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INTEGRABLE COUPLINGS AND BI-HAMILTONIAN STRUCTURES OF GENERALIZED ISOSPECTRAL-NONISOSPECTRAL C-KDV HIERARCHIES

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Abstract In this paper, we derive a combined isospectral-nonisospectral C-KdV hierarchy, which generates some physically significant equations such as the generalized variable-coefficient C-KdV and Burgers equations. For the nonisospectral C-KdV equation, we establish infinite conservation laws and analyze invariant solutions and point symmetries for both the Burgers equation and the classical C-KdV equation. A key contribution is the construction of a new 4×4 Lie algebra g which enables the formulation of two linear problems. Their compatibility conditions yield isospectral and nonisospectral C-KdV integrable couplings. Using the quadratic-form identity, we derive the bi-Hamiltonian structures for the isospectral hierarchy, while the component-trace identity provides bi-Hamiltonian structures for the nonisospectral hierarchy.

Keywords Isospectral and nonisospectral integrable hierarchy, bi-Hamiltonian structures, Lie algebra, conserved quantities, invariant solutions.

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1. Introduction

Integrable systems play pivotal roles in mathematics and theoretical physics. Prominent examples include the KdV and NLS equations (continuous systems) and the Toda lattice and Volterra lattices (discrete systems), all with significant physical applications. Such systems are commonly constructed via the Lax representation ($L_t = [A, L]$) or the zero curvature representation ($U_t - V_x + [U, V] = 0$) [2, 30], originating from the Lax-pair method introduced by Magri [25]. Tu later developed a systematic approach to generate soliton hierarchies by representing spectral problems via loop algebras and solving the associated zero curvature equations [38]. The Tu scheme has enabled the derivation of numerous integrable hierarchies, such as the KdV hierarchy, the AKNS hierarchy, the WKI hierarchy, and their Hamiltonian structures [8, 9, 13, 17, 39, 40, 52, 53]. However, most of the previous studies have focused on the isospectral problems, and relatively little research has been done on the nonisospectral problems, especially the time evolution of the spectral parameter λ as the polynomial form in λ, i.e., $\lambda_t = \sum_{i\geq 0} k_i(t)\lambda^i$. In [18], Ma once proposed a new simple scheme for generating nonisospectral flows from the zero curvature representation in the case of $\lambda_t = \lambda^n,\ n \geq 0$. In [32], Qiao applied the generalized Lax representations and considered an ordinary $N \times N$ nonisospectral

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problem $Ly = \lambda y$, $\lambda_{t_m} = \lambda^{(m+1)\eta}$, $m \geq 0$, $\lambda_{t_m} = \lambda^{m\eta}$, $m \leq 0$ to generate some nonisospectral soliton hierarchies. In [16], starting from the Lax pair $\Phi_x = U\Phi$, $\Phi_t = V\Phi$ and then using the compatibility condition, Li derived the nonisospectral AKNS hierarchy in the case of $\lambda_t = \sum_{i\geq 0}^{n} k_i(t)\lambda^{n-i}$. Further extensions by Chen et al. have enriched the theory of nonisospectral hierarchies [6, 7, 10, 11, 31, 51, 54, 56].

Integrable couplings have emerged as a significant research direction in mathematical physics since Fuchssteiner introduced the concept through the study of centerless Virasoro symmetry algebras. Ma pioneered perturbation-based constructions of integrable couplings [22, 23], while Guo and Zhang developed methods via matrix Lie algebra primitives [12]. Subsequently, Ma established a link between integrable couplings and semi-direct sums of Lie algebras [24]. Wang extended this framework by constructing integrable couplings for nonisospectral problems via a perturbation term ε [42]. Based on the above work, many examples of soliton hierarchies of integrable couplings have been obtained, including the AKNS hierarchy, the Kaup-Newell hierarchy, the WKI hierarchy, the Toda hierarchy, the Jaulent-Miodek hierarchy, the Glachette-Johnson hierarchy, and the Tu hierarchy [15, 19–21, 26, 27, 29, 33, 43–49].

In this paper, we generalize the Lie algebra constructed by Ma [24] to obtain a new 4×4 matrix Lie algebra. Based on its loop algebra, we introduce two distinct spectral problems (isospectral and nonisospectral) (3.1), (3.18). By means of the Tu scheme, we derive the isospectral integrable coupling system (3.8) and the nonisospectral integrable coupling system (3.26), from which several distinct nonlinear equations (3.9), (3.28) are reduced. These nonlinear equations contain new perturbation terms and nonisospectral terms. When setting $\varepsilon = 0$ and $k_i(t) = 0$, they can be reduced to the integrable couplings of the classical C-KdV equation [50]. Therefore, the results obtained in this paper are new generalized forms of the existing results. As applications, the invariant solutions and conservation laws of some reduced nonlinear evolution equations can be obtained by Lie symmetry analysis.

The organization of the paper is as follows. In Section 2, based on the nonisospectral C-KdV spectral problem, an isospectral plus nonisospectral integrable hierarchy of new nonlinear evolution equations is presented, including the classical C-KdV equations, generalized Burgers equations with variable coefficients. Then, we find that the integrable hierarchy of nonlinear evolution equations has Hamiltonian structures. In Section 3, a 4×4 matrix Lie algebra g is introduced. Based on Lie algebra g, we design a new linear spectral problem, and by considering whether the spectral parameters λ is evolution of time, we obtain isospectral and nonisospectral C-KdV integrable couplings. Then their bi-Hamiltonian structures are derived using quadratic constant equation and two sets of component-trace identity, respectively. In Section 4, infinite conservation laws of isospectral C-KdV equation with variable coefficient are obtained. In Section 5, we consider the Lie point symmetries and the invariant solutions of Burgers equation (2.21) and partial differential equations system (2.18). The last section contains a summary and discussion.

2. An isospectral-nonisospectral C-KdV soliton hierarchy and its Hamiltonian structures

2.1. An isospectral-nonisospectral C-KdV soliton hierarchy

In this section, we shall derive an isospectral-nonisospectral coupled KdV soliton hierarchy associated with the matrix Lie algebra A_1. This Lie algebra is simple and has a basis

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

whose commutator relations are

$$[f_1, f_2] = 2f_2,\ [f_1, f_3] = -2f_3,\ [f_2, f_3] = f_1.$$

The loop algebra corresponding to the simple Lie algebra A_1 is denoted by

$$f_i(n) = f_i\lambda^n,\ i = 1, 2, 3;\ n = 0, 1, 2, 3, \cdots ,$$

with the commutator

$$[f_1(n), f_2(m)] = 2f_2(n+m),\ [f_1(n), f_3(m)] = -2f_3(n+m),\ [f_2(n), f_3(m)] = f_1(n+m).$$

