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COMPARISON OF RESEARCH METHODS FOR DISEASE MODELS WITH TWO DIFFERENT RANDOM PERTURBATIONS UNDER THE INFLUENCE OF SANITATION AND AWARENESS

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Abstract Environmental hygiene and public awareness play a key role in controlling the spread of infectious diseases, and they are also very effective health intervention measures for the public. This paper studies the dynamical behaviors of a nonlinear mathematical model with health and publicity which controlled by investment budget. We compared and analyzed the research methods of two disease models generated under white noise and Ornstein-Uhlenbeck process perturbations. At first, we discuss the local stability of the endemic equilibrium by Lyapunov function method which avoids the tedious calculation process when studying the local stability of the positive solution of the models with dimensions greater than three. And then we conduct research on random models under the influence of white noise, we study the existence and uniqueness of positive solution. We get a critical value R^* which corresponding to the control reproduction number R_1 of the ordinary differential equation when we discuss the stationary distribution of the stochastic system. In addition, constructing a Lyapunov function is a method to obtain some sufficient conditions for the extinction of the disease. Finally, the numerical simulations illustrate our above theoretical results and several parameters have a significant impact on the model are pointed out. Specifically, we present the dynamic properties of the same model under Ornstein-Uhlenbeck process perturbations in the Appendix.

Keywords Logarithmic Ornstein-Uhlenbeck processes, white noises, local stability, stationary distribution, extinction.

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1. Introduction

People keep at a distance from diseases as if it were defiling and most diseases are caused by bacteria and viruses, which require us to pay attention to environmental sanitation. It is estimated that inadequate sanitation leads to about 432000 diarrhoeal deaths each year, which is the main factor in the outbreak of several neglected tropical diseases, including intestinal worms, schistosomiasis and trachoma. Poor sanitation also contributes to malnutrition and it is linked to transmission of diseases such as cholera, diarrhoea, dysentery, hepatitis A, typhoid and polio and exacerbates stunting [27]. The rate of sanitation coverage plays a vital role for better hygiene, similarly, safe drinking water is related to economic and social development. Over the last decades, various government and non-government organizations have made efforts

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to improve sanitation coverage and clean drinking water in order to slow down the spread of diseases [28].

In most developing countries, people face difficulties to prevent the spread of new and re-emerging infectious diseases due to the inadequate of medical equipment. Thus, they focus on the renewal of medical facilities to slow down diseases spread when the emergence of any new infectious disease. Awareness, which brings the behavioral changes among the population, can be seen as partial treatment at no cost. Apart from this, awareness regarding the spread of any infectious disease also reduces the economic burden required for medication [21]. Kumar, Srivastava and Takeuchi in [12] propose, the responses of person to the threat of disease are often related to their perception of risk, which could be influenced by the public and the private information disseminated widely by the media. Therefore, there are more interests among researchers in studying the impact of those behaviour influencing factors on the spread of infectious diseases. For other studies on epidemic models with media or awareness programs, we have read [4, 5, 18, 30] and the references contained therein in detail.

In addition, environmental noise is a key part of the ecosystem and an important factor of the population system. Because the random interference in the environment is considered, the random model with environmental noise is more suitable for describing the actual situation than the existing traditional one. It is of great biological and mathematical significance to consider both deterministic systems and stochastic infectious disease models. Many papers [9, 16, 19, 29, 37, 39, 40] have studied the effects of white noises on the population models and drew some reasonable conclusions. For instance, Jiang et al. obtained the stationary distribution and extinction of a kind of Logistic equation with random perturbation in [10]. The dynamic behavior of a disease model under random environmental noise interference was studied by Zhou et al. in [36], while considering the conscious population under the influence of media. Zhou et al. in [35] also considered environmental noise disturbances for disease models that can be dimensionally reduced and included in the isolation phase. Some meaningful inferences are reached in [8, 13, 15, 25, 33] and these papers provide some ideas for us to further discuss the infectious disease model, and combine theory with reality to draw some conclusions about the intention of disease transmission in the bargain.

In the following content of the paper, we introduce some mathematical models and main lemmas used in the paper in Section 2. And we construct some complex Lyapunov functions to prove our conclusions in the paper afterwards. Section 3 discuss the local stability of the endemic equilibrium with Lyapunov function method method. Next is the study of random models under white noise and Ornstein-Uhlenbeck processes interference, respectively. The existence and uniqueness of positive solution of those stochastic models is showed in Section 4 and Appendix A. We mainly investigate the existence of stationary distribution of stochastic systems in Section 5 and Appendix C, which obtains two critical values related to the basic reproduction number R_1. In addition, in Section 6 and Appendix B, the sufficient conditions for the extinction of diseases is derived. We finally verify the conclusion in this article through some numerical examples in Section 7.

2. Mathematical models

Disease researches are relatively complex experiment and extensive experimentation would be required to find the exact parameters, however, mathematical modelling can provide a predictive tool and the capacity to adjust variables without experiments. Papers [1, 14, 17, 23, 31, 32, 34] verified the viewpoint.

2.1. The ODE system and analysis

In this subsection, we consider a mathematical SIS model with immigration where infected individuals recover from infection to the susceptible class after a certain time. Besides, the protection information regarding the disease is disseminated through some social media. As a result, we study the mathematical disease models with medium [22]:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \left(\beta - \beta_1 \dfrac{k_1 M(t)}{p + k_1 M(t)}\right) S(t)I(t) - \eta \dfrac{B(t)}{L + B(t)} S(t) + vI(t) - dS(t), \\ \dfrac{dI(t)}{dt} = \left(\beta - \beta_1 \dfrac{k_1 M(t)}{p + k_1 M(t)}\right) S(t)I(t) + \eta \dfrac{B(t)}{L + B(t)} S(t) - (v + \alpha + d)I(t), \\ \dfrac{dB(t)}{dt} = \phi_1 I(t) + \phi B(t) - \phi_0 B(t) - \Phi \dfrac{(1 - k_1)M(t)}{q + (1 - k_1)M(t)} B(t), \\ \dfrac{dM(t)}{dt} = r\left(1 - \dfrac{M(t)}{K}\right) M(t) + \theta I(t)M(t), \end{cases} \tag{2.1}$$

where $S(t)$ is susceptible population and infected population at any time t expressed by $I(t)$. Let $B(t)$ be the density of bacteria shed in the environment. The variable $M(t)$ addresses the budget allocation by the government to warn people and for sanitation coverage. And the meanings of other parameters in this function are showed in the following table.

All of parameters in the Table 1 are positive and $f_1(M) = \beta_1 \frac{k_1 M}{p + k_1 M}$, $f_2(M) = \Phi \frac{(1-k_1)M}{q+(1-k_1)M}$. In this paper, for the feasibility of the system, we let $\phi_0 > \phi$, $0 < k_1 < 1$ and $\beta_1 < \beta$. Let $N(t)$ is the total population in the region under consideration, where $N(t) = S(t) + I(t)$. We use the following set to represent the feasible region of (2.1):

$$\Omega = \left\{ (S, I, B, M) \in \mathbb{R}_+^4 : \ 0 \le S + I \le \frac{\Lambda}{d},\ 0 \le B \le \frac{\phi_1 \Lambda}{d(\phi_0 - \phi)}, \right. \\ \left. 0 \le M \le \frac{K}{r}\left(r + \frac{\theta\Lambda}{d}\right) \right\}.$$

The description of the feasible equilibrium point of this model is as follows:

- The disease and budget-free equilibrium is $E_1\left(\frac{\Lambda}{d}, 0, 0, 0\right)$ and it is always feasible but unstable.
- The budget-free endemic equilibrium is $E_2(S_2, I_2, B_2, 0)$, this equilibrium is feasible if $R_0 > 1$ and it is unstable.
- The disease-free equilibrium $E_3\left(\frac{\Lambda}{d}, 0, 0, K\right)$ is always feasible and it is globally asymptotically stable in the region $\mathbb{R}_+^2$ of I and B plane if $R_1 < 1$.
- The interior equilibrium is $E^*(S^*, I^*, B^*, M^*)$, this equilibrium is feasible if $R_1 > 1$ and it is globally asymptotically stable in Ω provided the following inequalities are satisfied equation in Th5.3 of [22],

where,

$$R_0 = \frac{\beta\Lambda}{d(v + \alpha + d)} + \frac{\eta\phi_1\Lambda}{dL(v + \alpha + d)(\phi_0 - \phi)},$$

Table 1. Parameter description for the model system.

Parameters	Description
Λ	Immigrants from the susceptible groups
β	Contact rate of susceptible with infected individuals in absence of funds
β_1	Efficacy of budget allocation to reduce the contact rate via propagating awareness
k_1	Fractional constant determining the budget allocation to warn susceptibles via propagating awareness
p	Half saturation point for $f_1(M)$ as it attains half of its maximum possible value β_1 when budget allocation arrives at $\frac{p}{k_1}$
η	Transmission rate of susceptible to infected class due to interaction of susceptible with bacteria present in the environment
L	Half saturation constant
v	Recovery rate of human population
α	Disease induced death rate of human population
d	Natural death rate of human population
ϕ_1	Growth rate of bacteria due to increase in infected individuals
ϕ	Self-growth rate of bacteria
ϕ_0	Natural death rate of bacteria
Φ	Efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation
q	Half saturation point for $f_2(M)$ as it attains half of its maximum possible value Φ when budget allocation arrives at $\frac{q}{1-k_1}$
r	Intrinsic growth rate of budget allocation
K	Carrying capacity of budget allocation
θ	Per-capita growth rate of budget allocation due to increase in infected individuals

$$R_1 = \left(\beta - \beta_1 \frac{k_1 K}{p + k_1 K}\right) \frac{\Lambda}{d(v+\alpha+d)} + \frac{\eta\phi_1\Lambda}{dL(v+\alpha+d)(\phi_0 - \phi + \Phi\frac{(1-k_1)K}{q+(1-k_1)K})}, \tag{2.2}$$

and $S_2,\ I_2,\ B_2$ are the solution of $dS = dI = dB = 0$ in (2.1) when $M_2 = 0$. $S^*,\ I^*,\ B^*,\ M^*$ are positive and are the solution of $dS = dI = dB = dM = 0$ in (2.1).

The quantity R_0 is the basic reproduction quantity without budget and R_1 is the basic reproduction number with budget. It is easy to note that $R_1 < R_0$, indicating that the presence of awareness and sanitation coverage lowers the epidemic threshold and reduces the infection risk through budget allocation.

2.2. The stochastic model

As mentioned in the paper, compared with the deterministic infectious disease model without considering environmental white noise and Ornstein-Uhlenbeck process, it is closer to reality to consider the impact of environmental noise. After adding the inference of random white noises, we get the random system based on the system (2.1):

$$\begin{cases} dS(t) = \left[\Lambda - \left(\beta - \beta_1 \dfrac{k_1 M(t)}{p + k_1 M(t)}\right) S(t)I(t) - \eta \dfrac{B(t)}{L + B(t)} S(t) + vI(t) - dS(t)\right] dt \\ \qquad\quad + \sigma_1 S(t) dB_1(t), \\ dI(t) = \left[\left(\beta - \beta_1 \dfrac{k_1 M(t)}{p + k_1 M(t)}\right) S(t)I(t) + \eta \dfrac{B(t)}{L + B(t)} S(t) - (v + \alpha + d)I(t)\right] dt \\ \qquad\quad + \sigma_2 I(t) dB_2(t), \\ dB(t) = \left[\phi_1 I(t) + \phi B(t) - \phi_0 B(t) - \Phi \dfrac{(1-k_1)M(t)}{q + (1-k_1)M(t)} B(t)\right] dt + \sigma_3 B(t) dB_3(t), \\ dM(t) = \left[r\left(1 - \dfrac{M(t)}{K}\right) M(t) + \theta I(t)M(t)\right] dt + \sigma_4 M(t) dB_4(t). \end{cases} \tag{2.3}$$

where $B_j(t)$ are mutually independent standard Brownian motions and $\sigma_j^2 \geq 0,\ j = 1, 2, 3, 4$ represent the intensity of white noises.

