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THE MKDV-ZK LIMIT OF THE EULER-POISSON SYSTEM AT CRITICAL DENSITIES

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Abstract This paper focuses on the long wavelength limit for the Euler-Poisson system arising in plasma including three species in three dimensional case. The goal here is to deduce rigorously the modified Korteweg-de Vries-ZK (mKdV-ZK) equation, under the Gardner-Morikawa transform $\varepsilon^{1/2}(x_1 - Vt) \to x_1, \varepsilon^{1/2}x_2 \to x_2, \varepsilon^{1/2}x_3 \to x_3, \varepsilon^{3/2}t \to t$, as $\varepsilon \to 0$. By employing delicate energy method, we give uniform in ε estimate for the error between the mKdV-ZK equation and the Euler-Poisson system.

Keywords Euler-Poisson equation, long wavelength limit, modified Korteweg-de Vries-ZK (mKdV-ZK) equation.

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1. Introduction

Consider a three-dimensional general mixture of hot isothermal, warm adiabatic fluid and cold immobile background species in electrostatic plasmas [27]. This model includes three species. The cooler adiabatic species is described by the standard fluid equations

$$\begin{aligned} &\partial_t n_\alpha + \operatorname{div}(n_\alpha \boldsymbol{u}) = 0, \\ &\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + T_\alpha \frac{\nabla n_\alpha}{n_\alpha} = -\nabla\phi - \boldsymbol{e_1} \times \boldsymbol{u}, \end{aligned} \tag{1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ denotes the space-time position, and n_α, $\boldsymbol{u} = (u_1, u_2, u_3)$ are the density, velocity of the cooler adiabatic species, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The parameter T_α is the temperature of the fluid species.

The hot isothermal Boltzmann species is described by the so called isothermal Boltzmann relation

$$n_\beta = N_\beta e^\phi / T_\beta,$$

where T_β refers to the isothermal temperatures. The equilibrium quantities will be denoted by capital letters, say, N_β for the Boltzmann species. The electrostatic potential ϕ satisfies the Possion's equation

$$\triangle\phi = N_\beta e^{\phi/T_\beta} - n_\alpha - N_\delta, \tag{1.2}$$

where N_δ is the immobile background species. On the sluggish side, some immobile background species with constant density N_δ is considered, to allow for density imbalances between three species. These species can be used to simulate respectively unmagnetized ions when dealing with electron-acoustic solitons [28].

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There are many nonlinear dispersive partial differential equations, which can be formally derived from the Euler-Poisson system, such as the KdV equation, the KP-II equation and the ZK equation, these equations were extensively studied in recent decades. Guo and Pu [13] established rigourously such a derivation of KdV equation for ion Euler-Poisson system in one dimension. Pu also considered higher dimensional case, gave rigorous limit proof for KP-II equation and ZK equation under different scalings of the Euler-Poisson equation in [22]. Almost in the meantime, the KdV limit in 1D and the ZK equation in 3D from the Euler-Poisson system were established in [15] and [14], respectively. It is worth mentioning that Isaza, Lopez and Mejia gave derivations for KP-II equation from Cauchy equation in [10], Linares and Saut gave derivations for ZK equation from Cauchy equation in [17]. For other studies of KdV equation, KP-II equation and ZK equation, see [3, 5, 6, 15, 18, 20], to list only a few. For more researches of the Euler-Poisson equation and related singular limits, see [2, 4, 7, 11, 13, 14, 17, 19, 24] and the references therein.

It is thus of interest to revisit the whole field of electrostatic modes in magnetized plasmas in a general treatment that encompasses all the known dispersion laws and nonlinear equations, compared with earlier treatments that are often restricted to specific models. For more complicated plasma compositions, the mKdV-ZK equation with cubic nonlinearities will be used for special critical densities, at which the soliton character can switch from compressive to rarefactive or vice versa. The system of equations for the Alfvén wave can be reduced to the mKdV equation in [26]. In the vicinity of critical densities, mixed KdV-ZK equations with both quadratic and cubic nonlinearities can occur, and model double layers in plasmas. The formal derivations for $T_\alpha > 0$ of ordinary, modified and mixed KdV-ZK equations have been given in [27] for general mixtures of hot isothermal, warm adiabatic fluid and cold immobile background species. In particular, Pu and Xi [23] established the rigorous derivation of the mKdV equation for $T_\alpha \geq 0$. However, until now there have been no rigorous mathematical justifications of the mKdV-ZK limit, the purpose of this paper is to justify rigorously the limit, at least for well-prepared initial data. When $T_\alpha = 0$, the system (3.1) consisting of the density equation and the momentum equation is neither symmetric nor symmetrizable, our paper corresponds to the cold ion case $T_\alpha = 0$, we give not only the formal derivations, but also the rigorous justification. The result in this paper can be generalized to the case of $T_\alpha > 0$, the proof will be slightly simpler since in this case the system is Friedrich symmetrizable.

We organized this paper as follows. In Section 2, we formally derive the mKdV-ZK equation in 3D Euler-Poisson system from a different Gardner-Morikawa type transformation and state the main Theorem 2.1, then we write the remainder equations into a system (2.21) by introducing some new differential notations. In Section 3, we concentrate on the proof of Theorem 2.1. The key issue is to provide uniform in ε estimates for the remainders, here more delicate estimates are required. We also use continuity method to close the estimate, and introduce weighted Sobolev norms $\|\!|\cdot|\!\|_{s'}$, which enables us to combine the energy estimates in different levels.

Throughout this paper, we use $[a, b] = ab - ba$ to denote the commutator of a and b, the norm $\|\cdot\|_X$ to denote an X-norm, the $\|\cdot\|_{L^2}$ is replaced by $\|\cdot\|$ when $X = L^2$. We also use $\langle f, g\rangle = \int fg dx$ to denote the inner product of two L^2 functions.

2. Formal mKdV-ZK expansion and the main results

2.1. Formal mKdV-ZK expansion

By the classical Gardner-Morikawa transformation [25]

$$\varepsilon^{1/2}(x_1 - Vt) \to x_1, \varepsilon^{1/2}x_2 \to x_2, \varepsilon^{1/2}x_3 \to x_3, \varepsilon^{3/2}t \to t, \tag{2.1}$$

we obtain from (1.1)-(1.2) the parameterized system

$$\begin{cases} \varepsilon\partial_t n_\alpha - V\partial_{x_1} n_\alpha + \nabla\cdot(n_\alpha \boldsymbol{u}) = 0, \\ \varepsilon\partial_t \boldsymbol{u} - V\partial_{x_1}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + T_\alpha \dfrac{\nabla n_\alpha}{n_\alpha} = -\nabla\phi + \dfrac{1}{\varepsilon^{1/2}}\boldsymbol{u}\times\boldsymbol{e_1}, \\ \varepsilon\Delta\phi = N_\beta e^{\phi/T_\beta} - n_\alpha - N_\delta, \end{cases} \tag{2.2}$$

where $\boldsymbol{e}_1 = (1,0,0)^T$, ε denotes the amplitude of the initial disturbance and is assumed to be small compared with unity and V is the wave speed to be determined. We consider the following formal expansion:

$$\begin{cases} n_\alpha = N_\alpha(1 + \varepsilon^{\frac{1}{2}} n_\alpha^{(1)} + \varepsilon n_\alpha^{(2)} + \varepsilon^{\frac{3}{2}} n_\alpha^{(3)} + \varepsilon^2 n_\alpha^{(4)} + \cdots), \\ u_1 = \varepsilon^{\frac{1}{2}} u_1^{(1)} + \varepsilon u_1^{(2)} + \varepsilon^{\frac{3}{2}} u_1^{(3)} + \varepsilon^2 u_1^{(4)} + \cdots, \\ \phi = \varepsilon^{\frac{1}{2}} \phi^{(1)} + \varepsilon\phi^{(2)} + \varepsilon^{\frac{3}{2}}\phi^{(3)} + \varepsilon^2\phi^{(4)} + \cdots, \\ u_2 = \varepsilon u_2^{(1)} + \varepsilon^{\frac{3}{2}} u_2^{(2)} + \varepsilon^2 u_2^{(3)} + \varepsilon^{\frac{5}{2}} u_2^{(4)} + \cdots, \\ u_3 = \varepsilon u_3^{(1)} + \varepsilon^{\frac{3}{2}} u_3^{(2)} + \varepsilon^2 u_3^{(3)} + \varepsilon^{\frac{5}{2}} u_3^{(4)} + \cdots. \end{cases} \tag{2.3}$$

Plugging (2.3) into (2.2), we get a power series of ε, whose coefficients depend on $(n_\alpha^{(k)}, \boldsymbol{u}^{(k)}, \phi^{(k)})$ for $k \geq 1$.

2.1.1. Derivation of mKdV-ZKE for $n_\alpha^{(1)}$

From the power series of ε we thus obtained, we get at the order of ε the following coefficients.

At the order of ε^0: Setting the coefficient of ε^0 to be 0, we obtain

$$N_\beta - N_\alpha - N_\delta = 0,$$

which implies overall charge neutrality in equilibrium.

At the order of $\varepsilon^{\frac{1}{2}}$: Setting the coefficient of $\varepsilon^{\frac{1}{2}}$ to be 0, we obtain

$$\begin{cases} -V\partial_{x_1} n_\alpha^{(1)} + \partial_{x_1} u_1^{(1)} = 0, & \text{(a)} \\ -V\partial_{x_1} u_1^{(1)} + T_\alpha \partial_{x_1} n_\alpha^{(1)} = -\partial_{x_1}\phi^{(1)}, & \text{(b)} \\ \dfrac{N_\beta}{T_\beta}\phi^{(1)} = N_\alpha n_\alpha^{(1)}, & \text{(c)} \\ T_\alpha \partial_{x_2} n_\alpha^{(1)} = -\partial_{x_2}\phi^{(1)} + u_3^{(1)}, & \text{(d)} \\ T_\alpha \partial_{x_3} n_\alpha^{(1)} = -\partial_{x_3}\phi^{(1)} - u_2^{(1)}. & \text{(e)} \end{cases} \tag{2.4}$$

Write this equation in the matrix form

$$\begin{bmatrix} -V & 1 & 0 \\ T_\alpha & -V & 1 \\ N_\alpha & 0 & -\dfrac{N_\beta}{T_\beta} \end{bmatrix} \begin{bmatrix} \partial_{x_1} n_\alpha^{(1)} \\ \partial_{x_1} u_1^{(1)} \\ \partial_{x_1}\phi^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.5}$$

In order to get a nontrivial solution for $n_\alpha^{(1)}, u_1^{(1)}$, and $\phi^{(1)}$, we require the determinant of the coefficient matrix to vanish to obtain

$$V^2 = T_\alpha + \frac{N_\alpha T_\beta}{N_\beta}. \tag{2.6}$$

From (2.4a) and (2.4c), we can derive the relation

$$\begin{cases} u_1^{(1)} = V n_\alpha^{(1)}, & \text{(a)} \\ \phi^{(1)} = \dfrac{N_\alpha T_\beta}{N_\beta} n_\alpha^{(1)}. & \text{(b)} \end{cases} \tag{2.7}$$

Therefore, in order to determine $(n_\alpha^{(1)}, u_1^{(1)}, \phi^{(1)})$, we only need to determine $n_\alpha^{(1)}$.

From (2.4d) and (2.4e), we have

$$\begin{cases} u_2^{(1)} = -T_\alpha \partial_{x_3} n_\alpha^{(1)} - \partial_{x_3}\phi^{(1)} = -V^2 \partial_{x_3} n_\alpha^{(1)}, \\ u_3^{(1)} = T_\alpha \partial_{x_2} n_\alpha^{(1)} + \partial_{x_2}\phi^{(1)} = V^2 \partial_{x_2} n_\alpha^{(1)}, \end{cases} \tag{2.8}$$

where we have used (2.6) and (2.7).

At the order of ε: Setting the coefficient of ε to be 0, we obtain

$$\begin{cases} -V\partial_{x_1} n_\alpha^{(2)} + \partial_{x_1}(n_\alpha^{(1)} u_1^{(1)}) + \partial_{x_1} u_1^{(2)} + \partial_{x_2} u_2^{(2)} + \partial_{x_3} u_3^{(2)} = 0, & \text{(a)} \\ -V\partial_{x_1} u_1^{(2)} + u_1^{(1)}\partial_{x_1} u_1^{(1)} + T_\alpha\{\partial_{x_1} n_\alpha^{(2)} - n_\alpha^{(1)}\partial_{x_1} n_\alpha^{(1)}\} = -\partial_{x_1}\phi^{(2)}, & \text{(b)} \\ \dfrac{N_\alpha}{T_\beta}\phi^{(2)} + \dfrac{N_\alpha}{2T_\beta^2}(\phi^{(1)})^2 = N_\alpha n_\alpha^{(2)}, & \text{(c)} \\ -V\partial_{x_1} u_2^{(2)} + T_\alpha\{\partial_{x_2} n_\alpha^{(2)} - n_\alpha^{(1)}\partial_{x_2} n_\alpha^{(1)}\} = -\partial_{x_2}\phi^{(2)} + u_3^{(2)}, & \text{(d)} \\ -V\partial_{x_1} u_3^{(2)} + T_\alpha\{\partial_{x_3} n_\alpha^{(2)} - n_\alpha^{(1)}\partial_{x_3} n_\alpha^{(1)}\} = -\partial_{x_3}\phi^{(2)} - u_2^{(2)}. & \text{(e)} \end{cases} \tag{2.9}$$

To find the equation satisfied by $n_\alpha^{(1)}$, we take ∂_{x_1} of (2.9c), multiply (2.9a) with V, and then add them to (2.9b). We thus obtain

$$\mathcal{F}(n_\alpha^{(1)})^2 = 0, \tag{2.10}$$

and $\mathcal{F}$ is given by

$$\mathcal{F} = 3V^2 - T_\alpha - \frac{T_\beta N_\alpha^2}{N_\beta^2}. \tag{2.11}$$

For the generic case $\mathcal{F} \neq 0$, we have $n_\alpha^{(1)} = 0$, which reduces to the expansion studied in [13] and leads to the standard KdV expansion. But when the plasma is at critical densities, we consider the case $\mathcal{F} = 0$, i.e., we have

$$3V^2 = T_\alpha + \frac{T_\beta N_\alpha^2}{N_\beta^2}. \tag{2.12}$$

So that we will continue to work with $n_\alpha^{(1)}$. From (2.9a) and (2.9c) we can deduce the relation

$$\begin{cases} u_1^{(2)} = V n_\alpha^{(2)} - n_\alpha^{(1)} u_1^{(1)}, & \text{(a)} \\ \phi^{(2)} = \dfrac{N_\alpha T_\beta}{N_\beta} n_\alpha^{(2)} - \dfrac{1}{2T_\beta}(\phi^{(1)})^2. & \text{(b)} \end{cases} \tag{2.13}$$

