

EXTENDED FISHER-KOLMOGOROV EQUATION WITH SINGULARITIES IN THE COMPLEX DOMAIN

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Abstract In the article, the extended nonlinear differential Fisher-Kolmogorov equation in the complex domain is considered. Its stationary solutions are investigated for existence of a movable singular point of the algebraic type. The theorem of the existence of a movable singular point, and uniqueness of a solution in the neighborhood of such a point is proved. An analytical approximate solution is constructed, and the error of the found solution is estimated in analytical form, its numerical study is carried out. The absence of such a singular point in the real domain is demonstrated. A special approach for optimizing the error is proposed.

Keywords Extended Fisher-Kolmogorov equation, movable singular point, a priori estimate, analytical approximate solution.

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1. Introduction and basic equations

The extended Fisher-Kolmogorov equation (EFK)

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad (1.1)$$

was proposed as a higher-order model for bistable physical systems [15,31]. This equation occurs in many problems of physics, chemistry and biology [2,14–16,23,31,36–38].

The classical case ($\gamma = 0$) is also interesting. This equation arises in numerous applications in chemical kinetics and population dynamics; a classic example is sexual reproduction, where the reproduction rate is proportional to the square of the population density [39]. Its extensions are employed in diverse fields such as cancer modeling, the analysis of propagating crystallization and polymerization fronts, and geochemistry [4]. Furthermore, equations of type (1.1) are used to study a broad class of auto wave processes in active media, with applications spanning combustion and explosion theory, nuclear reactor theory, neurophysiology, cardiology, heat and mass transfer theory [8].

For stationary solutions, Eq. (1.1) reduces to an ordinary differential equation

$$-\gamma u'''' + u'' - u^3 + u = 0, \quad (1.2)$$

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the primes denote differentiation with respect to variable x . This ordinary differential equation is interesting by itself. For example, the nonlinear Schrödinger equation [9]

$$i \frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial t^2} - \frac{\partial^4 v}{\partial t^4} + |v|^2 v = 0$$

has the structure similar to Eq. (1.2). Equation (1.2) has been extensively studied in different approaches: By variational method, topological shooting method, methods of qualitative and asymptotic theory of differential equations, Hamiltonian method, Chebyshev wavelet collocation method, B-spline method, far-field/core decompositions and other [1, 3, 5, 6, 10–12, 21, 22, 24, 25, 31–35, 40].

It is known that for nonlinear differential equations and their solutions may exist singular points of different types. The issues of existence and uniqueness of the solution of Eq. (1.2) were considered in [6, 7, 18–20, 32].

In the present article, we will study behaviour of solutions of Eq. (1.2) in the complex domain. In particular, conditions for existence or absence of the movable singular points in solutions of Eq. (1.2), will be found. The obtained results are verified by the numerical study.

2. The basic method and main results

Let us consider Eq. (1.2) in the form

$$u'''' + au'' + u^3 - u = 0, \quad (2.1)$$

imposing the initial conditions

$$u(z_0) = u_0, \quad u'(z_0) = u_1, \quad u''(z_0) = u_2, \quad u'''(z_0) = u_3, \quad (2.2)$$

where a is a real-valued parameter. Below we will use the method described in [26–28], which is based on the modified Cauchy majorant method and the theory of generalized power series [17].

Assuming that the solutions of Eq. (2.1) have a movable singular point of algebraic type [13, 19, 20], we will prove existence and uniqueness of the solution of the problem (2.1), (2.2) in the form

$$u(z) = (z - z^*)^\rho \sum_{n=0}^{\infty} C_n (z - z^*)^n, \quad (2.3)$$

where z^* stands for the movable singular point, ρ is a parameter whose value will be determined below. The structure (2.3) means existence of a movable singular point z^* of algebraic type. The domain of existence of such solutions is divided into the analyticity domain and the neighborhood of the movable singular point. Below we will study behaviour of solutions only in the neighborhood of the movable singular point, and prove the theorem which specifies the structure of the solutions near the movable singular point.

Theorem 2.1. *Let z^* be a movable singular point of the problem (2.1)-(2.2), then there exists a unique solution of the form (2.3) in the domain*

$$|z - z^*| < \frac{1}{|1 + a + \lambda + C_0|}, \quad (2.4)$$

where a is the parameter of the Eq. (2.1), λ is a free parameter of the solution.