Based on our research on logarithmic Ornstein-Uhlenbeck processes, we chose β_2 as a variable, and the perturbed model is

$$\begin{cases} d\log\beta_2(t) = \omega\left(\log\beta - \log\beta_2(t)\right) dt + \sigma_5 dB_5(t), \\ dS(t) = \left[\Lambda - \left(\beta_2 - \beta_1 \dfrac{k_1 M(t)}{p + k_1 M(t)}\right) S(t)I(t) - \eta \dfrac{B(t)}{L + B(t)} S(t) + vI(t) - dS(t)\right] dt, \\ dI(t) = \left[\left(\beta_2 - \beta_1 \dfrac{k_1 M(t)}{p + k_1 M(t)}\right) S(t)I(t) + \eta \dfrac{B(t)}{L + B(t)} S(t) - (v + \alpha + d)I(t)\right] dt, \\ dB(t) = \left[\phi_1 I(t) + \phi B(t) - \phi_0 B(t) - \Phi \dfrac{(1-k_1)M(t)}{q + (1-k_1)M(t)} B(t)\right] dt, \\ dM(t) = \left[r\left(1 - \dfrac{M(t)}{K}\right) M(t) + \theta I(t)M(t)\right] dt, \end{cases} \tag{2.4}$$

where $\log\beta$ is the average value of $\log\beta_2(t)$, $\omega > 0$ is the symbol for the speed of reversion and $\sigma_5 > 0$ means the intensity of fluctuation, $B_5(t)$ is Brownian motions.

We define $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ as a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) in the paper. Denote

$$\mathbb{R}_+^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, \cdots, n\}.$$

2.3. Main lemmas

Lemma 2.1. [11] *Let $X(t)$ is a homogeneous Markov process in $d-$ dimensional Eucidean space, satisfying*

$$dX(t) = b(x)dt + \sum_{l=1}^{k} g_l(x) dB_l(t).$$

We let $A(x) = (a_{ij}(x))$ be a diffusion matrix, and $a_{ij}(x) = \sum_{l=1}^{k} g_l^i(x) g_l^j(x)$. Assume there exists a bounded open domain $\mathbb{R}^d$ (d dimensional Euclidean space) with regular boundary Γ, having the following properties:

(I): In the domain G and some neighbourhood, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(II): If $x \in \mathbb{R}^d \setminus G$, then mean time τ at which a path issuing from x reaches the set G is finite, and $\sup_{x\in G} E^x\tau < \infty$ for every compact subset $G \in \mathbb{R}^d$.

Then the Markov process $X(t)$ has a unique stationary distribution $\pi(\cdot)$.

Remark 2.1. In Lemma 2.1, according to the Chapter.3 in [6], Chapter.6 in [26] and Rayleigh's principle, we get the assumption (I) equivalent to that there exists a positive number M satisfying

$$\Sigma_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2,\ z \in K,\ \xi \in \mathbb{R}^d.$$

And according to the Theorem 3.13 in [38], the assumption (II) holds if there exists a nonnegative C^2-function V satisfying

$$LV \leq -1, \text{ for any } z \in \mathbb{R}^d \setminus G.$$

Lemma 2.2. [24] *Let $V(x(t))$ be a scalar function with continuous first partial derivatives, $x(t) \geq 0$. Assume that Ω_l is bounded and that within Ω_l:*

$$V(x) > 0 \textit{ for } x \neq 0;\quad \dot{V}(x) < 0 \textit{ for all } x \neq 0 \textit{ in } \Omega_l.$$

Then the origin is asymptotically stable, and above all, every solution in Ω_l tends to the origin as $t \to \infty$.

Lemma 2.3. [2, 3, 20] *Consider the d-dimensional stochastic differential equation*

$$dX(t) = b(X(t))dt + \sigma_r(X(t))dB_r(t),\ t \geq 0,\ \ X(t) \in \mathbb{R}^d, \tag{2.5}$$

where $b(X(t))$ and $\sigma_r(X(t))$ are Borel measurable, $B_r(t)$ is Brownian motion. Suppose that there exists a bounded closed set $\Gamma \subset \mathbb{R}^d$ and any initial value $X(0) \in \mathbb{R}^d$ satisfy the following condition, for transition probability $\mathbb{P}(t, X(0), \Gamma)$ of $X(t)$,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(s, X(0), \Gamma)ds > 0 \ a.s.$$

Then system (2.5) exists a stationary distribution.

Lemma 2.4. *Let $x = \log\beta_2(t)$ and we can see $x \sim N(\bar{x}, \frac{\sigma_5^2}{2\omega})$. And then define $n \geq 1$ is an integer,*

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta_2^n(m)dm &= \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{nx(m)}dm\\
&= \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\frac{\sqrt{2\omega}(x(m)-\bar{x})}{\sigma_5}\frac{n\sigma_5}{\sqrt{2\omega}}}e^{n\bar{x}}dm\\
&=\beta^n\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}e^{\frac{n\sigma_5 y}{\sqrt{2\omega}}}dy\\
&=\beta^n\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\frac{n\sigma_5}{\sqrt{2\omega}})^2}{2}}e^{\frac{n^2\sigma_5^2}{4\omega}}dy\\
&=\beta^n e^{\frac{n^2\sigma_5^2}{4\omega}},
\end{aligned}$$

where $y = \frac{\sqrt{2k}(x(t)-\bar{x})}{\sigma}$.

3. Local stability of disease-free equilibrium by LYP method

When we size up the local stability of the solution of the model, we generally determine whether the characteristic roots of its characteristic equations have negative real parts by Routh-Hurwitz criterion. However, for four dimensions models and other higher dimensions models, its calculation process is extremely complicated. Therefore, in order to avoid such problems, we consider a new method to judge the local stability of the solution. It exists a endemic equilibrium $E^*(S^*, I^*, B^*, M^*)$ of deterministic system (2.1) when $R_1 > 1$, at first, we introduce a theorem:

Theorem 3.1. *On the premise of* $R_1 > 1$, *if*

$$a_2 + \frac{a_1(\alpha + d)}{\alpha + 2d} - \frac{b_1 \phi_1 B^*}{I^*} - \frac{b_1 a_4 M^* B^*}{I^{*2}} > 0, \ \frac{a_3 r}{\theta K} - \frac{b_1 a_4 B^*}{M^*} > 0,$$

are true, and $a_1 = \frac{(v+\alpha+d)I^*}{S^*}$, $a_2 = \frac{\eta B^* S^*}{(L+B^*)I^*}$, $a_3 = \frac{\beta_1 k_1 p S^* I^*}{(p+k_1 M^*)^2}$, $a_4 = \frac{\Phi(1-k_1)qB^*}{(q+(1-k_1)M^*)^2}$, $b_1 = \frac{\eta L S^*}{\phi_1 (L+B^*)^2}$, *then the endemic equilibrium* E^* *is locally asymptotically stable.*

Proof. For the proof of this theorem, our general approach is to linearize the model (2.1) based on the positive equilibrium point at first, and then prove that the linearized equation is asymptotically stable, thus obtaining that the original model (2.1) is locally asymptotically stable. We first derive the linearized equation of model (2.1):

$$\begin{cases} \dfrac{d(S-S^*)}{dt} = -\dfrac{\Lambda + vI^*}{S^*}(S - S^*) - \left((\beta - \beta_1 \dfrac{k_1 M^*}{p + k_1 M^*})S^* - v\right)(I - I^*) \\ \qquad -\dfrac{\eta L S^*}{(L+B^*)^2}(B - B^*) + \dfrac{\beta_1 k_1 p S^* I^*}{(p + k_1 M^*)^2}(M - M^*), \\ \dfrac{d(I-I^*)}{dt} = \dfrac{(v+\alpha+d)I^*}{S^*}(S - S^*) - \dfrac{\eta B^* S^*}{(L+B^*)I^*}(I - I^*) + \dfrac{\eta L S^*}{(L+B^*)^2}(B - B^*) \\ \qquad -\dfrac{\beta_1 k_1 p S^* I^*}{(p + k_1 M^*)^2}(M - M^*), \\ \dfrac{d(B-B^*)}{dt} = \phi_1(I - I^*) - \dfrac{\phi_1 I^*}{B^*}(B - B^*) - \dfrac{(1-k_1)q\Phi B^*}{[q + (1-k_1)M^*]^2}(M - M^*), \\ \dfrac{d(M-M^*)}{dt} = \theta M^*(I - I^*) - \dfrac{rM^*}{K}(M - M^*). \end{cases} \tag{3.1}$$

At this moment in time, the stability of the positive equilibrium of system (2.1) is equivalent to the stability of zero solution of system (3.1). After calculation, we can get

$$\begin{aligned} \mathcal{L}\frac{(I-I^*)^2}{2} &= a_1(S-S^*)(I-I^*) - a_2(I-I^*)^2 + \frac{\eta L S^*}{(L+B^*)^2}(I-I^*)(B-B^*) \\ &\quad - a_3(I-I^*)(M-M^*), \\ \mathcal{L}\frac{(S-S^*+I-I^*)^2}{2} &= -d(S-S^*)^2 - (\alpha+d)(I-I^*)^2 - (\alpha+2d)(S-S^*)(I-I^*)), \\ \mathcal{L}\frac{(M-M^*)^2}{2} &= \theta M^*(I-I^*)(M-M^*) - \frac{rM^*}{K}(M-M^*)^2, \\ \mathcal{L}\frac{(B-B^*)^2}{2} &= \phi_1(I-I^*)(B-B^*) - \frac{\phi_1 I^*}{B^*}(B-B^*)^2 - a_4(M-M^*)(B-B^*), \end{aligned}$$

where

$$a_1 = \frac{(v+\alpha+d)I^*}{S^*},\ a_2 = \frac{\eta B^* S^*}{(L+B^*)I^*},\ a_3 = \frac{\beta_1 k_1 p S^* I^*}{(p+k_1 M^*)^2},\ a_4 = \frac{\Phi(1-k_1)qB^*}{(q+(1-k_1)M^*)^2}.$$

Define

$$W_1 = \frac{(I-I^*)^2}{2} + \frac{a_1(S-S^*+I-I^*)^2}{2(\alpha+2d)} + \frac{a_3(M-M^*)^2}{2\theta M^*}.$$

Using Itô′s formula, we get

$$\begin{aligned}\mathcal{L}W_1 = &-a_2(I-I^*)^2 + \frac{\eta L S^*}{(L+B^*)^2}(I-I^*)(B-B^*) - \frac{a_1 d}{\alpha+2d}(S-S^*)^2 \\ &- \frac{a_1(\alpha+d)}{\alpha+2d}(I-I^*)^2 - \frac{a_3 r}{\theta K}(M-M^*)^2.\end{aligned}$$

To eliminate $(I-I^*)(B-B^*)$, we deform the following equation

$$\begin{aligned}\mathcal{L}\frac{(B-B^*)^2}{2} = &\phi_1 I^* B^* \left[\left(\frac{I}{I^*}-1\right)\left(\frac{B}{B^*}-1\right) - \left(\frac{B}{B^*}-1\right)^2\right] \\ &- a_4(M-M^*)(B-B^*).\end{aligned}$$

Thus we construct a radially unbounded function

$$W = W_1 + \frac{b_1(B-B^*)^2}{2},\ b_1 = \frac{\eta L S^*}{\phi_1(L+B^*)^2}.$$

And

$$\begin{aligned}\mathcal{L}W = &-a_2(I-I^*)^2 - \frac{a_1 d}{\alpha+2d}(S-S^*)^2 - \frac{a_1(\alpha+d)}{\alpha+2d}(I-I^*)^2 - \frac{a_3 r}{\theta K}(M-M^*)^2 \\ &+ \frac{\eta L S^* I^* B^*}{(L+B^*)^2}\left(\frac{I}{I^*}-1\right)\left(\frac{B}{B^*}-1\right) - b_1 a_4(M-M^*)(B-B^*) \\ &+ b_1\phi_1 I^* B^* \left[\left(\frac{I}{I^*}-1\right)\left(\frac{B}{B^*}-1\right) - \left(\frac{B}{B^*}-1\right)^2\right] \\ = &-\left[a_2 + \frac{a_1(\alpha+d)}{\alpha+2d} - \frac{b_1\phi_1 B^*}{I^*}\right](I-I^*)^2 - \frac{a_1 d}{\alpha+2d}(S-S^*)^2 - \frac{a_3 r}{\theta K}(M-M^*)^2 \\ &- b_1\phi_1 I^* B^* \left(\frac{I}{I^*} - \frac{B}{B^*}\right)^2 - b_1 a_4(M-M^*)(B-B^*),\end{aligned}$$

where

$$\begin{aligned}&-(M-M^*)(B-B^*) \\ &= -B^* M^* \left(\frac{M}{M^*}-1\right)\left(\frac{B}{B^*}-1\right) \\ &= -B^* M^* \left[\left(\frac{M}{M^*}-1\right)\left(\frac{B}{B^*}-\frac{I}{I^*}\right) + \left(\frac{M}{M^*}-1\right)\left(\frac{I}{I^*}-1\right)\right]\end{aligned}$$

$$\leq B^*M^*\left(\frac{M}{M^*}-1\right)^2+\frac{B^*M^*}{2}\left(\frac{B}{B^*}-\frac{I}{I^*}\right)^2+\frac{B^*M^*}{2}\left(\frac{I}{I^*}-1\right)^2.$$