The coefficients of $\varepsilon^{\frac{3}{2}}$ and the mKdV-ZK equation for $n_\alpha^{(1)}$: Setting the coefficient of $\varepsilon^{\frac{3}{2}}$ to be 0, we obtain

$$\left\{\begin{aligned}
&\partial_t n_\alpha^{(1)} - V\partial_{x_1} n_\alpha^{(3)} + \partial_{x_1}(n_\alpha^{(1)} u_1^{(2)} + n_\alpha^{(2)} u_1^{(1)}) + \partial_{x_1} u_1^{(3)} + \partial_{x_2} u_2^{(2)} + \partial_{x_3} u_3^{(2)}\\
&+\{\partial_{x_2}(n_\alpha^{(1)} u_2^{(1)}) + \partial_{x_3}(n_\alpha^{(1)} u_3^{(1)})\} = 0, && \text{(a)}\\
&\quad \partial_t u_1^{(1)} - V\partial_{x_1} u_1^{(3)} + u_1^{(1)}\partial_{x_1} u_1^{(2)} + u_1^{(2)}\partial_{x_1} u_1^{(1)} + T_\alpha\{\partial_{x_1} n_\alpha^{(3)} - n_\alpha^{(1)}\partial_{x_1} n_\alpha^{(2)}\}\\
&\quad -(n_\alpha^{(2)} - (n_\alpha^{(1)})^2)\partial_{x_1} n_\alpha^{(1)}\}\\
&= -\partial_{x_1}\phi^{(3)} + \{u_2^{(1)}\partial_{x_2} u_1^{(1)} + u_3^{(1)}\partial_{x_3} u_1^{(1)}\}, && \text{(b)}\\
&\Delta\phi^{(1)} = \frac{N_\beta}{T_\beta}\phi^{(3)} + \frac{N_\beta}{T_\beta^2}\phi^{(1)}\phi^{(2)} + \frac{N_\beta}{3!T_\beta^3}(\phi^{(1)})^3 - N_\alpha n_\alpha^{(3)}, && \text{(c)}\\
&\quad -V\partial_{x_1} u_2^{(2)} + u_1^{(1)}\partial_{x_1} u_2^{(1)} + T_\alpha\{\partial_{x_2} n_\alpha^{(3)} - n_\alpha^{(1)}\partial_{x_2} n_\alpha^{(2)} - (n_\alpha^{(2)} - (n_\alpha^{(1)})^2)\partial_{x_2} n_\alpha^{(1)}\\
&= -\partial_{x_2}\phi^{(3)} + u_3^{(3)}, && \text{(d)}\\
&\quad -V\partial_{x_1} u_3^{(2)} + u_1^{(1)}\partial_{x_1} u_3^{(1)} + T_\alpha\{\partial_{x_3} n_\alpha^{(3)} - n_\alpha^{(1)}\partial_{x_3} n_\alpha^{(2)} - (n_\alpha^{(2)} - (n_\alpha^{(1)})^2)\partial_{x_3} n_\alpha^{(1)}\\
&= -\partial_{x_3}\phi^{(3)} - u_2^{(3)}. && \text{(e)}
\end{aligned}\right. \tag{2.14}$$

From (2.9d) and (2.9e), we can get

$$V\partial_{x_2} u_2^{(2)} + V\partial_{x_3} u_3^{(2)} = (V^4 + \frac{T_\beta^2 N_\alpha}{N_\beta^2})\partial_{x_1}(\partial_{x_2}^2 n_\alpha^{(1)} + \partial_{x_3}^2 n_\alpha^{(1)}), \tag{2.15}$$

where we have used (2.8) and (2.13).

Differentiating (2.14c) with respect to x_1 and multiplying the resultant equality with $\frac{T_\beta}{N_\beta}$, multiplying (2.14a) with V, and then adding them to (2.14b) together, we deduce that $n_\alpha^{(1)}$ satisfies the mKdV-ZK equation

$$\begin{aligned}
&\partial_t n_\alpha^{(1)} + \left(\frac{T_\beta N_\alpha^3}{2VN_\beta^3} - 3V + \frac{T_\alpha}{2V}\right)(n_\alpha^{(1)})^2\partial_{x_1} n_\alpha^{(1)} + \frac{T_\beta^2 N_\alpha}{2VN_\beta^2}\partial_{x_1}^3 n_\alpha^{(1)}\\
&+(\frac{V^3}{2} + \frac{T_\beta^2 N_\alpha}{2VN_\beta^2})\partial_{x_1}(\partial_{x_2}^2 n_\alpha^{(1)} + \partial_{x_3}^2 n_\alpha^{(1)}) = 0.
\end{aligned} \tag{2.16}$$

Here we have used the relation (2.6), (2.7), (2.8), (2.12) and (2.13), (2.15). We also note that the systems (2.16) and (2.7) for $(n_\alpha^{(1)}, u_1^{(1)}, \phi^{(1)})$ are self contained, which do not depend on $(n_\alpha^{(i)}, u_1^{(i)}, \phi^{(i)})$ for $i \geq 2$. Proceeding as above, by balancing of the coefficients at order $\varepsilon^{(k+2)/2}$ for $k \geq 2$, we obtain

$$\left\{\begin{aligned}
&u_1^{(k)} = Vn_\alpha^{(k)} + h^{(k-1)}, && \text{(a)}\\
&\phi^{(k)} = \frac{N_\alpha T_\beta}{N_\beta} n_\alpha^{(k)} + g^{(k-1)}, && \text{(b)}
\end{aligned}\right. \tag{2.17}$$

for some $h^{(k-1)}$ and $g^{(k-1)}$ depending only on $(n_\alpha^{(i)}, u_1^{(i)}, \phi^{(i)})$ for $1 \leq i \leq k-1$ and the linearized mKdV-ZK equation

$$\begin{aligned}&\partial_t n_\alpha^{(k)} + \left(\frac{T_\beta N_\alpha^3}{2VN_\beta^3} - 3V + \frac{T_\alpha}{2V}\right)\partial_{x_1}((n_\alpha^{(1)})^2 n_\alpha^{(k)}) + \frac{T_\beta^2 N_\alpha}{2VN_\beta^2}\partial_{x_1}^3 n_\alpha^{(k)}\\ &+ (\frac{V^3}{2} + \frac{T_\beta^2 N_\alpha}{2VN_\beta^2})\partial_{x_1}(\partial_{x_2}^2 n_\alpha^{(k)} + \partial_{x_3}^2 n_\alpha^{(k)}) = G^{(k-1)}.\end{aligned} \tag{2.18}$$

Here $G^{(k-1)}$ depends only on $n_\alpha^{(1)}, n_\alpha^{(2)}, ..., n_\alpha^{(k-1)}$. Recalling results in [5], we will assume hereafter that the solutions of $n_\alpha^{(k)}$ are sufficiently smooth on any time interval $[-\tau_*, \tau_*]$ for any $\tau_* > 0$. There may be regularity loss between successive steps, but we can still assume smoothness in finite steps in this paper.

For the solvability of $n_\alpha^{(k)}$ for $k \geq 1$, we have the following two propositions.

Proposition 2.1. (see [5]) *Let $\tilde{s} > \frac{1}{2}$, the Cauchy problem of mKdV-ZK equation (2.16) is locally well-posed in $H^{\tilde{s}}(\mathbb{R}^3)$.*

Proposition 2.2. *Let $k \geq 2$ and $\tilde{s} > \frac{1}{2}$, the Cauchy problem of the linearized inhomogeneous mKdV-ZK equation (2.18) is well-posed in $H^{\tilde{s}}(\mathbb{R}^3)$.*

The proof of Proposition 2.2 is standard.

2.2. Main result

To show that $n_\alpha^{(1)}$ converges to a solution of the mKdV-ZK equation as $\varepsilon \to 0$ in any finite time interval, we must make the above procedure rigorous. Let $(n_\alpha, \boldsymbol{u}, \phi)$ be a solution of the scaled system (2.2) of the following expansion

$$\begin{aligned}n_\alpha &= N_\alpha(1 + \varepsilon^{\frac{1}{2}} n_\alpha^{(1)} + \varepsilon n_\alpha^{(2)} + \varepsilon^{\frac{3}{2}} n_\alpha^{(3)} + \varepsilon^2 n_\alpha^{(4)} + \varepsilon^{\frac{5}{2}} n_\alpha^{(5)} + \varepsilon^{\frac{3}{2}} n_R^\varepsilon),\\ u_1 &= \varepsilon^{\frac{1}{2}} u_1^{(1)} + \varepsilon u_1^{(2)} + \varepsilon^{\frac{3}{2}} u_1^{(3)} + \varepsilon^2 u_1^{(4)} + \varepsilon^{\frac{5}{2}} u_1^{(5)} + \varepsilon^{\frac{3}{2}} {u_1}_R^\varepsilon,\\ \phi &= \varepsilon^{\frac{1}{2}} \phi^{(1)} + \varepsilon \phi^{(2)} + \varepsilon^{\frac{3}{2}} \phi^{(3)} + \varepsilon^2 \phi^{(4)} + \varepsilon^{\frac{5}{2}} \phi^{(5)} + \varepsilon^{\frac{3}{2}} \phi_R^\varepsilon,\\ u_2 &= \varepsilon u_2^{(1)} + \varepsilon^{\frac{3}{2}} u_2^{(2)} + \varepsilon^2 u_2^{(3)} + \varepsilon^{\frac{5}{2}} u_2^{(4)} + \varepsilon^3 u_2^{(5)} + \varepsilon^{\frac{3}{2}} {u_2}_R^\varepsilon,\\ u_3 &= \varepsilon u_3^{(1)} + \varepsilon^{\frac{3}{2}} u_3^{(2)} + \varepsilon^2 u_3^{(3)} + \varepsilon^{\frac{5}{2}} u_3^{(4)} + \varepsilon^3 u_3^{(5)} + \varepsilon^{\frac{3}{2}} {u_3}_R^\varepsilon,\end{aligned} \tag{2.19}$$

where $\boldsymbol{u}_R^\varepsilon = ({u_1}_R^\varepsilon, {u_2}_R^\varepsilon, {u_3}_R^\varepsilon)$, and $(n_\alpha^{(1)}, u_1^{(1)}, \phi^{(1)})$ satisfies (2.7) and (2.15), the other profiles $u_1^{(i)}$ and $\phi^{(i)}$ for $i = 2, 3, 4$ and $u_2^{(j)}$ and $u_3^{(j)}$ for $j = 1, 2, 3, 4$ are solved from the systems (2.4), (2.9) and (2.14). The $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ is the remainder.

For notational convenience, we denote $\boldsymbol{u} = (u_1, u_2, u_3)^T$, $\boldsymbol{u}_R^\varepsilon = ({u_1}_R^\varepsilon, {u_2}_R^\varepsilon, {u_3}_R^\varepsilon)^T$, and

$$\begin{aligned}\widetilde{n_\alpha} &= n_\alpha^{(1)} + \varepsilon^{\frac{1}{2}} n_\alpha^{(2)} + \varepsilon^1 n_\alpha^{(3)} + \varepsilon^{\frac{3}{2}} n_\alpha^{(4)} + \varepsilon^2 n_\alpha^{(5)},\\ \widetilde{\phi} &= \phi^{(1)} + \varepsilon^{\frac{1}{2}} \phi^{(2)} + \varepsilon^1 \phi^{(3)} + \varepsilon^{\frac{3}{2}} \phi^{(4)} + \varepsilon^2 \phi^{(5)},\\ \widetilde{\boldsymbol{u}} &= (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3)^T,\\ \widetilde{u}_1 &= u_1^{(1)} + \varepsilon^{\frac{1}{2}} u_1^{(2)} + \varepsilon^1 u_1^{(3)} + \varepsilon^{\frac{3}{2}} u_1^{(4)} + \varepsilon^2 u_1^{(5)},\\ \widetilde{u}_2 &= \varepsilon^{\frac{1}{2}} u_2^{(1)} + \varepsilon u_2^{(2)} + \varepsilon^{\frac{3}{2}} u_2^{(3)} + \varepsilon^2 u_2^{(4)} + \varepsilon^{\frac{5}{2}} u_2^{(5)},\\ \widetilde{u}_3 &= \varepsilon^{\frac{1}{2}} u_3^{(1)} + \varepsilon u_3^{(2)} + \varepsilon^{\frac{3}{2}} u_3^{(3)} + \varepsilon^2 u_3^{(4)} + \varepsilon^{\frac{5}{2}} u_3^{(5)}.\end{aligned} \tag{2.20}$$

After careful and tedious computations, we obtain the following remainder system for $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$

$$\begin{cases} \partial_t n_R^\varepsilon - \dfrac{V\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon} \cdot \nabla n_R^\varepsilon + \dfrac{n_\alpha}{\varepsilon} \nabla \cdot \boldsymbol{u}_R^\varepsilon + \dfrac{1}{\sqrt{\varepsilon}} n_R^\varepsilon \nabla \cdot \widetilde{\boldsymbol{u}} + \dfrac{1}{\sqrt{\varepsilon}} \boldsymbol{u}_R^\varepsilon \cdot \nabla \widetilde{n_\alpha} + \sqrt{\varepsilon} R_n = 0, \quad \text{(a)} \\ \partial_t \boldsymbol{u}_R^\varepsilon - \dfrac{V\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon} \cdot \nabla \boldsymbol{u}_R^\varepsilon + \dfrac{1}{\sqrt{\varepsilon}} \boldsymbol{u}_R^\varepsilon \nabla \cdot \widetilde{\boldsymbol{u}} + \dfrac{T_\alpha}{\varepsilon} \nabla n_R^\varepsilon - \dfrac{T_\alpha}{\sqrt{\varepsilon}} \left(\dfrac{\widetilde{n_\alpha} + \varepsilon n_R^\varepsilon}{n_\alpha} \right) \nabla n_R^\varepsilon \\ -\dfrac{T_\alpha \boldsymbol{p}}{\sqrt{\varepsilon} n_\alpha} n_R^\varepsilon - \dfrac{T_\alpha \sqrt{\varepsilon}}{n_\alpha} \boldsymbol{R}_T + \sqrt{\varepsilon} \boldsymbol{R}_u = -\dfrac{1}{\varepsilon} \nabla \phi_R^\varepsilon + \dfrac{1}{\varepsilon} \boldsymbol{u}_R^\varepsilon \times \boldsymbol{e}_1, \quad \text{(b)} \\ \varepsilon \Delta \phi_R^\varepsilon = \dfrac{N_\beta}{T_\beta} \phi_R^\varepsilon - N_\alpha n_R^\varepsilon + \dfrac{\sqrt{\varepsilon} N_\beta}{T_\beta^2} \phi^{(1)} \phi_R^\varepsilon + \varepsilon R_\phi, \quad \text{(c)} \end{cases} \tag{2.21}$$

where $\boldsymbol{u}_R^\varepsilon = (u_1{}_R^\varepsilon, u_2{}_R^\varepsilon, u_3{}_R^\varepsilon)$, $\widetilde{n_\alpha}, \widetilde{\boldsymbol{u}}$, and $\widetilde{\phi}$ are given in (2.20), and

$$\begin{cases} R_n = \partial_t n_\alpha^{(4)} + \sqrt{\varepsilon} \partial_t n_\alpha^{(5)} + \displaystyle\sum_{1 \leqslant i,j \leqslant 5; i+j \geqslant 6} \varepsilon^{i+j-6} \partial_{x_1} (n_\alpha^{(i)} u_1^{(j)}) \\ \qquad + \displaystyle\sum_{1 \leqslant i,l \leqslant 5; i+l \geqslant 6} \varepsilon^{i+l-6} \partial_{x_2} (n_\alpha^{(i)} u_2^{(l)}) + \displaystyle\sum_{1 \leqslant i,m \leqslant 5; i+m \geqslant 6} \varepsilon^{i+m-6} \partial_{x_3} (n_\alpha^{(i)} u_3^{(m)}), \\ \boldsymbol{p} = (p_1, p_2, p_3) = \boldsymbol{p}(n_\alpha^{(1)}, n_\alpha^{(2)}, n_\alpha^{(3)}, n_\alpha^{(4)}, n_\alpha^{(5)}), \\ \boldsymbol{R}_T = (R_{T1}, R_{T2}, R_{T3}) = \boldsymbol{R}_T(n_\alpha^{(1)}, n_\alpha^{(2)}, n_\alpha^{(3)}, n_\alpha^{(4)}, n_\alpha^{(5)}), \\ \boldsymbol{R}_{\boldsymbol{u}} = (R_{u1}, R_{u2}, R_{u3}) = \boldsymbol{R}_{\boldsymbol{u}}(\boldsymbol{u}^{(i)}, \phi^{(i)})(i = 1,2,3,4,5), \\ R_\phi = {R_\phi}' + \sqrt{\varepsilon} {R_\phi}''. \end{cases} \tag{2.22}$$

Here $R_\phi'' = \hat{R_\phi} - \left(\varepsilon^3 \Delta \phi^{(4)} + \varepsilon^{\frac{7}{2}} \Delta \phi^{(5)} \right)$, $\hat{R_\phi}$ depends only on $\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(5)}$, $R_\phi = F(\sqrt{\varepsilon} \phi_R^\varepsilon) \phi_R^\varepsilon + \sqrt{\varepsilon} R_\phi''$ depends only on ϕ_R^ε for some function of F depending on $\sqrt{\varepsilon} \phi_R^\varepsilon$, ${R_\phi}' = F(\sqrt{\varepsilon} \phi_R^\varepsilon) \phi_R^\varepsilon$.