Proof. After substituting the structure (2.3) into Eq. (2.1), we arrive at

$$\begin{aligned} & \sum_{n=0}^{\infty} (n + \rho)(n + \rho - 1)(n + \rho - 2)(n + \rho - 3)C_n(z - z^*)^{n+\rho-4} \\ &= -a \sum_{n=0}^{\infty} (n + \rho - 2)(n + \rho - 3)C_n(z - z^*)^{n+\rho-2} \\ & \quad - \sum_{n=0}^{\infty} C_n^{**}(z - z^*)^{n+3\rho} + \sum_{n=0}^{\infty} C_n(z - z^*)^{n+\rho}, \end{aligned} \tag{2.5}$$

where the notations are used

$$C_n^* = \sum_{i=0}^n C_i C_{n-i}, \quad C_n^{**} = \sum_{i=0}^n C_i^* C_{n-i}.$$

We require that the quantities $n + \rho - 4$ and $n + 3\rho$ be equal, this leads to

$$\rho = -2.$$

Then we obtain the recurrence relations for coefficients C_n :

For $n = 0$, from (2.5) it follows

$$120C_0 = -C_0^3 \Rightarrow C_0 = 0 \text{ or } C_0^2 = -120.$$

The value $C_0 = 0$ is of no interest, since it leads to the trivial solution. So, we consider the values

$$C_0 = \pm 2i\sqrt{30},$$

these two values generate two branches of the solution.

First consider the case $C_0 = 2i\sqrt{30}$. The case $C_0 = -2i\sqrt{30}$ is treated in a similar way. As C_0 are imaginary, the real solutions with the structure (2.3) do not exist.

For $n = 1$, we obtain

$$24C_1 = -C_1^{**} \text{ or } 24C_1 = -3C_0^2C_1 \Rightarrow C_1 = 0.$$

For $n = 2$, we get

$$0 = -6aC_0 + C_2^{**} \Rightarrow C_2 = -\frac{2a}{C_0}.$$

In a similar way, we find

$$C_3 = 0, \quad C_4 = \frac{1 - 3C_2^2}{3C_0}, \quad C_5 = 0.$$

For $n \geq 6$, the general recurrence formula has the structure

$$(n - 2)(n - 3)(n - 4)(n - 5)C_n = -a(n - 4)(n - 5)C_{n-2} - C_n^{**} + C_{n-4}. \tag{2.6}$$

From Eq. (2.6), when $n = 6, 7$ we obtain

$$C_6 = \frac{C_2 - C_2^3 - 2aC_4 - 6C_0C_2C_4}{24 + 3C_0^2}, \quad C_7 = 0.$$

When $n = 8$, from Eq. (2.6) it follows

$$C_8 = \frac{C_4 - 3C_4C_2^2 - 6C_0C_2C_6 - 3C_0C_4^2 - 12aC_6}{3(C_0^2 + 120)}. \tag{2.7}$$

Substituting the values of C_0, C_2, C_4, C_6 into (2.7), we get

$$0 \times C_8 = 0, \quad \text{let } C_8 = \lambda,$$

where λ is a free parameter through which the all non-vanishing coefficients C_{2k} with even indices are expressed.

At odd indices $n = 2k + 1$, from Eq. (2.6) we get (allowing for the structure of C_{2k+1}^{***})

$$C_{2k+1} = 0, \quad k = 0, 1, 2, \dots$$

For C_{2n+2} , from Eq. (2.6) we obtain

$$2n(2n - 1)(2n - 2)(2n - 3)C_{2n+2} = -a(2n - 1)(2n - 2)C_{2n} - C_{2n+2}^{**} + C_{2n-2},$$

whence it follows

$$\begin{aligned} & \left(2n(2n - 1)(2n - 2)(2n - 3) + 3C_0^2\right)C_{2n+2} \\ &= -a(2n - 1)(2n - 2)C_{2n} - \sum_{i=1}^n \left(\sum_{l=1}^i C_{2l} \times C_{2i+2-2l}\right)C_{2n+2-2i} \\ & \quad - C_0 \sum_{k=1}^n C_{2k} \times C_{2n+2-2k} + C_{2n-2}. \end{aligned} \tag{2.8}$$

Taking in mind for the above analysis of the coefficients C_0, \dots, C_8 , we can assume the following estimation for the coefficients C_{2n} :

$$|C_{2n}| \leq \frac{|1 + a + \lambda + C_0|^n}{(2n - 2)(2n - 3)(2n - 4)(2n - 5)}, \quad n \geq 4. \tag{2.9}$$