Then the spectral matrix U can be expressed by

$$U = \begin{pmatrix} \frac{q-\lambda}{2} & -r \\ 1 & \frac{\lambda-q}{2} \end{pmatrix} = \frac{1}{2}qf_1(0) - \frac{1}{2}f_1(1) - rf_2(0) + f_3(0), \tag{2.1}$$

which appears in the spectral problem

$$\varphi_x = U\varphi,\ \varphi = (\varphi_1, \varphi_2)^T , \tag{2.2}$$

where q, r are two potentials, and λ is a spectral parameter. Set

$$\varphi_t = V_1\varphi + V_2\varphi =: V\varphi, \tag{2.3}$$

where

$$V_1 = \sum_{i\geq 0} a_i f_1(-i) + b_i f_2(-i) + c_i f_3(-i),$$
$$V_2 = \sum_{j\geq 0} \bar{a}_j f_1(-j) + \bar{b}_j f_2(-j) + \bar{c}_j f_3(-j).$$

Taking the spectral evolution $\lambda_t = \sum_{j\geq 0} k_j(t)\lambda^{-j}$, then the compatibility of (2.2) and (2.3) reads that

$$\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda_t - V_x + [U, V] = 0, \tag{2.4}$$

which can be decomposed into the following form

$$\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda_t - V_{1,x} + [U, V_1] - V_{2,x} + [U, V_2] = 0. \tag{2.5}$$

According to the generalized Tu scheme, we first solve the isospectral stationary zero-curvature equation

$$V_{1,x} = [U, V_1], \tag{2.6}$$

which admits that

$$\begin{cases} a_{i+1,x} = -rc_{i+1} - b_{i+1}, \\ c_{i+1} = c_{i.x} + qc_i - 2a_i, \\ b_{i+1} - b_{i,x} + 2ra_i + qb_i,\ i = 1, 2, 3, \cdots . \end{cases} \tag{2.7}$$

Then, the nonisospectral stationary zero-curvature equation

$$V_{2,x}=[U,V_2]+\frac{\partial U}{\partial\lambda}\lambda_t,\ \lambda_t=\sum_{j\geq 0}k_j(t)\lambda^{-j}, \tag{2.8}$$

gives rise to

$$\begin{cases}\bar{a}_{j+1}=-(\partial^{-1}r\partial+\partial^{-1}rq)\bar{c}_j+\bar{b}_j-\partial^{-1}q\bar{b}_j-\frac{1}{2}xk_{j+1}(t)+\beta_{j+1}(t),\\ \bar{c}_{j+1}=\bar{c}_{j.x}+q\bar{c}_j-2\bar{a}_j,\\ \bar{b}_{j+1}-\bar{b}_{j,x}+2r\bar{a}_j+q\bar{b}_j,\ j=1,2,3,\cdots.\end{cases} \tag{2.9}$$

Taking $a_0=\frac{1}{2}\alpha_0(t),\ c_0=b_0=0$, we have in terms of (2.7) that

$$\begin{cases}c_1=-\alpha_0(t),\ b_1=\alpha_0(t)r,\ a_1=\alpha_1(t),\\ c_2=-\alpha_0(t)q-2\alpha_1(t),\ b_2=-\alpha_0(t)r_x+\alpha_0(t)qr+2\alpha_1(t)r,\\ a_2=\alpha_0(t)r+\alpha_2(t),c_3=-\alpha_0(t)q_x-2\alpha_0(t)r-\alpha_0(t)q^2-2\alpha_1(t)q-2\alpha_2(t),\\ a_3=-\alpha_0(t)r_x+2\alpha_0(t)qr+2\alpha_1(t)r+\alpha_3(t),\\ b_3=\alpha_0(t)r_{xx}-\alpha_0(t)(qr)_x+2\alpha_0(t)r^2-\alpha_0(t)qr_x+\alpha_0(t)q^2r-2\alpha_1(t)r_x\\ \qquad +2r\alpha_2(t)+2\alpha_1(t)rq,\\ \cdots,\end{cases}$$

where $\alpha_1(t),\ \alpha_2(t),\ \alpha_3(t)$ are integral constants. Choosing $\bar{a}_0=\frac{1}{2}\beta_0(t),\ \bar{c}_0=\bar{b}_0=0,\ k_0(t)=0$, then from Eq. (2.9) we have

$$\begin{cases}\bar{c}_1=-\beta_0(t),\ \bar{b}_1=\beta_0(t)r,\ \bar{a}_1=-\frac{1}{2}xk_1(t)+\beta_1(t),\\ \bar{b}_2=-\beta_0(t)r_x+\beta_0(t)qr-rxk_1(t)+2\beta_1(t)r,\\ \bar{c}_2=-\beta_0(t)q+xk_1(t)-2-\beta_1(t),\\ \bar{a}_2=\beta_0(t)r-\frac{1}{2}xk_2(t)+\beta_2(t),\\ \bar{c}_3=-\beta_0(t)q_x-2\beta_0(t)r-\beta_0(t)q^2-2\beta_1(t)q+(qx+1)k_1(t)+xk_2(t)-2\beta_2(t),\\ \bar{a}_3=-\beta_0(t)r_x+2\beta_1(t)r+2\beta_0(t)qr-xk_1(t)r-\frac{1}{2}xk_3(t)+\beta_3(t),\\ \bar{b}_3=\beta_0(t)r_{xx}-\beta_0(t)(qr)_x+2\beta_0(t)r^2-\beta_0(t)qr_x+\beta_0(t)q^2r+(r+r_xx-qrx)k_1(t)\\ \qquad -r_xk_2(t)-2\beta_1(t)r_x+2\beta_2(t)r+2\beta_1(t)qr,\\ \cdots,\end{cases}$$

where $\beta_1(t),\ \beta_2(t),\ \beta_3(t)$ are integral constants. Note that

$$V_{1,+}^{[n]}=\sum_{i=0}^{n}\left(a_if_1(n-i)+b_if_2(n-i)+c_if_3(n-i)\right)=\lambda^nV_1-V_{1,-}^{[n]},$$

$$V_{2,+}^{[m]} = \sum_{j=0}^{m} \left(\bar{a}_j f_1(m-j) + \bar{b}_j f_2(m-j) + \bar{c}_j f_3(m-j) \right) = \lambda^m V_2 - V_{2,-}^{[m]},$$

$$\lambda_{t,+}^{[m]} = \sum_{j=0}^{m} k_j(t)\lambda^{-j} = \lambda^m \sum_{j\geq 0} k_j(t)\lambda^{-j} - \sum_{j\geqslant m+1} = \lambda^m \lambda_t - \lambda_{t,-}^{[m]},$$

then the Eq. (2.6) can be decomposed into

$$-(V_{1,+}^{[n]})_x + [U, V_{1,+}^{[n]}] = (V_{1,-}^{n})_x - [U, V_{1,-}^{[n]}] = -c_{n+1} f_3(0) + b_{n+1} f_2(0),$$

and the Eq. (2.8) can be broken down into

$$\begin{aligned} \frac{\partial U}{\partial \lambda} \lambda_{t,+}^{[m]} - (V_{2,+}^{[m]})_x + [U, V_{2,+}^{[m]}] &= -\frac{\partial U}{\partial \lambda} \lambda_{t,-}^{[m]} + (V_{2,-}^{[m]})_x - [U, V_{2,-}^{[m]}] \\ &= -\bar{c}_{m+1} f_3(0) + \bar{b}_{m+1} f_2(0). \end{aligned}$$