And we give the coefficient of ε^2 and $\varepsilon^{\frac{5}{2}}$ for $\boldsymbol{p}$, $\boldsymbol{R}_T$.

At the order of ε^2:

$$\begin{aligned} &\partial_t u_1^{(2)} - V \partial_{x_1} u_1^{(4)} + u_1^{(1)} \partial_{x_1} u_1^{(3)} + u_1^{(2)} \partial_{x_1} u_1^{(2)} + u_1^{(3)} \partial_{x_1} u_1^{(1)} + T_\alpha \{ (\partial_{x_1} n_\alpha^{(4)} - \\ &n_\alpha^{(1)} \partial_{x_1} n_\alpha^{(3)} + n_\alpha^{(2)} \partial_{x_1} (n_\alpha^{(1)})^2 + \partial_{x_1} n_\alpha^{(1)} ((n_\alpha^{(1)})^3 - n_\alpha^{(3)}) + \partial_{x_1} n_\alpha^{(2)} ((n_\alpha^{(1)})^2 - n_\alpha^{(2)})) \} \\ = &- \partial_{x_1} \phi^{(4)} + \{ u_2^{(1)} \partial_{x_2} u_1^{(2)} + u_2^{(2)} \partial_{x_2} u_1^{(1)} + u_3^{(1)} \partial_{x_3} u_1^{(2)} + u_3^{(2)} \partial_{x_3} u_1^{(1)} \}. \end{aligned} \tag{2.23}$$

At the order of $\varepsilon^{\frac{5}{2}}$:

$$\begin{aligned} &\partial_t u_1^{(3)} - V \partial_{x_1} u_1^{(5)} + u_1^{(1)} \partial_{x_1} u_1^{(4)} + u_1^{(2)} \partial_{x_1} u_1^{(3)} + u_1^{(3)} \partial_{x_1} u_1^{(2)} + u_1^{(4)} \partial_{x_1} u_1^{(1)} + \\ &T_\alpha \{ (\partial_{x_1} n_\alpha^{(5)} - n_\alpha^{(1)} \partial_{x_1} n_\alpha^{(4)} - \partial_{x_1} n_\alpha^{(3)} ((n_\alpha^{(1)})^2 - n_\alpha^{(2)}) + \partial_{x_1} n_\alpha^{(1)} ((n_\alpha^{(1)})^4 + 2 n_\alpha^{(1)} n_\alpha^{(3)} \\ &- 3 (n_\alpha^{(1)})^2 n_\alpha^{(2)} + (n_\alpha^{(2)})^2 - n_\alpha^{(4)}) + \partial_{x_1} n_\alpha^{(2)} (2 n_\alpha^{(1)} n_\alpha^{(2)} - n_\alpha^{(3)})) \} \\ = &- \partial_{x_1} \phi^{(5)} + \{ u_2^{(1)} \partial_{x_2} u_1^{(3)} + u_2^{(2)} \partial_{x_2} u_1^{(2)} + u_2^{(3)} \partial_{x_2} u_1^{(1)} + u_3^{(1)} \partial_{x_3} u_1^{(3)} \\ &+ u_3^{(2)} \partial_{x_3} u_1^{(2)} + u_3^{(3)} \partial_{x_3} u_1^{(1)} \}. \end{aligned} \tag{2.24}$$

Then subtracting $\varepsilon^{\frac{1}{2}} \times (2.4b) + \varepsilon \times (2.9b) + \varepsilon^{\frac{3}{2}} \times (2.14b) + \varepsilon^2 \times (2.23) + \varepsilon^{\frac{5}{2}} \times (2.24)$ from the equation of (2.2b), we obtain the remainder equation (2.21b). We derive the remainder terms of the pressure term $\frac{T_\alpha \partial_{x_1} n_\alpha}{n_\alpha}$. After subtracting, we obtain

$$
\begin{aligned}
& T_\alpha \varepsilon^{\frac{3}{2}} \partial_{x_1} n_R^\varepsilon - T_\alpha \varepsilon^2 \left(\frac{\widetilde{n_\alpha} + \varepsilon n_R^\varepsilon}{n_\alpha} \right) \partial_{x_1} n_R^\varepsilon - T_\alpha \{ \varepsilon^{\frac{1}{2}} \partial_{x_1} n_\alpha^{(1)} + \varepsilon (\partial_{x_1} n_\alpha^{(2)} - n_\alpha^{(1)} \partial_{x_1} n_\alpha^{(1)}) \\
& + \varepsilon^{\frac{3}{2}} (\partial_{x_1} n_\alpha^{(3)} + \partial_{x_1} n_\alpha^{(1)} ((n_\alpha^{(1)})^2 - n_\alpha^{(2)}) - n_\alpha^{(1)} \partial_{x_1} n_\alpha^{(2)}) \\
& + \varepsilon^2 (\partial_{x_1} n_\alpha^{(4)} - n_\alpha^{(1)} \partial_{x_1} n_\alpha^{(3)} + n_\alpha^{(2)} \partial_{x_1} (n_\alpha^{(1)})^2 + \partial_{x_1} n_\alpha^{(1)} ((n_\alpha^{(1)})^3 - n_\alpha^{(3)}) \\
& + \partial_{x_1} n_\alpha^{(2)} ((n_\alpha^{(1)})^2 - n_\alpha^{(2)})) \\
& + \varepsilon^{\frac{5}{2}} (\partial_{x_1} n_\alpha^{(5)} - n_\alpha^{(1)} \partial_{x_1} n_\alpha^{(4)} - \partial_{x_1} n_\alpha^{(3)} ((n_\alpha^{(1)})^2 - n_\alpha^{(2)}) + \partial_{x_1} n_\alpha^{(1)} ((n_\alpha^{(1)})^4 + 2 n_\alpha^{(1)} n_\alpha^{(3)} \\
& - 3 (n_\alpha^{(1)})^2 n_\alpha^{(2)} + (n_\alpha^{(2)})^2 - n_\alpha^{(4)}) + \partial_{x_1} n_\alpha^{(2)} (2 n_\alpha^{(1)} n_\alpha^{(2)} - n_\alpha^{(3)})) \}.
\end{aligned}
$$

After being divided by $\varepsilon^{\frac{5}{2}}$, the above equation can be rearranged into

$$
T_\alpha \frac{1}{\varepsilon} \partial_{x_1} n_R^\varepsilon - T_\alpha \frac{1}{\sqrt{\varepsilon}} \left(\frac{\widetilde{n_\alpha} + \varepsilon n_R^\varepsilon}{n_\alpha} \right) \partial_{x_1} n_R^\varepsilon - T_\alpha \frac{p_1}{\sqrt{\varepsilon} n_\alpha} n_R^\varepsilon - T_\alpha \frac{\sqrt{\varepsilon} R_{T_1}}{n_\alpha},
$$

where p_1 and R_{T_1} are finite combinations of $n_\alpha^{(1)}$, $n_\alpha^{(2)}$, and $n_\alpha^{(3)}$ only. p_2, p_3 and R_{T_2}, R_{T_3} can be obtained similarly. The expression of R_u depends only on $u^{(i)}$ and $\phi^{(i)}$ and can be derived similarly to the derivation of R_n.

We also give some basic estimates for the remainder term R_ϕ in the following.

Lemma 2.1. *Let $k \geq 0$ be an integer; then there exists a constant $1 \leq C_1 \leq C_1(\sqrt{\varepsilon} \| \phi_R^\varepsilon \|_{H^\delta})$ such that*

$$
\| R_\phi \|_{H^k} \leq C_1(\sqrt{\varepsilon} \| \phi_R^\varepsilon \|_{H^\delta})(1 + \| \phi_R^\varepsilon \|_{H^k}) \tag{2.25}
$$

and

$$
\| \partial_t R_\phi \|_{H^k} \leq C_1(\sqrt{\varepsilon} \| \phi_R^\varepsilon \|_{H^\delta})(1 + \| \partial_t \phi_R^\varepsilon \|_{H^k}), \tag{2.26}
$$

where $\delta = max(2, k-1)$. Furthermore, the constant $C_1(\cdot)$ can be chosen to be nondecreasing.

The proof of Lemma 2.1 is straightforward because we note that H^2 is an algebra in $\mathbb{R}^3$.

Corollary 2.1. *Let $k \geq 0$ be an integer; then there exists a constant $1 \leq C_1 = C_1(\sqrt{\varepsilon} \| \phi_R^\varepsilon \|_{H^\delta})$ such that*

$$
\| \nabla R_\phi \|_{H^k} \leq C_1(\sqrt{\varepsilon} \| \phi_R^\varepsilon \|_{H^\delta})(1 + \| \nabla \phi_R^\varepsilon \|_{H^k}) \tag{2.27}
$$

and

$$
\| \partial_t \nabla R_\phi \|_{H^k} \leq C_1(\sqrt{\varepsilon} \| \phi_R^\varepsilon \|_{H^\delta})(1 + \| \partial_t \nabla \phi_R^\varepsilon \|_{H^k}), \tag{2.28}
$$

where $\delta = max(2, k-1)$. Furthermore, the constant $C_1(\cdot)$ can be chosen to be nondecreasing.

In view of above, we may assume thereafter that the known profiles $(\widetilde{n_\alpha}, \widetilde{\boldsymbol{u}}, \widetilde{\phi})$ are smooth enough such that there exist some $C > 0$ and some $s \geq 4$,

$$
\sup_{[0, \tau_*]} \| (\widetilde{n_\alpha}, \widetilde{\boldsymbol{u}}, \widetilde{\phi}), R_n, \boldsymbol{R_u}, \boldsymbol{R}_T \|_{H^s} \leq C, \tag{2.29}
$$

where τ_* is the existence time.

So, our key mathematical difficulty is to derive estimates for the remainders $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ uniformly in ε.

We first introduce the following ε-dependent norms. We denote

$$
\vert\!\vert\!\vert (n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) \vert\!\vert\!\vert_{s'}^2 = \vert\!\vert\!\vert n_R^\varepsilon \vert\!\vert\!\vert_{s'}^2 + \vert\!\vert\!\vert \boldsymbol{u}_R^\varepsilon \vert\!\vert\!\vert_{s'}^2 + \vert\!\vert\!\vert \phi_R^\varepsilon \vert\!\vert\!\vert_{s'}^2, \tag{2.30}
$$

where

$$\begin{cases} \|\!|n_R^\varepsilon|\!\|_{s'}^2 = \|n_R^\varepsilon\|_{H^{s'}}^2, \\ \|\!|\boldsymbol{u}_R^\varepsilon|\!\|_{s'}^2 = \|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2 + \varepsilon\|\nabla \boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2, \\ \|\!|\phi_R^\varepsilon|\!\|_{s'}^2 = \|\phi_R^\varepsilon\|_{H^{s'}}^2 + \varepsilon\|\nabla\phi_R^\varepsilon\|_{H^{s'}}^2 + \varepsilon^2\|\Delta\phi_R^\varepsilon\|_{H^{s'}}^2, \end{cases}$$

where $\|\cdot\|_{H^{s'}}$ is the standard Sobolev norm. And the ∇,Δ reduce to $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$.

Then the main theorem is stated in the following.

Theorem 2.1. *Let* $s \geq 4$ *be such that* (2.29) *holds and let* $(n_\alpha^{(i)}, \boldsymbol{u}^{(i)}, \phi^{(i)}) \in H^s$ *for* $1 \leq i \leq 6$ *be solutions constructed on the interval* $[0, \tau_*)$ *with smooth initial data* $(n_{\alpha 0}^{(i)}, \boldsymbol{u}_0^{(i)}, \phi_0^{(i)})$. *Let* $4 \leq s' < s$ *and assume that the initial data* $(n_{\alpha 0}, \boldsymbol{u}_0, \phi_0)$ *for the Euler-Poisson system* (1.1)-(1.2) *has the expansion of the form* (2.19) *and* $(n_R^\varepsilon, u_R^\varepsilon, \phi_R^\varepsilon)|_{t=0} = (n_{R0}^\varepsilon, u_{R0}^\varepsilon, \phi_{R0}^\varepsilon)$ *satisfy* (2.21). *Then for any* $0 < \tau_0 < \tau_*$, *there exist* $\varepsilon_0 > 0$ *and* $C_{\tau_0} > 0$ *such that when* $0 < \varepsilon < \varepsilon_0$, *the solutions of the Euler-Poisson system* (1.1)-(1.2) *with initial data* $(n_{\alpha 0}, \boldsymbol{u}_0, \phi_0)$ *can be expressed in the expansion* (2.19), *such that the solutions* $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ *of* (2.21) *satisfy, when* $T_\alpha = 0$,

$$\sup_{[0,\tau_0]} \|\!|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2 \leq C_{\tau_0}(1 + \|\!|(n_{R0}^\varepsilon, u_{R0}^\varepsilon, \phi_{R0}^\varepsilon)|\!\|_{s'}^2), \tag{2.31}$$

where $\|\!|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2 = \|\!|n_R^\varepsilon|\!\|_{s'}^2 + \|\!|\boldsymbol{u}_R^\varepsilon|\!\|_{s'}^2 + \|\!|\phi_R^\varepsilon|\!\|_{s'}^2$ *is defined in* (2.30).

Remark 2.1. When $T_\alpha = 0$, the system of (3.1) for ϕ_R^ε does not match the common structure of Friedrichs symmetric systems. To overcome this difficulty, we need to combine the energy estimates with the delicate structure of the Poisson equation carefully. This is why we introduce the norm $\|\!|\cdot|\!\|_{s'}$.

3. Uniform energy estimates

In this section, we dedicate to the proof of Theorem 2.1, which requires a combination of energy method and analysis of remainder equation (2.21). In what follows, we take the rescaling $\tau = t/\sqrt{\varepsilon}$ and for simplicity we drop the bars. Throughout this section, we renormalize $N_\alpha = 2$ and all the other constants to be 1. As $T_\alpha = 0$, from (2.21) we obtain the following new remainder equation

$$\begin{cases} \dfrac{1}{\sqrt{\varepsilon}}\partial_t n_R^\varepsilon - \dfrac{\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon}\cdot\nabla n_R^\varepsilon + \dfrac{n_\alpha}{\varepsilon}\nabla\cdot\boldsymbol{u}_R^\varepsilon + \dfrac{1}{\sqrt{\varepsilon}} n_R^\varepsilon \nabla\cdot\widetilde{\boldsymbol{u}} + \dfrac{1}{\sqrt{\varepsilon}}\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{n_\alpha} + \sqrt{\varepsilon}R_n = 0, & \text{(a)} \\ \dfrac{1}{\sqrt{\varepsilon}}\partial_t \boldsymbol{u}_R^\varepsilon - \dfrac{\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon}\cdot\nabla\boldsymbol{u}_R^\varepsilon + \dfrac{1}{\sqrt{\varepsilon}}\boldsymbol{u}_R^\varepsilon\nabla\cdot\widetilde{\boldsymbol{u}} + \sqrt{\varepsilon}\boldsymbol{R}_u = -\dfrac{1}{\varepsilon}\nabla\phi_R^\varepsilon + \dfrac{1}{\varepsilon}\boldsymbol{u}_R^\varepsilon \times \boldsymbol{e}_1, & \text{(b)} \\ \varepsilon\Delta\phi_R^\varepsilon = \phi_R^\varepsilon - 2n_R^\varepsilon + \sqrt{\varepsilon}\phi^{(1)}\phi_R^\varepsilon + \varepsilon R_\phi. & \text{(c)} \end{cases} \tag{3.1}$$

Now, we need to give estimates uniformly in ε for system (3.1). The continuity method will be employed. We also assume that (3.1) has smooth solutions in a small time τ_ε dependent on ε. Let $\tilde{C}$ be a constant, which will be determined later, much larger than the bound of $\|\!|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)(0)|\!\|_{s'}$, such that on $[0, \tau_\varepsilon]$

$$\sup_{[0,\tau_\varepsilon]} \|\!|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'} \leq \tilde{C}. \tag{3.2}$$

We will prove that $\tau_\varepsilon > \tau_0$ as $\varepsilon \to 0$ for some $0 < \tau_0 < \tau_*$. Recalling the expressions for n_α and $\boldsymbol{u}$, we immediately know that there exists some $\varepsilon_1 = \varepsilon_1(\tilde{C}) > 0$ such that on $[0, \tau_\varepsilon]$,

$$1/2 < n_\alpha < 3/2, \quad |\boldsymbol{u}| \leq 1/2, \tag{3.3}$$

for all $0 < \varepsilon < \varepsilon_1$. To smooth the presentation, we first show several lemmas (Lemmas 3.1-3.3), giving some basic estimates between the remainders (Lemma 3.1) using the Poisson equation (3.1c) and their time derivatives (Lemmas 3.2 and 3.3).

