) It is evident that condition  $(H_1)'$  is weaker than condition  $(H_1)$ , as it no longer requires strict increasing.

In [4], under the conditions  $(K_1)$ ,  $K(t, x) \leq \nabla K(t, x) \cdot x \leq 2K(t, x)$  for all  $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ ,  $(H_2)$  and, (AR) condition, Izydorek and Janczewska investigated the existence of periodic solution and homoclinic solution for second order Hamiltonian system

$$u''(t) - \nabla K(t, u) + \nabla H(t, u) = f(t),$$

where  $f$  is a continuous and bounded function, and satisfies other suitable condition. Furthermore, Liu and Guo [9] considered the periodic solution for perturbed second order Hamiltonian system

$$\begin{cases} u''(t) - \nabla K(u) + \nabla H(u) = \nabla G(t, u), \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$

where  $K$  satisfies  $(K_1)$  and (AR) condition,  $H$  satisfies (AR) condition and other suitable condition, and  $G$  is subquadratic at the infinity. Subsequently, they [10] studied the multiplicity of periodic solution for perturbed second order Hamiltonian system

$$\begin{cases} u''(t) - \nabla K(t, u) + \nabla H(t, u) = \nabla G(t, u), \\ u(0) = u(T), u'(0) = u'(T), \end{cases}$$

where  $K$  satisfies  $(K_1)$  and (AR) condition,  $H$  satisfies local superquadratic condition, and  $G$  is subquadratic at the infinity. Meanwhile, monotonicity conditions  $(K_2)$  and  $(H_1)$  are a standard assumption in the study of ground state solutions, see [12, 15, 16].

At the end of this section, we present the main results.

**Theorem 1.1.** *Assume the conditions  $(K_1)$ - $(K_2)$  and  $(H_1)$ - $(H_4)$  hold. Then the problem (1.2) has at least one ground state rotating periodic solution; moreover, if  $K(t, x)$  and  $H(t, x)$  are even in  $x$ , then the problem (1.2) possesses infinitely many solutions.*

**Theorem 1.2.** *Assume the conditions  $(K_1)$ - $(K_2)$ ,  $(H_1)'$  and  $(H_2)$ - $(H_4)$  hold. Then the problem (1.2) has at least one ground state rotating periodic solution.*

The paper is organized as follows: Section 2 introduces some preliminary concepts. In Section 3, we prove our main results.

## 2. Preliminaries

In this section, we give some Banach spaces, basic concepts, and lemmas.

Throughout the paper,  $a \cdot b$  denotes the inner product for  $a, b \in \mathbb{R}^N$  and  $|a|$  denotes the Euclidean norm on  $\mathbb{R}^N$ . Let  $I$  be the identity operator and  $O(N)$  denote the orthogonal group on  $\mathbb{R}^N$ .

Let us review some Banach spaces:

- $C = C^0([0, T], \mathbb{R}^N)$  with the norm

$$\|u\|_0 = \max_{t \in [0, T]} |u(t)|;$$

- $C^1 = C^1([0, T], \mathbb{R}^N)$  with the norm

$$\|u\|_1 = \max\{\|u\|_0, \|u'\|_0\};$$

- $L^p = L^p((0, T); \mathbb{R}^N)$  ( $1 \leq p < +\infty$ ) with the norm

$$\|u\|_{L^p} = \left( \int_0^T |u(t)|^p dt \right)^{\frac{1}{p}}.$$

For convenience, set

$$C_Q^1 = \{u \in C^1([0, T], \mathbb{R}^N) : u(T) = Qu(0), u'(T) = Qu'(0)\},$$

where  $Q \in O(N)$ .

Let  $E := \{u \in H^1([0, T]; \mathbb{R}^N) : u(T) = Qu(0)\} \subset H^1$  with the inner product

$$\langle u, v \rangle = \int_0^T [u(t) \cdot v(t) + u'(t) \cdot v'(t)] dt,$$

and corresponding norm  $\|u\|^2 = \langle u, u \rangle$ , where  $Q \in O(N)$ .

**Lemma 2.1** ([18]). *The space  $E$  is a Hilbert space, and the embedding  $E \hookrightarrow C$  and  $E \hookrightarrow L^2$  are compact. Then there exists a constant  $c_1 > 0$  such that  $\|u\|_0 \leq c_1 \|u\|_E$  for  $\forall u \in E$ .*

Define the functional  $J : E \rightarrow \mathbb{R}$  by

$$J(u) = \int_0^T \left[ \frac{1}{2} (|u'(t)|^2) + K(t, u) - H(t, u) \right] dt, \quad \forall u \in E. \tag{2.1}$$

From  $K, H \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R})$ , it follows that  $J \in C^1(E, \mathbb{R})$  and

$$J'(u)v = \int_0^T [u'(t) \cdot v'(t) + \nabla K(t, u) \cdot v(t) - \nabla H(t, u) \cdot v(t)] dt, \quad \forall u, v \in E. \tag{2.2}$$

Assume  $u \in E$  is a critical point of  $J$ , that is,  $J'(u)v = 0$  for any  $v \in E$ . By the Proposition 3.4 in [18], we obtain that  $u$  is a classical solution to (1.2), which allows us to apply the variational method to investigate the existence of rotating periodic solutions to the system (1.1).

**Definition 2.1.** The  $u \in E$  is called a ground state rotating periodic solution to (1.1), if  $u$  is a nontrivial critical point of  $J$  with least possible energy, that is,  $u \in E_0$  and

$$J(u) = \inf_{v \in E_0} J(v),$$

where  $E_0 = \{v \in E \setminus \{0\} \mid J'(v)w = 0, \forall w \in E\}$ .

**Definition 2.2** ([13]). Let  $E$  be a Banach space and the functional  $J \in C^1(E, \mathbb{R})$ . If a sequence  $\{u_n\} \subset E$  satisfies the conditions:  $\{J(u_n)\}$  is bounded, and  $J'(u_n) \rightarrow 0$ , then the sequence  $\{u_n\}$  is called a Palais-Smale (PS for short) sequence. If every (PS) sequence has a convergent subsequence, then the functional  $J$  is said to satisfy the Palais-Smale condition, denoted as the (PS) condition.

**Lemma 2.2** ([16]). Let  $E$  be an infinite-dimensional Banach space. If the functional  $J \in C^1(\mathcal{S}, \mathbb{R})$  is bounded from below, even, and satisfies the (PS) condition, then the functional  $J$  has infinitely many pairs of critical points, where  $\mathcal{S} = \{u \in E : \|u\|_E = 1\}$ .

### 3. Proof of the main results

In this section, we prove the main results. First, we define a Nehari manifold.

Define the functional  $\varphi : E \rightarrow \mathbb{R}$  by

$$\varphi(u) = J'(u)u = \int_0^T [|u'(t)|^2 + \nabla K(t, u) \cdot u(t) - \nabla H(t, u) \cdot u(t)] dt. \quad (3.1)$$

Define Nehari manifold  $\mathcal{N}$  by

$$\mathcal{N} = \{u \in E \setminus \{0\} \mid \varphi(u) = 0\}.$$

Next, we give some lemmas.