Comprehensive factors were considered,

$$\begin{aligned}\mathcal{L}W\leq&-\left(a_2+\frac{a_1(\alpha+d)}{\alpha+2d}-\frac{b_1\phi_1B^*}{I^*}-\frac{b_1a_4M^*B^*}{I^{*2}}\right)(I-I^*)^2-\frac{a_1d}{\alpha+2d}(S-S^*)^2\\&-\left(\frac{a_3r}{\theta K}-\frac{b_1a_4B^*}{M^*}\right)(M-M^*)^2-\left(b_1\phi_1I^*B^*-\frac{b_1a_4B^*M^*}{2}\right)\left(\frac{I}{I^*}-\frac{B}{B^*}\right)^2,\end{aligned}$$

where $b_1\phi_1I^*B^*-\frac{b_1a_4M^*B^*}{2}>\frac{\Phi(1-k_1)M^*B^*}{q+(1-k_1)M^*}(1-\frac{q}{2(q+(1-k_1)M^*)})>0$, and

$$\begin{aligned}&a_2+\frac{a_1(\alpha+d)}{\alpha+2d}-\frac{b_1\phi_1B^*}{I^*}-\frac{b_1a_4M^*B^*}{I^{*2}}\\&=\frac{\eta B^*S^*}{(L+B^*)I^*}+\frac{(v+d+\alpha)I^*(\alpha+d)}{S^*(\alpha+2d)}-\frac{\eta LS^*\phi_1B^*}{\phi_1I^*(L+B^*)^2}-\frac{\eta LS^*B^{*2}M^*\Phi q(1-k_1)}{\phi_1I^{*2}(L+B^*)^2[q+(1-k_1)M^*]^2},\\&\frac{a_3r}{\theta K}-\frac{b_1a_4B^*}{M^*}=\frac{\beta_1k_1prS^*I^*}{\theta k(p+k_1M^*)^2}-\frac{\eta Lq\Phi(1-k_1)S^*B^{*2}}{\phi_1(L+B^*)^2[q+(1-k_1)M^*]^2M^*}.\end{aligned}$$

By Lemma 2.2, when the following conditions

$$a_2+\frac{a_1(\alpha+d)}{\alpha+2d}-\frac{b_1\phi_1B^*}{I^*}-\frac{b_1a_4M^*B^*}{I^{*2}}>0,\ \frac{a_3r}{\theta K}-\frac{b_1a_4B^*}{M^*}>0,$$

are true, the positive equilibrium of system (2.1) is locally asymptotically stable. □

Remark 3.1. When ϕ_1 is as large as possible, the condition in the above theorem remains true. That is to say, as the number of infected individuals increases, the bacterial growth rate ϕ_1 increases, and the disease equilibrium point is locally asymptotically stable, which is consistent with reality.

4. Existence and uniqueness of positive solution of random system (2.3)

$S,\ I,\ B,\ M$ in random system (2.3) are positive because they are different kinds of individual numbers. Prior to studying the dynamic behavior of the model, we consider the existence and uniqueness of positive solutions at first.

Theorem 4.1. *For any initial value* $(S(0),I(0),B(0),M(0))\in\mathbb{R}_+^4$, *the system (2.3) has a unique global positive solution* $(S(t),I(t),B(t),M(t))\in\mathbb{R}_+^4$ *for all* $t\geq0$ *when* $r\geq\sigma_4^2$.

Proof. For the proof of the existence and uniqueness of positive solutions, we have a relatively mature demonstration method. For further understanding of this method, refer to the complete proof in [35]. Here we give the construction of C^2-function $U(\mathbb{R}_+^4\to\mathbb{R})$ bases on the specificity of $M(t)$:

$$U(S,I,B,M)=(S-1-\ln S)+(I-1-\ln I)+(B-1-\ln B)+\left(\frac{1}{M}-1+\ln M\right).$$

By

$$u-1-\ln u\geq 0 \text{ for any } u>0 \text{ and } \frac{1}{v}-1+\ln v\geq 0 \text{ for any } v>0,$$

we can explain that the function is nonnegative. Applying Itô$'$s formula to U, we have

$$\begin{aligned}
LU(S,I,B,M) ={}& \left(1-\frac{1}{S}\right)\left(\Lambda-\left(\beta-\beta_1\frac{k_1M}{p+k_1M}\right)SI-\eta\frac{B}{L+B}S+vI-dS\right)\\
&+\left(1-\frac{1}{I}\right)\left(\left(\beta-\beta_1\frac{k_1M}{p+k_1M}\right)SI+\eta\frac{B}{L+B}S-(v+\alpha+d)I\right)\\
&+\frac{\sigma_2^2}{2}+\left(1-\frac{1}{B}\right)\left(\phi_1I+\phi B-\phi_0B-\Phi\frac{(1-k_1)M}{q+(1-k_1)M}B\right)+\frac{\sigma_3^2}{2}\\
&+\left(\frac{1}{M}-\frac{1}{M^2}\right)\left(r\left(1-\frac{M}{K}\right)M+\theta IM\right)-\frac{\sigma_4^2}{2}+\frac{\sigma_4^2}{M}+\frac{\sigma_1^2}{2}\\
\leq{}& \Lambda+\eta+2d+\Sigma_{i=1}^3\frac{\sigma_i^2}{2}+v+\alpha+\phi_0+\Phi+\frac{r}{K}+r+(\phi_1+\beta+\theta)I\\
&-\frac{r-\sigma_4^2}{M}\\
\leq{}& \Lambda+\eta+2d+\Sigma_{i=1}^3\frac{\sigma_i^2}{2}+v+\alpha+\phi_0+\Phi+\frac{r}{K}+r+(\phi_1+\beta+\theta)I\\
={}& M_1+(\phi_1+\beta+\theta)I,
\end{aligned}$$

among above function

$$M_1=\Lambda+\eta+2d+\Sigma_{i=1}^3\frac{\sigma_i^2}{2}+v+\alpha+\phi_0+\Phi+\frac{r}{K}+r.$$

In addition, we can see that $I\leq 2(I-1-\log I)+2\log 2\leq 2(U+\log 2)$, thus we have

$$LU\leq M_1+2(\phi_1+\beta+\theta)(U+\log 2)\leq\lambda(1+U),$$

where

$$\lambda=\max\{M_1+2(\phi_1+\beta+\theta)\log 2, 2(\phi_1+\beta+\theta)\}.$$

□

5. The existence of stationary distribution of system (2.3)

The existence of stationary distribution has always been an important part of the research on stochastic epidemic models and population models, and it also shows the persistence of the diseases. Denote

$$\begin{aligned}
R^* ={}&\frac{\Lambda\beta pr+\Lambda k_1K(\beta-\beta_1)(r-\sigma_4^2)}{k_1K(d+\frac{\sigma_1^2}{2})(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})(v+\alpha+d+\frac{\sigma_2^2}{2})}\\
&+\frac{\Lambda\eta\phi_1}{(d+\frac{\sigma_1^2}{2})L(v+\alpha+d+\frac{\sigma_2^2}{2})(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}+\frac{\sigma_3^2}{2})}.
\end{aligned}$$

Theorem 5.1. *Let us suppose,* $R^* > 1$ *and* $r \geq \sigma_4^2$*, then the system (2.3) has a stationary distribution.*

Proof. We prove the existence and ergodicity of stationary distribution need to verify the conditions in Lemma 2.1. Firstly, we write the diffusion matrix of system (2.3),

$$\operatorname{diag}(\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 B^2, \sigma_4^2 M^2).$$

Since it is a diagonal matrix whose elements on the diagonal are greater than zero, it is positive definite. Hence, condition (I) holds. Next, the key point is to verify the condition (II).

Define a C^2-function $\tilde{V}(S,I,B,M) = WV_1 + V_2 + V_3$, where

$$\begin{aligned} V_1 = &-(a_1+a_2+a_3)\ln S - \ln I - a_4 \ln B + a_5 \frac{B}{\phi_0 - \phi} + (a_6+a_7+a_9)\ln M + a_8 \frac{1}{M} \\ &+ a_{10} B, \\ V_2 = &-\ln S - \ln B + \frac{1}{M},\ V_3 = \frac{1}{2+\theta}(S+I+a_{11}B)^{2+\theta} + M, \end{aligned}$$

and the parameters $W,\ \theta,$ and $a_i,\ i=1,2,\ldots,11$ are given at soon. We can know that the function $\tilde{V}$ exists a minimum value $\tilde{V}_{min}$, then the C^2-function V

$$V(S,I,B,M) = \tilde{V}(S,I,B,M) - \tilde{V}_{\min} \geq 0.$$

With the Itô$'$s formula, we obtain

$$\begin{aligned} LV_1 = &(a_1+a_2+a_3)\left(-\frac{\Lambda}{S} + \left(\beta - \beta_1 \frac{k_1 M}{p+k_{1M}}\right) I + \eta \frac{B}{L+B} - v\frac{I}{S} + d + \frac{\sigma_1^2}{2}\right) \\ &+ \left(-\frac{\beta p}{p+k_1M}S - \frac{(\beta-\beta_1)k_1M}{p+k_1M}S - \eta\frac{B}{L+B}\frac{S}{I} + v + \alpha + d + \frac{\sigma_2^2}{2}\right) \\ &+ a_4\left(-\phi_1\frac{I}{B} + \Phi\frac{(1-k_1)M}{q+(1-k_1)M} + \phi_0 - \phi + \frac{\sigma_3^2}{2}\right) \\ &+ a_5\left(\frac{\phi_1}{\phi_0-\phi}I - (L+B) + L - \frac{\Phi}{\phi_0-\phi}\frac{(1-k_1)M}{q+(1-k_1)M}B\right) \\ &+ (a_6+a_7+a_9)\left(-r\frac{M}{K} + \theta I + r - \frac{\sigma_4^2}{2}\right) + a_8\left(-\frac{r-\sigma_4^2}{M} - \frac{\theta I}{M} + \frac{r}{K}\right) \\ &+ a_{10}\left(\phi_1 I - (\phi_0-\phi)B - \Phi\frac{(1-k_1)M}{q+(1-k_1)M}B\right) \\ \leq &-\left(a_1\frac{\Lambda}{S} + \frac{\beta p}{p+k_1M}S + a_6 r\frac{p+k_1M}{k_1K}\right) + a_1\left(d+\frac{\sigma_1^2}{2}\right) + a_6\left(r - \frac{\sigma_4^2}{2} + \frac{rp}{k_1K}\right) \\ &-\left(a_2\frac{\Lambda}{S} + \frac{(\beta-\beta_1)k_1M}{p+k_1M}S + a_7 r\frac{p+k_1M}{k_1K} + a_8\frac{r-\sigma_4^2}{M}\right) + a_2\left(d+\frac{\sigma_1^2}{2}\right) \\ &+ a_8\frac{r}{K} + a_7\left(r - \frac{\sigma_4^2}{2} + \frac{rp}{k_1K}\right) - \left(a_3\frac{\Lambda}{S} + \eta\frac{B}{L+B}\frac{S}{I} + a_4\frac{\phi_1 I}{B} + a_5(L+B)\right) \\ &+ a_3\left(d+\frac{\sigma_1^2}{2}\right) + a_4\Phi\frac{(1-k_1)M}{q+(1-k_1)M} - a_9 r\frac{M}{K} + a_9 r + a_4\left(\phi_0 - \phi + \frac{\sigma_3^2}{2}\right) \\ &+ a_5 L + \left(v+\alpha+d+\frac{\sigma_2^2}{2}\right) + (a_1+a_2+a_3)\left(\beta I + \eta\frac{B}{L}\right) + a_5\frac{\phi_1}{\phi_0-\phi}I \end{aligned}$$

$$+ (a_6 + a_7 + a_9)\theta I + a_{10}\phi_1 I - a_{10}(\phi_0 - \phi)B.$$

Let

$$\begin{aligned}
a_1 &= \frac{\Lambda\beta pr}{k_1K(d+\frac{\sigma_1^2}{2})^2(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})},\ a_4 = \frac{\Lambda\eta\phi_1}{(d+\frac{\sigma_1^2}{2})L(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}+\frac{\sigma_3^2}{2})^2},\\
a_3 &= \frac{\Lambda\eta\phi_1}{(d+\frac{\sigma_1^2}{2})^2L(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}+\frac{\sigma_3^2}{2})},\ a_2 = \frac{\Lambda(\beta-\beta_1)(r-\sigma_4^2)}{(d+\frac{\sigma_1^2}{2})^2(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})},\\
a_5 &= \frac{\Lambda\eta\phi_1}{(d+\frac{\sigma_1^2}{2})L^2(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}+\frac{\sigma_3^2}{2})},\ a_6 = \frac{\Lambda\beta pr}{k_1K(d+\frac{\sigma_1^2}{2})(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})^2},\\
a_7 &= \frac{\Lambda(\beta-\beta_1)(r-\sigma_4^2)}{(d+\frac{\sigma_1^2}{2})(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})^2},\ a_8 = \frac{\Lambda(\beta-\beta_1)(r-\sigma_4^2)K}{r(d+\frac{\sigma_1^2}{2})(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})},\\
a_9 &= a_4\Phi\frac{(1-k_1)qK}{(q+(1-k_1)K)^2r},\ a_{10} = \frac{(a_1+a_2+a_3)\eta}{(\phi_0-\phi)L}.
\end{aligned}$$