Let

$$V_1^{[n]} = V_{1,+}^{[n]} + \frac{1}{2} c_{n+1} f_1(0), \;\; V_2^{[n]} = V_{2,+}^{[n]} + \frac{1}{2} \bar{c}_{m+1} f_1(0).$$

One infers that

$$\begin{aligned} & - (V_{1,+}^{[n]})_x - (V_{2,+}^{[m]})_x + [U, V_{1,+}^{[n]}] + [U, V_{2,+}^{[m]}] + \frac{\partial U}{\partial \lambda} \lambda_{t,+}^{[m]} \\ = & \, (V_{1,-}^{[n]})_x + (V_{2,-}^{[m]})_x - [U, V_{1,-}^{[n]}] - [U, V_{2,-}^{[m]}] - \frac{\partial U}{\partial \lambda} \lambda_{t,-}^{[m]} \\ = & - \frac{1}{2}(c_{n+1,x} + \bar{c}_{m+1,x}) f_1(0) - (a_{n+1,x} + \bar{a}_{m+1,x}) f_2(0) - \frac{1}{2} k_{m+1}(t) f_2(0). \end{aligned}$$

Therefore, the generalized zero-curvature equation

$$\frac{\partial U}{\partial u} u_t + \frac{\partial U}{\partial \lambda} \lambda_t - V_{1,x}^{[n]} + [U, V_1^{[n]}] - V_{2,x}^{[m]} + [U, V_2^{[m]}] = 0, \tag{2.10}$$

admits the isospectral-nonisospectral C-KdV soliton hierarchy as follows

$$u_{t_{n,m}} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_{n,m}} = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} + \bar{a}_{m+1} \\ -c_{n+1} - \bar{c}_{m+1} \end{pmatrix} + \begin{pmatrix} 0 \\ -\dfrac{1}{2} \end{pmatrix} k_{m+1}(t). \tag{2.11}$$

Using Eq. (2.7), we deduce the following recursion relation

$$\begin{aligned} \begin{pmatrix} a_{n+1} \\ -c_{n+1} \end{pmatrix} &= \begin{pmatrix} \partial^{-1} q \partial - \partial\, r + \partial^{-1} r \partial \\ 2 & \partial + q \end{pmatrix} \begin{pmatrix} a_n \\ -c_n \end{pmatrix} + \alpha_{n+1}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= (\frac{1}{2}\alpha_0(t) L^{n+1} + \alpha_1(t) L^n + \cdots + \alpha_n(t) L + \alpha_{n+1}(t)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &=: P_{n+1}(L, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.12}$$

where

$$P_{n+1}(L,t) = \frac{1}{2}\alpha_0(t)L^{n+1} + \alpha_1(t)L^n + \cdots + \alpha_n(t)L + \alpha_{n+1}(t),$$
$$L = \begin{pmatrix} \partial^{-1}q\partial - \partial\, r + \partial^{-1}r\partial \\ 2 \qquad\qquad \partial + q \end{pmatrix}.$$

Using the Eq. (2.9), we find

$$\begin{aligned}\begin{pmatrix} \bar{a}_{m+1} \\ -\bar{c}_{m+1} \end{pmatrix} &= \begin{pmatrix} \partial^{-1}q\partial - \partial\, r + \partial^{-1}r\partial \\ 2 \qquad\qquad \partial + q \end{pmatrix} \begin{pmatrix} \bar{a}_m \\ -\bar{c}_m \end{pmatrix} + \beta_{m+1}(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{2}\partial^{-1}q \\ 0 \end{pmatrix} k_m(t) + \begin{pmatrix} -\frac{1}{2}x \\ 0 \end{pmatrix} k_{m+1}(t) \\ &=: Q_{m+1}(L,t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + R_m(L,t)\begin{pmatrix} \frac{1}{2}\partial^{-1}q \\ 0 \end{pmatrix} + S_{m+1}\begin{pmatrix} -\frac{1}{2}x \\ 0 \end{pmatrix},\end{aligned} \tag{2.13}$$

where

$$Q_{m+1}(L,t) = \frac{1}{2}\beta_0(t)L^{m+1} + \alpha_1(t)L^m + \alpha_2(t)L^{m-1} + \cdots + \alpha_m(t)L + \alpha_{m+1}(t),$$
$$R_m(L,t) = k_0(t)L^m + k_1(t)L^{m-1} + \cdots + k_{m-1}(t)L + k_m(t),$$
$$S_{m+1}(L,t) = k_1(t)L^m + k_2(t)L^{m-1} + \cdots + k_m(t)L + k_{m+1}(t).$$

Taking

$$\Phi = JLJ^{-1} = \begin{pmatrix} \partial + \partial q\partial^{-1} & 0 \\ \partial r\partial^{-1} + r & q - \partial \end{pmatrix},$$

then we find $JL^n = \Phi^n J$. Hence, by using Eqs. (2.12) and (2.13) the integrable hierarchy (2.11) can be written as

$$\begin{aligned}\begin{pmatrix} q \\ r \end{pmatrix}_t &= J(P_{n+1}(L,t) + Q_{m+1}(L,t))\begin{pmatrix} 1 \\ 0 \end{pmatrix} + JR_m(L,t)\begin{pmatrix} \frac{1}{2}\partial^{-1}q \\ 0 \end{pmatrix} \\ &\quad - JS_{m+1}(L,t)\begin{pmatrix} \frac{1}{2}x \\ 0 \end{pmatrix} - \frac{k_{m+1}(t)}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &=: (P_{n+1}(\Phi,t) + Q_{m+1}(\Phi,t))\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{-R_m(\Phi,t)}{2}\begin{pmatrix} 0 \\ q \end{pmatrix} \\ &\quad + \frac{S_{m+1}(\Phi,t)}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{k_{m+1}(t)}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.\end{aligned} \tag{2.14}$$

In what follows, we consider some reductions of the integrable hierarchy (2.14) as follows:

Case 1. When $n=1,\ m=1$, the integrable hierarchy (2.14) reduces to a simple linear partial differential equation system as follows:

$$\begin{cases} q_t=-(\alpha_0(t)+\beta_0(t))q_x+k_1(t), \\ r_t=-(\alpha_0(t)+\beta_0(t))r_x. \end{cases} \tag{2.15}$$

Case 2. When $n=2,\ m=0$, we get

$$\begin{cases} q_t=-\alpha_0(t)q_{xx}-2\alpha_0(t)r_x-2\alpha_0(t)qq_x-2\alpha_1(t)q_x, \\ r_t=\alpha_0(t)r_{xx}-2\alpha_0(t)(qr)_x-2\alpha_1(t)r_x-\dfrac{1}{2}k_1(t), \end{cases} \tag{2.16}$$

which is a nonisospectral C-KdV equation with variable coefficients. When $\alpha_0(t)=-1,\ \alpha_1(t)=1,\ k_1(t)=0$, we get

$$\begin{cases} q_t=q_{xx}+2r_x+2qq_x-2q_x, \\ r_t=-r_{xx}+2q_xr+2r_xq-2r_x, \end{cases} \tag{2.17}$$

which is a generalized form for the equation derived by D Levi [15]. When $\alpha_0(t)=-1,\ \alpha_1(t)=0,\ k_1(t)=0$, the Eq. (2.16) can be reduced to the classical C-KdV equation

$$\begin{cases} q_t=q_{xx}+2r_x+2qq_x, \\ r_t=-r_{xx}+2q_xr+2r_xq. \end{cases} \tag{2.18}$$