**Lemma 3.1.** Suppose  $(K_2)$  and  $(H_1)$  hold. Then

$$\nabla K(t, x) \cdot x \leq 2K(t, x) \quad (3.2)$$

and

$$\nabla H(t, x) \cdot x > 2H(t, x), \quad (3.3)$$

for  $x \in \mathbb{R}^N \setminus \{0\}$  and  $t \in [0, T]$

**Proof.** For  $x \in \mathbb{R}^N \setminus \{0\}$ , set  $s = |x|$  and  $y = \frac{x}{|x|}$ . Then we have  $|y| = 1$  and  $x = sy$ . It follows from  $(K_2)$  that

$$\begin{aligned} K(t, x) &= K(t, sy) \\ &= \int_0^s \nabla K(t, \tau y) \cdot y d\tau \\ &= \int_0^s \frac{\nabla K(t, \tau y) \cdot y}{\tau} \tau d\tau \\ &\geq \frac{\nabla K(t, sy) \cdot y}{s} \int_0^s \tau d\tau \\ &= \frac{1}{2} \nabla K(t, sy) \cdot sy \\ &= \frac{1}{2} \nabla K(t, x) \cdot x, \end{aligned}$$

which implies

$$2K(t, x) \geq \nabla K(t, x) \cdot x.$$

Similarly, we can prove the (3.3).

As  $x = 0$ , it is obvious that

$$K(t, 0) = 0 = \nabla K(t, 0) \cdot 0$$

and

$$H(t, 0) = 0 = \nabla H(t, 0) \cdot 0.$$

□

**Lemma 3.2.** *Assume the conditions  $(K_1)$ - $(K_2)$  and  $(H_1)$ - $(H_4)$  hold. Then for any given  $u \in E \setminus \{0\}$ , there is unique  $s_u > 0$  such that*

$$s_u u \in \mathcal{N}.$$

**Proof.** For any given  $u \in E \setminus \{0\}$ , let  $g(s) := J(su)$ ,  $s > 0$ . It is easy to show

$$g'(s) = J'(su)u = \frac{1}{s}J'(su)su.$$

Hence,

$$g'(s) = 0 \Leftrightarrow su \in \mathcal{N}.$$

We first prove that there exist constants  $\rho, \sigma > 0$  such that  $J(u) > 0$  for  $\forall u \in B_\rho(0) \setminus \{0\}$ , and  $J(u) \geq \sigma$  for  $\forall u \in \partial B_\rho(0)$ , where  $B_\rho(0) := \{u \in E \mid \|u\|_E < \rho\}$ .

Let  $b_1 = \min\{\frac{1}{2}, a_1\}$ . By  $(H_2)$ , for  $\varepsilon = \frac{b_1}{2}$ , there is a  $\delta > 0$  such that

$$H(t, x) \leq \varepsilon|x|^2, \tag{3.4}$$

for  $|x| \leq \delta$  and  $t \in [0, T]$ . For  $u \in E \setminus \{0\}$  with  $\|u\|_E \leq \frac{\delta}{c_1}$ , we have  $0 < \|u\|_0 \leq \delta$  by the Lemma 2.1. According to (2.1),  $(H_1)$ , and (3.4), we get that

$$\begin{aligned} J(u) &= \int_0^T [\frac{1}{2}(|u'(t)|^2) + K(t, u) - H(t, u)]dt \\ &\geq \int_0^T [\frac{1}{2}(|u'(t)|^2) + a_1|u(t)|^2 - \varepsilon|u(t)|^2]dt \\ &\geq b_1\|u\|_E^2 - \varepsilon\|u\|_E^2. \end{aligned} \tag{3.5}$$

Hence,  $J(u) \geq \frac{b_1}{2}\|u\|_E^2$ . Set  $\rho = \frac{\delta}{c_1}$  and  $\sigma = \frac{b_1\delta^2}{2c_1^2}$ . Then  $J(u) \geq \sigma$  for  $u \in \partial B_\rho(0)$ . Due to (3.5), 0 is a strict local minimum of the functional  $J$ . Therefore, there is  $s_1 > 0$  such that  $g(s) > 0$  when  $0 < s < s_1$ .

Next, we prove that for any given  $u \in E \setminus \{0\}$ , there exists  $s_2 > 0$  such that  $g(s) < 0$  when  $s > s_2$ .

For given  $u \in E \setminus \{0\}$ , we have

$$\delta_1 = \frac{1}{T} \int_0^T |u(t)|^2 dt > 0.$$

Choose  $\delta_2 \in (0, \delta_1)$ . Write

$$\Omega_1 = \left\{ t \in [0, T] \mid |u(t)|^2 \geq \delta_2 \right\}$$

and  $\Omega_2 = [0, T] \setminus \Omega_1$ . Obviously,  $\text{meas}(\Omega_1) \geq \delta_3 > 0$ . Set  $\Omega_3 = \{t \in \Omega_2 \mid u(t) \equiv 0\}$  and  $\Omega_4 = \Omega_2 \setminus \Omega_3$ .

Let  $b_2 = \max\{\frac{1}{2}, a_2\}$  and  $M = \frac{2b_2\|u\|_E^2}{\delta_2\delta_3}$ . By  $(H_4)$ , there is  $r_M > 0$  such that

$$H(t, x) \geq M|x|^2,$$

for  $|x| \geq r_M$  and  $t \in [0, T]$ . Let

$$g_1(s) = \frac{s^2}{2} \int_0^T |u'(t)|^2 dt + \int_0^T K(t, su) dt - \int_{\Omega_1} H(t, su) dt$$

and

$$g_2(s) = - \int_{\Omega_2} H(t, su) dt.$$

Then,

$$g(s) = g_1(s) + g_2(s).$$

Take  $s_3 > \frac{r_M}{\sqrt{\delta_2}}$ . Then as  $t \in \Omega_1$ , we get  $s_3 |u(t)| \geq r_M$ . Combined with condition  $(K_1)$ , we obtain

$$g_1(s) \leq b_2 \|u\|_E^2 s^2 - M\delta_2\delta_3 s^2 = -b_2 \|u\|_E^2 s^2,$$

when  $s > s_3$ . Then condition  $H(t, 0) = 0$  yields  $\int_{\Omega_3} H(t, su) dt = 0$ . According to the continuity of  $H(t, x)$  and  $(H_4)$ , there are  $b_3, b_4 > 0$  such that

$$H(t, x) \geq b_3|x|^2 - b_4,$$

for  $(t, x) \in [0, T] \times \mathbb{R}^N$ . Then,

$$g_2(s) \leq -b_3 s^2 \int_{\Omega_4} |u(t)|^2 dt + b_4 \text{meas}(\Omega_4) < b_4 T.$$

Hence,

$$g(s) = g_1(s) + g_2(s) < -b_2 \|u\|_E^2 s^2 + b_4 T.$$

Set  $s_2 = \max\{s_3, \sqrt{\frac{b_4 T}{b_2 \|u\|_E^2}} + 1\}$ . Then  $g(s) < 0$ , when  $s \geq s_2$ . Then we obtain that the function  $g(s)$  takes the maximum value in  $(0, s_2)$ , namely, there is  $s_u$  such that

$$g(s_u) = \max_{s>0} g(s) = \max_{s>0} J(su),$$

which means  $g'(s_u) = 0$ , that is,  $s_u u \in \mathcal{N}$ .