Due to the function

$$h(M) = a_4\Phi\frac{(1-k_1)M}{q+(1-k_1)M} - a_9 r\frac{M}{K} + a_9 r \le h(K) = a_4\Phi\frac{(1-k_1)K}{q+(1-k_1)K}.$$

In consequence we obtain

$$\begin{aligned}
LV_1 \le& -\left(a_1\frac{\Lambda}{S} + \frac{\beta p}{p+k_1M}S + a_6 r\frac{p+k_1M}{k_1K}\right) + a_1\left(d+\frac{\sigma_1^2}{2}\right) + a_6\left(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K}\right)\\
&-\left(a_2\frac{\Lambda}{S} + \frac{(\beta-\beta_1)k_1M}{p+k_1M}S + a_7 r\frac{p+k_1M}{k_1K} + a_8\frac{r-\sigma_4^2}{M}\right) + a_2\left(d+\frac{\sigma_1^2}{2}\right) + a_8\frac{r}{K}\\
&+ a_7\left(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K}\right) - \left(a_3\frac{\Lambda}{S} + \eta\frac{B}{L+B}\frac{S}{I} + a_4\frac{\phi_1 I}{B} + a_5(L+B)\right)\\
&+ a_4\Phi\frac{(1-k_1)K}{q+(1-k_1)K} + a_4\left(\phi_0-\phi+\frac{\sigma_3^2}{2}\right) + a_5L + \left(v+\alpha+d+\frac{\sigma_2^2}{2}\right)\\
&+ (a_1+a_2+a_3)\beta I + a_5\frac{\phi_1}{\phi_0-\phi}I + (a_6+a_7+a_9)\theta I + a_{10}\phi_1 I + a_3\left(d+\frac{\sigma_1^2}{2}\right)\\
\le& -3\left(\frac{a_1a_6\Lambda\beta pr}{k_1K}\right)^{\frac{1}{3}} + a_1\left(d+\frac{\sigma_1^2}{2}\right) + a_6\left(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K}\right)\\
&-4\left(\frac{a_2a_7a_8\Lambda(\beta-\beta_1)r(r-\sigma_4^2)}{K}\right)^{\frac{1}{4}} + a_2\left(d+\frac{\sigma_1^2}{2}\right) + a_7\left(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K}\right) + a_8\frac{r}{K}\\
&-4(a_3a_4a_5\Lambda\eta\phi_1)^{\frac{1}{4}} + a_3\left(d+\frac{\sigma_1^2}{2}\right) + a_4\left(\Phi\frac{(1-k_1)K}{q+(1-k_1)K} + \phi_0-\phi+\frac{\sigma_3^2}{2}\right) + a_5L\\
&+\left(v+\alpha+d+\frac{\sigma_2^2}{2}\right) + \left((a_1+a_2+a_3)\beta + a_5\frac{\phi_1}{\phi_0-\phi} + (a_6+a_7+a_9)\theta + a_{10}\phi_1\right)I\\
=& -\frac{\Lambda\beta pr + \Lambda k_1K(\beta-\beta_1)(r-\sigma_4^2)}{k_1K(d+\frac{\sigma_1^2}{2})(r-\frac{\sigma_4^2}{2}+\frac{rp}{k_1K})} - \frac{\Lambda\eta\phi_1}{(d+\frac{\sigma_1^2}{2})L(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}+\frac{\sigma_3^2}{2})}
\end{aligned}$$

$$\begin{aligned}
&+\left(v+\alpha+d+\frac{\sigma_2^2}{2}\right)+AI\\
=&-\left(v+\alpha+d+\frac{\sigma_2^2}{2}\right)(R^*-1)+AI\\
:=&-\varphi+AI, \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
&A=(a_1+a_2+a_3)\beta+a_5\frac{\phi_1}{\phi_0-\phi}+(a_6+a_7+a_9)\theta+a_{10}\phi_1,\\
&\varphi=\left(v+\alpha+d+\frac{\sigma_2^2}{2}\right)(R^*-1).
\end{aligned}$$

In the same way, one can get

$$\begin{aligned}
LV_2=&-\frac{\Lambda}{S}+\left(\beta-\beta_1\frac{k_1M}{p+k_1M}\right)I+\eta\frac{B}{L+B}-v\frac{I}{S}+d+\frac{\sigma_1^2}{2}-\phi_1\frac{I}{B}\\
&+\Phi\frac{(1-k_1)M}{q+(1-k_1)M}+\phi_0-\phi+\frac{\sigma_3^2}{2}-\frac{r-\sigma_4^2}{M}-\theta\frac{I}{M}+\frac{r}{K} \qquad (5.2)\\
\leq&-\frac{\Lambda}{S}+\beta I+\eta+d+\frac{\sigma_1^2}{2}-\phi_1\frac{I}{B}+\Phi+\phi_0-\phi+\frac{\sigma_3^2}{2}-\frac{r-\sigma_4^2}{M}+\frac{r}{K}.
\end{aligned}$$

Let $N_1=S+I+a_{11}B$, $a_{11}=\frac{\alpha}{\phi_1}$ and $\theta+1=\frac{\min\{d,\phi_0-\phi\}}{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}$, then

$$\begin{aligned}
LV_3=&N_1^{\theta+1}\left(\Lambda-dS-(\alpha+d)I+a_{11}(\phi-\phi_0)B+a_{11}\phi_1I-a_{11}\Phi\frac{(1-k_1)M}{q+(1-k_1)M}B\right)\\
&+\frac{(\theta+1)N_1^{\theta}}{2}\left(\sigma_1^2S^2+\sigma_2^2I^2+a_{11}^2\sigma_3^2B^2\right)+r\left(1-\frac{M}{K}\right)M+\theta IM\\
\leq&N_1^{\theta+1}\left(\Lambda-\min\{d,\alpha+d-a_{11}\phi_1,a_{11}(\phi_0-\phi)\}N_1\right)\\
&+\frac{(\theta+1)N_1^{\theta+2}}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)-\frac{r}{K}M^2+rM+\theta IM\\
=&\Lambda N_1^{\theta+1}-\left(\min\{d,a_{11}(\phi_0-\phi)\}-\frac{\theta+1}{2}\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\right)\right)N_1^{\theta+2}-\frac{r}{K}M^2+rM \qquad (5.3)\\
&+\theta IM\\
\leq&\Lambda N_1^{\theta+1}-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{4}N_1^{\theta+2}\\
&-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{4}\left(S^{\theta+2}+I^{\theta+2}+a_{11}^{\theta+2}B^{\theta+2}\right)-\frac{r}{K}M^2+rM+\theta IM.
\end{aligned}$$

By (5.1)-(5.3), then

$$\begin{aligned}
LV\leq&-W\varphi+(WA+\beta)I-\frac{\Lambda}{S}+\eta+d+\frac{\sigma_1^2}{2}-\phi_1\frac{I}{B}+\Phi+\phi_0-\phi+\frac{\sigma_3^2}{2}-\frac{r-\sigma_4^2}{M}\\
&+\frac{r}{K}+\Lambda N_1^{\theta+1}-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{4}\left(S^{\theta+2}+I^{\theta+2}+a_{11}^{\theta+2}B^{\theta+2}\right)\\
&-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{4}N_1^{\theta+2}-\frac{r}{K}M^2+rM+\theta IM
\end{aligned}$$

$$\begin{aligned}&= - W\varphi + (WA+\beta)I - \frac{\Lambda}{S} - \phi_1\frac{I}{B} - \frac{r-\sigma_4^2}{M} - \frac{r}{2K}M^2\\&\quad - \frac{\min\{d, a_{11}(\phi_0-\phi)\}}{4}\left(S^{\theta+2} + I^{\theta+2} + a_{11}^{\theta+2}B^{\theta+2}\right) + C,\end{aligned}$$

where

$$\begin{aligned}C = \sup_{(S,I,B,M)\in\mathbb{R}_+^4} \{&\eta + d + \frac{\sigma_1^2}{2} + \Phi + \phi_0 - \phi + \frac{\sigma_3^2}{2} + \frac{r}{K} + \Lambda N_1^{\theta+1}\\&- \frac{\min\{d, a_{11}(\phi_0-\phi)\}}{4}N_1^{\theta+2} - \frac{r}{2K}M^2 + rM + \theta IM\}.\end{aligned}$$

There is a large enough M to satisfy the inequality,

$$-W\varphi + C \le -2. \tag{5.4}$$

And

$$D = \left\{(S,I,B,M)\in\mathbb{R}_+^4 : \epsilon \le S \le \frac{1}{\epsilon}, \epsilon \le I \le \frac{1}{\epsilon}, \epsilon^2 \le B \le \frac{1}{\epsilon^2}, \epsilon \le M \le \frac{1}{\epsilon}\right\},$$

is a bounded closed set, where $0 < \epsilon < 1$ is sufficiently small such that the following conditions hold

$$- \frac{\Lambda}{\epsilon} + F \le -1, \tag{5.5}$$

$$(WA+\beta)\epsilon - W\varphi + C \le -1, \tag{5.6}$$

$$- \phi_1\frac{1}{\epsilon} + F \le -1, \tag{5.7}$$

$$- \frac{r-\sigma_4^2}{\epsilon} + F \le -1, \tag{5.8}$$

$$- \frac{\min\{d, a_{11}(\phi_0-\phi)\}}{8\epsilon^{\theta+2}} + F \le -1, \tag{5.9}$$

$$- \frac{\min\{d, a_{11}(\phi_0-\phi)\}a_{11}^{\theta+2}}{8\epsilon^{\theta+2}} + F \le -1, \tag{5.10}$$

$$- \frac{r}{2K\epsilon^2} + F \le -1, \tag{5.11}$$

and

$$F = \max_{I\in(0,+\infty)}\left\{(WA+\beta)I - \frac{\min\{d, a_{11}(\phi_0-\phi)\}}{8}I^{\theta+2}\right\} - W\varphi + C.$$

For convenience, we divide $\mathbb{R}_+^4 \setminus D$ into the following eight domains, for $(S,I,B,M)\in\mathbb{R}_+^4$,

$$\begin{aligned}&D_1 = \{0<S<\epsilon\}, D_2 = \{0<I<\epsilon\}, D_3 = \left\{0<B<\epsilon^2, I\ge\epsilon\right\}, D_4 = \{0<M<\epsilon\},\\&D_5 = \left\{S \ge \frac{1}{\epsilon}\right\}, D_6 = \left\{I \ge \frac{1}{\epsilon}\right\}, D_7 = \left\{B \ge \frac{1}{\epsilon^2}\right\}, D_8 = \left\{M \ge \frac{1}{\epsilon}\right\}.\end{aligned}$$

Next, we will prove that $LV(S,I,B,M) \le -1$ for any $(S,I,B,M)\in\mathbb{R}_+^4$ on the above eight domains.