When $r=0,\ -\alpha_0(t)=f(t),\ -2\alpha_1(t)=g(t)$, the Eq. (2.16) reduces to the generalized Burgers equation with variable coefficients

$$q_t=f(t)q_{xx}+2f(t)qq_x+g(t)q_x. \tag{2.19}$$

When $g(t)=0,\ q=u$, the Eq. (2.16) can be reduced to the classical Burgers equation with variable coefficients (see [33])

$$u_t-f(t)u_{xx}-2f(t)uu_x=0. \tag{2.20}$$

When $f(t)=1,\ g(t)=0,\ q=u$, the generalized Burgers equation with variable coefficients (2.19) becomes the classical Burgers equation (see [5])

$$u_t-u_{xx}-2uu_x=0, \tag{2.21}$$

which is the lowest order approximation for the one dimensional propagation of weak shock waves in the fluid [41].

Case 3. When $n=0,\ m=2$, we obtain a nonisospectral nonlinear evolution equation with variable coefficients from the C-KdV soliton hierarchy as follows:

$$\begin{cases} q_t=-\beta_0(t)q_{xx}-2\beta_0(t)r_x-2\beta_0(t)qq_x-2\beta_1(t)q_x \\ \qquad +(q+q_xx)k_1(t)+k_2(t), \\ r_t=\beta_0(t)r_{xx}-2\beta_0(t)(qr)_x-2\beta_1(t)r_x+(r+r_xx)k_1(t), \end{cases} \tag{2.22}$$

which is the nonisospectral form of the Eq. (2.16).

Case 4. When $n = 1,\ m = 2$, we have from the soliton hierarchy (2.14) that

$$\begin{cases} q_t = -\beta_0(t)q_{xx} - 2\beta_0(t)qq_x - 2\beta_0(t)r_x - (2\beta_1(t) + \alpha_0(t))q_x \\ \qquad +(q + q_x x)k_1(t) + k_2(t), \\ r_t = \beta_0(t)r_{xx} - 2\beta_0(t)(qr)_x - (2\beta_1(t) + \alpha_0(t))r_x + (r + r_x x)k_1(t). \end{cases} \tag{2.23}$$

Case 5. When $n = 2,\ m = 1$, we have from (2.14) that

$$\begin{cases} q_t = -\alpha_0(t)q_{xx} - 2\alpha_0(t)r_x - 2\alpha_0(t)qq_x - (2\alpha_1(t) \\ \qquad +\beta_0(t))q_x + k_1(t), \\ r_t = \alpha_0(t)r_{xx} - 2\alpha_0(t)(qr)_x - (2\beta_1(t) + \alpha_0(t))r_x + \frac{1}{2}k_2(t). \end{cases} \tag{2.24}$$

Remark. Eqs. (2.23) and (2.24) are the generalized form of the soliton equation from D Levi [15], but they are different from the Eq. (2.22). In addition, Eqs. (2.23) and (2.24) are the different evolution equations,because they have different nonisospectral terms and coefficients.

2.2. Hamiltonian structures of the isospectral-nonisospectral C-KdV soliton hierarchy (2.14)

To furnish Hamiltonian structures, we use the variational identity or trace identity. Denoting the trace of the square matrices U and V by $< U, V >$, that is $< U, V >= tr(UV)$. Since

$$\frac{\partial U}{\partial q} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix},\ \frac{\partial U}{\partial r} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},\ \frac{\partial U}{\partial \lambda} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

we have

$$\begin{aligned} &< V_1, \frac{\partial U}{\partial q} >= \sum_{i\geq 0} a_i\lambda^{-i},\ < V_1, \frac{\partial U}{\partial r} >= -\sum_{i\geq 0} c_i\lambda^{-i}, \\ &< V_1, \frac{\partial U}{\partial \lambda} >= -\sum_{i\geq 0} a_i\lambda^{-i}, < V_2, \frac{\partial U}{\partial q} >= \sum_{j\geq 0} \bar{a}_j\lambda^{-j}, \\ &< V_2, \frac{\partial U}{\partial r} >= -\sum_{j\geq 0} \bar{c}_j\lambda^{-j},\ < V_2, \frac{\partial U}{\partial \lambda} >= -\sum_{j\geq 0} \bar{a}_j\lambda^{-j}. \end{aligned} \tag{2.25}$$

Substituting the results (2.25) into the corresponding trace identity

$$\frac{\delta}{\delta u} < V_i, \frac{\partial u}{\partial \lambda} >= \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} < V_i, \frac{\partial U}{\partial q} > \\ < V_i, \frac{\partial U}{\partial r} > \end{pmatrix},\ i = 1, 2,$$

and balancing the coefficients of the powers of λ, we get that

$$\frac{\delta}{\delta u}(a_{n+1}) = (\gamma - n)\begin{pmatrix} a_n \\ -c_n \end{pmatrix},\ \frac{\delta}{\delta u}(-\bar{a}_{m+1}) = (\gamma - m)\begin{pmatrix} \bar{a}_m \\ -\bar{c}_m \end{pmatrix}.$$

The case of $n=0,\ m=0$ tells $\gamma=0.$ and thus we have

$$\frac{\delta}{\delta u}(\frac{a_{n+1}}{n})=\begin{pmatrix}a_n\\-c_n\end{pmatrix}=\frac{\delta H_{n+1}}{\delta u},\ \frac{\delta}{\delta u}(\frac{\bar{a}_{m+1}}{m})=\begin{pmatrix}\bar{a}_m\\-\bar{c}_m\end{pmatrix}=\frac{\delta H_{m+1}}{\delta u}.$$

Therefore, the Hamiltonian structures of the isospectral and nonisospectral soliton hierarchy (2.14) is

$$u_{t_{n,m}}=J\left(\frac{\delta H_{n+1}}{\delta u}+\frac{\delta H_{m+1}}{\delta u}\right)+k_{m+1}(t)\begin{pmatrix}0\\-\frac{1}{2}\end{pmatrix},$$

where the Hamiltonian functional is $\frac{\delta H_{n+1}}{\delta u}+\frac{\delta H_{m+1}}{\delta u}$ and Hamiltonian operator is $J=\begin{pmatrix}0&-\partial\\-\partial&0\end{pmatrix}$.

3. Isospectral and nonisospectral integrable couplings of the C-KdV soliton hierarchy and their bi-Hamiltonian structures

In this section, we extend the 2×2 matrix Lie algebra A_1 to the 4×4 matrix Lie algebra g in order to obtain the extended linear spectral problem. In what follows, we can derive the integrable couplings system of original equation hierarchy (2.14) by using the Tu scheme.