Let us prove this inequality. With the use of (2.8) we obtain

$$\begin{aligned} |C_{2n+2}| &\leq \frac{1}{2n(2n - 1)(2n - 2)(2n - 3) - 360} - a(2n - 2)(2n - 3) \\ &\quad \times \frac{|1 + a + \lambda + C_0|^n}{(2n - 2)(2n - 3)(2n - 4)(2n - 5) - 360} \\ &\quad - \sum_{i=1}^n \left(\sum_{l=1}^i \frac{|1 + a + \lambda + C_0|^n}{(2l - 2)(2l - 3)(2l - 4) - 360} \right. \\ &\quad \left. \times \frac{|1 + a + \lambda + C_0|^{i-l}}{(2i - 2l - 2)(2i - 2l - 3)(2i - 2l - 4)(2i - 2l - 5) - 360} \right) \\ &\quad \times \frac{|1 + a + \lambda + C_0|^{n-i}}{(2n - 2i - 2)(2n - 2i - 3)(2n - 2i - 4)(2n - 2i - 5) - 360} \\ &\quad - C_0 \times \sum_{k=1}^n \frac{|1 + a + \lambda + C_0|^k}{(2k - 2)(2k - 3)(2k - 4) - 360} \end{aligned}$$

$$\begin{aligned} & \times \frac{|1 + a + \lambda + C_0|^{n-k}}{(2n - 2k + 2)(2n - 2k + 1)(2n - 2k)(2n - 2k - 1) - 360} \\ & + \frac{|1 + a + \lambda + C_0|^{n-1}}{(2n - 4)(2n - 5)(2n - 6)(2n - 7) - 360}. \end{aligned}$$

By straightforward computations, we get

$$\begin{aligned} |C_{2n+2}| & \leq \frac{1}{2n(2n - 1)(2n - 2)(2n - 3) - 360} \\ & \times \left| -\frac{a|1 + a + \lambda + C_0|^n}{1} - |1 + a + \lambda + C_0|^n \right. \\ & \times \sum_{i=1}^n \left(\sum_{l=1}^i \frac{1}{(2l - 2)(2l - 3)(2l - 4)(2l - 5) - 360} \right. \\ & \times \left. \frac{1}{(2i - 2l - 2)(2i - 2l - 3)(2i - 2l - 4) - 360} \right) \\ & \times \frac{1}{(2n - 2i)(2n - 2i - 1)(2n - 2i - 2)(2n - 2i - 3) - 360} \\ & - C_0|1 + a + \lambda + C_0|^n \times \sum_{k=1}^n \frac{1}{(2k - 2)(2k - 3)(2k - 4)(2k - 5) - 360} \\ & \times \frac{1}{(2n - 2k)(2n - 2k - 1)(2n - 2k - 2)(2n - 2k - 3) - 360} \\ & \left. + \frac{|1 + a + \lambda + C_0|^{n-1}}{(2n - 4)(2n - 5)(2n - 6)(2n - 7) - 360} \right| \\ & \leq \frac{|1 + a + \lambda + C_0|^{n+1}}{2n(2n - 1)(2n - 2)(2n - 3) - 360}. \end{aligned}$$

Let us consider the series

$$\sum_{n=2}^{\infty} v_{2n} |z - z^*|^{2n-2}, \tag{2.10}$$

where

$$v_{2n} = \frac{|1 + a + \lambda + C_0|^n}{(2n - 2)(2n - 3)(2n - 4)(2n - 5) - 360}.$$

This series is majorizing one for the series

$$\sum_{n=2}^{\infty} C_{2n} |z - z^*|^{2n-2}. \tag{2.11}$$

Taking into account the convergence condition [19] for the series (2.10) in the domain

$$|z - z^*| < \frac{1}{\sqrt{|1 + a + \lambda + C_0|}} = r, \tag{2.12}$$

we conclude that the series (2.11) is absolutely convergent in the domain (2.12). □

Remark 2.1. From the structure of the solution (2.3) at $\rho = -2$, we conclude that here we have the movable singular point of the algebraic type (the singularity z^* is a pole of the second order).

Remark 2.2. In contrast, in the real domain, solutions of Eq. (2.1) do not have movable points of the algebraic type.