Case 1. With any $(S,I,B,M)\in D_1$,

$$LV \le - W\varphi + C + (WA+\beta)I - \frac{\min\{d, a_{11}(\phi_0-\phi)\}}{8}I^{\theta+2} - \frac{\Lambda}{S}$$

$$
\begin{aligned}
&\leq -\frac{\Lambda}{S}+F\\
&\leq -\frac{\Lambda}{\epsilon}+F\\
&\leq -1 \text{ by (5.4) and (5.5).}
\end{aligned}
$$

Case 2. With any $(S,I,B,M)\in D_2$,

$$
LV\leq -W\varphi+C+(WA+\beta)I\leq -W\varphi+C+(WA+\beta)\epsilon\leq -1 \text{ by (5.4) and (5.6).}
$$

Case 3. With any $(S,I,B,M)\in D_3$,

$$
\begin{aligned}
LV\leq& -W\varphi+C+(WA+\beta)I-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{8}I^{\theta+2}-\phi_1\frac{I}{B}\\
\leq& -\phi_1\frac{I}{B}+F\\
\leq& -\frac{\phi_1}{\epsilon}+F\\
\leq& -1 \text{ by (5.4) and (5.7).}
\end{aligned}
$$

Case 4. With any $(S,I,B,M)\in D_4$,

$$
\begin{aligned}
LV\leq& -W\varphi+C+(WA+\beta)I-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{8}I^{\theta+2}-\frac{r-\sigma_4^2}{M}\\
\leq& -\frac{r-\sigma_4^2}{M}+F\\
\leq& -\frac{r-\sigma_4^2}{\epsilon}+F\\
\leq& -1 \text{ by (5.4) and (5.8).}
\end{aligned}
$$

Case 5. With any $(S,I,B,M)\in D_5$, (5.4) and (5.9),

$$
\begin{aligned}
LV\leq& -W\varphi+C+(WA+\beta)I-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{8}(I^{\theta+2}+S^{\theta+2})\\
\leq& -\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{8}S^{\theta+2}+F\\
\leq& -\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{8\epsilon^{\theta+2}}+F\\
\leq& -1.
\end{aligned}
$$

Case 6. With any $(S,I,B,M)\in D_6$, (5.4) and (5.9).

$$
\begin{aligned}
LV\leq& -W\varphi+C+(WA+\beta)I-\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{4}I^{\theta+2}\\
\leq& -\frac{\min\{da_{11}(\phi_0-\phi)\}}{8}I^{\theta+2}+F\\
\leq& -\frac{\min\{d,a_{11}(\phi_0-\phi)\}}{8\epsilon^{\theta+2}}+F\\
\leq& -1.
\end{aligned}
$$

Case 7. With any $(S, I, B, M) \in D_7$,

$$\begin{aligned}
LV \leq & -W\varphi + C + (WA + \beta)I - \frac{\min\{d, a_{11}(\phi_0 - \phi)\}}{8}(I^{\theta+2} + a_{11}^{\theta+2}B^{\theta+2}) \\
\leq & -\frac{\min\{d, a_{11}(\phi_0 - \phi)\}a_{11}^{\theta+2}}{8}B^{\theta+2} + F \\
\leq & -\frac{\min\{d, a_{11}(\phi_0 - \phi)\}a_{11}^{\theta+2}}{8\epsilon^{\theta+2}} + F \\
\leq & -1 \text{ by (5.4) and (5.10).}
\end{aligned}$$

Case 8. With any $(S, I, B, M) \in D_8$,

$$\begin{aligned}
LV \leq & -W\varphi + C + (WA + \beta)I - \frac{\min\{d, a_{11}(\phi_0 - \phi)\}}{8}I^{\theta+2} - \frac{r}{2K}M^2 \\
\leq & -\frac{r}{2K}M^2 + F \\
\leq & -\frac{r}{2K\epsilon^2} + F \\
\leq & -1 \text{ by (5.4) and (5.11).}
\end{aligned}$$

Consequently, from above eight situations, we obtain that for a sufficient small ϵ, $LV \leq -1$ for any $(S, I, B, M) \in \mathbb{R}_+^4 \setminus D$. Now the condition (II) of Lemma 2.1 is tested. Hence the system (2.3) has a stationary distribution. □

6. The extinction of random system (2.3)

It is of great significance to study the extinction of diseases for the prevention and control of infectious diseases. In this section, we will show the sufficient conditions for the extinction of the disease.

Theorem 6.1. *Let $(S(t), I(t), B(t), M(t))$ be the solution of the system (2.3) with any initial value $(S(0), I(0), B(0), M(0)) \in \Omega$. If $R_0 < 1$, then*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\ln(I(t) + aB(t) + bB(t))}{t} \\
&\leq -\frac{1}{2}(v + \alpha + d)(1 - R_0)\min\{1, \frac{\phi_0 - \phi}{\phi_1(a+b)}\} - \min\{\frac{\sigma_2^2}{4}, \frac{\sigma_3^2}{8}\} \\
&< 0 \ a.s.,
\end{aligned}$$

that is to say that the disease will become extinct. And $a = \frac{\eta\Lambda}{dL(\phi_0 - \phi)}$, $b = \frac{1}{2\phi_1}(v + \alpha + d)(1 - R_0)$.

Proof. We first define function $P(t) = I(t) + aB(t) + bB(t)$ and $a = \frac{\eta\Lambda}{dL(\phi_0 - \phi)}$, $b = \frac{1}{2\phi_1}(v + \alpha + d)(1 - R_0)$, with the Itô$'$s formula and $R_0 < 1$,

$$\begin{aligned}
L\ln P = & \frac{\left(\beta - \beta_1\frac{k_1 M}{p + k_1 M}\right)SI + \eta\frac{B}{L+B}S - (v + \alpha + d)I + (a+b)\left(\phi_1 I + (\phi - \phi_0)B\right)}{I + (a+b)B} \\
& - \frac{(a+b)\Phi\frac{(1-k_1)M}{q + (1-k_1)M}B}{I + aB + bB} - \frac{\sigma_2^2 I^2 + a^2\sigma_3^2 B^2 + b^2\sigma_3^2 B^2}{2(I + aB + bB)^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\beta SI + \frac{\eta}{L}BS - (v+\alpha+d)I + (a+b)\phi_1 I - (a+b)(\phi_0-\phi)B}{I+(a+b)B} \\
&\quad - \frac{\sigma_2^2 I^2 + (a^2+b^2)\sigma_3^2 B^2}{2(I+(a+b)B)^2} \\
&\leq \frac{(\beta\frac{\Lambda}{d} + a\phi_1 - (v+\alpha+d))I + (\frac{\eta\Lambda}{Ld} - (a+b)(\phi_0-\phi))B + b\phi_1 I}{I+(a+b)B} \\
&\quad - \frac{\sigma_2^2 I^2 + (a^2+b^2)\sigma_3^2 B^2}{4(I^2+(a+b)^2B^2)} \\
&\leq -\frac{\frac{1}{2}(v+\alpha+d)(1-R_0)I - b(\phi_0-\phi)B}{I+(a+b)B} - \min\{\frac{\sigma_2^2}{4}, \frac{\sigma_3^2}{8}\} \\
&\leq -\frac{1}{2}(v+\alpha+d)(1-R_0)\min\{1, \frac{\phi_0-\phi}{\phi_1(a+b)}\} - \min\{\frac{\sigma_2^2}{4}, \frac{\sigma_3^2}{8}\}.
\end{aligned}
$$

For this reason,

$$
\begin{aligned}
d\ln P \leq & (-\frac{1}{2}(v+\alpha+d)(1-R_0)\min\{1, \frac{\phi_0-\phi}{\phi_1(a+b)}\} - \min\{\frac{\sigma_2^2}{4}, \frac{\sigma_3^2}{8}\})dt \\
& + \frac{\sigma_2 I dB_2(t) + (a+b)\sigma_3 B dB_3(t)}{P}.
\end{aligned} \tag{6.1}
$$

Integrating (6.1) from 0 to t on both sides, one can see

$$
\begin{aligned}
\frac{\ln P(t)}{t} \leq & \frac{\ln P(0)}{t} - \frac{1}{2}(v+\alpha+d)(1-R_0)\min\{1, \frac{\phi_0-\phi}{\phi_1(a+b)}\} - \min\{\frac{\sigma_2^2}{4}, \frac{\sigma_3^2}{8}\} \\
& + \frac{M_2(t) + (a+b)M_3(t)}{t},
\end{aligned} \tag{6.2}
$$

where

$$
M_2(t) := \int_0^t \frac{\sigma_2 I(s)}{P(s)} dB_2(s), \ M_3(t) := \int_0^t \frac{\sigma_3 B(s)}{P(s)} dB_3(s),
$$

they are local martingales whose quadratic variation are $\langle M_2, M_2\rangle_t \leq \sigma_2^2 t$ and $\langle M_3, M_3\rangle_t \leq \sigma_3^2 t$. Applying the strong law of large numbers yields

$$
\lim_{t\to\infty} \frac{M_2(t)}{t} = 0 \text{ a.s. and } \lim_{t\to\infty} \frac{M_3(t)}{t} = 0 \text{ a.s.}
$$

Then, we take the superior limit on both sides of (6.2),

$$
\limsup_{t\to\infty} \frac{\ln P(t)}{t} \leq -\frac{1}{2}(v+\alpha+d)(1-R_0)\min\{1, \frac{\phi_0-\phi}{\phi_1(a+b)}\} - \min\{\frac{\sigma_2^2}{4}, \frac{\sigma_3^2}{8}\} \text{ a.s.}
$$

If $R_0 < 1$, we can have

$$
\limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0, \ \limsup_{t\to\infty} \frac{\ln B(t)}{t} < 0,
$$

which shows that

$$
\lim_{t\to\infty} I(t) = 0, \ \lim_{t\to\infty} B(t) = 0.
$$

That is to say, when $R_0 < 1$, the $I(t), B(t)$ of the disease will tend to extinction. □

7. Numerical examples

In Sections 5 and 6, we have drawn the conditions for the persistence and extinction of diseases. To make our conclusion more convincing, by using the Milstein's Higher Order Method mentioned in [7], we obtain the following discretization form of system (2.3),

$$\begin{aligned}
S_{i+1} &= S_i + \left(\Lambda - \left(\beta - \beta_1 \frac{k_1 M_i}{p + k_1 M_i}\right) S_i I_i - \eta \frac{B_i}{L + B_i} S_i + v I_i - d S_i\right) \Delta t \\
&\quad + \sigma_1 S_i \xi_{1,i} \sqrt{\Delta t} + \frac{\sigma_1^2}{2} S_i (\xi_{1,i}^2 - 1) \Delta t, \\
I_{i+1} &= I_i + \left(\left(\beta - \beta_1 \frac{k_1 M_i}{p + k_1 M_i}\right) S_i I_i + \eta \frac{B_i}{L + B_i} S_i - (v + \alpha + d) I_i\right) \Delta t \\
&\quad + \sigma_2 I_i \xi_{2,i} \sqrt{\Delta t} + \frac{\sigma_2^2}{2} I_i (\xi_{2,i}^2 - 1) \Delta t, \\
B_{i+1} &= B_i + \left(\phi_1 I_i + \phi(t) - \phi_0 B_i - \Phi \frac{(1 - k_1) M_i}{q + (1 - k_1) M_i} B_i\right) \Delta t + \sigma_3 B_i \xi_{3,i} \sqrt{\Delta t} \\
&\quad + \frac{\sigma_3^2}{2} B_i (\xi_{3,i}^2 - 1) \Delta t, \\
M_{i+1} &= M_i + \left(r \left(1 - \frac{M_i}{K}\right) M_i + \theta I_i M_i\right) \Delta t + \sigma_4 M_i \xi_{4,i} \sqrt{\Delta t} + \frac{\sigma_4^2}{2} M_i (\xi_{4,i}^2 - 1) \Delta t,
\end{aligned}$$

where $\Delta t > 0$ is time increment, $\xi_{1,i}$, $\xi_{2,i}$ are the Gaussian random variables which meet the distribution $N(0,1)$.

Given the initial value at first, $(S(0), I(0), B(0), M(0)) = (0.5, 0.5, 0.5, 0.5)$, we consider the value of the parameter as follows:

$$\begin{aligned}
&\Lambda = 0.5,\ \beta = 0.3,\ \beta_1 = 0.2,\ p = 0.5,\ r = 0.5,\ K = 0.5,\ k_1 = 0.5,\ L = 0.5,\ \phi = 0.2, \\
&\phi_0 = 0.5,\ \phi_1 = 0.5,\ \Phi = 0.5,\ q = 0.5,\ \theta = 0.5.
\end{aligned}$$

In addition, we can get different results by changing the values of other parameters.

Example 7.1. $d = 0.5,\ v = 0.9,\ \eta = 0.8,\ \alpha = 0.1,\ \sigma_i^2 = 0.1,\ i = 1, 2, 3, 4$ and after some calculations we obtain

$$R^* = 1.4312 > 1.$$

Under this group of data, we get the Figure 1, which implies that the system (2.3) is stable. In the left half of it, the blue lines represent the change trend of the random system and the red lines indicate the change trend of the corresponding ordinary system, we can see the curve keeps moving irregularly. The pictures on the right mean the density of the stochastic system (2.3). This is consistent with our theorem.