We denote the Lie algebra $g=\text{span}\{g_i,\ i=1,2,\cdots,6\}$, which has a set of bases

$$g_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\ g_2=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\ g_3=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},$$

$$g_4=\begin{pmatrix}0&0&\varepsilon&0\\0&0&0&-\varepsilon\\1&0&0&0\\0&-1&0&0\end{pmatrix},\ g_5=\begin{pmatrix}0&0&0&\varepsilon\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\ g_6=\begin{pmatrix}0&0&0&0\\0&0&\varepsilon&0\\0&0&0&0\\1&0&0&0\end{pmatrix},$$

with the commutation relations

$$\begin{aligned}&[g_1,g_2]=2g_2,\ [g_1,g_3]=-2g_3,\ [g_1,g_4]=0,\ [g_1,g_5]=2g_5,\ [g_1,g_6]=-2g_6,\\&[g_2,g_3]=g_1,\ [g_2,\ g_4]=-2g_5,\ [g_2,g_5]=0,\ [g_2,g_6]=g_4,\ [g_3,g_4]=2g_6,\\&[g_3,g_5]=-g_4,\ [g_3,g_6]=0,\ [g_4,g_5]=2\varepsilon g_2,\ [g_4,g_6]=-2\varepsilon g_3,\ [g_5,g_6]=\varepsilon g_1.\end{aligned}$$

The corresponding loop algebra is defined as

$$\widetilde{g}=\text{span}\{g_i(n),\ i=1,2,\cdots,6,\ n=1,2,\cdots\},\ g_i(n)=g_i\lambda^n,$$

along with the commutator

$$[g_1(m),g_2(n)]=2g_2(m+n),\ [g_1(m),g_3(n)]=-2g_3(m+n),$$

$$
\begin{aligned}
&[g_1(m), g_5(n)] = 2g_5(m+n),\ [g_1(m), g_6(n)] = -2g_6(m+n),\\
&[g_2(m), g_4(n)] = -2g_5(m+n),\ [g_2(m), g_6(n)] = g_4(m+n),\\
&[g_3(m), g_4(n)] = 2g_6(m+n),\ [g_3(m), g_5(n)] = -g_4(m+n),\\
&[g_4(m), g_5(n)] = 2\varepsilon g_2(m+n),\ [g_4(m), g_6(n)] = -2\varepsilon g_3(m+n),\\
&[g_5(m), g_6(n)] = \varepsilon g_1(m+n),\ [g_2(m), g_3(n)] = g_1(m+n),\\
&[g_1(m), g_4(n)] = 0,\ [g_2(m), g_5(n)] = 0,\ [g_3(m), g_6(n)] = 0.
\end{aligned}
$$

3.1. An isospectral integrable couplings of C-KdV equation

Based on the loop algebra $\widetilde{g}$, we introduce the isospectral problem

$$
\begin{cases}
\Phi_x = \bar{U}\Phi,\\
\Phi_t = \bar{V}_1\Phi,\\
\lambda_t = 0,
\end{cases}
\tag{3.1}
$$

where

$$
\begin{aligned}
\bar{U} &= \frac{q}{2}g_1(0) - \frac{1}{2}g_1(1) - rg_2(0) + g_3(0) + \frac{p}{2}g_4(0) - \frac{1}{2}g_4(1) - sg_5(0) + g_6(0),\\
\bar{V}_1 &= \sum_{i\geq 0} a_i g_1(-i) + b_i g_2(-i) + c_i g_3(-i) + d_i g_4(-i) + e_i g_5(-i) + f_i g_6(-i),
\end{aligned}
$$

and $\bar{u} = (q, r, p, s)^T$ is potential, $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)^T$ is eigenfunction, for convenience, we set

$$
\begin{aligned}
a &= \sum_{i\geq 0} a_i\lambda^{-i},\ b = \sum_{i\geq 0} b_i\lambda^{-i},\ c = \sum_{i\geq 0} c_i\lambda^{-i},\\
d &= \sum_{i\geq 0} d_i\lambda^{-i},\ e = \sum_{i\geq 0} e_i\lambda^{-i},\ f = \sum_{i\geq 0} f_i\lambda^{-i}.
\end{aligned}
$$

In terms of Tu scheme, by solving the stationary zero curvature equation corresponding to $\bar{V}_1$

$$
\bar{V}_{1,x} = [\bar{U}, \bar{V}_1],
\tag{3.2}
$$

we obtain the recurrence relations

$$
\begin{cases}
a_{i,x} = -rc_i - b_i - \varepsilon s f_i - \varepsilon e_i,\\
b_{i,x} = 2ra_i + 2\varepsilon s d_i + qb_i + \varepsilon p e_i - b_{i+1} - \varepsilon e_{i+1},\\
c_{i,x} = 2a_i + c_{i+1} - qc_i + 2\varepsilon d_i - \varepsilon p f_i + \varepsilon f_{i+1},\\
d_{i,x} = -sc_i - b_i - rf_i - e_i,\\
e_{i,x} = 2rd_i + 2sa_i + pb_i + qe_i - b_{i+1} - e_{i+1},\\
f_{i,x} = 2a_i + c_{i+1} - pc_i + 2d_i - qf_i + f_{i+1},
\end{cases}
$$

which has an equivalent form

$$\begin{cases} a_{i,x} = & -rc_i - b_i - \varepsilon s f_i - \varepsilon e_i, \\ d_{i,x} = & -sc_i - b_i - rf_i - e_i, \\ b_{i+1} = & \frac{1}{\varepsilon - 1}[b_{i,x} - \varepsilon e_{i,x} + 2(\varepsilon s - r)a_i + (\varepsilon p - q)b_i + 2\varepsilon(r - s)d_i \\ & +\varepsilon(q - p)e_i], \\ e_{i+1} = & \frac{1}{\varepsilon - 1}[e_{i,x} - b_{i,x} + 2(r - s)a_i + (q - p)b_i + 2(\varepsilon s - r)d_i \\ & +(\varepsilon p - q)e_i], \\ c_{i+1} = & \frac{1}{\varepsilon - 1}[\varepsilon f_{i,x} - c_{i,x} + 2(1 - \varepsilon)a_i + (\varepsilon p - q)c_i + \varepsilon(q - p)f_i], \\ f_{i+1} = & \frac{1}{\varepsilon - 1}[c_{i,x} - f_{i,x} + (q - p)c_i + 2(1 - \varepsilon)d_i + (\varepsilon p - q)f_i]. \end{cases} \tag{3.3}$$

Take

$$a_0 = d_0 = \frac{1}{2},\ b_0 = c_0 = e_0 = f_0 = 0,$$

and impose the integration conditions

$$a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = d_i|_{u=0} = e_i|_{u=0} = f_i|_{u=0} = 0,\ i \geq 1.$$