The results of Theorem 2.1 allow us to construct an analytical approximate solution for the problem (2.1)-(2.2) in the neighborhood of a movable singular point z^* in the following form

$$y_{2N}(z) = (z - z^*)^{-2} \sum_{n=0}^N C_{2n}(z - z^*)^{2n}. \tag{2.13}$$

Theorem 2.2. Let z^* be a movable singular point of the algebraic type of problem (2.1)-(2.2), then for the analytical approximate solution (2.13), in the domain (2.12) the following error estimate is valid

$$\begin{aligned} \Delta y_{2N}(z) &\leq \frac{|1 + a + \lambda + C_0|^{N+2} |z - z^*|^{2N}}{2N(2N - 1)(2N - 2)(2N - 3) - 360} \\ &\quad \times \frac{1}{1 - |1 + a + \lambda + C_0| |z - z^*|^2} \quad (N \in \mathbb{N}). \end{aligned} \tag{2.14}$$

Proof. Let us estimate the error between the exact and approximate solutions

$$\begin{aligned} \Delta y_{2N}(z) &= |y(z) - y_{2N}| \\ &= \left| \sum_{n=0}^{\infty} C_{2n}(z - z^*)^{2n-2} - \sum_{n=0}^N C_{2n}(z - z^*)^{2n-2} \right| \\ &\leq \left| \sum_{n=N+1}^{\infty} C_{2n}(z - z^*)^{2n-2} \right| \\ &\leq \sum_{n=N+1}^{\infty} \frac{|1 + a + \lambda + C_0|^n \times |z - z^*|^{2n-2}}{(2n - 2)(2n - 3)(2n - 4)(2n - 5) - 360} \\ &\leq \frac{|1 + a + \lambda + C_0|^{N+2} |z - z^*|^{2N}}{(2N(2N - 1)(2N - 2)(2N - 3) - 360)(1 - |1 + a + \lambda + C_0| |z - z^*|^2)}. \end{aligned}$$

Thus, an error estimate of the approximate solution is obtained. □

In the next section the obtained results will be verified by numerical simulation.

3. Numerical simulation

Let the parameter a be

$$a = 1. \tag{3.1}$$

Consider the initial problem (2.1)-(2.2):

$$\begin{aligned} z_0 = 1, \quad u(z_0) = u_0 = 0.5i, \quad u'(z_0) = u_1 = 0.5, \\ u''(z_0) = u_2 = 1 - 0.5i, \quad u'''(z_0) = u_3 = 0.5 + 0.5i. \end{aligned} \tag{3.2}$$

Let us find the value of the singular point using the algorithm given in [29, 30]. The algorithm for finding movable singular points is based on checking the necessary and sufficient conditions for the existence of a movable singular point of algebraic type [13], the theory of regularization of a movable singular point and introduction of two three-dimensional spaces with coordinates $(x, y, \Re(u))$ and $(x, y, \Im(u))$, characterizing the real and imaginary parts of the variable $z = x + iy$ and the function u . To find the movable singular point of problem (2.1)-(2.2), the second phase space with coordinate $\Im(z^*) = 0$ was fixed.

For the parameter (3.1) and the initial conditions (3.2), we find

$$z^* = 5.94276477612.$$

The values of the coefficients $C_n (n = 0, 2, 4, \dots, 18)$ are

$$\begin{aligned} C_0 &= 2\sqrt{30}i, \quad C_2 = \frac{i}{\sqrt{30}}, \quad C_4 = -\frac{11i}{60\sqrt{30}}, \quad C_6 = \frac{i}{420\sqrt{30}}, \quad C_8 = \lambda, \\ C_{10} &= -\frac{3\lambda}{220} + \frac{17i}{30240000\sqrt{30}}, \quad C_{12} = -\frac{\lambda}{9360} - \frac{89993i}{148599360000\sqrt{30}}, \\ C_{14} &= \frac{\lambda}{756000} + \frac{77i}{69984000000\sqrt{30}}, \\ C_{16} &= -\frac{i\sqrt{\frac{15}{2}}\lambda^2}{1972} + \frac{70387\lambda}{426377952000} - \frac{2584063i}{351645525504000000\sqrt{30}}, \\ C_{18} &= \frac{227i\sqrt{\frac{3}{10}}\lambda^2}{9212720} - \frac{76733\lambda}{20577866400000} + \frac{1135320931i}{15681321934387200000000\sqrt{30}}. \end{aligned} \tag{3.3}$$

In the 3D space $(z, \Re(\lambda), \Im(\lambda))$, we draw a region G , defined by the inequality (2.12), at $a = 1, C_0 = 2\sqrt{30}i$ (Figure 1).