Example 7.2. $d = 0.9,\ v = 0.5,\ \alpha = 0.8,\ \eta = 0.1,\ \sigma_i^2 = 0.9,\ i = 1, 2, 3, 4$. We calculate

$$R_0 = 0.1599 < 1.$$

For the time being, we can verify intuitively $I(t)$ and $B(t)$ will die out in Figure 2. On the left, the blue lines indicate the phenomenon under white noises and the red lines show the phenomenon of ODE system. The pictures on the right mean the density of the stochastic system (2.3).

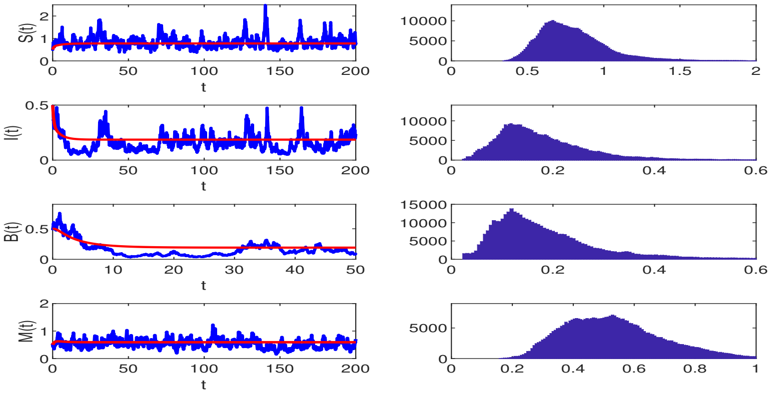

Figure 1. The stationary distribution of system (2.3) with initial value $(S(0), I(0), B(0), M(0)) = (0.5, 0.5, 0.5, 0.5)$.

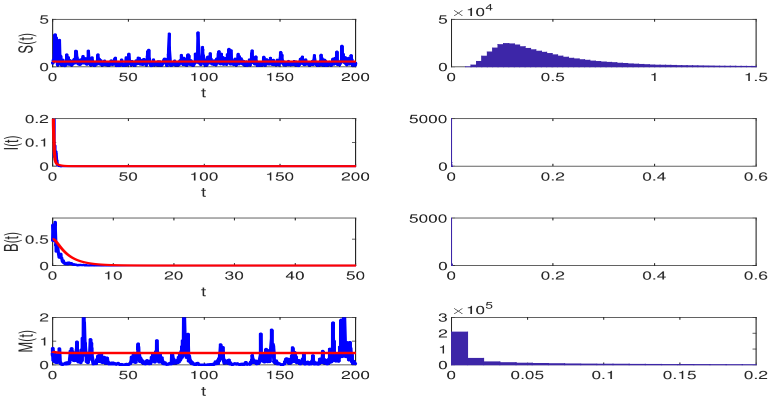

Figure 2. The extinction of system (2.3) with initial value $(S(0), I(0), B(0), M(0)) = (0.5, 0.5, 0.5, 0.5)$.

Example 7.3. In the process of data construction, we observe that the value of $\alpha,\ v,\ d$ have a great influence on the model. So let's focus on the change of the value of $\alpha,\ v,\ d$.

For $R_0,\ R_1$ and R^*, when $\eta = 0.5,\ \sigma_i^2 = 0.1,\ i = 1, 2, 3, 4$ are remain constant, then we obtained the following three figures, with $d = 0.9, d = 0.5$ and $d = 0.1$ from left to right. It can be concluded that as d increases, "$R_0,\ R_1$ and R^*" decreases, while as v and α increases, "$R_0,\ R_1$ and R^*" increases.

Example 7.4. For system 2.3, we also consider three cases. We assume $d = 0.5,\ v = 0.5,\ \eta = 0.8,\ \sigma_i^2 = 0.1,\ i = 1, 2, 3, 4$ are fixed values, with different values α, we obtain the influence of $I(t)$ and $B(t)$ under the α in Figure 4. Second case, the phenomenon of system (2.3) when $\alpha = 0.5,\ v = 0.5,\ \eta = 0.8,\ \sigma_i^2 = 0.1,\ i = 1, 2, 3, 4$ are unchanged and the change of d is indicated in Figure 5. Similarly, Figure 6 express the variety of $I(t),\ B(t)$ with $d = 0.5,\ \alpha = 0.5,\ \eta = 0.8,\ \sigma_i^2 = 0.1,\ i = 1, 2, 3, 4$. We can observe the phenomenon of the blue lines and red lines mean $I(t)$ and $B(t)$ will be persistent, but the the phenomenon of the yellow lines and green lines shows $I(t)$ and $B(t)$ will become extinct.

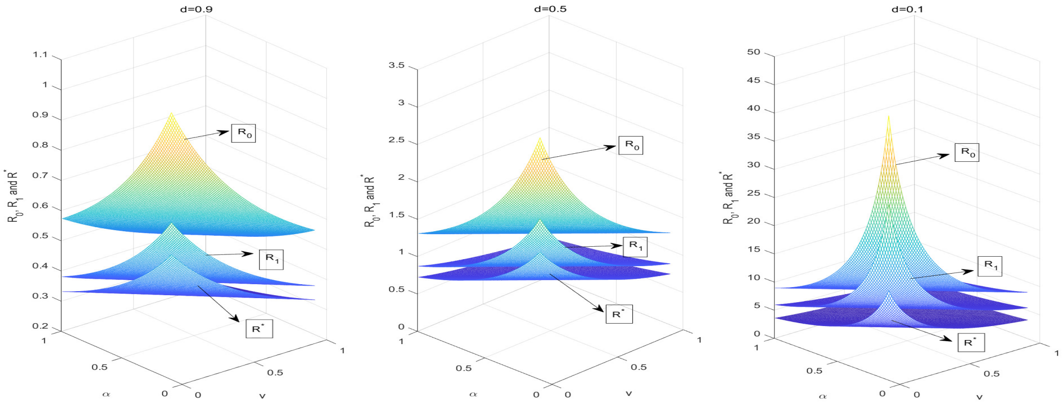

Figure 3. The trend of R_0, R_1 and R^* changing with v and α under the value $d = 0.9$, $d = 0.5$ and $d = 0.1$.

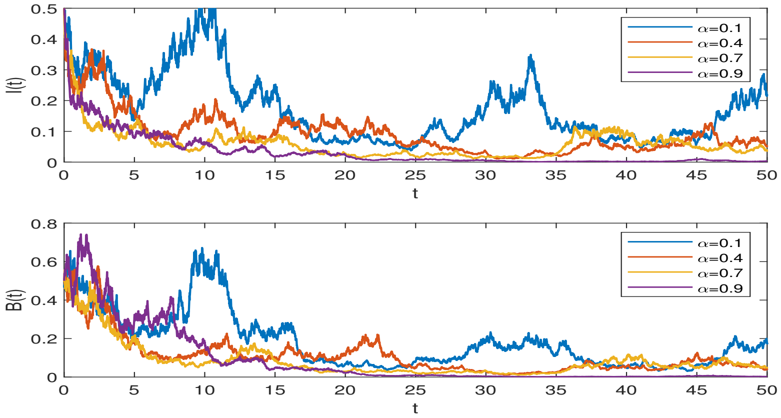

Figure 4. The influence of system (2.3) under the different α, the blue lines, red lines, yellow lines and green lines indicate the change of $I(t)$ and $B(t)$ when $\alpha = 0.1$, $\alpha = 0.4$, $\alpha = 0.7$, $\alpha = 0.9$, respectively.

On the basis of the above research, we give a Remark:

Remark 7.1. For the disease models studied in the paper and based on the appropriate parameter values, changing the values of parameters α, d, v will affect the state of disease. In other words, when other parameters are fixed, the larger d is, the more the disease will become extinct, while the smaller the d is, the disease will last. And α, v have the same status.

8. Discussion and conclusion

In the artical, we consider the unpredictability of the environmental factors, white noises influence the transmission of the diseases. In this reason, we mainly investigate the dynamics of a system with white noises. To begin with, by using the method of Lyapunov function, we obtain the local stability condition of its positive solution of the deterministic model (2.1). This method greatly reduces the complexity of operation. And then we obtain that the stochastic model (2.3) has a unique global positive solution under the condition of $r \geq \sigma_4^2$ by building

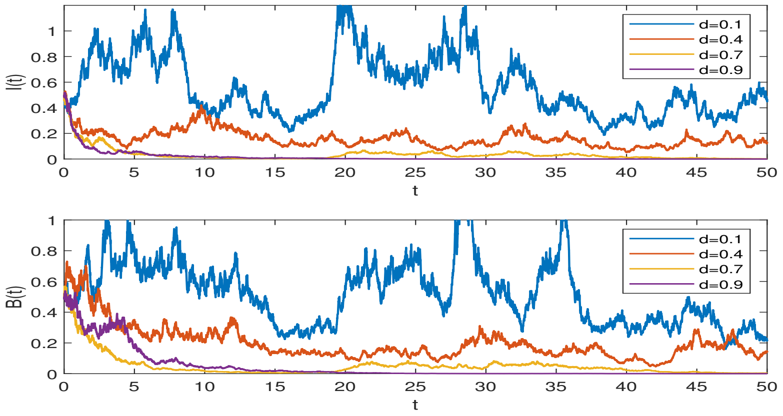

Figure 5. The influence of system (2.3) under the different d, the blue lines, red lines, yellow lines and green lines indicate the change of $I(t)$ and $B(t)$ when $d = 0.1,\ d = 0.4,\ d = 0.7,\ d = 0.9$, respectively.

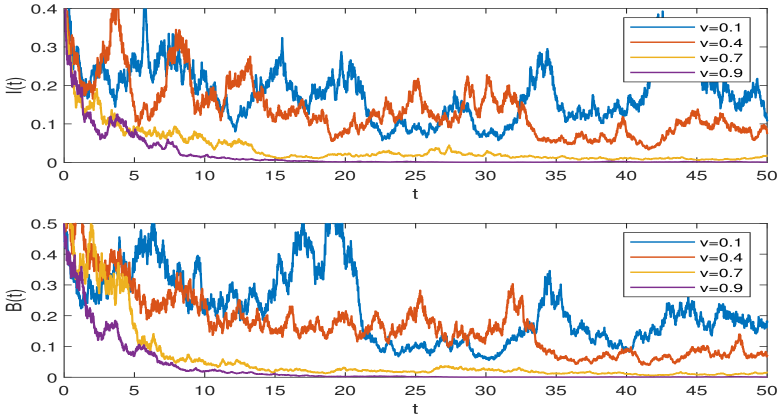

Figure 6. The influence of system (2.3) under the different v, the blue lines, red lines, yellow lines and green lines indicate the change of $I(t)$ and $B(t)$ when $v = 0.1,\ v = 0.4,\ v = 0.7,\ v = 0.9$, respectively.

a suitable Lyapunov function. In addition, we verify this stochastic system has a stationary distribution when

$$R^* > 1 \text{ and } r \geq \sigma_4^2,$$

illustrated in Figure 1. And the $I(t)$ and $B(t)$ of the diseases will be extinction when $R_0 < 1$ illustrated in Figure 2. In particular, for the systems (2.1) and (2.3), parameters α, d, v in those models have a great influence on the disease when other parameters with suitable values, illustrated in Figure 3 to Figure 6.

The conclusions we obtained in the paper enrich the research of the deterministic one and provide us new insights in controlling the spread of the diseases. For complex systems, constructing a suitable Lyapunov function is a difficult step in this paper and we discuss the local stability of the endemic equilibrium by Lyapunov function method which avoids the tedious calculation process when studying the local stability of the positive solution of the high-dimensional model.

Data availability statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Statements and declarations. The authors declare that they have no conflict of interest.

Appendix A

In this part, we investigate the existence and uniqueness of global positive solutions for model (2.4), which can be compared with the Section 4 in the article.

Theorem 8.1. *For any initial value* $(\beta_2(0), S(0), I(0), B(0), M(0)) \in \mathbb{R}_+^5$*, the system (2.4) has a unique global positive solution* $(\beta_2(t), S(t), I(t), B(t), M(t)) \in \mathbb{R}_+^5$ *for all* $t \geq 0$.