Hence, we have in terms of Eq. (3.3) that

$$\begin{aligned} a_1 =& 0,\ b_1 = r,\ c_1 = -1,\ d_1 = 0,\ e_1 = s,\ f_1 = -1, \\ a_2 =& r,\ b_2 = \frac{1}{\varepsilon - 1}[r_x - \varepsilon s_x + (\varepsilon p - q)r + \varepsilon(q - p)s],\ c_2 = -q, \\ d_2 =& s,\ e_2 = \frac{1}{\varepsilon - 1}[s_x - r_x + qr - pr + \varepsilon ps - qs],\ f_2 = -p, \\ a_3 =& \frac{1}{\varepsilon - 1}[r_x - \varepsilon s_x + 2\varepsilon pr - 2qr - 2\varepsilon ps + 2\varepsilon qs], \\ d_3 =& \frac{1}{\varepsilon - 1}[-r_x + s_x - 2pr + 2qr + 2\varepsilon ps - 2qs], \\ c_3 =& \frac{1}{\varepsilon - 1}[q_x - \varepsilon p_x + q^2 + \varepsilon p^2 - 2\varepsilon pq] - 2r, \\ f_3 =& \frac{1}{\varepsilon - 1}[p_x - q_x - q^2 - \varepsilon p^2 + 2pq] - 2s, \\ b_3 =& \frac{1}{(\varepsilon - 1)^2}[(1 + \varepsilon)r_{xx} - 2\varepsilon s_{xx} + 2\varepsilon p_x r + 4\varepsilon p r_x - (1 + \varepsilon)q_x r - 2(1 + \varepsilon)q r_x + 2\varepsilon q_x s \\ & + 4\varepsilon q s_x - \varepsilon(\varepsilon + 1)p_x s - 2\varepsilon(\varepsilon + 1)p s_x + \varepsilon(\varepsilon + 1)p^2 r - 4\varepsilon pqr + 2\varepsilon(\varepsilon + 1)pqs \\ & + (\varepsilon + 1)q^2 r - 2\varepsilon q^2 s - 2\varepsilon^2 p^2 s + 2(1 - \varepsilon)r^2 + 2\varepsilon(1 - \varepsilon)s^2 + 4\varepsilon(\varepsilon - 1)rs], \\ e_3 =& \frac{1}{(\varepsilon - 1)^2}[-2r_{xx} + (\varepsilon + 1)s_{xx} + 2q_x r + 4q r_x - (1 + \varepsilon)p_x r - 2(1 + \varepsilon)p r_x \\ & + 2\varepsilon p_x s + 4\varepsilon p s_x - (1 + \varepsilon)q_x s - 2(\varepsilon + 1)q s_x + 2(\varepsilon + 1)pqr - 4pqs - 2q^2 r \\ & + \varepsilon(\varepsilon + 1)p^2 r + (\varepsilon + 1)q^2 s - 2\varepsilon p^2 r + 2(\varepsilon - 1)r^2 + 2\varepsilon(\varepsilon - 1)s^2 - 4(\varepsilon - 1)sr], \\ & \cdots . \end{aligned}$$

Noting

$$\begin{aligned}\bar{V}_{1,+}^{[n]} =& \sum_{i=0}^{n}(a_i g_1(n-i) + b_i g_2(n-i) + c_i g_3(n-i) + d_i g_4(n-i)\\ &+ e_i g_5(n-i) + f_i g_6(n-i))\\ =:& \lambda^n \bar{V}_1 - \bar{V}_{1,-}^{[n]},\end{aligned}$$

we calculate that

$$\begin{aligned}-(\bar{V}_{1,+}^{[n]})_x + [\bar{U}, \bar{V}_{1,+}^{[n]}] =& (\bar{V}_{1,+}^{[n]})_x - [\bar{U}, \bar{V}_{1,-}^{[n]}]\\ =& (b_{n+1} + \varepsilon e_{n+1}) g_2(0) + (b_{n+1} + e_{n+1}) g_5(0)\\ &- (c_{n+1} + \varepsilon f_{n+1}) g_3(0) - (c_{n+1} + f_{n+1}) g_6(0).\end{aligned} \tag{3.4}$$

Taking

$$\bar{V}_1^{[n]} = \bar{V}_{1,+}^{[n]} + \frac{1}{2} c_{n+1} g_1(0) + \frac{1}{2} f_{n+1} g_4(0), \tag{3.5}$$

then a direct calculation shows that

$$\begin{aligned}&-(\bar{V}_{1,+}^{[n]})_x + [\bar{U}, \bar{V}_{1,+}^{[n]}]\\ =& - a_{n+1,x} g_2(0) - d_{n+1,x} g_5(0) - \frac{1}{2} c_{n+1,x} g_1(0) - \frac{1}{2} f_{n+1,x} g_4(0).\end{aligned} \tag{3.6}$$

Therefore, the zero curvature equation

$$\frac{\partial \bar{U}}{\partial \bar{u}} u_t - \bar{V}_{1,x}^{[n]} + [\bar{U}, \bar{V}_1^{[n]}] = 0 \tag{3.7}$$

admits the isospectral integrable couplings hierarchy of C-KdV equation as follows:

$$\bar{u}_{t_n} = \begin{pmatrix} q \\ r \\ p \\ s \end{pmatrix}_{t_n} = \begin{pmatrix} c_{n+1,x} \\ -a_{n+1,x} \\ f_{n+1,x} \\ -d_{n+1,x} \end{pmatrix} = \begin{pmatrix} \partial & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 \\ 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & -\partial \end{pmatrix} \begin{pmatrix} c_{n+1} \\ a_{n+1} \\ f_{n+1} \\ d_{n+1} \end{pmatrix} = J_1 \begin{pmatrix} c_{n+1} \\ a_{n+1} \\ f_{n+1} \\ d_{n+1} \end{pmatrix}. \tag{3.8}$$

When $n = 2$, the first isospectral integrable couplings system in (3.8) is

$$\begin{cases} q_t = \dfrac{1}{\varepsilon - 1}(q_{xx} - \varepsilon p_{xx} + 2qq_x + 2\varepsilon pp_x - 2\varepsilon qp_x - 2\varepsilon pq_x) - r_x, \ \varepsilon \neq 1, \\ r_t = \dfrac{1}{\varepsilon - 1}(\varepsilon s_x - r_x - 2\varepsilon(pr)_x + 2(qr)_x + 2\varepsilon(ps)_x - 2(qs)_x), \ \varepsilon \neq 1, \\ p_t = \dfrac{1}{\varepsilon - 1}(p_{xx} - q_{xx} - 2qq_x - 2\varepsilon pp_x + 2(pq)_x) - s_x, \ \varepsilon \neq 1, \\ s_t = \dfrac{1}{\varepsilon - 1}(r_{xx} - s_{ss} + 2(pr)_x - 2(qr)_x - 2\varepsilon(ps)_x + 2(qs)_x), \ \varepsilon \neq 1. \end{cases} \tag{3.9}$$

The Eq. (3.9) is a novel isospectral integrable couplings of the C-KdV equation with a perturbation term ε. And we find that it can be reduced to the C-KdV integrable coupling [50] if we take $\varepsilon = 0$.