For example, let it be

$$\lambda = 5 + 5i, \tag{3.4}$$

then the inequality (2.12) takes the form

$$|z_1 - z^*| < r = 0.239577. \tag{3.5}$$

Let us consider a point z_1 near the z^*

$$z_1 = 5.8, \tag{3.6}$$

which satisfies the inequality (3.5). Substituting (3.4), (3.6) and $N = 4$ into the formula (2.14), we obtain the inequality

$$\Delta_{y_{2N}}(z) \leq 5.66984 \cdot 10^{-3}.$$

In order to optimize this error estimate, we set an a posteriori error estimate Δ_1 of solution (2.13) as

$$\Delta_1 = 10^{-6}. \tag{3.7}$$

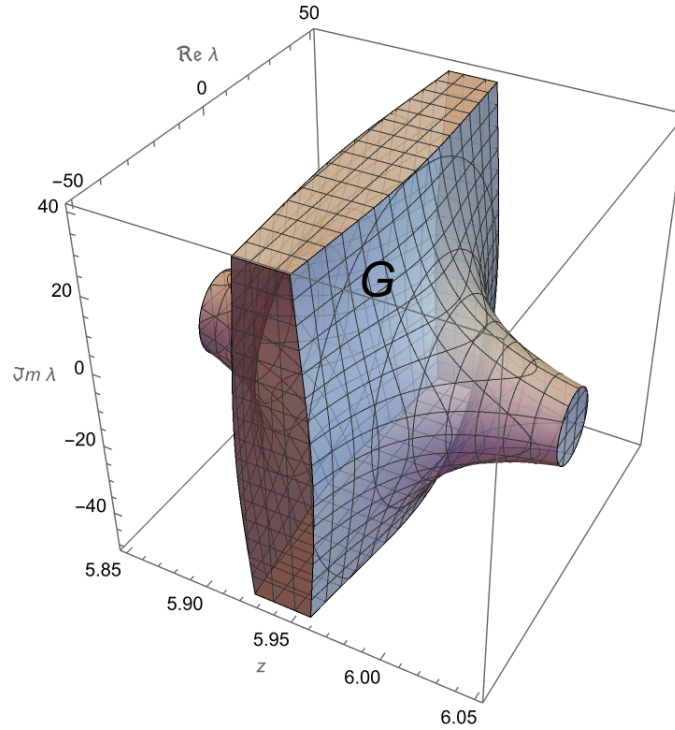


Figure 1. Set of points $(z, \Re(\lambda), \Im(\lambda))$, satisfying (2.12).

Substituting the values (3.4), (3.6), (3.7) into formula (2.14) we obtain an inequality for determining the value of N :

$$\frac{1.55063 \cdot 0.142765^{2N} \left(49 + (5 + 2\sqrt{30})^2\right)^{\frac{N+2}{2}}}{2N(2N - 3)(2N - 2)(2N - 1) - 360} \leq 10^{-6}.$$

The smallest number satisfying this inequality is $N = 9$. In the solution (2.13) we choose $N = 9$. Therefore, the desired solution has the form

$$y_{18}(z) = (z - z^*)^{-2} \sum_{n=0}^9 C_{2n}(z - z^*)^{2n}, \tag{3.8}$$

where the coefficients C_i ($i = \overline{0, 18}$) are determined by the formulas (3.3) and (3.4). In order to estimate the value of $\Delta y_8(z_1)$ at the point (3.6), we should have the inequalities

$$|y(z_1) - y_{18}(z_1)| \leq |y(z_1) - y_8(z_1)| + \left| \sum_{N=5}^9 C_{2N}(z_1 - z^*)^{2N} \right| \leq 10^{-6}. \tag{3.9}$$

In (3.9), the sum of terms from $N = 5$ to $N = 9$ does not exceed the required accuracy Δ_1 (this sum is less than $3.39204 \cdot 10^{-10}$). Consequently, the accuracy of the found value $y_8(z_1)$, according to the accepted estimate (3.7), satisfies the inequality $\Delta y_8(z_1) \leq 10^{-6}$. This posterior estimate can also be additionally refined. The results of the numerical experiment for $N = 4$ are given in Table 1.

Table 1. Results of the numerical experiment.

z_1	$y_8(z_1)$	$\Delta y_8(z_1)$	Δ_1
5.8	$2.92441 \cdot 10^{-5} + 537.645i$	$5.66984 \cdot 10^{-3}$	10^{-6}

Explanations.

The quantity $y_8(z_1)$ is the value of the analytical approximate solution (2.13) at point z_1 ; $\Delta y_8(z_1)$ is the a priori error estimate obtained from Theorem 2.2.