Lemma 3.1. *Let* $(n^\varepsilon_R, \boldsymbol{u}^\varepsilon_R, \phi^\varepsilon_R)$ *be a solution to* (3.1) *and* $0 \leq k \leq s' < s$ *be an integer. There exist some* $0 < \varepsilon_1 < 1$ *and* C, $C_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_1$,

$$\|n^\varepsilon_R\|^2_{H^k} \leq C\|\phi^\varepsilon_R\|^2_{H^k} + C\varepsilon\|\nabla\phi^\varepsilon_R\|^2_{H^k} + C\varepsilon^2\|\Delta\phi^\varepsilon_R\|^2_{H^k} + CC_0^2\varepsilon, \tag{3.4}$$

$$\|\phi^\varepsilon_R\|^2_{H^k} + \varepsilon\|\nabla\phi^\varepsilon_R\|^2_{H^k} + \varepsilon^2\|\Delta\phi^\varepsilon_R\|^2_{H^k} \leq C\|n^\varepsilon_R\|^2_{H^k} + CC_0^2\varepsilon. \tag{3.5}$$

Proof. The proof is standard and hence omitted. □

Lemma 3.2. *Let* $(n^\varepsilon_R, \boldsymbol{u}^\varepsilon_R, \phi^\varepsilon_R)$ *be a solution to (3.1) and* $s' < s$ *be an integer. Then for any* $1 \leq k \leq s'$, *there exist some* $0 < \varepsilon_1 < 1$ *and* $C > 0$ *such that for every* $0 < \varepsilon < \varepsilon_1$,

$$\|\sqrt{\varepsilon}\partial_t n^\varepsilon_R\|^2_{H^{k-1}} \leq C\{1 + \|\boldsymbol{u}^\varepsilon_R\|^2_{H^{\delta+1}} + \|n^\varepsilon_R\|^2_{H^{\delta+1}}\}, \tag{3.6}$$

or equivalently,

$$\|\sqrt{\varepsilon}\partial_t n^\varepsilon_R\|^2_{H^{k-1}} \leq C\{1 + \|\boldsymbol{u}^\varepsilon_R\|^2_{H^{\delta+1}} + |\!|\!|\phi^\varepsilon_R|\!|\!|^2_{\delta+1}\}, \tag{3.7}$$

where $\delta = \max\{2, k-1\}$.

Proof. We can complete the proof by multiplying (3.1a) by ε, and then take the H^{k-1} norm. □

Lemma 3.3. *Let* $(n^\varepsilon_R, \boldsymbol{u}^\varepsilon_R, \phi^\varepsilon_R)$ *be a solution to (3.1) and* $s' < s$ *be an integer. Then for any* $1 \leq k \leq s'$, *there exist some* $0 < \varepsilon_1 < 1$ *and* $C > 0$ *such that for every* $0 < \varepsilon < \varepsilon_1$,

$$|\!|\!|\partial_t\phi^\varepsilon_R|\!|\!|^2_{k-1} \leq C\|\partial_t n^\varepsilon_R\|^2_{H^{k-1}} + C\sqrt{\varepsilon}\|\phi^\varepsilon_R\|^2_{H^{k-1}} + CC_0^2\varepsilon. \tag{3.8}$$

Proof. We take ∂_t of (3.1c) and then take the H^{k-1} inner product with $\partial_t\phi^\varepsilon_R$ to obtain

$$\begin{aligned}
&\varepsilon\|\nabla\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} + \|\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} \\
=&\langle 2\partial_t n^\varepsilon_R, \partial_t\phi^\varepsilon_R\rangle_{H^{k-1}} - \sqrt{\varepsilon}\langle\partial_t(\phi^{(1)}\phi^\varepsilon_R), \partial_t\phi^\varepsilon_R\rangle_{H^{k-1}} - \varepsilon\langle\partial_t R_\phi, \partial_t\phi^\varepsilon_R\rangle_{H^{k-1}} \\
\leq& 4\|\partial_t n^\varepsilon_R\|^2_{H^{k-1}} + \frac{1}{4}\|\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} + C\varepsilon\|\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} \\
&+ C\varepsilon\|\phi^\varepsilon_R\|^2_{H^{k-1}} + 2\varepsilon^2\|\partial_t R_\phi\|^2_{H^{k-1}}.
\end{aligned}$$

By (2.26) in Lemma 2.1, there exists some $\varepsilon_1 = \varepsilon_1(\tilde{C}) > 0$ such that when $\varepsilon < \varepsilon_1$, we have

$$\varepsilon\|\nabla\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} + \|\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} \leq C\|\partial_t n^\varepsilon_R\|^2_{H^{k-1}} + C\varepsilon\|\phi^\varepsilon_R\|^2_{H^{k-1}} + CC_0^2\varepsilon^2 \tag{3.9}$$

for some universal constant $C > 0$.

On the other hand, by taking ∂_t of (3.1c) and then taking the H^{k-1} inner product with $\varepsilon\partial_t\Delta\phi^\varepsilon_R$, we obtain

$$\begin{aligned}
&\varepsilon^2\|\partial_t\Delta\phi^\varepsilon_R\|^2_{H^{k-1}} + \varepsilon\|\partial_t\nabla\phi^\varepsilon_R\|^2_{H^{k-1}} \\
\leq& C\|\partial_t n^\varepsilon_R\|^2_{H^{k-1}} + C\varepsilon^2(\|\partial_t\phi^\varepsilon_R\|^2_{H^{k-1}} + \|\phi^\varepsilon_R\|^2_{H^{k-1}}) + CC_0^2\varepsilon
\end{aligned} \tag{3.10}$$

for $\varepsilon < \varepsilon_1$ with some $0 < \varepsilon_1 < 1$.

Adding (3.9) and (3.10), by choosing ε_1 sufficiently small such that $C_1^2 \leq \frac{1}{2}$, we obtain

$$\begin{aligned}&\varepsilon^2 \|\partial_t \Delta \phi_R^\varepsilon\|_{H^{k-1}}^2 + \varepsilon \|\partial_t \nabla \phi_R^\varepsilon\|_{H^{k-1}}^2 + \|\partial_t \phi_R^\varepsilon\|_{H^{k-1}}^2 \\ \leq & C\|\partial_t n_R^\varepsilon\|_{H^{k-1}}^2 + C\varepsilon \|\phi_R^\varepsilon\|_{H^{k-1}}^2 + CC_0^2 \varepsilon.\end{aligned} \tag{3.11}$$

□

From Lemma 3.3, we can get Corollary 3.1 and Corollary 3.2.

Corollary 3.1. *Under the same assumptions of Lemma 3.3, we have*

$$|\!|\!| \sqrt{\varepsilon} \partial_t \phi_R^\varepsilon |\!|\!|_{k-1}^2 \leq C\{1 + \|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{\delta+1}}^2\}, \tag{3.12}$$

or equivalently

$$|\!|\!| \sqrt{\varepsilon} \partial_t \phi_R^\varepsilon |\!|\!|_{k-1}^2 \leq C\{1 + \|\boldsymbol{u}_R^\varepsilon\|_{H^{\delta+1}}^2 + |\!|\!| \phi_R^\varepsilon |\!|\!|_{\delta+1}^2\}, \tag{3.13}$$

where $\delta = \max\{2, k-1\}$.

Proof. The proof is complete by multiplying (3.8) with ε^2 and then using Lemma 3.2. □

Corollary 3.2. *Let α be a multi-index with $|\alpha| = k$; then*

$$\varepsilon \|\varepsilon \partial_t \nabla \partial_x^\alpha \phi_R^\varepsilon\|_{L^2}^2 \leq C\|\varepsilon \partial_t \Delta \phi_R^\varepsilon\|_{H^{k-1}}^2.$$

Proof. By the Riesz theorem [17], we have

$$\|\sqrt{\varepsilon} \partial_{x_i} \nabla f\|_{L^2} \leq C\|\Delta f\|_{L^2}$$

for those f that make sense. The proof is complete by letting $f = \varepsilon \partial_t \phi_R^\varepsilon$. □

In the following, we first prove Proposition 3.1, which gives the s' order energy estimates of the remainders. Then we give the $s'+1$ order energy estimates of the remainders with weights ŠÅ in Proposition 3.2. By combining Propositions 3.1 and 3.2, we can finish the proof of Theorem 2.1.

Proposition 3.1. *Let $s' \geq 4$ be an integer and $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1). Then for any integer $0 \leq k \leq s'$, there holds*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha \boldsymbol{u}_R^\varepsilon\|_{L^2}^2 + \frac{1}{4}\frac{d}{dt}\int \frac{1 + \sqrt{\varepsilon}\phi^{(1)} + \varepsilon^{\frac{3}{2}}\phi_R^\varepsilon}{n_\alpha} |\partial_x^\alpha \phi_R^\varepsilon|^2 + \frac{1}{4}\frac{d}{dt}\int \frac{\varepsilon}{n_\alpha}|\nabla \partial_x^\alpha \phi_R^\varepsilon|^2 \\ \leq & CC_1(1 + \varepsilon |\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2)\{1 + |\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2\},\end{aligned} \tag{3.14}$$

where α is any multi-index with $|\alpha| = k \leq s'$ and $|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2$ is defined in (2.30).

Proof. We take ∂_x^α of (3.1b) and then take L^2 inner product with $\partial_x^\alpha \boldsymbol{u}_R^\varepsilon$ to obtain

$$\begin{aligned}&\frac{1}{2\sqrt{\varepsilon}}\frac{d}{dt}\|\partial_x^\alpha \boldsymbol{u}_R^\varepsilon\|_{L^2}^2 \\ =& \frac{1}{\varepsilon}\langle \partial_{x_1}\partial_x^\alpha \boldsymbol{u}_R^\varepsilon, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle - \frac{1}{\varepsilon}\langle \partial_x^\alpha(\boldsymbol{u} \cdot \nabla \boldsymbol{u}_R^\varepsilon), \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle - \frac{1}{\sqrt{\varepsilon}}\langle \partial_x^\alpha(\boldsymbol{u}_R^\varepsilon \cdot \nabla \widetilde{\boldsymbol{u}}), \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle \\ &- \sqrt{\varepsilon}\langle \partial_x^\alpha \boldsymbol{R}_{\boldsymbol{u}}, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle + \frac{1}{\varepsilon}\langle \partial_x^\alpha \boldsymbol{u}_R^\varepsilon \times \boldsymbol{e}_1, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle - \frac{1}{\varepsilon}\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle \\ =:& F_1 + \cdots + F_6.\end{aligned} \tag{3.15}$$

F_1 and F_5 are easy to deal with. We can get $F_1 = 0$ and $F_5 = 0$ by integrating by parts.

Estimate of F_2. Using the commutator, we decompose F_2 as

$$F_2 = -\frac{1}{\varepsilon}\langle \boldsymbol{u}\cdot\nabla\partial_x^\alpha \boldsymbol{u}_R^\varepsilon, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle - \frac{1}{\varepsilon}\langle [\partial_x^\alpha, \boldsymbol{u}]\cdot\nabla \boldsymbol{u}_R^\varepsilon, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle := F_{21} + F_{22}. \tag{3.16}$$

Since

$$-\langle \boldsymbol{u}\cdot\nabla\partial_x^\alpha \boldsymbol{u}_R^\varepsilon, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle = \frac{1}{2}\langle \nabla\cdot\boldsymbol{u}\partial_x^\alpha \boldsymbol{u}_R^\varepsilon, \partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle,$$

so we have

$$\begin{aligned} |F_{21}| \le & C\frac{1}{\sqrt{\varepsilon}}\|\nabla(\widetilde{\boldsymbol{u}} + \varepsilon\boldsymbol{u}_R^\varepsilon)\|_{L^\infty}\|\partial_x^\alpha \boldsymbol{u}_R^\varepsilon\|_{L^2}^2 \\ \le & C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^3})\|\partial_x^\alpha \boldsymbol{u}_R^\varepsilon\|_{L^2}^2. \end{aligned} \tag{3.17}$$

Here we used (2.20) and sobolev embedding.