Proof. Here we give the construction of C^2-function $F(\mathbb{R}_+^5 \to \mathbb{R})$ bases on the specificity of $M(t)$:

$$\begin{aligned} F(\beta_2, S, I, B, M) =& (S - 1 - \ln S) + (I - 1 - \ln I) + (B - 1 - \ln B) \\ &+ \left(\frac{1}{M} - 1 + \ln M\right) + (\beta_2 - 1 - \ln \beta_2). \end{aligned}$$

We can explain that the function is nonnegative. Applying Itô$'$s formula to U, we have

$$\begin{aligned} LF =& \left(1 - \frac{1}{S}\right)\left(\Lambda - \left(\beta_2 - \beta_1 \frac{k_1 M}{p + k_1 M}\right) SI - \eta \frac{B}{L + B} S + vI - dS\right) \\ &+ \left(1 - \frac{1}{I}\right)\left(\left(\beta_2 - \beta_1 \frac{k_1 M}{p + k_1 M}\right) SI + \eta \frac{B}{L + B} S - (v + \alpha + d) I\right) \\ &+ \left(1 - \frac{1}{B}\right)\left(\phi_1 I + \phi B - \phi_0 B - \Phi \frac{(1 - k_1) M}{q + (1 - k_1) M} B\right) \\ &+ \left(\frac{1}{M} - \frac{1}{M^2}\right)\left(r\left(1 - \frac{M}{K}\right) M + \theta I M\right) + \beta_2 \left(\omega \log \beta - \omega \log \beta_2 + \frac{1}{2}\sigma_5^2\right) \\ &- \omega(\log \beta - \log \beta_2) \\ \leq& \Lambda + \eta + 2d + v + \alpha + \phi_0 + \Phi + \frac{r}{K} + r + \beta_1 S + (\phi_1 + \theta) I + \beta_2 I \\ &- \omega(\log \beta - \log \beta_2) + \beta_2 \left(\omega \log \beta - \omega \log \beta_2 + \frac{1}{2}\sigma_5^2\right). \end{aligned}$$

Define $N_0 = \max\left\{\frac{\Lambda}{d}, S(0) + I(0)\right\}$, thereby

$$S + I \leq N_0 = \begin{cases} \dfrac{\Lambda}{d}, & S(0) + I(0) \leq \dfrac{\Lambda}{d}, \\ S(0) + I(0), & S(0) + I(0) > \dfrac{\Lambda}{d}. \end{cases}$$

As a consequence,

$$\begin{aligned} \mathcal{L}F \leq& \Lambda + \eta + 2d + v + \alpha + \phi_0 + \Phi + \frac{r}{K} + r + (\beta_1 + \phi_1 + \theta) N_0 + \beta_2 N_0 \\ &- \omega(\log \beta - \log \beta_2) + \beta_2 \left(\omega \log \beta - \omega \log \beta_2 + \frac{1}{2}\sigma_5^2\right) \\ =& \Lambda + \eta + 2d + v + \alpha + \phi_0 + \Phi + \frac{r}{K} + r + (\beta_1 + \phi_1 + \theta) N_0 + f(\beta_2), \end{aligned}$$

where $f(\beta_2) = \beta_2 N_0 - \omega(\log\beta - \log\beta_2) + \beta_2\left(\omega\log\beta - \omega\log\beta_2 + \frac{1}{2}\sigma_5^2\right)$ and $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = -\infty$, so it is a function with a supremum.

$$\mathcal{L}F \le \Lambda + \eta + 2d + v + \alpha + \phi_0 + \Phi + \frac{r}{K} + r + (\beta_1 + \phi_1 + \theta)N_0 + \sup_{\beta_2\in\mathbb{R}_+}\{f(\beta_2)\} := \bar{M}.$$

□

- And then we can define the feasible region Γ of the stochastic system:

$$\Gamma = \left\{(\beta_2, S, I, B, M) \in \mathbb{R}_+^5 :\ S + I \le \frac{\Lambda}{d},\ B \le \frac{\phi_1\Lambda}{d(\phi_0 - \phi)},\ M \le \frac{K}{r}\left(r + \frac{\theta\Lambda}{d}\right)\right\}.$$

Appendix B

In this part, we investigate the extinction of system (2.4), which can be compared with the Section 6 in the article.

Theorem 8.2. *Let $(\beta_2(t), S(t), I(t), B(t), M(t))$ be the solution of the system (2.4) with any initial value $(\beta_2(0), S(0), I(0), B(0), M(0)) \in \Gamma$. If $R_0^s = R_0 + \frac{\Lambda}{d(v+\alpha+d)}\beta(e^{\frac{\sigma_5^2}{\omega}} - 2e^{\frac{\sigma_5^2}{4\omega}} + 1)^{\frac{1}{2}} < 1$, then the disease will be extinct.*

Proof. We first define function $G(t) = I(t) + c_1B(t) + c_2B(t)$ and $c_1 = \frac{\eta\Lambda}{dL(\phi_0-\phi)}$, $c_2 = \frac{1}{2\phi_1}(v + \alpha + d)(1 - R_0^s)$, with the Itô′s formula and $R_0^s < 1$, we have,

$$\begin{aligned}
L\ln G =& \frac{\left(\beta_2 - \beta_1\frac{k_1M}{p+k_1M}\right)SI + \eta\frac{B}{L+B}S - (v+\alpha+d)I + (c_1+c_2)\left(\phi_1 I + (\phi-\phi_0)B\right)}{I + (c_1+c_2)B}\\
&- \frac{(c_1+c_2)\Phi\frac{(1-k_1)M}{q+(1-k_1)M}B}{I + (c_1+c_2)B}\\
\le& \frac{\beta_2 SI + \frac{\eta}{L}BS - (v+\alpha+d)I + (c_1+c_2)\phi_1 I - (c_1+c_2)(\phi_0-\phi)B}{I + (c_1+c_2)B}\\
\le& \frac{(\beta_2\frac{\Lambda}{d} + c_1\phi_1 - (v+\alpha+d))I + (\frac{\eta\Lambda}{Ld} - (c_1+c_2)(\phi_0-\phi))B + c_2\phi_1 I}{I + (c_1+c_2)B}\\
\le& -\frac{\frac{1}{2}(v+\alpha+d)(1-R_0)I - c_2(\phi_0-\phi)B}{I + (c_1+c_2)B} + \frac{\Lambda}{d} \mid \beta - \beta_2 \mid\\
\le& -\frac{1}{2}(v+\alpha+d)(1-R_0)\min\{1, \frac{\phi_0-\phi}{\phi_1(c_1+c_2)}\} + \frac{\Lambda}{d} \mid \beta - \beta_2 \mid .
\end{aligned}$$

Integrating above equation from 0 to t on both sides, one can see

$$\begin{aligned}
\frac{\ln G(t)}{t} \le& \frac{\ln G(0)}{t} - \frac{1}{2}(v+\alpha+d)(1-R_0)\min\{1, \frac{\phi_0-\phi}{\phi_1(c_1+c_2)}\}\\
&+ \frac{\Lambda}{td}\int_0^t \mid \beta - \beta_2(m) \mid dm.
\end{aligned} \tag{8.1}$$

On account of a equation: $\lim_{t\to\infty}\frac{1}{t}\int_0^t |\beta_2(m) - \beta| dm \le \beta(e^{\frac{\sigma_5^2}{\omega}} - 2e^{\frac{\sigma_5^2}{4\omega}} + 1)^{\frac{1}{2}}$, then we take the upper limit on both sides of the above equation (8.1),

$$\limsup_{t\to\infty}\frac{\ln G(t)}{t} \le -\frac{1}{2}(v+\alpha+d)(1-R_0^s)\min\{1, \frac{\phi_0-\phi}{\phi_1(c_1+c_2)}\} \text{ a.s.},$$

where $R_0^s = R_0 + \frac{\Lambda}{d(v+\alpha+d)}\beta(e^{\frac{\sigma_5^2}{\omega}} - 2e^{\frac{\sigma_5^2}{4\omega}} + 1)^{\frac{1}{2}} < 1$. If $R_0^s < 1$, we can have

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} < 0, \ \limsup_{t\to\infty} \frac{\ln B(t)}{t} < 0,$$

which shows that

$$\lim_{t\to\infty} I(t) = 0, \ \lim_{t\to\infty} B(t) = 0.$$

That is to say, when $R_0^s < 1$, the $I(t), B(t)$ of the disease will tend to extinction. □

Appendix C

In this part, we investigate the stationary distribution of system (2.4), which can be compared with the Section 5 in the article.

Theorem 8.3. *Let us suppose* $R_1^s > 1$, $\varphi > 0$*, then the system (2.4) has a stationary distribution in* Γ*, where*

$$\begin{aligned} R_1^s =& \frac{e^{\frac{\sigma_5^2}{12\omega}}\Lambda\beta p}{(k_1K+p)d(v+\alpha+d)} + \frac{\Lambda(e^{\frac{\sigma_5^2}{16\omega}}\beta - \beta_1)k_1K}{(k_1K+p)d(v+\alpha+d)} \\ &+ \frac{\Lambda\eta\phi_1}{dL(\phi_0 - \phi + \Phi\frac{(1-k_1)K}{q+(1-k_1)K})(v+\alpha+d)}, \\ \varphi =& \frac{1}{2}\left(v+\alpha+d\right)\left(R_1^s - 1\right). \end{aligned}$$

Proof. We prove the existence of stationary distribution according to Lemma 2.3. Firstly, we define a C^2-function $\tilde{H}(\beta_2, S, I, B, M) = WH_1 + H_2$, where

$$\begin{aligned} H_1 =& -(b_1+b_2+b_3)\ln S - \ln I - b_4\ln B + b_5\frac{B}{\phi_0 - \phi} + (b_6+b_7+b_9)\ln M + b_8\frac{1}{M} \\ &+ b_{10}B, \\ H_2 =& -\ln S - \ln B + \frac{1}{M} + \beta_2 - 1 - \ln\beta_2, \end{aligned}$$

and the parameters W, and b_i, $i = 1, 2, \ldots, 10$ are given at soon. We can know that the function $\tilde{H}$ exists a minimum value $\tilde{H}_{\min}$, then the C^2-function H

$$H(\beta_2, S, I, B, M) = \tilde{H}(\beta_2, S, I, B, M) - \tilde{H}_{\min} \geq 0.$$

With the Itô$'$s formula, we obtain

$$\begin{aligned} LH_1 =& (b_1+b_2+b_3)\left(-\frac{\Lambda}{S} + \left(\beta_2 - \beta_1\frac{k_1M}{p+k_{1M}}\right)I + \eta\frac{B}{L+B} - v\frac{I}{S} + d\right) \\ &- \left(\frac{\beta_2 p}{p+k_1M}S + \frac{(\beta_2-\beta_1)k_1M}{p+k_1M}S + \eta\frac{B}{L+B}\frac{S}{I} - v - \alpha - d\right) \\ &- b_8\left(\frac{r}{M} + \frac{\theta I}{M} - \frac{r}{K}\right) + b_4\left(-\phi_1\frac{I}{B} + \Phi\frac{(1-k_1)M}{q+(1-k_1)M} + \phi_0 - \phi\right) \end{aligned}$$

$$
\begin{aligned}
&+ b_{10}\left(\phi_1 I - (\phi_0-\phi)B - \Phi\frac{(1-k_1)M}{q+(1-k_1)M}B\right)\\
&+ b_5\left(\frac{\phi_1}{\phi_0-\phi}I - (L+B) + L - \frac{\Phi}{\phi_0-\phi}\frac{(1-k_1)M}{q+(1-k_1)M}B\right)\\
&+ (b_6+b_7+b_9)\left(r - r\frac{M}{K} + \theta I\right)\\
\le & -\left(b_1\frac{\Lambda}{S} + \frac{\beta_2 p}{p+k_1M}S + b_6 r\frac{p+k_1M}{k_1K}\right) + b_1 d + b_6\left(r+\frac{rp}{k_1K}\right)\\
&-\left(b_2\frac{\Lambda}{S} + \frac{(\beta_2-\beta_1)k_1M}{p+k_1M}S + b_7 r\frac{p+k_1M}{k_1K} + b_8\frac{r}{M}\right) + b_2 d + b_8\frac{r}{K}\\
&+ b_7\left(r+\frac{rp}{k_1K}\right) - \left(b_3\frac{\Lambda}{S} + \eta\frac{B}{L+B}\frac{S}{I} + b_4\frac{\phi_1 I}{B} + b_5(L+B)\right) + b_3 d\\
&+ b_4\Phi\frac{(1-k_1)M}{q+(1-k_1)M} - b_9 r\frac{M}{K} + b_9 r\\
&+ b_4\left(\phi_0-\phi\right) + b_5 L + (v+\alpha+d) + (b_1+b_2+b_3)\left(\beta_2 I + \eta\frac{B}{L}\right) + b_5\frac{\phi_1}{\phi_0-\phi}I\\
&+ (b_6+b_7+b_9)\theta I + b_{10}\phi_1 I - b_{10}(\phi_0-\phi)B.
\end{aligned}
$$