Figure 2 shows a graphical interpretation of the solution $\Im m(y_8)$ in 3-dimensional space $(x, y, \Im m(y_8))$. The movable singular point has coordinates $(x = 5.94276477612, y = 0)$. Theorems 2.1 and 2.2 are valid in the neighborhood of the point z^* with radius $r = 0.239577$.

Figure 3 shows the graph $\Im m(y_8(x))$, which is a projection of the surface shown in Figure 2 onto the plane $x = x, y = 0, z = \Im m(y_8)$. It is evident that in the neighborhood of the point z^* (second-order poles), the branches of the solution tend to infinity.

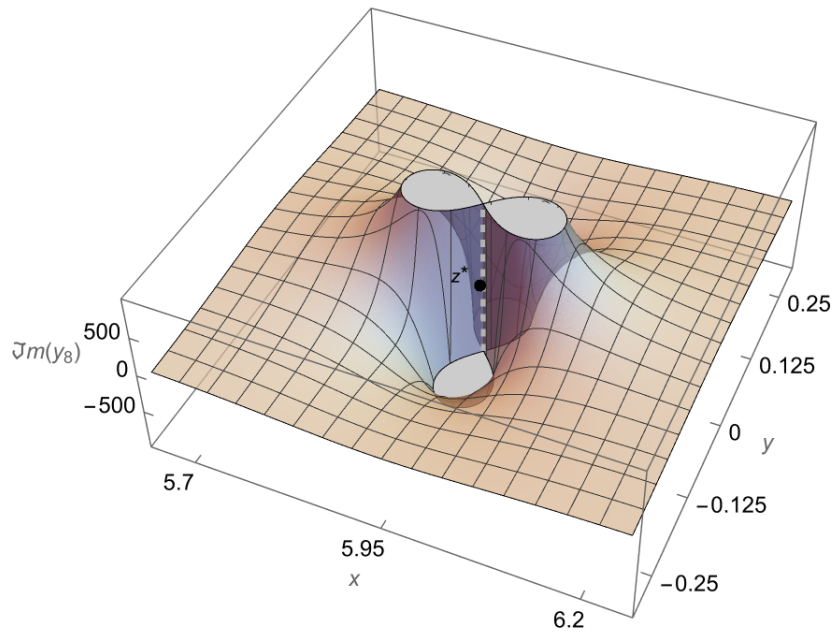


Figure 2. Solution $\Im m(y_8)$ in the complex neighborhood of the point z^* (black point).

4. Conclusions

The main results of the paper are summarized as follows:

A theorem of existence and uniqueness of the solution of the Fisher-Kolmogorov equation in the neighborhood of a movable singular point of algebraic type in a complex domain is proved.

It is shown that in the real domain this equation does not have movable singular points of the algebraic type.

A formula is given for calculating the domain in which the error estimates for the constructed analytical approximate solution are valid.

An analytical approximate solution is constructed in the neighborhood of a movable singular point and an a priori error estimate is found.

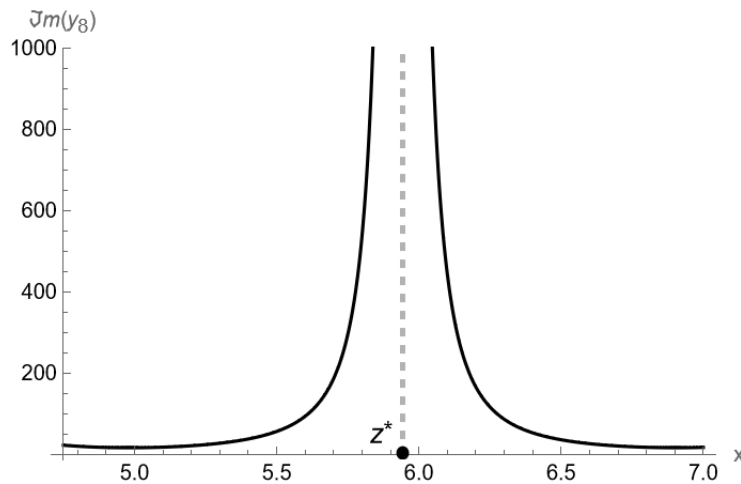


Figure 3. Plot $\Im m(y_8)(x)$ in the neighborhood of the point z^* .

The obtained results are accompanied by an additional numerical study. The method for optimizing an a priori estimate is presented.

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