On the other hand, recalling (2.20), by commutator estimates in [26], we have

$$\begin{aligned} \|[\partial_x^\alpha, \boldsymbol{u}]\cdot\nabla\boldsymbol{u}_R^\varepsilon\|_{L^2} &\le C(\|\nabla\boldsymbol{u}_R^\varepsilon\|_{\dot{H}^{k-1}}\|\nabla\boldsymbol{u}\|_{L^\infty} + \|\boldsymbol{u}\|_{\dot{H}^k}\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}) \\ &\le C\sqrt{\varepsilon}(\|\boldsymbol{u}_R^\varepsilon\|_{\dot{H}^k}(1+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}) + \sqrt{\varepsilon}\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^k})) \\ &\le C\sqrt{\varepsilon}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}})\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}, \end{aligned}$$

where $k \le s'$ and $s' \ge 3$. So we have

$$|F_{22}| \le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}})\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2. \tag{3.18}$$

We can estimate F_2 by combining (3.17) and (3.18)

$$|F_2| \le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}})\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2. \tag{3.19}$$

Estimate of F_3. By multiplicative estimates and (2.27), we have

$$\begin{aligned} \|\partial_x^\alpha(\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{\boldsymbol{u}})\|_{L^2} &\le C(\|\boldsymbol{u}_R^\varepsilon\|_{\dot{H}^k}\|\nabla\widetilde{\boldsymbol{u}}\|_{L^\infty} + \|\boldsymbol{u}_R^\varepsilon\|_{L^\infty}\|\nabla\widetilde{\boldsymbol{u}}\|_{\dot{H}^k}) \\ &\le C\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}, \end{aligned}$$

so we can get

$$|F_3| \le C\frac{1}{\sqrt{\varepsilon}}\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2. \tag{3.20}$$

F_4 is Similar to F_3, we obtain

$$|F_4| \le C\|\boldsymbol{u}_R^\varepsilon\|_{H^k}^2 + C\varepsilon. \tag{3.21}$$

Estimate of F_6. The estimate for F_6 is not straightforward and is very delicate since we need to use the structure of the remainder system (3.1) very carefully. By integration by parts, we have

$$F_6 = \frac{1}{\varepsilon}\langle \partial_x^\alpha \phi_R^\varepsilon, \nabla\cdot\partial_x^\alpha \boldsymbol{u}_R^\varepsilon\rangle,$$

where $|\alpha| = k$. Taking ∂_x^α of (3.1a) with $|\alpha| = k$, we have

$$\begin{aligned}\frac{1}{\varepsilon}\nabla\cdot\partial_x^\alpha \boldsymbol{u}_R^\varepsilon =&\frac{1}{n_\alpha}\left\{\frac{\boldsymbol{e}_1-\boldsymbol{u}}{\varepsilon}\cdot\nabla\partial_x^\alpha n_R^\varepsilon - \frac{1}{\sqrt{\varepsilon}}\partial_t\partial_x^\alpha n_R^\varepsilon - \left[\partial_x^\alpha, \frac{\boldsymbol{u}}{\varepsilon}\right]\cdot\nabla n_R^\varepsilon\right.\\ &\left. - \left[\partial_x^\alpha,\frac{n_\alpha}{\varepsilon}\right]\nabla\cdot\boldsymbol{u}_R^\varepsilon - \partial_x^\alpha\Big(\frac{1}{\sqrt{\varepsilon}}n_R^\varepsilon\nabla\cdot\widetilde{\boldsymbol{u}} + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{n_\alpha}+\sqrt{\varepsilon}R_n\Big)\right\}.\end{aligned}\tag{3.22}$$

Then, F_6 is decomposed as

$$\begin{aligned}F_6 =& \left\langle \partial_x^\alpha\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{\varepsilon n_\alpha}\cdot\nabla\partial_x^\alpha n_R^\varepsilon\right\rangle - \left\langle\partial_x^\alpha\phi_R^\varepsilon, \frac{1}{\sqrt{\varepsilon}n_\alpha}\partial_t\partial_x^\alpha n_R^\varepsilon\right\rangle\\ &- \left\langle\partial_x^\alpha\phi_R^\varepsilon, \frac{1}{\varepsilon n_\alpha}[\partial_x^\alpha,\boldsymbol{u}]\cdot\nabla n_R^\varepsilon\right\rangle - \left\langle\partial_x^\alpha\phi_R^\varepsilon, \frac{1}{\varepsilon n_\alpha}[\partial_x^\alpha, n]\nabla\cdot\boldsymbol{u}_R^\varepsilon\right\rangle\\ &- \left\langle\partial_x^\alpha\phi_R^\varepsilon, \frac{1}{n_\alpha}\partial_x^\alpha\Big(\frac{1}{\sqrt{\varepsilon}}n_R^\varepsilon\nabla\cdot\widetilde{\boldsymbol{u}} + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{n_\alpha}+\sqrt{\varepsilon}R_n\Big)\right\rangle\\ =:& F_{61}+\cdots+F_{65}.\end{aligned}\tag{3.23}$$

We first bound $F_{63} - F_{65}$ and start with F_{63}. By commutator estimates, we have

$$\begin{aligned}\|[\partial_x^\alpha,\boldsymbol{u}]\cdot\nabla n_R^\varepsilon\|_{L^2} &\le C(\|\nabla n_R^\varepsilon\|_{\dot H^{k-1}}\|\nabla\boldsymbol{u}\|_{L^\infty} + \|\boldsymbol{u}\|_{\dot H^k}\|\nabla n_R^\varepsilon\|_{L^\infty})\\ &\le C\sqrt{\varepsilon}(\|n_R^\varepsilon\|_{\dot H^k}(1+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}) + \|\nabla n_R^\varepsilon\|_{L^\infty}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^k}))\\ &\le C\sqrt{\varepsilon}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}})(\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}+\|n_R^\varepsilon\|_{H^{s'}}),\end{aligned}\tag{3.24}$$

where $k \le s'$ and $s' \ge 3$. Therefore, from (3.3), we obtain

$$\begin{aligned}|F_{63}| &\le C\frac{1}{\sqrt{\varepsilon}}\|\partial_x^\alpha\phi_R^\varepsilon\|_{L^2}\|[\partial_x^\alpha,\boldsymbol{u}]\cdot\nabla n_R^\varepsilon\|_{L^2}\\ &\le C\frac{1}{\sqrt{\varepsilon}}[(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon)\|_{H^{s'}}^2 + C\|\phi_R^\varepsilon\|_{H^{s'}}^2].\end{aligned}\tag{3.25}$$

F_{64} behaves like F_{63}, we have

$$|F_{64}| \le C\frac{1}{\sqrt{\varepsilon}}[(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon)\|_{H^{s'}}^2 + C\|\phi_R^\varepsilon\|_{H^{s'}}^2].\tag{3.26}$$

By multiplicative estimates, we have

$$\|\partial_x^\alpha(\frac{1}{\sqrt{\varepsilon}}n_R^\varepsilon\nabla\cdot\widetilde{\boldsymbol{u}} + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{n_\alpha}+\sqrt{\varepsilon}R_n)\| \le C\frac{1}{\sqrt{\varepsilon}}\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon)\|_{H^{s'}} + C\sqrt{\varepsilon},\tag{3.27}$$

it follows that

$$|F_{65}| \le C\frac{1}{\sqrt{\varepsilon}}[(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon)\|_{H^{s'}}^2 + C\|\phi_R^\varepsilon\|_{H^{s'}}^2].\tag{3.28}$$

Summarizing (3.25), (3.26) and (3.28), we have

$$|F_{63}+F_{64}+F_{65}| \le C\frac{1}{\sqrt{\varepsilon}}[\|\phi_R^\varepsilon\|_{H^{s'}}^2 + C(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)\|(n_R^\varepsilon,\boldsymbol{u}_R^\varepsilon)\|_{H^{s'}}^2].$$

We estimate F_{61} and F_{62} in Lemma 3.4 and Lemma 3.5, respectively. □

Lemma 3.4. *Let* $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ *be a solution to (3.1) and* $0 \le k \le s'$ *be an integer; then*

$$|F_{61}| \le C\frac{1}{\sqrt{\varepsilon}}(1 + C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^{s'}}) + \varepsilon\|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon)\|_{H^4}^2)(1 + \|\phi_R^\varepsilon\|_{H^{s'}}^2 + \varepsilon\|\nabla\phi_R^\varepsilon\|_{H^{s'}}^2). \tag{3.29}$$

Proof. Let α be an multi-index such that $|\alpha| = k \le s'$. Taking ∂_x^α of (3.1c), we have

$$2\partial_x^\alpha n_R^\varepsilon = \partial_x^\alpha\phi_R^\varepsilon - \varepsilon\Delta\partial_x^\alpha\phi_R^\varepsilon + \sqrt{\varepsilon}\partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) + \varepsilon\partial_x^\alpha R_\phi. \tag{3.30}$$

Then F_{61} in (3.23) is decomposed as

$$\begin{aligned} F_{61} &= \left\langle \partial_x^\alpha\phi_R^\varepsilon, \frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\cdot\nabla\partial_x^\alpha\phi_R^\varepsilon \right\rangle - \varepsilon\left\langle \partial_x^\alpha\phi_R^\varepsilon, \frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\cdot\nabla\Delta\partial_x^\alpha\phi_R^\varepsilon \right\rangle \\ &\quad + \sqrt{\varepsilon}\left\langle \partial_x^\alpha\phi_R^\varepsilon, \frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\cdot\nabla\partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) \right\rangle + \varepsilon\left\langle \partial_x^\alpha\phi_R^\varepsilon, \frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\cdot\nabla\partial_x^\alpha R_\phi \right\rangle \\ &=: F_{611} + F_{612} + F_{613} + F_{614}. \end{aligned}$$

We first bound F_{611}. By integrating by parts, we have

$$F_{611} = -\frac{1}{4}\left\langle \partial_x^\alpha\phi_R^\varepsilon, \nabla\cdot\left(\frac{\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon n_\alpha}\right)\partial_x^\alpha\phi_R^\varepsilon \right\rangle. \tag{3.31}$$

We can get the follows by assumptions (3.2) and (3.3), and Sobolev embedding

$$\begin{aligned} \left\|\nabla\cdot\left(\frac{\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon n_\alpha}\right)\right\|_{L^\infty} &\le C\frac{1}{\sqrt{\varepsilon}}(1 + \varepsilon\|\nabla n_R^\varepsilon\|_{L^\infty} + \varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}) \\ &\le C\frac{1}{\sqrt{\varepsilon}}(1 + \varepsilon\|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon)\|_{H^3}), \end{aligned} \tag{3.32}$$

then we use the Hölder inequality can obtain

$$|F_{611}| \le C\frac{1}{\sqrt{\varepsilon}}(1 + \varepsilon\|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon)\|_{H^3}^2)\|\partial_x^\alpha\phi_R^\varepsilon\|^2. \tag{3.33}$$

Estimate of F_{612}. By integration by parts twice, we get the upper bound consists of three parts,

$$\begin{aligned} F_{612} &= \varepsilon\left\langle \nabla\partial_x^\alpha\phi_R^\varepsilon, \frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\Delta\partial_x^\alpha\phi_R^\varepsilon \right\rangle + \varepsilon\left\langle \partial_x^\alpha\phi_R^\varepsilon, \nabla\cdot\left(\frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\right)\Delta\partial_x^\alpha\phi_R^\varepsilon \right\rangle \\ &= -\varepsilon\left\langle \nabla\partial_x^\alpha\phi_R^\varepsilon, \nabla\left(\frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\right)\nabla\partial_x^\alpha\phi_R^\varepsilon \right\rangle - \frac{\varepsilon}{4}\left\langle \nabla\partial_x^\alpha\phi_R^\varepsilon, \nabla\cdot\left(\frac{\boldsymbol{e}_1 - \boldsymbol{u}}{\varepsilon n_\alpha}\right)\nabla\partial_x^\alpha\phi_R^\varepsilon \right\rangle \\ &\quad - \varepsilon\left\langle \partial_x^\alpha\phi_R^\varepsilon, \nabla\nabla\cdot\left(\frac{\boldsymbol{e}_1 - \boldsymbol{u}}{2\varepsilon n_\alpha}\right)\nabla\partial_x^\alpha\phi_R^\varepsilon \right\rangle \\ &=: F_{6121} + F_{6122} + F_{6123}. \end{aligned} \tag{3.34}$$

We can estimate F_{6121} and F_{6121} from (3.34)

$$F_{6121}, F_{6122} \le C\frac{1}{\sqrt{\varepsilon}}(1 + \varepsilon\|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon)\|_{H^3})\varepsilon\|\nabla\partial_x^\alpha\phi_R^\varepsilon\|^2. \tag{3.35}$$

Similar to (3.32), we have

$$\left|\nabla\nabla\cdot\left(\frac{\boldsymbol{e}_1-\boldsymbol{u}}{2\varepsilon n_\alpha}\right)\right| \leq C(1+\sqrt{\varepsilon}|\nabla^2 n_R^\varepsilon|+\sqrt{\varepsilon}|\nabla\nabla\cdot\boldsymbol{u}_R^\varepsilon|+\varepsilon|\nabla n_R^\varepsilon| \\ +\varepsilon|\nabla\cdot\boldsymbol{u}_R^\varepsilon|+\varepsilon^2|\nabla n_R^\varepsilon|^2+\varepsilon^2|\nabla\cdot\boldsymbol{u}_R^\varepsilon|^2), \tag{3.36}$$

where we used (3.3). Then we get

$$|F_{6123}|\leq C\sqrt{\varepsilon}(1+\varepsilon\|(n_R^\varepsilon,\boldsymbol{u}_R^\varepsilon)\|_{H^4}^2)(\|\partial_x^\alpha\phi_R^\varepsilon\|^2+\varepsilon\|\nabla\partial_x^\alpha\phi_R^\varepsilon\|^2). \tag{3.37}$$

So, we can estimate F_{612} by combining (3.35) and (3.37)

$$|F_{612}|\leq C\sqrt{\varepsilon}(1+\varepsilon\|(n_R^\varepsilon,\boldsymbol{u}_R^\varepsilon)\|_{H^4}^2)(\|\partial_x^\alpha\phi_R^\varepsilon\|^2+\varepsilon\|\nabla\partial_x^\alpha\phi_R^\varepsilon\|^2). \tag{3.38}$$

Estimate of F_{613}. By integrating by parts, we have

$$\begin{aligned} F_{613} &= \frac{\sqrt{\varepsilon}}{2}\left\langle\partial_x^\alpha\phi_R^\varepsilon,\left(\frac{(\boldsymbol{e}_1-\boldsymbol{u})\phi^{(1)}}{\varepsilon n_\alpha}\right)\cdot\nabla\partial_x^\alpha\phi_R^\varepsilon\right\rangle+\frac{\sqrt{\varepsilon}}{2}\left\langle\partial_x^\alpha\phi_R^\varepsilon,\frac{(\boldsymbol{e}_1-\boldsymbol{u})}{\varepsilon n_\alpha}[\nabla\partial_x^\alpha,\phi^{(1)}]\phi_R^\varepsilon\right\rangle \\ &=: F_{6131}+F_{6132}. \end{aligned} \tag{3.39}$$

To bound the term of F_{6131}, by integrating by parts,

$$F_{6131}=-\frac{\sqrt{\varepsilon}}{4}\left\langle\partial_x^\alpha\phi_R^\varepsilon,\nabla\cdot\left(\frac{(\boldsymbol{e}_1-\boldsymbol{u})\phi^{(1)}}{\varepsilon n_\alpha}\right)\partial_x^\alpha\phi_R^\varepsilon\right\rangle.$$

Hence, using the Hölder inequality and (3.32), we obtain

$$|F_{6131}|\leq C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|(n_R^\varepsilon,\boldsymbol{u}_R^\varepsilon)\|_{H^3})\|\partial_x^\alpha\phi_R^\varepsilon\|^2.$$

On the other hand, by the commutator estimate, we know

$$\begin{aligned} \|[\nabla\partial_x^\alpha,\phi^{(1)}]\phi_R^\varepsilon\|_{L^2} &\leq C(\|\phi_R^\varepsilon\|_{H^k}\|\phi^{(1)}\|_{L^\infty})+\|\phi^{(1)}\|_{H^k}\|\phi_R^\varepsilon\|_{L^\infty} \\ &\leq C\|\phi_R^\varepsilon\|_{H^{s'}}, \end{aligned} \tag{3.40}$$

so we have

$$|F_{6132}|\leq C\frac{1}{\sqrt{\varepsilon}}\|\phi_R^\varepsilon\|_{H^{s'}}^2.$$

Therefore, we obtain

$$|F_{613}|\leq C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|n_R^\varepsilon\|_{H^3}^2+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^3}^2)\|\phi_R^\varepsilon\|_{H^k}^2. \tag{3.41}$$

Estimate of F_{614}. By the Hölder inequality and (2.27) in Corollary 2.1, we have

$$|F_{614}|\leq C\frac{1}{\sqrt{\varepsilon}}[\|\partial_x^\alpha\phi_R^\varepsilon\|^2+C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^{s'}})(1+\varepsilon\|\nabla\phi_R^\varepsilon\|_{H^{s'}}^2)], \tag{3.42}$$

where $|\alpha|=k\leq s'$. So the proof of Lemma 3.4 is complete by adding (3.33), (3.38), (3.41) and (3.42). □

Lemma 3.5. *Let $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1) and $0 \le k \le s'$ be an integer; then*

$$\begin{aligned}F_{62} \le &-\frac{1}{4}\frac{d}{dt}\int \frac{1+\sqrt{\varepsilon}\phi^{(1)}+\varepsilon^{\frac{3}{2}}\phi_R^\varepsilon}{\sqrt{\varepsilon}n_\alpha}|\partial_x^\alpha \phi_R^\varepsilon|^2 - \frac{1}{4}\frac{d}{dt}\int \frac{\sqrt{\varepsilon}}{n_\alpha}|\nabla \partial_x^\alpha \phi_R^\varepsilon|^2 \\ &+ CC_1\frac{1}{\sqrt{\varepsilon}}(C_1+\varepsilon\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{s'}}^2)\{1+\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{s'}}^2\},\end{aligned} \tag{3.43}$$

where $|\alpha| = k \le s'$.