Let

$$
\begin{aligned}
&b_1 = \frac{e^{\frac{\sigma_5^2}{12\omega}}\Lambda\beta p}{(k_1K+p)d^2},\ b_6 = \frac{e^{\frac{\sigma_5^2}{12\omega}}\Lambda\beta p k_1 K}{(k_1K+p)^2 dr},\ b_2 = \frac{(e^{\frac{\sigma_5^2}{16\omega}}\beta-\beta_1)\Lambda k_1 K}{(k_1K+p)d^2},\\
&b_8 = \frac{\Lambda(e^{\frac{\sigma_5^2}{16\omega}}\beta-\beta_1)k_1K^2}{rd(k_1K+p)},\ b_3 = \frac{\Lambda\eta\phi_1}{d^2L(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K})},\\
&b_5 = \frac{\Lambda\eta\phi_1}{dL^2(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K})},\ b_9 = b_4\Phi\frac{(1-k_1)Kq}{r(q+(1-k_1)K)^2},\ b_{10} = \frac{(b_1+b_2+b_3)\eta}{L(\phi_0-\phi)},\\
&b_4 = \frac{\Lambda\eta\phi_1}{dL(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K})^2},\ b_7 = \frac{\Lambda(e^{\frac{\sigma_5^2}{16\omega}}\beta-\beta_1)k_1^2K^2}{(k_1K+p)^2dr}.
\end{aligned}
$$

Due to the function

$$
h(M) = b_4\Phi\frac{(1-k_1)M}{q+(1-k_1)M} - b_9 r\frac{M}{K} + b_9 r \le h(K) = b_4\Phi\frac{(1-k_1)K}{q+(1-k_1)K},
$$

and $(b_1+b_2+b_3)\xi\frac{\Lambda}{d}e^{\frac{\sigma_5^2}{\omega}}\beta^2 = \frac{1}{2}(R_1^s-1)(v+\alpha+d)$,

$$
\begin{aligned}
h_1(\beta_2) = {}& (b_1+b_2+b_3)\xi\frac{\Lambda}{d}\left(\beta_2^2 - e^{\frac{\sigma_5^2}{\omega}}\beta^2\right) + 3\left(\frac{b_1b_6\Lambda pr}{k_1K}\right)^{\frac{1}{3}}\left(e^{\frac{\sigma_5^2}{12\omega}}\beta^{\frac{1}{3}} - \beta_2^{\frac{1}{3}}\right)\\
&+ 4\left(\frac{b_2b_7b_8\Lambda r^2}{K}\right)^{\frac{1}{4}}\left((e^{\frac{\sigma_5^2}{16\omega}}\beta-\beta_1)^{\frac{1}{4}} - (\beta_2-\beta_1)^{\frac{1}{4}}\right),
\end{aligned}
$$

we obtain

$$
LH_1 \le -3\left(\frac{b_1b_6\Lambda\beta_2 pr}{k_1K}\right)^{\frac{1}{3}} + b_1 d + b_6\left(r+\frac{rp}{k_1K}\right) - 4\left(\frac{b_2b_7b_8\Lambda(\beta_2-\beta_1)r^2}{K}\right)^{\frac{1}{4}}
$$

$$
\begin{aligned}
&+ b_2 d + b_7\left(r + \frac{rp}{k_1K}\right)\\
&+ b_8\frac{r}{K} - 4(b_3b_4b_5\Lambda\eta\phi_1)^{\frac{1}{4}} + b_3 d + b_4\left(\Phi\frac{(1-k_1)K}{q+(1-k_1)K} + \phi_0 - \phi\right) + b_5 L\\
&+ (v+\alpha+d) + \left((b_1+b_2+b_3)\beta_2 + b_5\frac{\phi_1}{\phi_0-\phi} + (b_6+b_7+b_9)\theta + b_{10}\phi_1\right) I\\
\le &- 3\left(\frac{b_1b_6e^{\frac{\sigma_5^2}{4\omega}}\Lambda\beta pr}{k_1K}\right)^{\frac{1}{3}} + b_1 d + b_6\left(r + \frac{rp}{k_1K}\right) - 4\left(\frac{b_2b_7b_8\Lambda(e^{\frac{\sigma_5^2}{4\omega}}\beta - \beta_1)r^2}{K}\right)^{\frac{1}{4}}\\
&+ b_2 d + b_7\left(r + \frac{rp}{k_1K}\right) + b_8\frac{r}{K} - 4(b_3b_4b_5\Lambda\eta\phi_1)^{\frac{1}{4}} + b_3 d + b_5 L\\
&+ b_4\left(\Phi\frac{(1-k_1)K}{q+(1-k_1)K} + \phi_0 - \phi\right) + (v+\alpha+d)\\
&+ \left(\frac{b_1+b_2+b_3}{4\xi} + b_5\frac{\phi_1}{\phi_0-\phi} + (b_6+b_7+b_9)\theta + b_{10}\phi_1\right) I + (b_1+b_2+b_3)\xi\frac{\Lambda}{d}e^{\frac{\sigma_5^2}{\omega}}\beta^2\\
&+ (b_1+b_2+b_3)\xi\frac{\Lambda}{d}\left(\beta_2^2 - e^{\frac{\sigma_5^2}{\omega}}\beta^2\right) + 3\left(\frac{b_1b_6\Lambda pr}{k_1K}\right)^{\frac{1}{3}}\left(e^{\frac{\sigma_5^2}{12\omega}}\beta^{\frac{1}{3}} - \beta_2^{\frac{1}{3}}\right)\\
&+ 4\left(\frac{b_2b_7b_8\Lambda r^2}{K}\right)^{\frac{1}{4}}\left((e^{\frac{\sigma_5^2}{4\omega}}\beta - \beta_1)^{\frac{1}{4}} - (\beta_2 - \beta_1)^{\frac{1}{4}}\right)\\
= &- \left(\frac{e^{\frac{\sigma_5^2}{12\omega}}\Lambda\beta p}{(k_1K+p)d} + \frac{\Lambda(e^{\frac{\sigma_5^2}{4\omega}}\beta - \beta_1)k_1K}{(k_1K+p)d} + \frac{\Lambda\eta\phi_1}{dL(\phi_0 - \phi + \Phi\frac{(1-k_1)K}{q+(1-k_1)K})}\right)\\
&+ (v+\alpha+d) + AI + h_1(\beta_2) + (b_1+b_2+b_3)\xi\frac{\Lambda}{d}e^{\frac{\sigma_5^2}{\omega}}\beta^2\\
= &- \frac{1}{2}(v+\alpha+d)(R_1^s - 1) + AI + h_1(\beta_2)\\
:= &- \varphi + AI + h_1(\beta_2),
\end{aligned}
$$

where

$$
A = \frac{b_1+b_2+b_3}{4\xi} + b_5\frac{\phi_1}{\phi_0-\phi} + (b_6+b_7+b_9)\theta + b_{10}\phi_1,\ \ \varphi = \frac{1}{2}(v+\alpha+d)(R_1^s - 1).
$$

In the same way, one can get

$$
\begin{aligned}
LH_2 = &- \frac{\Lambda}{S} + \left(\beta_2 - \beta_1\frac{k_1M}{p+k_1M}\right) I + \eta\frac{B}{L+B} - v\frac{I}{S} + d - \phi_1\frac{I}{B} + \phi_0 - \phi\\
&+ \Phi\frac{(1-k_1)M}{q+(1-k_1)M} - \frac{r}{M} - \theta\frac{I}{M} + \frac{r}{K} + \beta_2\left(\omega\log\beta - \omega\log\beta_2 + \frac{1}{2}\sigma_5^2\right)\\
&- \omega(\log\beta - \log\beta_2)\\
\le &- \frac{\Lambda}{S} - \phi_1\frac{I}{B} - \frac{r}{M} + \eta + d + \Phi + \phi_0 + \frac{r}{K} + \beta_2\left(\omega\log\beta - \omega\log\beta_2 + \frac{1}{2}\sigma_5^2 + \frac{\Lambda}{d}\right)\\
&- \omega(\log\beta - \log\beta_2).
\end{aligned}
$$

Let $h_2(\beta_2) = \beta_2\left(\omega \log \beta - \omega \log \beta_2 + \frac{1}{2}\sigma_5^2 + \frac{\Lambda}{d}\right) - \omega(\log \beta - \log \beta_2)$. Therefore, it can be seen that

$$\begin{aligned}LH \le& - W\varphi + WAI - \frac{\Lambda}{S} - \phi_1\frac{I}{B} - \frac{r}{M} + \eta + d + \Phi + \phi_0 + \frac{r}{K} + Wh_1(\beta_2) + h_2(\beta_2)\\ \le& - W\varphi + WAI - \frac{\Lambda}{S} - \phi_1\frac{I}{B} - \frac{r}{M} + Wh_1(\beta_2) + C\\ :=& \, g(\beta_2, S, I, B, M) + Wh_1(\beta_2),\end{aligned}$$

where

$$C = \eta + d + \Phi + \phi_0 + \frac{r}{K} + \sup_{\beta_2 \in \mathbb{R}_+}\{h_2(\beta_2)\}.$$

There is a large enough M to satisfy the inequality,

$$-W\varphi + C \le -2. \tag{8.2}$$

And

$$D = \left\{(\beta_2, S, I, B, M) \in \mathbb{R}_+^5 : \epsilon \le \beta_2 \le \frac{1}{\epsilon}, \epsilon \le S, \epsilon \le I, \epsilon^2 \le B, \epsilon \le M\right\},$$

is a bounded closed set, where $0 < \epsilon < 1$ is sufficiently small. With (8.2) we see

$$g(\beta_2, S, I, B, M) \le \begin{cases} -2 + WA\dfrac{\Lambda}{\mu} < -1, \text{ as } \beta_2 \to 0^+ \text{ or } \beta_2 \to \infty,\\ -2 + WA\dfrac{\Lambda}{\mu} - \frac{\Lambda}{\epsilon} \to -\infty, \text{ as } S \to 0^+,\\ -2 + WA\epsilon < -1, \text{ as } I \to 0^+,\\ -2 + WA\dfrac{\Lambda}{\mu} - \frac{\phi_1}{\epsilon} \to -\infty, \text{ as } B \to 0^+ \text{ and } I \nrightarrow 0^+,\\ -2 + WA\dfrac{\Lambda}{\mu} - \frac{r}{\epsilon} \to -\infty, \text{ as } M \to 0^+ \text{ and } I \nrightarrow 0^+.\end{cases}$$

In the final analysis, for a sufficient small ϵ, $g(\beta_2, S, I, B, M) \le -1$ for any $(\beta_2, S, I, B, M) \in \Gamma \setminus D$. So we can easy to see that for any $(\beta_2, S, I, B, M) \in \Gamma$, $g(\beta_2, S, I, B, M) \le C_1$, which C_1 is a positive constant.

In the same instant,

$$\begin{aligned}0 \le& \frac{\mathbb{E}(H(t))}{t}\\ =& \frac{\mathbb{E}(H(0))}{t} + \frac{1}{t}\int_0^t \mathbb{E}(\mathcal{L}H(m))dm\\ \le& \frac{\mathbb{E}(H(0))}{t} + \frac{1}{t}\int_0^t \mathbb{E}(g(m))dm + \frac{W}{t}\int_0^t \mathbb{E}(h_1(\beta_2(m)))dm,\end{aligned}$$

and by Lemma 2.4, $\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_2^{\frac{1}{n}}(m)dm = e^{\frac{\sigma_5^2}{4n^2\omega}}\beta$ and $\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_2^2(m)dm = e^{\frac{\sigma_5^2}{\omega}}\beta^2$. In this reason,

$$0 \le \liminf_{t\to\infty} \frac{\mathbb{E}(H(0))}{t} + \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}(g(m))dm$$

$$
\begin{aligned}
&= \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}(g(m))dm \\
&= \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(\mathbb{E}(g(m))\mathbf{1}_{(\beta_2,S,I,B,M)\in D} + \mathbb{E}(g(m))\mathbf{1}_{(\beta_2,S,I,B,M)\in D^c}\right) dm \\
&\le \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(C_1\mathbb{P}_D - \mathbb{P}_{D^c}\right) dm \\
&\le -1 + (1+C_1)\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}_D dm.
\end{aligned}
$$

The moral is that

$$
0 < \frac{1}{1+C_1} \le \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}_D dm,
$$

where $\mathbb{P}_D$ is the transition probability of $(\beta_2, S, I, B, M) \in D$. Consequently, the stochastic system (3.1) on Γ has a stationary distribution by the Lemma 2.3. This completes the proof. □

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