3.2. Bi-Hamiltonian structures of the isospectral C-KdV integrable couplings hierarchy (3.8)

Through calculations, we found that the trace identity cannot generate the Hamiltonian structures of the above integrable couplings system. Therefore, in order to apply the quadratic constant equation we represent the matrix Lie algebra g as column forms. Let us consider the Lie algebra g. For $a, b \in g$, let $a = \sum_{i=1}^{6} a_i g_i,\ b = \sum_{i=1}^{6} b_i g_i$, we see that

$$\begin{aligned}
[a,b] =&(a_2b_3 - a_3b_2 + \varepsilon a_5b_6 - \varepsilon a_6b_5)g_1 + (2a_1b_2 - 2a_2b_1 + 2\varepsilon a_4b_5 - 2\varepsilon a_5b_4)g_2\\
&+ (2a_3b_1 - 2a_1b_3 - 2\varepsilon a_4b_6 + 2\varepsilon a_6b_4)g_3 + (a_2b_6 - a_6b_2 + a_5b_3 - a_3b_5)g_4\\
&+ (2a_1b_5 - 2a_5b_1 + 2a_4b_2 - 2a_2b_4)g_5 + (2a_6b_1 - 2a_1b_6 + 2a_3b_4 - 2a_4b_3)g_6\\
=:&(a_2b_3 - a_3b_2 + \varepsilon a_5b_6 - \varepsilon a_6b_5, 2a_1b_2 - 2a_2b_1 + 2\varepsilon a_4b_5 - 2\varepsilon a_5b_4,\\
&2a_3b_1 - 2a_1b_3 - 2\varepsilon a_4b_6 + 2\varepsilon a_6b_4, a_2b_6 - a_6b_2 + a_5b_3 - a_3b_5,\\
&2a_1b_5 - 2a_5b_1 + 2a_4b_2 - 2a_2b_4, 2a_6b_1 - 2a_1b_6 + 2a_3b_4 - 2a_4b_3)^T \in R^6,
\end{aligned} \tag{3.10}$$

where R^6 stands for the real linear space on R. Now put (3.10) into another form

$$[a,b]^T = (a_1,a_2,a_3,a_4,a_5,a_6)\begin{pmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6\\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0\\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4\\ 0 & 2\varepsilon b_5 & -2\varepsilon b_6 & 0 & 2b_2 & -2b_3\\ \varepsilon b_6 & -2\varepsilon b_4 & 0 & b_3 & -2b_1 & 0\\ -\varepsilon b_5 & 0 & 2\varepsilon b_4 & -b_2 & 0 & 2b_1 \end{pmatrix} =: a^T R(b). \tag{3.11}$$

When we use the quadratic-form identity, a constant symmetric matrix F should be applied, which satisfies the matrix equation

$$R(b)F = -(R(b)F)^T.$$

After a simple calculation, one has

$$F = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1 & 0\\ 2 & 0 & 0 & 2\varepsilon & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & \varepsilon\\ 0 & 1 & 0 & 0 & \varepsilon & 0 \end{pmatrix}.$$

Define the linear functional

$$\{a,b\} = a^T F b,\ a,b \in \tilde{g}.$$

Consider a linear map σ:

$$\sigma : g \longrightarrow R^6,\ \bar{U} = \begin{pmatrix} a & b & \varepsilon e & \varepsilon f \\ c & -a & \varepsilon d & -\varepsilon e \\ e & f & a & b \\ d & -e & c & -a \end{pmatrix} \longrightarrow (a, b, c, e, f, d)^T \in R^6,$$

then σ is an isomorphism from the Lie algebra g to R^6. It is verified that R_6 is also a Lie algebra equipped with the commutation operation (3.11). The corresponding loop algebra with R^6 is defined by $\widetilde{R^6}$, along with the commutator $[a(n), b(m)] = [a, b]\lambda^{m+n},\ m, n \in \mathbb{Z}$. Employing $\widetilde{R^6}$, we consider the isospectral Lax pair

$$\begin{cases} \Phi_x = \bar{U}\Phi, \\ \Phi_t = \bar{V}_1^{[n]}\Phi, \\ \lambda_t = 0, \end{cases} \tag{3.12}$$

where

$$\bar{U} = (\frac{q-\lambda}{2}, -r, 1, \frac{p-\lambda}{2}, -s, 1)^T,$$
$$\bar{V}_1^{[n]} = \sum_{i=0}^{n}(a_i, b_i, c_i, d_i, e_i, f_i)^T\lambda^{n-i} + \frac{1}{2}(c_{n+1}, 0, 0, f_{n+1}, 0, 0)^T.$$

It is easy to find that the compatibility condition of (3.12), i.e., the zero curvature equation (3.7) also generates the soliton equation hierarchy (3.8). A direct computation gives

$$\begin{aligned} &\left\{\bar{V}_1, \frac{\partial \bar{U}}{\partial q}\right\} = a + d,\ \left\{\bar{V}_1, \frac{\partial \bar{U}}{\partial r}\right\} = -c - f,\ \left\{\bar{V}_1, \frac{\partial \bar{U}}{\partial p}\right\} = a + \varepsilon d, \\ &\left\{\bar{V}_1, \frac{\partial \bar{U}}{\partial s}\right\} = -c - \varepsilon f,\ \left\{\bar{V}_1, \frac{\partial \bar{U}}{\partial \lambda}\right\} = -2a - (1+\varepsilon)d. \end{aligned} \tag{3.13}$$

Substituting the results of (3.13) into the quadratic-form identity

$$\frac{\delta}{\delta \bar{u}}\left\{\bar{V}_1, \frac{\delta \bar{U}}{\delta \lambda}\right\} = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\{\bar{V}_1, \frac{\delta \bar{U}}{\delta \bar{u}}\right\}$$

and balancing the coefficients of each power of λ gives rise to

$$\frac{\delta}{\delta \bar{u}}\left(-2a_{n+1} - (\varepsilon+1)d_{n+1}\right) = (\gamma - n)\begin{pmatrix} a_n + d_n \\ -c_n - f_n \\ a_n + \varepsilon d_n \\ -c_n - \varepsilon f_n \end{pmatrix}. \tag{3.14}$$

In terms of the initial values of Eq. (3.14), we find that $\lambda = 0$ when $n = 0$. Hence, we have

$$\frac{\delta}{\delta \bar{u}}\left(\frac{-2a_{n+2}-(\varepsilon+1)d_{n+2}}{n+1}\right) = \begin{pmatrix} a_{n+1}+d_{n+1} \\ -c_{n+1}-f_{n+1} \\ a_{n+1}+\varepsilon d_{n+1} \\ -c_{n+1}-\varepsilon f_{n+1} \end{pmatrix} = \frac{\delta \bar{H}_{n+1}}{\delta \bar{u}}.$$