Proof. Taking ∂_x^α of (3.1c), we have

$$2\partial_x^\alpha n_R^\varepsilon = \partial_x^\alpha \phi_R^\varepsilon - \varepsilon \Delta \partial_x^\alpha \phi_R^\varepsilon + \sqrt{\varepsilon}\partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) + \varepsilon^{\frac{3}{2}}\partial_x^\alpha[(\phi_R^\varepsilon)^2] + \varepsilon \partial_x^\alpha \overline{R}_\phi.$$

Here $\overline{R}_\phi$ satisfies the same estimates of Lemma 2.1. Inserting this into F_{62} which defined in (3.23), we have

$$\begin{aligned}F_{62} =& -\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{2\sqrt{\varepsilon}n_\alpha}\partial_t \partial_x^\alpha \phi_R^\varepsilon \right\rangle + \varepsilon \left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{2\sqrt{\varepsilon}n_\alpha}\partial_t \Delta \partial_x^\alpha \phi_R^\varepsilon \right\rangle \\ &- \sqrt{\varepsilon}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{2\sqrt{\varepsilon}n_\alpha}\partial_t \partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) \right\rangle - \varepsilon^{\frac{3}{2}}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{2\sqrt{\varepsilon}n_\alpha}\partial_t \partial_x^\alpha[(\phi_R^\varepsilon)^2] \right\rangle \\ &- \varepsilon \left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{2\sqrt{\varepsilon}n_\alpha}\partial_t \partial_x^\alpha \overline{R}_\phi \right\rangle \\ =:& F_{621}+F_{622}+F_{623}+F_{624}+F_{625}.\end{aligned} \tag{3.44}$$

Estimate of F_{621}. By integration by parts in time, we obtain

$$F_{621} = -\frac{1}{4\sqrt{\varepsilon}}\frac{d}{dt}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{n_\alpha}\partial_x^\alpha \phi_R^\varepsilon \right\rangle + \frac{1}{4\sqrt{\varepsilon}}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \partial_t\left(\frac{1}{n_\alpha}\right)\partial_x^\alpha \phi_R^\varepsilon \right\rangle, \tag{3.45}$$

we know that

$$\left\|\partial_t\left(\frac{1}{n_\alpha}\right)\right\|_{L^\infty} \le C\sqrt{\varepsilon}(1+\varepsilon\|\partial_t n_R^\varepsilon\|_{L^\infty}), \tag{3.46}$$

so we have

$$\begin{aligned}F_{621} \le& -\frac{1}{4\sqrt{\varepsilon}}\frac{d}{dt}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{n_\alpha}\partial_x^\alpha \phi_R^\varepsilon \right\rangle + C(1+\varepsilon\|\partial_t n_R^\varepsilon\|_{L^\infty})\|\partial_x^\alpha \phi_R^\varepsilon\|^2 \\ \le& -\frac{1}{4\sqrt{\varepsilon}}\frac{d}{dt}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{n_\alpha}\partial_x^\alpha \phi_R^\varepsilon \right\rangle + C(1+\varepsilon\|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon)\|_{H^3})\|\phi_R^\varepsilon\|_{H^{s'}}^2,\end{aligned} \tag{3.47}$$

where in the second inequality, we have used (3.6) in Lemma 3.2.

Estimate of F_{622}. By first integrating by parts in space and then in time, we have

$$\begin{aligned}F_{622} =& -\frac{1}{4}\frac{d}{dt}\left\langle \nabla \partial_x^\alpha \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{n_\alpha}\nabla \partial_x^\alpha \phi_R^\varepsilon \right\rangle + \frac{\sqrt{\varepsilon}}{4}\left\langle \nabla \partial_x^\alpha \phi_R^\varepsilon, \partial_t\left(\frac{1}{n_\alpha}\right)\nabla \partial_x^\alpha \phi_R^\varepsilon \right\rangle \\ &- \frac{\sqrt{\varepsilon}}{2}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \nabla\left(\frac{1}{n_\alpha}\right)\cdot \partial_t \nabla \partial_x^\alpha \phi_R^\varepsilon \right\rangle.\end{aligned} \tag{3.48}$$

We can estimate the second term on the RHS of F_{622} by using (3.46)

$$\left|\frac{1}{4}\left\langle \nabla \partial_x^\alpha \phi_R^\varepsilon, \partial_t\left(\frac{\sqrt{\varepsilon}}{n_\alpha}\right)\nabla \partial_x^\alpha \phi_R^\varepsilon \right\rangle\right| \le C\varepsilon(1+\varepsilon\|\partial_t n_R^\varepsilon\|_{L^\infty})\|\nabla \partial_x^\alpha \phi_R^\varepsilon\|^2. \tag{3.49}$$

The third term on the RHS of F_{622} by using Hölder inequality

$$\begin{aligned}&\left|\frac{\sqrt{\varepsilon}}{2}\langle\partial_x^\alpha\phi_R^\varepsilon,\nabla\left(\frac{1}{n_\alpha}\right)\cdot\partial_t\nabla\partial_x^\alpha\phi_R^\varepsilon\rangle\right|\\&\le C(1+\varepsilon\|\nabla n_R^\varepsilon\|_{L^\infty}^2)\|\partial_x^\alpha\phi_R^\varepsilon\|^2+C\varepsilon^2\|\varepsilon\partial_t\Delta\phi_R^\varepsilon\|_{H^{k-1}}^2,\end{aligned}\tag{3.50}$$

where we have used Corollary 3.2. So we can estimate F_{622} by using (3.12) in Corollary 3.1 and Lemmas 3.1-3.2,

$$\begin{aligned}F_{622}\le&-\frac{1}{4}\frac{d}{dt}\left\langle\nabla\partial_x^\alpha\phi_R^\varepsilon,\frac{\sqrt{\varepsilon}}{n_\alpha}\nabla\partial_x^\alpha\phi_R^\varepsilon\right\rangle\\&+C(1+\varepsilon\|(\boldsymbol{u}_R^\varepsilon,n_R^\varepsilon,\phi_R^\varepsilon)\|_{H^3}^2)\{1+\|(\boldsymbol{u}_R^\varepsilon,n_R^\varepsilon,\phi_R^\varepsilon)\|_{H^{s'}}^2\}.\end{aligned}\tag{3.51}$$

Estimate of F_{623}. By integration by parts

$$\begin{aligned}F_{623}=&-\left\langle\partial_x^\alpha\phi_R^\varepsilon,\frac{\phi^{(1)}}{2n_\alpha}\partial_t\partial_x^\alpha\phi_R^\varepsilon\right\rangle-\left\langle\partial_x^\alpha\phi_R^\varepsilon,\frac{1}{2n_\alpha}[\partial_x^\alpha,\phi^{(1)}]\partial_t\phi_R^\varepsilon\right\rangle\\&-\left\langle\partial_x^\alpha\phi_R^\varepsilon,\frac{1}{2n_\alpha}\partial_x^\alpha(\partial_t\phi^{(1)}\phi_R^\varepsilon)\right\rangle\\=:&\,F_{6231}+F_{6232}+F_{6233}.\end{aligned}\tag{3.52}$$

For the first term on the RHS, by integration by parts, we have

$$F_{6231}=-\frac{1}{4}\frac{d}{dt}\left\langle\partial_x^\alpha\phi_R^\varepsilon,\frac{\phi^{(1)}}{n_\alpha}\partial_x^\alpha\phi_R^\varepsilon\right\rangle+\frac{1}{4}\left\langle\partial_x^\alpha\phi_R^\varepsilon,\partial_t\left(\frac{\phi^{(1)}}{n_\alpha}\right)\partial_x^\alpha\phi_R^\varepsilon\right\rangle,\tag{3.53}$$

by direct computation, we have

$$\left\|\partial_t\left(\frac{\phi^{(1)}}{n_\alpha}\right)\right\|_{L^\infty}\le C(1+\varepsilon\|\partial_t n_R^\varepsilon\|_{L^\infty}),$$

so we have

$$F_{6231}\le-\frac{1}{4}\frac{d}{dt}\left\langle\partial_x^\alpha\phi_R^\varepsilon,\frac{\phi^{(1)}}{n_\alpha}\partial_x^\alpha\phi_R^\varepsilon\right\rangle+C(1+\varepsilon\|\partial_t n_R^\varepsilon\|_{L^\infty})\|\partial_x^\alpha\phi_R^\varepsilon\|^2,\tag{3.54}$$

and by commutator estimates, we have

$$\begin{aligned}\|[\partial_x^\alpha,\phi^{(1)}]\partial_t\phi_R^\varepsilon\|_{L^2}&\le C(\|\partial_t\phi_R^\varepsilon\|_{H^{k-1}}\|\phi^{(1)}\|_{L^\infty}+\|\partial_t\phi_R^\varepsilon\|_{L^\infty}\|\phi^{(1)}\|_{H^{k-1}})\\&\le C(\|\partial_t\phi_R^\varepsilon\|_{H^{k-1}}+\|\partial_t\phi_R^\varepsilon\|_{H^2}),\end{aligned}$$

therefore, by using (3.12) in Corollary 3.1, we have

$$\begin{aligned}F_{6232}&\le C\|\partial_x^\alpha\phi_R^\varepsilon\|^2+C\varepsilon^2\|\partial_t\phi_R^\varepsilon\|_{H^\delta}^2\\&\le C\{1+\|(n_R^\varepsilon,\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon\|_{H^{\delta+1}}^2\},\end{aligned}\tag{3.55}$$

where $\delta=\max\{2,k-1\}$. By multiplicative estimates, we have

$$F_{6233}\le C\varepsilon\|\phi_R^\varepsilon\|_{H^{s'}}^2.\tag{3.56}$$

Combining the upper bounds, and using (3.6) in Lemma 3.2, we obtain

$$\begin{aligned} F_{623} \leq & -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{\phi^{(1)}}{n_\alpha}\partial_x^\alpha \phi_R^\varepsilon \right\rangle \\ & + C(1+\varepsilon\|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon)\|_{H^3}^2)\{1+\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{s'}}^2\}, \end{aligned} \tag{3.57}$$

where $s' \geq 3$.

Estimate of F_{624}. We have

$$\begin{aligned} F_{624} &= -\frac{\varepsilon}{2}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{\phi_R^\varepsilon}{n_\alpha}\partial_t \partial_x^\alpha \phi_R^\varepsilon \right\rangle - \frac{\varepsilon}{2}\left\langle \partial_x^\alpha \phi_R^\varepsilon, \frac{1}{n_\alpha}[\partial_x^\alpha, \phi_R^\varepsilon]\partial_t \phi_R^\varepsilon \right\rangle \\ &=: F_{6241} + F_{6242}. \end{aligned}$$

By integrating by parts in time, we have

$$F_{6241} = -\frac{\varepsilon}{4}\frac{d}{dt}\int \left(\frac{\phi_R^\varepsilon}{n_\alpha}\right)|\partial_x^\alpha \phi_R^\varepsilon|^2 dx - \frac{\varepsilon}{4}\frac{d}{dt}\int \left(\frac{\phi_R^\varepsilon}{n_\alpha}\right)|\partial_x^\alpha \phi_R^\varepsilon|^2 dx.$$

From Lemma 3.2 and Corollary 3.1, we have

$$\left\|\partial_t \left(\frac{\phi_R^\varepsilon}{n_\alpha}\right)\right\|_{L^\infty} \leq C\sqrt{\varepsilon}(1+\|\boldsymbol{u}_R^\varepsilon\|_{H^3}+\|n_R^\varepsilon\|_{H^3}+\|\phi_R^\varepsilon\|_{H^3}).$$

Therefore, we obtain

$$F_{6241} \leq -\frac{\varepsilon}{4}\frac{d}{dt}\int \left(\frac{\phi_R^\varepsilon}{n_\alpha}\right)|\partial_x^\alpha \phi_R^\varepsilon|^2 dx + C\sqrt{\varepsilon}(1+\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^3})\|\phi_R^\varepsilon\|_{H^k}^2. \tag{3.58}$$

On the other hand, by the commutator estimate, we have

$$\begin{aligned} \|[\partial_x^\alpha, \phi_R^\varepsilon]\partial_t \phi_R^\varepsilon\|_{L^2} &\leq C(\|\partial_t \phi_R^\varepsilon\|_{H^{k-1}}\|\phi_R^\varepsilon\|_{L^\infty}+\|\partial_t \phi_R^\varepsilon\|_{L^\infty}\|\phi_R^\varepsilon\|_{H^k}) \\ &\leq C\{\|\phi_R^\varepsilon\|_{H^2}(1+\|\partial_t n_R^\varepsilon\|_{H^{k-1}}+\|\phi_R^\varepsilon\|_{H^{k-1}}) \\ &\quad + \|\phi_R^\varepsilon\|_{H^k}(1+\|\partial_t n_R^\varepsilon\|_{H^2}+\|\phi_R^\varepsilon\|_{H^2})\}, \end{aligned}$$

it then follows that

$$|F_{6242}| \leq C(1+\varepsilon\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{s'}})\|\phi_R^\varepsilon\|_{H^{s'}}^2, \tag{3.59}$$

where we have used Lemma 3.2 and Lemma 3.3. Combining the upper bounds for F_{624} yields

$$F_{624} \leq -\frac{\varepsilon}{4}\frac{d}{dt}\int \left(\frac{\phi_R^\varepsilon}{n_\alpha}\right)|\partial_x^\alpha \phi_R^\varepsilon|^2 dx + C(1+\varepsilon\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{s'}})\|\phi_R^\varepsilon\|_{H^{s'}}^2. \tag{3.60}$$

Estimate of F_{625}. By (2.26) in Lemma 2.1, we have

$$\begin{aligned} F_{625} &\leq C\|\partial_x^\alpha \phi_R^\varepsilon\|^2 + \varepsilon\|\partial_t \partial_x^\alpha \overline{R}_\phi\|^2 \\ &\leq C\|\partial_x^\alpha \phi_R^\varepsilon\|^2 + C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^\delta})(1+\varepsilon\|\varepsilon\partial_t \phi_R^\varepsilon\|_{H^k}^2), \end{aligned} \tag{3.61}$$

where $\delta = \max\{2, k-1\}$. By Corollary 3.1, when $0 < \varepsilon < \varepsilon_1$, we have

$$\varepsilon\|\varepsilon\partial_t \phi_R^\varepsilon\|_{H^k}^2 \leq C\{1+\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2+\|n_R^\varepsilon\|_{H^{s'}}^2+\varepsilon\|\phi_R^\varepsilon\|_{H^{k-1}}^2\},$$

so we have

$$F_{625} \leq C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^\delta})\{1+\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2+\|n_R^\varepsilon\|_{H^{s'}}^2+\|\phi_R^\varepsilon\|_{H^{s'}}^2\}, \tag{3.62}$$

where $\delta = \max\{2, k-1\} \leq s'-1$. Adding (3.47), (3.51), (3.57), (3.61), and (3.62), we have (3.43). □

The proof of Proposition 3.1 is completed.

Proposition 3.2. *Let $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1). Then*

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}_R^\varepsilon\|_{\dot H^{s'}}^2 + \frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla\phi_R^\varepsilon, \frac{\varepsilon(1+\sqrt{\varepsilon}\phi^{(1)}+\varepsilon^{\frac{3}{2}}\phi_R^\varepsilon)}{n_\alpha}\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle \\ &+\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha\Delta\phi_R^\varepsilon, \frac{\varepsilon^2}{n_\alpha}\partial_x^\alpha\Delta\phi_R^\varepsilon\right\rangle \\ &\le C(1+C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^3}))(1+\varepsilon\vert\!\vert\!\vert(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\vert\!\vert\!\vert_{s'}^2)(1+\vert\!\vert\!\vert(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\vert\!\vert\!\vert_{s'}^2),\end{aligned}\tag{3.63}$$

where $|\alpha| = s'$ for any $s' \ge 4$ and $\vert\!\vert\!\vert(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\vert\!\vert\!\vert_{s'}^2$ is defined in (2.30).