Finally, we prove the uniqueness of  $s_u$ .

By calculation, we get that

$$\begin{aligned} g'(s) &= J'(su)u \\ &= \int_0^T [s|u'(t)|^2 + \nabla K(t, su) \cdot u - \nabla H(t, su) \cdot u] dt \\ &= s \left( \int_0^T [|u'(t)|^2 + \frac{\nabla K(t, su) \cdot u}{s} - \frac{\nabla H(t, su) \cdot u}{s}] dt \right). \end{aligned} \tag{3.6}$$

In the light of  $(K_2)$  and  $(H_1)$ , the mapping

$$\left( \int_0^T [|u'(t)|^2 + \frac{\nabla K(t, su) \cdot u}{s} - \frac{\nabla H(t, su) \cdot u}{s}] dt \right)$$

is strictly monotonically decreasing with respect to  $s$  on  $(0, +\infty)$ , which yields the uniqueness of  $s_u$ . □

**Lemma 3.3.** *Set  $x \in \mathbb{R}^N \setminus \{0\}$ . If the conditions  $(K_2)$  and  $(H_1)$  hold, then*

$$\begin{aligned} f(s) &= \nabla K(t, x) \cdot \left( \frac{s^2 - 1}{2} x \right) - K(t, sx) + K(t, x) \\ &\quad - \nabla H(t, x) \cdot \left( \frac{s^2 - 1}{2} x \right) + H(t, sx) - H(t, x) \\ &\geq 0, \quad \forall t \in [0, T]. \end{aligned}$$

For  $s \geq 0$ ; moreover, if  $s \neq 1$ , then  $f(s) > 0$ .

**Proof.** Through calculation, we obtain that  $f(1) = 0$  and

$$f(0) = K(t, x) - \frac{1}{2} \nabla K(t, x) \cdot x + \frac{1}{2} \nabla H(t, x) \cdot x - H(t, x).$$

It follows from Lemma 3.1 that  $f(0) > 0$ . Differentiating  $f(s)$  with respect to  $s$  results in

$$\begin{aligned} f'(s) &= \nabla K(t, x) \cdot sx - \nabla K(t, sx) \cdot x + \nabla H(t, sx) \cdot x - \nabla H(t, x) \cdot sx \\ &= s \left( \nabla K(t, x) \cdot x - \frac{\nabla K(t, sx) \cdot x}{s} \right) + s \left( \frac{\nabla H(t, sx) \cdot x}{s} - \nabla H(t, x) \cdot sx \right). \end{aligned}$$

Due to the conditions  $(K_2)$  and  $(H_1)$ , we have that

$$s \left( \nabla K(t, x) \cdot x - \frac{\nabla K(t, sx) \cdot x}{s} \right) \leq 0$$

and

$$s \left( \frac{\nabla H(t, sx) \cdot x}{s} - \nabla H(t, x) \cdot sx \right) < 0,$$

as  $0 < s < 1$ , and

$$s \left( \nabla K(t, x) \cdot x - \frac{\nabla K(t, sx) \cdot x}{s} \right) \geq 0$$

and

$$s \left( \frac{\nabla H(t, sx) \cdot x}{s} - \nabla H(t, x) \cdot sx \right) > 0,$$

as  $s > 1$ . Therefore,  $f'(s) < 0$ , as  $0 < s < 1$ ;  $f'(s) > 0$ , as  $s > 1$ . It is easy to get  $f'(1) = 0$ . Hence,  $f(s)$  attains minimum value at  $s = 1$ , which implies  $f(s) > 0$ , when  $s \geq 0$  and  $s \neq 1$ . □

**Lemma 3.4.** *Assume that all the conditions of Theorem 1.1 are satisfied. Then*

$$J(u) \geq J(su), \quad s \geq 0$$

and

$$J(u) \geq \sigma,$$

for  $u \in \mathcal{N}$ , where  $\sigma$  is given in Lemma 3.2.

**Proof.** Because  $u \in \mathcal{N}$ , we have  $J'(u)u = 0$ , i.e.,

$$\int_0^T [|u'(t)|^2 + \nabla K(t, u) \cdot u(t) - \nabla H(t, u) \cdot u(t)] dt = 0,$$

which yields

$$J(u) - J(su) = \int_0^T f(s) ds.$$

By Lemma 3.3, we obtain that  $J(u) - J(su) \geq 0$ . Clearly, there exists  $s_1 > 0$  such that  $s_1u \in \partial B_\rho(0)$ . By combining the proof of Lemma 3.2, we can conclude that

$$J(u) \geq J(s_1u) \geq \sigma.$$

□

**Lemma 3.5.** *Assume that all the conditions of Theorem 1.1 hold. Then the functional  $J$  is coercive on  $\mathcal{N}$ .*

**Proof.** By contradiction, assume that there exists a sequence  $\{u_n\} \subset \mathcal{N}$  such that  $\sigma \leq J(u_n) \leq b_5$  as  $\|u_n\|_E \rightarrow \infty$ , where  $b_5 > 0$ .

It follows from (2.1) that

$$\begin{aligned} \int_0^T H(t, u_n) dt &= \int_0^T [\frac{1}{2}|u'_n|^2 + K(t, u_n)] dt - J(u_n) \\ &\leq b_2 \|u_n\|_E^2 + b_5, \end{aligned} \tag{3.7}$$

which means

$$\int_0^T \frac{H(t, u_n)}{\|u_n\|_E^2} dt \leq b_2 + o(1), \tag{3.8}$$

for sufficiently large  $n$ . On the other hand, because  $H(t, x) \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R})$ , there exists  $b_6 > 0$  such that

$$|\nabla H(t, x) \cdot x - \mu H(t, x)| \leq b_6,$$

for  $|x| \leq r_1$  and  $t \in [0, T]$ . Combining the condition  $(H_3)$ , we have

$$\nabla H(t, x) \cdot x - \mu H(t, x) \geq -a_3|x|^\gamma - b_6,$$

for  $(t, x) \in [0, T] \times \mathbb{R}^N$ . Due to  $\varphi(u_n) = 0$ ,  $(K_1)$  and Lemma 3.1, we obtain that

$$\begin{aligned} J(u_n) &= \int_0^T [\frac{1}{2}(|u'_n(t)|^2) + K(t, u_n) - H(t, u_n)] dt \\ &\geq \int_0^T [\frac{1}{2}(|u'_n(t)|^2) + K(t, u_n) - \frac{1}{\mu} \nabla H(t, u_n) \cdot u_n] dt - \frac{a_3}{\mu} \int_0^T |u_n(t)|^\gamma dt - \frac{b_6 T}{\mu} \\ &\geq \frac{\mu - 2}{2\mu} \int_0^T |u'_n(t)|^2 dt + \frac{\mu - 2}{\mu} \int_0^T K(t, u_n) dt - \frac{a_3}{\mu} \int_0^T |u_n(t)|^\gamma dt - \frac{b_6 T}{\mu} \\ &\geq b_7 \|u_n\|_E^2 - \frac{a_3 T}{\mu} \|u_n\|_0^\gamma - \frac{b_6 T}{\mu}, \end{aligned} \tag{3.9}$$

where  $b_7 = \min\{\frac{\mu-2}{2\mu}, \frac{(\mu-2)a_1}{\mu}\}$ . Noting that  $J(u_n) \leq b_5$ , we have

$$b_5 \geq b_7 \|u_n\|_E^2 - \frac{a_3 T}{\mu} \|u_n\|_0^\gamma - \frac{b_6 T}{\mu}. \tag{3.10}$$

If  $a_3 = 0$  or  $0 \leq \gamma < 2$ , then  $\|u_n\|_E$  is bounded by (3.10), which is a contradiction.