Proof. Let α be any multi-index with $|\alpha| = k \le s'$ for any $4 \le s' < s$. We take ∂_x^α of (3.1b) and then take the inner product with $\varepsilon\Delta\partial_x^\alpha\boldsymbol{u}_R^\varepsilon$ in $L^2(\mathbb{R}^3)$. By integrating by parts, we obtain

$$\begin{aligned}\frac{\sqrt{\varepsilon}}{2}\frac{d}{dt}\|\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\|^2 &= \langle\partial_{x_1}\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon, \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle - \langle\partial_x^\alpha(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}_R^\varepsilon), \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle \\ &\quad - \langle\partial_x^\alpha(\boldsymbol{u}\cdot\nabla^2\boldsymbol{u}_R^\varepsilon), \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle - \sqrt{\varepsilon}\langle\partial_x^\alpha(\nabla\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{\boldsymbol{u}}), \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle \\ &\quad - \sqrt{\varepsilon}\langle\partial_x^\alpha(\boldsymbol{u}_R^\varepsilon\cdot\nabla^2\widetilde{\boldsymbol{u}}), \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle - \varepsilon^{3/2}\langle\partial_x^\alpha\nabla\boldsymbol{R}_{\boldsymbol{u}}, \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle \\ &\quad + \langle\boldsymbol{u}_R^\varepsilon\times\boldsymbol{e}_1, \boldsymbol{u}_R^\varepsilon\rangle_{\dot H^k} - \langle\partial_x^\alpha\Delta\phi_R^\varepsilon, \partial_x^\alpha\nabla\cdot\boldsymbol{u}_R^\varepsilon\rangle \\ &=: I_1 + \cdots + I_8.\end{aligned}\tag{3.64}$$

First, I_1 and I_7 vanish by integration by parts.

We bound I_2 by using the commutator estimate and Sobolev embedding,

$$\begin{aligned}|I_2| &\le |\sqrt{\varepsilon}\langle\partial_x^\alpha(\nabla\widetilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{u}_R^\varepsilon), \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle| + |\varepsilon^{3/2}\langle\partial_x^\alpha(\nabla\boldsymbol{u}_R^\varepsilon\cdot\nabla\boldsymbol{u}_R^\varepsilon), \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle| \\ &\le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{H^2})(\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{H^k}^2).\end{aligned}$$

Estimate of I_3. By using commutator and then integrating by parts, we have

$$\begin{aligned}I_3 &= \frac{1}{2}\langle\nabla\cdot\boldsymbol{u}\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon, \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle - \langle[\partial_x^\alpha, \boldsymbol{u}]\cdot\nabla^2\boldsymbol{u}_R^\varepsilon, \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle \\ &=: I_{31} + I_{32}.\end{aligned}$$

For I_{31}, we have

$$\begin{aligned}I_{31} &\le \langle\nabla\cdot\boldsymbol{u}\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon, \partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\rangle \\ &\le C\sqrt{\varepsilon}(1+\varepsilon\|\nabla\cdot\boldsymbol{u}_R^\varepsilon\|_{L^\infty})\|\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\|^2 \\ &\le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^3})(\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{H^k}^2).\end{aligned}$$

The estimate of I_{32} shares some similarities with that for I_{31} and starts by writing

$$\begin{aligned}I_{32} \le& C(\|\nabla^2\boldsymbol{u}_R^\varepsilon\|_{H^{k-1}}\|\boldsymbol{u}\|_{L^\infty} + \|\nabla^2\boldsymbol{u}_R^\varepsilon\|_{L^\infty}\|\boldsymbol{u}\|_{H^k})\|\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\| \\ \le& C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{L^\infty})(\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{H^k}^2) \\ &+ C\sqrt{\varepsilon}\|\nabla^2\boldsymbol{u}_R^\varepsilon\|_{L^\infty}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^k})\|\partial_x^\alpha\nabla\boldsymbol{u}_R^\varepsilon\|_{L^2} \\ \le& C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^4})(\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2 + \varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2),\end{aligned}$$

where $k \le s'$ and $s' \ge 4$. So by using (2.30), we have

$$I_3 \le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^4})|\!|\!|\boldsymbol{u}_R^\varepsilon|\!|\!|_{s'}^2.$$

The estimates of I_4 and I_5 share some similarities with that for I_2 and I_3 and starts by writing

$$I_4, I_5 \le C\frac{1}{\sqrt{\varepsilon}}|\!|\!|\boldsymbol{u}_R^\varepsilon|\!|\!|_{s'}^2.$$

As we know $\boldsymbol{R_u}$ depends only on $\tilde{\boldsymbol{u}}$, so we have

$$I_6 \le C(1+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2).$$

Thus we have obtained the following bound

$$\sum_{i=1}^{7}|I_i| \le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^4})(1+|\!|\!|\boldsymbol{u}_R^\varepsilon|\!|\!|_{s'}^2). \tag{3.65}$$

Bound the term of I_8. We take ∂_x^α of (3.1a),

$$\begin{aligned}\frac{1}{\varepsilon}\nabla\cdot\partial_x^\alpha\boldsymbol{u}_R^\varepsilon =&\frac{1}{n_\alpha}\left\{\frac{\boldsymbol{e}_1-\boldsymbol{u}}{\varepsilon}\cdot\nabla\partial_x^\alpha n_R^\varepsilon-\frac{1}{\sqrt{\varepsilon}}\partial_t\partial_x^\alpha n_R^\varepsilon-\left[\partial_x^\alpha,\frac{\boldsymbol{u}}{\varepsilon}\right]\cdot\nabla n_R^\varepsilon\right.\\&\left.-\left[\partial_x^\alpha,\frac{n_\alpha}{\varepsilon}\right]\nabla\cdot\boldsymbol{u}_R^\varepsilon-\frac{1}{\sqrt{\varepsilon}}\partial_x^\alpha(n_R^\varepsilon\nabla\cdot\tilde{\boldsymbol{u}}+\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{n_\alpha}+\varepsilon R_n)\right\},\end{aligned} \tag{3.66}$$

where $|\alpha| = k$. Therefore, I_8 is decomposed as

$$\begin{aligned}I_8 =&-\left\langle\partial_x^\alpha\Delta\phi_R^\varepsilon,\frac{\boldsymbol{e}_1-\boldsymbol{u}}{n_\alpha}\cdot\nabla\partial_x^\alpha n_R^\varepsilon\right\rangle+\left\langle\partial_x^\alpha\Delta\phi_R^\varepsilon,\frac{\sqrt{\varepsilon}}{n_\alpha}\partial_t\partial_x^\alpha n_R^\varepsilon\right\rangle\\&+\left\langle\partial_x^\alpha\Delta\phi_R^\varepsilon,\frac{1}{n_\alpha}[\partial_x^\alpha,\boldsymbol{u}]\cdot\nabla n_R^\varepsilon\right\rangle+\left\langle\partial_x^\alpha\Delta\phi_R^\varepsilon,\frac{1}{n_\alpha}[\partial_x^\alpha,n]\nabla\cdot\boldsymbol{u}_R^\varepsilon\right\rangle\\&+\left\langle\partial_x^\alpha\Delta\phi_R^\varepsilon,\frac{\sqrt{\varepsilon}}{n_\alpha}\partial_x^\alpha(n_R^\varepsilon\nabla\cdot\tilde{\boldsymbol{u}}+\boldsymbol{u}_R^\varepsilon\cdot\nabla\widetilde{n_\alpha}+\varepsilon R_n)\right\rangle\\=:&I_{81}+\cdots+I_{85}.\end{aligned} \tag{3.67}$$

Recalling (2.19), we have

$$\begin{aligned}|I_{83}| \le&\left|\left\langle\partial_x^\alpha\Delta\phi_R^\varepsilon,\frac{1}{n_\alpha}[\partial_x^\alpha,\boldsymbol{u}]\nabla n_R^\varepsilon\right\rangle\right|\\\le&C\|\partial_x^\alpha\Delta\phi_R^\varepsilon\|_{L^2}^2\{\|\nabla n_R^\varepsilon\|_{H^{k-1}}\|\nabla\boldsymbol{u}\|_{L^\infty}+\|\nabla n_R^\varepsilon\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{H^{k-1}}\}\\\le&C\varepsilon\|\partial_x^\alpha\Delta\phi_R^\varepsilon\|_{L^2}\{(1+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty})\|n_R^\varepsilon\|_{H^3}+(1+\varepsilon\|\boldsymbol{u}_R^\varepsilon\|_{H^k})\|n_R^\varepsilon\|_{H^k}\\\le&C\varepsilon^2\|\Delta\phi_R^\varepsilon\|_{H^{s'}}^2+C(1+\varepsilon^2\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)\|n_R^\varepsilon\|_{H^{s'}}^2,\end{aligned} \tag{3.68}$$

where we used (3.3). The estimates for I_{84} and I_{85} are similar to I_{83}, so we know

$$|I_{84}|+|I_{85}| \le C\varepsilon^2\|\Delta\phi_R^\varepsilon\|_{H^{s'}}^2+C(1+\varepsilon^2\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)\|(n_R^\varepsilon,\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\|_{H^{s'}}^2, \tag{3.69}$$

where $s' \ge 4$. Adding (3.68) and (3.69) together and using Lemma 3.1, we have

$$|I_{83}|+|I_{84}|+|I_{85}| \le C(1+\varepsilon^2\|\boldsymbol{u}_R^\varepsilon\|_{H^{s'}}^2)|\!|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)|\!|\!|_{s'}^2. \tag{3.70}$$

We complete the proof of Proposition 3.2 when we estimate I_{81} and I_{82} in Lemmas 3.6 and 3.7. $\square$

Lemma 3.6. *Let* $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ *be a solution to* (3.1). *Then*

$$|I_{81}| \le C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^3})(1+\varepsilon\|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2)\|\!|\phi_R^\varepsilon|\!\|_{s'}^2.$$

Proof. We take ∂_x^α of (3.1c), we have

$$2\partial_x^\alpha n_R^\varepsilon = \partial_x^\alpha \phi_R^\varepsilon - \varepsilon\Delta\partial_x^\alpha\phi_R^\varepsilon + \sqrt{\varepsilon}\partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) + \varepsilon\partial_x^\alpha R_\phi.$$

So I_{81} is divided into

$$\begin{aligned} I_{81} &= -\left\langle \partial_x^\alpha\Delta\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{n_\alpha}\cdot\nabla\partial_x^\alpha n_R^\varepsilon\right\rangle \\ &= -\left\langle \partial_x^\alpha\Delta\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{2n_\alpha}\cdot\nabla\partial_x^\alpha \phi_R^\varepsilon\right\rangle + \varepsilon\left\langle \partial_x^\alpha\Delta\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{2n_\alpha}\cdot\nabla\Delta\partial_x^\alpha \phi_R^\varepsilon\right\rangle \\ &\quad - \frac{\sqrt{\varepsilon}}{2}\left\langle \partial_x^\alpha\Delta\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{n_\alpha}\cdot\nabla\partial_x^\alpha (\phi^{(1)}\phi_R^\varepsilon)\right\rangle - \varepsilon\left\langle \partial_x^\alpha\Delta\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{2n_\alpha}\cdot\nabla\partial_x^\alpha R_\phi\right\rangle \\ &=: I_{811}+I_{812}+I_{813}+I_{814}, \end{aligned}$$

where $|\alpha| = k \le s'$ for $s' \ge 3$.

We first bound I_{811}. By integrating by parts twice, we have

$$\begin{aligned} I_{811} &= -\left\langle \partial_x^\alpha\nabla\cdot\nabla\phi_R^\varepsilon, \frac{\boldsymbol{e}_1-\boldsymbol{u}}{2n_\alpha}\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle \\ &= -\frac{1}{2}\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \nabla\cdot\left(\frac{\boldsymbol{e}_1-\boldsymbol{u}}{2n_\alpha}\right)\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle + \left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \nabla\left(\frac{\boldsymbol{e}_1-\boldsymbol{u}}{n_\alpha}\right)\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle. \end{aligned}$$

Recalling (2.19), we have

$$\left\|\nabla\cdot\left(\frac{\boldsymbol{e}_1-\boldsymbol{u}}{n_\alpha}\right)\right\|_{L^\infty} \le C\sqrt{\varepsilon}(1+\varepsilon\|\nabla n_R^\varepsilon\|_{L^\infty}+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}), \tag{3.71}$$

therefore, using Lemma 3.1, we have

$$\begin{aligned} |I_{811}| &\le C\sqrt{\varepsilon}(1+\varepsilon(\|n_R^\varepsilon\|_{H^3}+\|\boldsymbol{u}_R^\varepsilon\|_{H^3}))\|\nabla\partial_x^\alpha\phi_R^\varepsilon\|^2 \\ &\le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)|\!\|_{s'})\|\!|\phi_R^\varepsilon|\!\|_{s'}^2. \end{aligned} \tag{3.72}$$

I_{812} and I_{813} behave like I_{811}, we can get

$$|I_{812}| \le C\frac{1}{\sqrt{\varepsilon}}(1+\varepsilon\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)|\!\|_{s'})\|\!|\phi_R^\varepsilon|\!\|_{s'}^2, \tag{3.73}$$

and

$$|I_{813}| \le C\varepsilon\|\nabla\phi_R^\varepsilon\|_{H^k}^2(1+\varepsilon\|\nabla n_R^\varepsilon\|_{L^\infty}+\varepsilon\|\nabla\boldsymbol{u}_R^\varepsilon\|_{L^\infty}).$$

We now bound I_{814}. Using (2.25) in Lemma 2.1, we have

$$\begin{aligned} |I_{814}| &\le C\varepsilon\|\Delta\partial_x^\alpha\phi_R^\varepsilon\|\|\partial_x^\alpha\nabla R_\phi\| \\ &\le C\varepsilon\|\Delta\phi_R^\varepsilon\|_{H^{s'}}^2 + \varepsilon C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^{s'}})\|\nabla\phi_R^\varepsilon\|_{H^{s'}}^2. \end{aligned} \tag{3.74}$$

Combining all the upper bounds, we obtain

$$|I_{81}| \le C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^3})(1+\varepsilon\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)|\!\|_{s'}^2)\|\!|\phi_R^\varepsilon|\!\|_{H^{s'}}^2,$$

where $\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)|\!\|_{s'}^2$ is given in (2.30). The proof of Lemma 3.6 is complete. □

Lemma 3.7. *Let $(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)$ be a solution to (3.1). Then*

$$\begin{aligned} I_{82} \le &-\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{n_\alpha}\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle - \frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\varepsilon^{3/2}}{n_\alpha}\partial_x^\alpha \Delta \phi_R^\varepsilon \right\rangle \\ &- \frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \left(\frac{\varepsilon \phi^{(1)}}{n_\alpha}\right)\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle - \frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \frac{\varepsilon^{3/2}\phi_R^\varepsilon}{n_\alpha}\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle \\ &+ C(C_1(\sqrt{\varepsilon}\tilde{C}) + \varepsilon |\!|\!| (\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) |\!|\!|_3^2)\{1 + |\!|\!| (\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) |\!|\!|_{s'}^2\}, \end{aligned}$$

where $|\alpha| = s'$ for $s' \ge 4$.