Next assume that  $a_3 \neq 0$  and  $\gamma = 2$ .

Let  $v_n = \frac{u_n}{\|u_n\|_E}$ . Then  $\{v_n\}$  is bounded in  $E$ . Hence, there exist  $v_1 \in E$  and subsequence of the sequence  $\{v_n\}$ , which is still denoted as  $\{v_n\}$ , such that  $v_n \rightharpoonup v_1$  in  $E$ , and  $v_n \rightarrow v_1$  in  $C$ .

According to (3.10), there is  $b_8 > 0$  such that

$$\|v_1\|_0 = \lim_{n \rightarrow \infty} \|v_n\|_0 = \lim_{n \rightarrow \infty} \frac{\|u_n\|_0}{\|u_n\|_E} \geq b_8, \tag{3.11}$$

which shows  $v_1 \neq 0$ . Set

$$\Omega_5 = \left\{ t \in [0, T] \mid |v_1(t)| \geq \frac{b_8}{2} \right\}$$

and  $\Omega_6 = [0, T] \setminus \Omega_5$ . Due to the continuity of  $v_1(t)$  and (3.11), there exists  $\delta_4 > 0$  such that  $\text{meas}(\Omega_5) \geq \delta_4$ . Because  $v_n \rightarrow v_1$  in  $C$ , there is  $N > 0$  such that as  $n > N$ ,  $v_n(t) \geq \frac{b_8}{4}$  in  $\Omega_5$ . Hence, when  $n \rightarrow \infty$ , we have

$$|u_n| = |v_n| \|u_n\|_E \rightarrow +\infty$$

in  $\Omega_5$ . From  $(H_4)$ , there is a constant  $b_9 > 0$  such that as  $|x| \geq b_9$ , one has

$$H(t, x) > 0, \tag{3.12}$$

for  $t \in [0, T]$ . Together with Fatou Lemma, we obtain that

$$\liminf_{n \rightarrow \infty} \int_{\Omega_5} \frac{H(t, u_n)}{\|u_n\|_E^2} dt \geq \int_{\Omega_5} \liminf_{n \rightarrow \infty} \frac{H(t, u_n)}{|u_n|^2} |v_n|^2 dt \rightarrow +\infty.$$

By (3.12) and  $(H_2)$ , there are constants  $b_{10}, b_{11} > 0$  such that

$$H(t, x) \geq -b_{10}|x|^2 - b_{11},$$

for  $(t, x) \in [0, T] \times \mathbb{R}^N$ , which implies

$$\int_{\Omega_6} \frac{H(t, u_n)}{\|u_n\|_E^2} dt \geq \int_{\Omega_6} \frac{-b_{10}|u_n|^2 - b_{11}}{\|u_n\|_E^2} dt \geq \int_0^T \frac{-b_{10}|u_n|^2 - b_{11}}{\|u_n\|_E^2} dt \geq -b_{10} - \frac{b_{11}}{\|u_n\|_E^2}.$$

Hence,

$$\liminf_{n \rightarrow \infty} \int_0^T \frac{H(t, u_n)}{\|u_n\|_E^2} dt = \liminf_{n \rightarrow \infty} \left( \int_{\Omega_5} \frac{H(t, u_n)}{\|u_n\|_E^2} dt + \int_{\Omega_6} \frac{H(t, u_n)}{\|u_n\|_E^2} dt \right) \rightarrow +\infty, \tag{3.13}$$

which contradicts (3.8). Therefore, the assumption is false, and this lemma holds. □

**Lemma 3.6.** *Assume that all the conditions of Theorem 1.1 hold. If  $\mathcal{V} \subset E \setminus \{0\}$  is a compact subset, then there is  $R > 0$  such that  $J|_{E(u) \setminus B_R} \leq 0$ , for all  $u \in \mathcal{V}$ , where  $E(u) = \{ru : r \in \mathbb{R}\}$  and  $B_R = \{u \in E : \|u\|_E < R\}$ .*

**Proof.** Without loss of generality, we may assume that the elements  $u$  in  $\mathcal{V}$  satisfy  $\|u\|_E = 1$ . By contradiction, suppose there exist sequences  $\{u_n\} \subset \mathcal{V}$  and  $\{s_n\} \subset \mathbb{R}$  such that  $v_n := s_n u_n \in E(u_n)$  and  $J(v_n) > 0$  for any  $n$ , and  $|s_n| \rightarrow +\infty, n \rightarrow \infty$ . Taking a subsequence, we may assume that  $u_n \rightarrow u \in E$ . It follows from (3.7) that

$$0 \leq \frac{J(v_n)}{s_n^2} \leq b_2 - \int_0^T \frac{H(t, s_n u_n)}{|s_n u_n|^2} |u_n|^2 dt.$$

Similar to the proof of (3.13), we have

$$0 \leq b_2 - \int_0^T \frac{H(t, s_n u_n)}{|s_n u_n|^2} |u_n|^2 dt \rightarrow -\infty,$$

which is a contradiction. Therefore, the assumption is false, and the original proposition is proven. □

Define mappings  $\tilde{m} : E \setminus \{0\} \rightarrow \mathcal{N}$  and  $m : \mathcal{S} \rightarrow \mathcal{N}$  by

$$\tilde{m}(u) := s_u u \quad \text{and} \quad m := \tilde{m}|_{\mathcal{S}}.$$

Define functional  $\tilde{\phi} : E \setminus \{0\} \rightarrow \mathbb{R}$  and  $\phi : \mathcal{S} \rightarrow \mathbb{R}$  by

$$\tilde{\phi}(u) := J(\tilde{m}(u)) \quad \text{and} \quad \phi := \tilde{\phi}|_{\mathcal{S}}.$$

**Lemma 3.7.** *Assume that all the conditions of Theorem 1.1 hold. Then*

(i) *the mapping  $\tilde{m}$  is continuous;*

(ii) *the mapping  $m$  is a homeomorphism, and its inverse is  $m^{-1}(u) = \frac{u}{\|u\|_E}$ .*

**Proof.** (i) Assume that  $u_n \rightarrow u \neq 0$ . It follows from Lemma 3.2 that  $\tilde{m}(tu) = \tilde{m}(u)$  for any constant  $t > 0$ . Hence, we may suppose that  $u_n \in \mathcal{S}$ . It is sufficient to show that  $\tilde{m}(u_n)$  has a subsequence, still denoted as  $\tilde{m}(u_n)$ , which converges to  $\tilde{m}(u)$ , that is,  $\tilde{m}(u_n) \rightarrow \tilde{m}(u)$ . Set  $\tilde{m}(u_n) = s_n u_n$ . According to Lemmas 3.2 and 3.6, we have that  $\{s_n\}$  is bounded, and satisfies  $s_n \geq \frac{\delta}{c_1}$  for all  $n$ . Hence, there is a subsequence such that  $s_n \rightarrow s > 0$ . Because  $\mathcal{N}$  is closed and  $\tilde{m}(u_n) \rightarrow su$ , then  $su \in \mathcal{N}$ . From Lemma 3.2, we deduce that  $\tilde{m}(u) = s_u u = su$ , which yields the mapping  $\tilde{m}$  is continuous.

(ii) The conclusion can be directly derived from the Lemmas 3.2 and 3.4, and conclusion (i). □

**Lemma 3.8.** *Assume that all the conditions of Theorem 1.1 hold. Then  $\tilde{\phi} \in C^1(E \setminus \{0\}, \mathbb{R})$  and*

$$\tilde{\phi}'(u)v = \frac{\|\tilde{m}(u)\|_E}{\|u\|_E} J'(\tilde{m}(u))v, \quad u, v \in E, \quad u \neq 0.$$

**Proof.** Let  $u \in E \setminus \{0\}$  and  $v \in E$ . By the Lemmas 3.2 and 3.4, and the Mean Value Theorem, we obtain

$$\begin{aligned} \tilde{\phi}(u + tv) - \tilde{\phi}(u) &= J(s_{u+tv}(u + tv)) - J(s_u u) \\ &\leq J(s_{u+tv}(u + tv)) - J(s_{u+tv} u) \\ &= J'(s_{u+tv}(u + \theta tv))s_{u+tv} tv, \end{aligned}$$

where  $|t|$  is sufficiently small and  $\theta \in (0, 1)$ . Similarly, we get that

$$\begin{aligned} \tilde{\phi}(u + tv) - \tilde{\phi}(u) &= J(s_{u+tv}(u + tv)) - J(s_u u) \\ &\geq J(s_u(u + tv)) - J(s_u u) \\ &= J'(s_u(u + \eta tv))s_u tv, \end{aligned}$$

where  $\eta \in (0, 1)$ . From Lemma 3.7, it follows that  $u \mapsto s_u$  is continuous. Therefore, we have that

$$\lim_{t \rightarrow 0} \frac{\tilde{\phi}(u + tv) - \tilde{\phi}(u)}{t} = s_u J'(s_u u)v = \frac{\|\tilde{m}(u)\|_E}{\|u\|_E} J'(\tilde{m}(u))v.$$

Hence, the Gâteaux-derivative of  $\tilde{\phi}$  is linearly bounded with respect to  $v$ , and is continuous with respect to  $u$ , namely,  $\tilde{\phi} \in C^1(E \setminus \{0\}, \mathbb{R})$ . □

**Lemma 3.9.** *Assume that all the conditions of Theorem 1.1 hold. Then*

(i)  $\phi \in C^1(\mathcal{S}, \mathbb{R})$  and

$$\phi'(u)v = \|m(u)\|_E J'(m(u))v, \quad v \in T_u \mathcal{S} = \{w \in E : \langle u, w \rangle_E = 0\}.$$

(ii)  $\{u_n\} \subset \mathcal{S}$  is a (PS) sequence of  $\phi$  if and only if  $\{m(u_n)\} \subset \mathcal{N}$  is a (PS) sequence of  $J$ .

(iii)  $u \in \mathcal{S}$  is a critical point of  $\phi$  if and only if  $m(u) \in \mathcal{N}$  is a critical point of  $J$ , with the corresponding critical values being the same, and moreover, there is

$$\inf_{\mathcal{S}} \phi = \inf_{\mathcal{N}} J.$$

(iv) If  $J$  is an even functional, then  $\phi$  is also an even functional.

**Proof.** (i) Conclusion (i) follows directly from Lemma 3.8.

(ii) Assume that  $\{u_n\} \in \mathcal{S}$  satisfies  $\sup_n \phi(u_n) = \sup_n J(m(u_n)) < +\infty$ . Let  $v_n := m(u_n) \in \mathcal{N}$ .

For any  $n$ , we have that

$$E = E(u_n) \oplus T_{u_n} \mathcal{S} = E(v_n) \oplus T_{u_n} \mathcal{S}$$

and  $J'(v_n)w = 0, \forall w \in E(v_n)$ , which yields  $J'(v_n) \in T_{u_n} \mathcal{S}$ . By further combining Conclusion (i), we obtain that

$$\|\phi'(u_n)\|_E = \sup_{w \in T_{u_n} \mathcal{S}, \|w\|_E=1} \phi'(u_n)w = \sup_{w \in T_{u_n} \mathcal{S}, \|w\|_E=1} \|v_n\|_E J'(v_n)w = \|v_n\|_E \|J'(v_n)\|_E. \tag{3.14}$$

Due to Lemmas 3.2 and 3.5, we get that  $\frac{\delta}{c_1} \leq \|v_n\|_E \leq \sup_n \|v_n\|_E < +\infty$ . Hence,  $\{u_n\} \subset \mathcal{S}$  is a (PS) sequence of  $\phi$  if and only if  $\{m(u_n)\} \subset \mathcal{N}$  is a (PS) sequence of  $J$ .

(iii) From (3.14), we know that  $\phi'(u) = 0$  if and only if  $J'(m(u)) = 0$ . The remaining conclusions are evident.

(iv) If  $J$  is an even functional, then  $s_u = s_{-u}$ . Hence,  $m(-u) = -m(u)$ . From the definition of  $\phi$ , we get that  $\phi$  is an even functional. □

**Lemma 3.10.** *Assume that all the conditions of Theorem 1.1 hold. Then the functional  $J$  satisfies the (PS) condition on the space  $\mathcal{N}$ , and the functional  $\phi$  satisfies the (PS) condition on the space  $\mathcal{S}$ .*

**Proof.** Assume that  $\{v_n\} \subset \mathcal{N}$  is a (PS) sequence of the functional  $J$ . From Lemma 3.5, we have that  $\{v_n\}$  is bounded. Therefore, there exist a subsequence still denoted by  $\{v_n\}$  and  $v_1 \in E$  such that  $v_n \rightharpoonup v_1$  in  $E$  and  $v_n \rightarrow v_1$  in  $L^2$ . According to the standard discussion, it follows that  $v'_n \rightarrow v'_1$  in  $L^2$ . Then we get that

$$\|v_n - v_1\|_E^2 = \|v_n - v_1\|_{L^2}^2 + \|v'_n - v'_1\|_{L^2}^2 \rightarrow 0, \quad n \rightarrow +\infty.$$

Hence,  $J$  satisfies the (PS) condition on the space  $\mathcal{N}$ .