Proof. Recalling I_{82} in (3.67). Taking ∂_x^α of (3.1c), we have

$$2\partial_x^\alpha n_R^\varepsilon = \partial_x^\alpha \phi_R^\varepsilon - \varepsilon \Delta \partial_x^\alpha \phi_R^\varepsilon + \sqrt{\varepsilon}\partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) + \varepsilon^{\frac{3}{2}}\partial_x^\alpha[(\phi_R^\varepsilon)^2] + \varepsilon \partial_x^\alpha \overline{R}_\phi.$$

So I_{82} is decomposed as,

$$\begin{aligned} I_{82} = &\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{2n_\alpha}\partial_t \partial_x^\alpha \phi_R^\varepsilon \right\rangle - \varepsilon \left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{2n_\alpha}\partial_t \Delta \partial_x^\alpha \phi_R^\varepsilon \right\rangle \\ &+ \sqrt{\varepsilon}\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{2n_\alpha}\partial_t \partial_x^\alpha(\phi^{(1)}\phi_R^\varepsilon) \right\rangle + \varepsilon^{\frac{3}{2}}\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{2n_\alpha}\partial_t \partial_x^\alpha[(\phi_R^\varepsilon)^2] \right\rangle \\ &+ \varepsilon \left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{2n_\alpha}\partial_t \partial_x^\alpha \overline{R}_\phi \right\rangle \\ =: &\, I_{821} + I_{822} + I_{823} + I_{824} + I_{825}. \end{aligned} \tag{3.75}$$

Estimate of I_{821}. By integration by parts, we obtain

$$\begin{aligned} I_{821} = &-\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{2n_\alpha}\partial_t \partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle - \left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \nabla\left(\frac{\sqrt{\varepsilon}}{2n_\alpha}\right)\partial_t \partial_x^\alpha \phi_R^\varepsilon \right\rangle \\ =: &\, I_{8211} + I_{8212}. \end{aligned} \tag{3.76}$$

By integrate in time, we have

$$I_{8211} = -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{n_\alpha}\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle + \frac{1}{4}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \partial_t\left(\frac{\sqrt{\varepsilon}}{n_\alpha}\right)\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle,$$

then from (3.46), we get

$$I_{8211} \le -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{n_\alpha}\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle + C(1 + \varepsilon |\!|\!| (\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) |\!|\!|_3) |\!|\!| \phi_R^\varepsilon |\!|\!|_{s'}^2, \tag{3.77}$$

where we have used Lemma 3.2 and Corollary 3.1.

The estimate of I_{8212} is pretty much like I_{8211}, and thus

$$I_{8212} \le C(1 + \varepsilon |\!|\!| \phi_R^\varepsilon |\!|\!|_3) |\!|\!| (\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) |\!|\!|_{s'}^2.$$

Therefore, we have

$$I_{821} \le -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \frac{\sqrt{\varepsilon}}{n_\alpha}\partial_x^\alpha \nabla \phi_R^\varepsilon \right\rangle + C(1 + \varepsilon |\!|\!| (\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) |\!|\!|_3) |\!|\!| (\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon) |\!|\!|_{s'}^2. \tag{3.78}$$

Estimate of I_{822}. By integration by parts in time, we obtain

$$I_{822} = -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\varepsilon^{3/2}}{n_\alpha}\partial_x^\alpha \Delta \phi_R^\varepsilon \right\rangle + \frac{1}{4}\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \partial_t\left(\frac{\varepsilon^{3/2}}{n_\alpha}\right)\partial_x^\alpha \Delta \phi_R^\varepsilon \right\rangle.$$

Similar to (3.78),

$$I_{822} \le -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \Delta \phi_R^\varepsilon, \frac{\varepsilon^{3/2}}{n_\alpha}\partial_x^\alpha \Delta \phi_R^\varepsilon \right\rangle + C(1+\varepsilon \|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)\|\!|_3)\|\!|\phi_R^\varepsilon\|\!|_{s'}^2. \tag{3.79}$$

Estimate of I_{823}. By integrating by parts and using the commutator, we have

$$\begin{aligned} I_{823} = &-\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \nabla\left(\frac{\varepsilon}{2n_\alpha}\right)\partial_x^\alpha(\phi^{(1)}\partial_t\phi_R^\varepsilon)\right\rangle - \left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \nabla\left(\frac{\varepsilon}{2n_\alpha}\right)\partial_x^\alpha(\partial_t\phi^{(1)}\phi_R^\varepsilon)\right\rangle \\ &-\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \left(\frac{\varepsilon\phi^{(1)}}{2n_\alpha}\right)\partial_t\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle - \left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \left(\frac{\varepsilon}{2n_\alpha}\right)[\partial_x^\alpha\nabla, \phi^{(1)}]\partial_t\phi_R^\varepsilon\right\rangle \\ &-\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \left(\frac{\varepsilon}{2n_\alpha}\right)\partial_x^\alpha\nabla(\partial_t\phi^{(1)}\phi_R^\varepsilon)\right\rangle \\ =:\ & I_{8231} + I_{8232} + I_{8233} + I_{8234} + I_{8235}. \end{aligned}$$

Using the multiplicative estimates, we obtain

$$I_{8231} \le C\sqrt{\varepsilon}(1+\varepsilon\|\nabla n_R^\varepsilon\|_{L^\infty})\|\partial_x^\alpha\nabla\phi_R^\varepsilon\|_{L^2}\{\|\partial_t\phi_R^\varepsilon\|_{H^k}\|\phi^{(1)}\|_{L^\infty} + \|\partial_t\phi_R^\varepsilon\|_{L^\infty}\|\phi^{(1)}\|_{H^k}\}.$$

We deal with I_{8231} first, because of

$$\|\partial_t\phi_R^\varepsilon\|_{H^k} \le \|\partial_t\phi_R^\varepsilon\|_{H^{k-1}} + \|\partial_t\nabla\phi_R^\varepsilon\|_{H^{k-1}}. \tag{3.80}$$

We can obtain

$$I_{8231} \le C(1+\varepsilon\|\!|\phi_R^\varepsilon\|\!|_3)\{1+\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\|\!|_{s'}^2\}.$$

Similarly, we have

$$I_{8232} \le C(1+\varepsilon\|\!|\phi_R^\varepsilon\|\!|_3)(1+\|\!|\phi_R^\varepsilon\|\!|_{s'}^2).$$

I_{8233} and I_{8234} can be dealt with similarly, so we obtain

$$I_{8233} \le -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \left(\frac{\varepsilon\phi^{(1)}}{n_\alpha}\right)\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle + C(1+\varepsilon\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\|\!|_3)\|\!|\phi_R^\varepsilon\|\!|_{s'}^2,$$

and

$$I_{8234} \le C(1+\varepsilon\|\!|\phi_R^\varepsilon\|\!|_3)\{1+\|\!|(\boldsymbol{u}_R^\varepsilon,\phi_R^\varepsilon)\|\!|_{s'}^2\}.$$

We now bound I_{8235}. By using multiplicative estimates, we have

$$\begin{aligned} I_{8235} = &-\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \left(\frac{\varepsilon}{2n_\alpha}\right)\partial_x^\alpha(\partial_t\phi^{(1)}\nabla\phi_R^\varepsilon)\right\rangle - \left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \left(\frac{\varepsilon}{2n_\alpha}\right)\partial_x^\alpha(\nabla\partial_t\phi^{(1)}\phi_R^\varepsilon)\right\rangle \\ \le\ & C\varepsilon^2\|\partial_x^\alpha\nabla\phi_R^\varepsilon\|_{L^2}\{\|\nabla\phi_R^\varepsilon\|_{H^k}\|\partial_t\phi^{(1)}\|_{L^\infty} + \|\phi_R^\varepsilon\|_{L^\infty}\|\partial_t\phi^{(1)}\|_{H^{k+1}} \\ &+ \|\phi_R^\varepsilon\|_{H^k}\|\partial_t\nabla\phi^{(1)}\|_{L^\infty} + \|\phi_R^\varepsilon\|_{L^\infty}\|\partial_t\nabla\phi^{(1)}\|_{H^{k+1}}\} \\ \le\ & C\varepsilon\|\partial_x^\alpha\nabla\phi_R^\varepsilon\|_{L^2}^2 + C\{\varepsilon\|\nabla\phi_R^\varepsilon\|_{H^k}^2 + \|\phi_R^\varepsilon\|_{H^k}^2 + \|\phi_R^\varepsilon\|_{H^3}^2\} \\ \le\ & C\|\!|\phi_R^\varepsilon\|\!|_{s'}^2, \end{aligned} \tag{3.81}$$

where $k \le s'$ and $s' \ge 3$.

So we can estimate I_{823} by adding above

$$\begin{aligned} I_{823} \le & -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha \nabla \phi_R^\varepsilon, \left(\frac{\varepsilon\phi^{(1)}}{n_\alpha}\right)\partial_x^\alpha \nabla\phi_R^\varepsilon \right\rangle \\ & + C(1+\varepsilon|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_3)\{1+|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2\}. \end{aligned} \tag{3.82}$$

Estimate of I_{824}. Recalling (3.76), we have

$$\begin{aligned} I_{824} =& \left\langle \partial_x^\alpha \Delta\phi_R^\varepsilon, \frac{\varepsilon^2}{2n_\alpha}\partial_x^\alpha(\phi_R^\varepsilon \partial_t \phi_R^\varepsilon)\right\rangle \\ =& -\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2\phi_R^\varepsilon}{2n_\alpha}\partial_t\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle - \left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2}{2n_\alpha}[\partial_x^\alpha\nabla, \phi_R^\varepsilon]\partial_t\phi_R^\varepsilon\right\rangle \\ & -\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2}{2n_\alpha}\partial_x^\alpha(\nabla\phi_R^\varepsilon\partial_t\phi_R^\varepsilon)\right\rangle \\ =:& \, I_{8241} + I_{8242} + I_{8243}. \end{aligned} \tag{3.83}$$

For the term I_{8241}, by integrating by parts in time, we obtain

$$I_{8241} = -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2\phi_R^\varepsilon}{n_\alpha}\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle + \frac{1}{4}\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \partial_t\left(\frac{\varepsilon^2\phi_R^\varepsilon}{n_\alpha}\right)\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle,$$

and we have

$$\left\|\partial_t\left(\frac{\varepsilon\phi_R^\varepsilon}{n_\alpha}\right)\right\|_{L^\infty} \le C + C\sqrt{\varepsilon}|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_3^2,$$

where we used Hölder inequality, Lemmas 3.1-3.2, and Corollary 3.1. Therefore I_{8241} is estimated as

$$I_{8241} \le -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2\phi_R^\varepsilon}{n_\alpha}\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle + C(1+\varepsilon|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_3^2)\{\varepsilon\|\partial_x^\alpha\nabla\phi_R^\varepsilon\|_{L^2}^2\}.$$

To bound the term of I_{8242}, by commutator estimates, we have

$$\begin{aligned} I_{8242} =& -\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2}{2n_\alpha}\partial_x^\alpha(\nabla\phi_R^\varepsilon\partial_t\phi_R^\varepsilon)\right\rangle \\ \le& \, C\varepsilon\|\partial_x^\alpha\nabla\phi_R^\varepsilon\|_{L^2}\{\|\partial_t\phi_R^\varepsilon\|_{H^k}\|\nabla\phi_R^\varepsilon\|_{L^\infty} + \|\partial_t\phi_R^\varepsilon\|_{L^\infty}\|\nabla\phi_R^\varepsilon\|_{H^k}\} \\ \le& \, C(1+\varepsilon\|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)\|_3^2)\|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)\|_{s'}^2, \end{aligned}$$

by using Lemma 3.1 and Corollary 3.1, then we have

$$I_{8242} \le C(1+\varepsilon|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_3^2)|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2.$$

I_{8243} behaves like I_{8242}, we obtain

$$I_{8243} \le C(1+\varepsilon|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_3^2)|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2.$$

Therefore we can estimate I_{824} by adding above

$$I_{824} \le -\frac{1}{4}\frac{d}{dt}\left\langle \partial_x^\alpha\nabla\phi_R^\varepsilon, \frac{\varepsilon^2\phi_R^\varepsilon}{n_\alpha}\partial_x^\alpha\nabla\phi_R^\varepsilon\right\rangle + C(1+\varepsilon|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_3^2)|\!|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!|\!|_{s'}^2. \tag{3.84}$$

Estimate of I_{825}. By (2.26) in Lemma 2.1, recall that I_{825} is defined in (3.76), we have

$$\begin{aligned} I_{825} &\leq C\varepsilon\|\partial_x^\alpha \Delta\phi_R^\varepsilon\|^2 + \varepsilon^2\|\partial_t\partial_x^\alpha R_\phi\|^2 \\ &\leq C\varepsilon\|\partial_x^\alpha \Delta\phi_R^\varepsilon\|^2 + C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^\delta})(1+\varepsilon\|\varepsilon\partial_t\phi_R^\varepsilon\|_{H^k}^2) \\ &\leq C_1(\sqrt{\varepsilon}\|\phi_R^\varepsilon\|_{H^\delta})\{1 + \|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2\}, \end{aligned} \tag{3.85}$$

where $\delta = \max\{2, k-1\} \leq s'-1$, and we have used (3.12) in Corollary 3.1. The proof of Lemma 3.7 is completed. □

So the proof of Proposition 3.2 is completed.

By combining Proposition 3.1 and Proposition 3.2, we are now ready to proof Theorem 2.1.

Proof of Theorem 2.1. From (3.2), there exists some $\varepsilon_1 > 0$ such that $1/2 \leq 1 + \sqrt{\varepsilon}\phi^{(1)} + \varepsilon^{\frac{3}{2}}\phi_R^\varepsilon \leq 3/2$. By adding inequalities (3.14) and (3.63), then integrating over $[0,t]$ and taking summation over $|\alpha| = k$ for $0 \leq k \leq s'$, we obtain

$$\|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2 \leq CC_\varepsilon(0) + CC_1\int_0^t (C_1 + \varepsilon\|\!|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2)\{1 + \|\!|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2\}dr.$$

Here $C_\varepsilon(0) = \|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)(0)|\!\|_{s'}^2$. We know that there exists some constant $0 < \varepsilon_0 < \varepsilon_1$ such that $\varepsilon\|(\boldsymbol{u}_R^\varepsilon, n_R^\varepsilon, \phi_R^\varepsilon)\|_{H^{s'}}^2 \leq 1$. As $C_1 = C_1(\sqrt{\varepsilon}\|n_R^\varepsilon\|_{H^{s'}})$ is nondecreasing, we have $C_1 \leq C_1(1)$ as $0 < \varepsilon < \varepsilon_0$. So, there exists some constant $C_3 > 1$ such that

$$\|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2 \leq C_3C_\varepsilon(0) + C_3\int_0^t \{1 + \|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2\}dr.$$

On the other hand, from Lemma 3.1, here exists some constant $C_4 > 1$ such that for any $0 < \varepsilon < \varepsilon_0$,

$$\|n_R^\varepsilon\|_{H^{s'}}^2 \leq C_4(1 + \|\!|\phi_R^\varepsilon|\!\|_{s'}^2). \tag{3.86}$$

Let $C_0' = \sup_{0<\varepsilon<1} C_\varepsilon(0)$. Given $0 < \tau_0 < \tau_*$, we let $\tilde{C}$ in (3.2) satisfy $\tilde{C} \geq 2C_4(1 + C_3C_0')e^{C_3\tau_0}$; then we use the Gronwall inequality,

$$\sup_{0\leq t\leq\tau_0} \|\!|(\boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}^2 \leq (1 + C_3C_0')e^{C_3\tau_0} \leq \tilde{C},$$

and from (3.86)

$$\sup_{0\leq t\leq\tau_0} \|n_R^\varepsilon\|_{H^{s'}}^2 \leq C_4\{1 + (1 + C_3C_0')e^{C_3\tau_0}\} \leq \tilde{C}.$$

It is then standard to obtain uniform estimates for $\|\!|(n_R^\varepsilon, \boldsymbol{u}_R^\varepsilon, \phi_R^\varepsilon)|\!\|_{s'}$ independent of ε by using the continuity method. So the proof of Theorem 2.1 is complete. □

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