Assume that  $\{u_n\} \subset \mathcal{S}$  is a (PS) sequence of the functional  $\phi$ . Due to Lemma 3.9(ii), we know that  $\{m(u_n)\} \subset \mathcal{N}$  is a (PS) sequence of the functional  $\phi$ . Thus, the sequence  $\{m(u_n)\}$  has a convergent subsequence still denoted by  $\{m(u_n)\}$ , i.e.,

$$m(u_n) \rightarrow v_0, \quad n \rightarrow +\infty.$$

From Lemma 3.7, it follows that

$$u_n \rightarrow m^{-1}(v_0), \quad n \rightarrow +\infty.$$

Hence,  $\phi$  satisfies the (PS) condition on the space  $\mathcal{S}$ . □

**Proof. (Proof of Theorem 1.1)** Set  $\omega = \inf_{\mathcal{N}} J$ . By Lemmas 3.4 and 3.9, we obtain that

$$\inf_{\mathcal{S}} \phi = \inf_{\mathcal{N}} J = \omega \geq \sigma. \tag{3.15}$$

Let  $\{u_n\} \subset \mathcal{S}$  be a minimizing sequence for  $\phi$ . According to Ekeland variational principle [3], we may suppose the sequence  $\{u_n\} \subset \mathcal{S}$  satisfies

$$\phi'(u_n) \rightarrow 0 \quad \text{and} \quad \phi(u_n) \rightarrow \omega, \quad n \rightarrow +\infty.$$

Thus,  $\{u_n\}$  is the (PS) sequence of the functional  $\phi$ . Due to Lemma 3.10, there exists a convergent subsequence of  $\{u_n\}$  that converges to  $u_0$ . Hence,  $u_0$  is a critical point of  $\phi$ . According to Lemma 3.9, we obtain that  $m(u_0) \in \mathcal{N}$  is a critical point of  $J$ . Obviously,  $J(m(u_0)) \geq \omega$ . Since  $E$  is compactly embedded into  $C$ , the functional  $J$  is weakly lower semi-continuous. From  $m(u_n) \rightarrow m(u_0)$ , we obtain that

$$\omega \leq J(m(u_0)) \leq \varliminf_{n \rightarrow \infty} J(m(u_n)) \leq \lim_{n \rightarrow \infty} J(m(u_n)) = \omega.$$

Hence,  $J(m(u_0)) = \omega$ . If  $v$  is a nontrivial critical point of the functional  $J$ , then we have that  $\varphi(v) = 0$ , that is  $v \in \mathcal{N}$ . Hence, the critical point  $m(u_0)$  is a ground state rotating periodic solution.

Further, if  $K(t, x)$  and  $H(t, x)$  are both even functions in  $x$ , then  $J$  is an even functional. According to Lemma 3.9,  $\phi$  is also an even functional. It follows from (3.15) that  $\phi$  is bounded from below. By Lemma 2.2, the functional  $\phi$  has infinitely many pairs of critical points. Furthermore, due to Lemma 3.9, the functional  $J$  also has infinitely many pairs of critical points. □

Based on Theorem 1.1, we prove Theorem 1.2 using the method of perturbation.

Let

$$H_n(t, x) = \frac{1}{n}|x|^\mu + H(t, x),$$

where the definition of  $\mu$  is given in  $(H_3)$  and  $n \geq 1$ . Obviously,

$$H_1(t, x) \geq H_n(t, x) \geq H(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^N \tag{3.16}$$

and

$$\nabla H_n(t, x) = \frac{\mu}{n}|x|^{\mu-2}x + \nabla H(t, x). \tag{3.17}$$

**Lemma 3.11.** *Assume that the conditions  $(K_1)$ - $(K_3)$ ,  $(H_1)'$  and  $(H_2)$ - $(H_4)$  hold. For any given  $n$ , the rotating periodic boundary value problem*

$$\begin{cases} u''(t) - \nabla K(t, u) + \nabla H_n(t, u) = 0, \\ u(T) = Qu(0), u'(T) = Qu'(0), \end{cases} \tag{3.18}$$

has at least one ground state solution.

**Proof.** From (3.17), we have

$$\frac{\nabla H_n(t, sx) \cdot x}{s} = \frac{\mu}{n}|x|^\mu s^{\mu-2} + \frac{\nabla H(t, sx) \cdot x}{s},$$

for  $x \in \mathbb{R}^N \setminus \{0\}$  and  $s \in (0, +\infty)$ . Because  $\mu > 2$ , the mapping  $s \rightarrow \frac{\mu}{n}|x|^\mu s^{\mu-2}$  is strictly increasing on  $(0, +\infty)$ . Furthermore, combining with  $(H_1)'$ , we get that  $H_n(t, x)$  satisfies the condition  $(H_1)$ . From  $\mu > 2$  and  $(H_2)$ , it follows that as  $|x| \rightarrow 0$ ,

$$\nabla H_n(t, x) = o(|x|),$$

uniformly for  $t \in [0, T]$ . Due to (3.16) and  $(H_4)$ , we get that

$$\lim_{|x| \rightarrow +\infty} \frac{H_n(t, x)}{|x|^2} = +\infty, \quad t \in [0, T].$$

Through simple calculations, we derive that

$$\begin{aligned} \nabla H_n(t, x) \cdot x - \mu H_n(t, x) &= \frac{\mu}{n}|x|^\mu + \nabla H(t, x) \cdot x - \frac{\mu}{n}|x|^\mu - \mu H(t, x) \\ &= \nabla H(t, x) \cdot x - \mu H(t, x). \end{aligned}$$

According to  $(H_3)$ , we get that

$$\nabla H_n(t, x) \cdot x - \mu H_n(t, x) \geq -a_3|x|^\gamma,$$

for  $|x| \geq r_1$  and  $t \in [0, T]$ . It follows from Theorem 1.1 that the lemma holds. □

Define a functional  $J_n : E \rightarrow \mathbb{R}$  by

$$\begin{aligned} J_n(u) &= \int_0^T \left[ \frac{1}{2}(|u'(t)|^2) + K(t, u) - H_n(t, u) \right] dt \\ &= \int_0^T \left[ \frac{1}{2}(|u'(t)|^2) + K(t, u) - \frac{1}{n}|u|^\mu - H(t, u) \right] dt, \quad \forall u \in E. \end{aligned} \tag{3.19}$$

Then

$$\begin{aligned} J'_n(u)v &= \int_0^T [u' \cdot v' + \nabla K(t, u) \cdot v - \nabla H_n(t, u) \cdot v] dt \\ &= \int_0^T [u' \cdot v' + \nabla K(t, u) \cdot v - \frac{\mu}{n}|u|^{\mu-2}u \cdot v - \nabla H(t, u) \cdot v] dt, \quad \forall u, v \in E. \end{aligned} \tag{3.20}$$

Define a functional  $\varphi_n : E \rightarrow \mathbb{R}$  by

$$\varphi_n(u) = \int_0^T [|u'|^2 + \nabla K(t, u) \cdot u - \nabla H_n(t, u) \cdot u] dt$$

and set

$$\mathcal{N}_n = \{u \in E \setminus \{0\} \mid \varphi_n(u) = 0\}.$$

Let

$$\omega_n = \inf_{u \in \mathcal{N}_n} J_n(u) = \inf_{u \in E \setminus \{0\}} \sup_{s > 0} J_n(su).$$

By Lemma 3.11, there exists  $y_n = s_n u_n \in \mathcal{N}_n$  such that  $J_n(y_n) = \omega_n$ .

**Lemma 3.12.** *Assume that all the conditions of Theorem 1.2 hold. Then  $\{\omega_n\}$  is a bounded sequence.*

**Proof.** For any given  $u \in E \setminus \{0\}$ , it is clear that

$$J_n(u) \geq J_1(u), \quad n \geq 1.$$

Due to  $\nabla H_1(t, x) = o(|x|)$ ,  $|x| \rightarrow 0$ , for  $\varepsilon = \frac{b_1}{2} > 0$ , there is  $\delta > 0$  such that as  $|x| \leq \delta$ , we have

$$H_1(t, x) \leq \varepsilon |x|^2,$$

uniformly for  $t \in [0, T]$ , where  $b_1 = \min\{\frac{1}{2}, a_1\}$ . For  $u \in E \setminus \{0\}$  satisfying  $\|u\|_E \leq \frac{\delta}{c_1}$ , we have  $0 < \|u\|_0 \leq \delta$ . Then

$$\begin{aligned} J_1(u) &= \int_0^T \left[ \frac{1}{2} (|u'(t)|^2) + K(t, u) - H_1(t, u) \right] dt \\ &\geq b_1 \|u\|_E^2 - \varepsilon \|u\|_E^2. \end{aligned}$$

Let  $\rho = \frac{\delta}{c_1}$ ,  $\sigma = \frac{b_1 \delta^2}{2c_1^2}$  and  $B_\rho(0) := \{u \in E \mid \|u\|_E < \rho\}$ . For  $u \in \partial B_\rho(0)$ , we have that  $J_1(u) \geq \sigma$ .

For given  $u \in E \setminus \{0\}$ , there exists  $s_0 > 0$  such that  $\|s_0 u\|_E = \rho$ . Hence,

$$\sup_{s > 0} J_1(su) \geq \sigma > 0,$$

which implies

$$\sup_{s > 0} J_n(su) \geq \sup_{s > 0} J_1(su) \geq \sigma.$$

Therefore,

$$\omega_n = \inf_{u \in E \setminus \{0\}} \sup_{s > 0} J_n(su) \geq \sigma.$$

On the other hand, let  $v_1(t) = \alpha \sin(\frac{2\pi}{T}t)$ , where  $\alpha = (1, 0, \dots, 0) \in \mathbb{R}^N$ . It is clear that  $v_1 \in E \setminus \{0\}$ . From (3.16), we have that

$$\omega_n = \inf_{u \in E \setminus \{0\}} \sup_{s > 0} J_n(su) \leq \sup_{s > 0} J_n(sv_1) \leq \sup_{s > 0} J(sv_1).$$

By Lemma 3.2, there is  $s_{v_1}$  such that  $J(s_{v_1} v_1) = \sup_{s > 0} J(sv_1) > 0$ , which yields  $\omega_n \leq M_1 = J(s_{v_1} v_1)$ . Hence,  $\{\omega_n\}$  is a bounded sequence. □

**Lemma 3.13.** *Assume that all the conditions of Theorem 1.2 hold. Then  $J'_n(y_n) = 0, \forall n \in \mathbb{N}$ .*

**Proof.** This result can be obtained from Lemma 3.11. □

**Lemma 3.14.** *Assume that all the conditions of Theorem 1.2 hold. Then  $\{y_n\}$  is bounded in  $E$ .*

**Proof.** By contradiction, suppose that  $\{y_n\}$  is unbounded in  $E$ , that is,  $\|y_n\|_E \rightarrow \infty$  as  $n \rightarrow \infty$ . According to (3.19) and Lemma 3.12, we obtain that

$$\begin{aligned} \int_0^T H_n(t, y_n) dt &= \int_0^T \left[ \frac{1}{2} |y'_n|^2 + K(t, y_n) \right] dt - J_n(y_n) \\ &\leq b_2 \|y_n\|_E^2. \end{aligned}$$

This means that when  $n$  is sufficiently large, we have that

$$\int_0^T \frac{H_n(t, y_n)}{\|y_n\|_E^2} dt \leq b_2. \tag{3.21}$$

On the other hand, since  $H_n(t, x) \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R})$ , there is  $M_2 > 0$  such that

$$|\nabla H_n(t, x) \cdot x - \mu H_n(t, x)| \leq M_2,$$

for  $|x| \leq r_1$  and  $t \in [0, T]$ . From the condition  $(H_3)$ , it follows that

$$\nabla H_n(t, x) \cdot x - \mu H_n(t, x) \geq -a_3 |x|^\gamma - M_2,$$

for  $(t, x) \in [0, T] \times \mathbb{R}^N$ . Similar to the process of obtaining (3.13), we prove that

$$\liminf_{n \rightarrow \infty} \int_0^T \frac{H_n(t, y_n)}{\|y_n\|_E^2} dt = \liminf_{n \rightarrow \infty} \left( \int_{\Omega_1} \frac{H_n(t, y_n)}{\|y_n\|_E^2} dt + \int_{\Omega_2} \frac{H_n(t, y_n)}{\|y_n\|_E^2} dt \right) \rightarrow +\infty,$$

which contradicts (3.21). Hence,  $\{y_n\}$  is bounded in  $E$ . Then there exist subsequence, still denoted by  $\{y_n\}$ , and  $z_0 \in E$  such that  $y_n \rightharpoonup z_0$ . □

**Lemma 3.15.** *Assume that all the conditions of Theorem 1.2 hold. Then as  $n \rightarrow \infty$ , one has*

$$J'_n(y_n) \rightarrow J'(z_0),$$

in  $E^*$ , where  $E^*$  is the dual space of  $E$ .

**Proof.** For any  $v \in E$ , we have that

$$|(J'_n(y_n) - J'(z_0)) v| \leq |(J'_n(y_n) - J'(y_n)) v| + |(J'(y_n) - J'(z_0)) v|. \tag{3.22}$$

Next, we estimate the two terms on the right-hand side of the above expression. And we get that

$$|(J'_n(y_n) - J'(y_n)) v| \leq \frac{\mu}{n} \int_0^T |y_n|^{\mu-1} |v| dt \tag{3.23}$$

and

$$\begin{aligned} |(J'(y_n) - J'(z_0)) v| &\leq |\langle y_n - z_0, v \rangle_E| + \left| \int_0^T (y_n - z_0) \cdot v dt \right| \\ &\quad + \left| \int_0^T (\nabla K(t, y_n) - \nabla K(t, z_0)) \cdot v dt \right| \\ &\quad + \left| \int_0^T (\nabla V(t, y_n) - \nabla V(t, z_0)) \cdot v dt \right|. \end{aligned} \tag{3.24}$$

Due to Lemma 3.13, we get that  $y_n \rightharpoonup z_0$  in  $E$  and  $y_n \rightarrow z_0$  in  $C$  as  $n \rightarrow \infty$ . Combined with (3.23) and (3.24), we obtain that

$$|(J'_n(y_n) - J'(y_n))v| \rightarrow 0$$

and

$$|(J'(y_n) - J'(z_0))v| \rightarrow 0,$$

when  $n \rightarrow \infty$ . It follows from (3.22) that

$$|(J'_n(y_n) - J'(z_0))v| \rightarrow 0, \quad n \rightarrow \infty.$$

Hence,  $J'_n(y_n) \rightarrow J'(z_0)$  in  $E^*$  as  $n \rightarrow \infty$ . □

**Lemma 3.16.** *Assume that all the conditions of Theorem 1.2 hold. Then  $y_n \rightarrow z_0$  in  $E$  as  $n \rightarrow \infty$ .*

**Proof.** According to Lemma 3.15 and  $y_n \rightharpoonup z_0$  in  $E$ , we obtain that

$$(J'_n(y_n) - J'(z_0))(y_n - z_0) \rightarrow 0.$$

Through calculation, we get that

$$\begin{aligned} (J'_n(y_n) - J'(z_0))(y_n - z_0) &= \int_0^T (y_n - z_0)' \cdot (y_n - z_0)' dt \\ &\quad + \int_0^T (\nabla K(t, y_n) - \nabla K(t, z_0)) \cdot (y_n - z_0) dt \\ &\quad - \frac{\mu}{n} \int_0^T |y_n|^\mu dt + \frac{\mu}{n} \int_0^T |y_n|^{\mu-2} y_n \cdot z_0 dt \\ &\quad - \int_0^T (\nabla H(t, y_n) - \nabla H(t, z_0)) \cdot (y_n - z_0) dt. \end{aligned}$$

Because  $y_n \rightarrow z_0$  in  $C$  as  $n \rightarrow \infty$ , then

$$\frac{\mu}{n} \int_0^T |y_n|^\mu dt \rightarrow \frac{\mu}{n} \int_0^T |z_0|^\mu dt$$

and

$$\frac{\mu}{n} \int_0^T |y_n|^{\mu-2} y_n \cdot z_0 dt \rightarrow \frac{\mu}{n} \int_0^T |z_0|^\mu dt.$$

It is clear that

$$\frac{\mu}{n} \int_0^T |z_0|^\mu dt \rightarrow 0, \quad n \rightarrow \infty.$$

By the continuity of  $\nabla K$ , we have that  $|\nabla K(t, y_n) - \nabla K(t, z_0)| \rightarrow 0$  as  $n \rightarrow \infty$ , uniformly for  $t \in [0, T]$ . Then

$$\begin{aligned} \left| \int_0^T [(\nabla V(t, y_n) - \nabla V(t, z_0)) \cdot (y_n - z_0)] dt \right| &\leq \int_0^T |\nabla V(t, y_n) - \nabla V(t, z_0)| dt \|y_n - z_0\|_0 \\ &\rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

which implies

$$\int_0^T (\nabla K(t, y_n) - \nabla K(t, z_0)) \cdot (y_n - z_0) dt \rightarrow 0, \quad n \rightarrow \infty.$$

Similarly, we obtain that

$$\int_0^T (\nabla H(t, y_n) - \nabla H(t, z_0)) \cdot (y_n - z_0) dt \rightarrow 0,$$

when  $n \rightarrow \infty$ . So,

$$\|y'_n - z'_0\|_{L^2} \rightarrow 0.$$

Since  $y_n \rightarrow z_0$  in  $L^2$ , we obtain that

$$\|y_n - z_0\|_E^2 = \|y'_n - z'_0\|_{L^2}^2 + \|y_n - z_0\|_{L^2}^2 \rightarrow 0, \quad n \rightarrow \infty.$$

Hence,  $y_n \rightarrow z_0$  in  $E$  as  $n \rightarrow \infty$ . □

**Lemma 3.17.** *Assume that all the conditions of Theorem 1.2 hold. Then  $J_n(y_n) \rightarrow J(z_0)$  as  $n \rightarrow \infty$ .*

**Proof.** By calculation, we obtain that

$$\begin{aligned} |J_n(y_n) - J(z_0)| &\leq |J_n(y_n) - J(y_n)| + |J(y_n) - J(z_0)| \\ &\leq \frac{1}{n} \int_0^T |y_n|^\mu dt + |J(y_n) - J(z_0)|. \end{aligned}$$

According to the continuity of the functional  $J$  and Lemma 3.16, we deduce that

$$\frac{1}{n} \int_0^T |y_n|^\mu dt \rightarrow 0$$

and

$$|J(y_n) - J(z_0)| \rightarrow 0,$$

when  $n \rightarrow \infty$ . Hence,  $J_n(y_n) \rightarrow J(z_0)$  as  $n \rightarrow \infty$ . □

**Proof. (Proof of Theorem 1.2)** By Lemma 3.11, there exists  $y_n \in \mathcal{N}_n$  such that

$$J_n(y_n) = \omega_n, \quad n = 1, 2, 3, \dots$$

Due to Lemma 3.16, there exist a subsequence of  $\{y_n\} \subset E$ , still denoted by  $\{y_n\}$ , and  $z_0 \in E$  such that  $y_n \rightarrow z_0$ . It follows from Lemmas 3.13 and 3.15 that  $J'(z_0) = 0$ . Hence,  $z_0$  is the solution to problem (1.2). Let

$$\omega = \inf_{u \in E \setminus \{0\}} \sup_{s > 0} J(su).$$

According to

$$\inf_{u \in E \setminus \{0\}} \sup_{s > 0} J_n(su) \leq \inf_{u \in E \setminus \{0\}} \sup_{s > 0} J(su)$$

and Lemma 3.17, we obtain that

$$J(z_0) = \lim_{n \rightarrow \infty} J_n(y_n) = \lim_{n \rightarrow \infty} \omega_n \leq \omega.$$

Let  $z = \frac{z_0}{\|z_0\|_E}$  and  $g(s) = J(sz)$ . Because  $g'(s) = J'(z_0)z = 0$  as  $s = \|z_0\|_E$ , we get that

$$g(\|z_0\|_E) = \sup_{s>0} g(s) = \sup_{s>0} J\left(s \frac{z_0}{\|z_0\|_E}\right)$$

by the proof process of Lemma 3.2. Hence,

$$\omega = \inf_{u \in E \setminus \{0\}} \sup_{s>0} J(su) \leq \sup_{s>0} J\left(s \frac{z_0}{\|z_0\|_E}\right) = J(z_0),$$

which yields

$$J(z_0) = \omega.$$

Thus,  $z_0$  is the ground state rotating periodic solution to problem (1.2).  $\square$

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