From the recursion relationship (3.3), we have

$$\begin{pmatrix} a_{n+1}+d_{n+1} \\ -c_{n+1}-f_{n+1} \\ a_{n+1}+\varepsilon d_{n+1} \\ -c_{n+1}-\varepsilon f_{n+1} \end{pmatrix} = \frac{1}{\varepsilon-1}\begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}\begin{pmatrix} a_n+d_n \\ -c_n-f_n \\ a_n+\varepsilon d_n \\ -c_n-\varepsilon f_n \end{pmatrix} = \frac{1}{\varepsilon-1}\bar{L}\begin{pmatrix} a_n+d_n \\ -c_n-f_n \\ a_n+\varepsilon d_n \\ -c_n-\varepsilon f_n \end{pmatrix},$$

where

$$\begin{aligned}
L_{11} =& \partial - \partial^{-1}q\partial + \varepsilon\partial^{-1}p\partial,\ L_{12} = \varepsilon s + \varepsilon\partial^{-1}s\partial - r - \partial^{-1}r\partial, \\
L_{13} =& -\partial + \partial^{-1}q\partial - \partial^{-1}p\partial,\ L_{14} = r + \partial^{-1}r\partial - s - \partial^{-1}s\partial, \\
L_{21} =& 2(\varepsilon-1),\ L_{22} = \varepsilon p - q - \partial,\ L_{23} = 0,\ L_{24} = q - p + \partial, \\
L_{31} =& \varepsilon(\partial^{-1}q\partial - \partial^{-1}p\partial - \partial),\ L_{32} = \varepsilon(r + \partial^{-1}r\partial - s - \partial^{-1}s\partial), \\
L_{33} =& \partial - \partial^{-1}q\partial + \varepsilon\partial^{-1}p\partial,\ L_{34} = \varepsilon s + \varepsilon\partial^{-1}s\partial - r - \partial^{-1}r\partial, \\
L_{41} =& 0,\ L_{42} = \varepsilon(\partial + q - p),\ L_{43} = 2(\varepsilon-1),\ L_{44} = \varepsilon p - q - \partial.
\end{aligned}$$

Therefore, we derive the bi-Hamiltonian structures of the hierarchy (3.8) as follows:

$$\bar{u}_{t_n} = \bar{k}_n = \begin{pmatrix} c_{n+1,x} \\ -a_{n+1,x} \\ f_{n+1,x} \\ -d_{n+1,x} \end{pmatrix} = \bar{J}\begin{pmatrix} a_{n+1}+d_{n+1} \\ -c_{n+1}-f_{n+1} \\ a_{n+1}+\varepsilon d_{n+1} \\ -c_{n+1}-\varepsilon f_{n+1} \end{pmatrix} = \bar{J}\frac{\delta \bar{H}_{n+1}}{\delta \bar{u}} = \bar{J}\bar{L}\frac{\delta \bar{H}_n}{\delta \bar{u}}, \tag{3.15}$$

with the Hamiltonian functional

$$\bar{H}_n = \frac{-2a_{n+2}-(\varepsilon+1)d_{n+2}}{n+1},$$

and the first Hamiltonian operator

$$\bar{J} = \frac{1}{\varepsilon-1}\begin{pmatrix} 0 & -\varepsilon\partial & 0 & \partial \\ -\varepsilon\partial & 0 & \partial & 0 \\ 0 & \partial & 0 & -\partial \\ \partial & 0 & -\partial & 0 \end{pmatrix},$$

and the second Hamiltonian operator

$$M = \bar{J}\bar{L} = \frac{1}{(\varepsilon-1)^2}\begin{pmatrix} \varepsilon M_1 & M_2 \\ M_2 & M_1 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 2(1-\varepsilon)\partial & 2\partial^2 + 2\partial q - (\varepsilon+1)\partial p \\ -2\partial^2 + 2q\partial - (\varepsilon+1)p\partial & -(\varepsilon+1)(\partial s + s\partial) + 2(\partial r + r\partial) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 2(\varepsilon-1)\partial & -(\varepsilon+1)(\partial^2 + \partial q) + 2\varepsilon\partial p \\ (\varepsilon+1)(\partial^2 - q\partial) & -(\varepsilon+1)(r\partial + \partial r) + 2\varepsilon(s\partial + \partial s) \end{pmatrix}.$$

As we all know that a bi-Hamiltonian structure property means that M and $\bar{J}$ constitute a Hamiltonian pair, or, $\bar{M} = \mu\bar{J} + \nu M$, for $\forall \mu, \nu \in \mathbb{R}$, is a Hamiltonian operator. As a direct result of the bi-Hamiltonian structure, we can conclude that the soliton hierarchy is integrable in the sense of Liouville, and thus we can obtain the Abelian algebra of symmetries

$$[\bar{K}_i, \bar{K}_j] = \bar{K}_i^{'}(\bar{u})[\bar{K}_j] - \bar{K}_j^{'}(\bar{u})[\bar{K}_i], \ i, j \geq 0, \tag{3.16}$$

and the component Abelian algebras of conserved functionals

$$\begin{aligned} \{\bar{H}_i, \bar{H}_j\}_{\bar{J}} &= \int (\frac{\delta \bar{H}_i}{\delta \bar{u}})^T \bar{J}\frac{\delta \bar{H}_j}{\delta \bar{u}} dx = 0, \\ \{\bar{H}_i, \bar{H}_j\}_{M} &= \int (\frac{\delta \bar{H}_i}{\delta \bar{u}})^T M\frac{\delta \bar{H}_j}{\delta \bar{u}} dx = 0. \end{aligned} \tag{3.17}$$

3.3. A nonisospectral integrable coupling of C-KdV hierarchy

According to the loop algebra $\widehat{g}$, we introduce a spectral problem

$$\begin{cases} \Psi_x = \bar{U}\Psi, \\ \Psi_t = \bar{V}_2\Psi, \end{cases} \tag{3.18}$$

where

$$\begin{aligned} \bar{U} &= \frac{q}{2}g_1(0) - \frac{1}{2}g_1(1) - rg_2(0) + g_3(0) + \frac{p}{2}g_4(0) - \frac{1}{2}g_4(1) - sg_5(0) + g_6(0), \\ \bar{V}_2 &= \sum_{j\geq 0} \bar{a}_j g_1(-j) + \bar{b}_j g_2(-j) + \bar{c}_j g_3(-j) + \bar{d}_j g_4(-j) + \bar{e}_j g_5(-j) + \bar{f}_j g_6(-j), \end{aligned}$$

along with a spectral evolution $\lambda_t = \sum_{j\geq 0} k_j(t)\lambda^{-j}$. Solving the nonisospectral stationary zero curvature equation corresponding to spectral problem (3.18)

$$\bar{V}_{2,x} = [\bar{U}, \bar{V}_2] + \frac{\partial \bar{U}}{\partial \lambda}\lambda_t, \tag{3.19}$$

and balancing the powers of λ, we obtain the following recurrence relations

$$\begin{cases} \bar{a}_{j,x} = -r\bar{c}_j - \bar{b}_j - \varepsilon s\bar{f}_j - \varepsilon\bar{e}_j - \dfrac{1}{2}k_j(t), \\ \bar{b}_{j,x} = 2r\bar{a}_j + 2\varepsilon s\bar{d}_j + q\bar{b}_j + \varepsilon p\bar{e}_j - \bar{b}_{j+} - \varepsilon\bar{e}_{j+1}, \\ \bar{c}_{j,x} = 2\bar{a}_j + \bar{c}_{j+1} - q\bar{c}_j + 2\varepsilon\bar{d}_j - \varepsilon p\bar{f}_j + \varepsilon\bar{f}_{j+1}, \\ \bar{d}_{j,x} = -s\bar{c}_j - \bar{b}_j - r\bar{f}_j - \bar{e}_j - \dfrac{1}{2}k_j(t), \\ \bar{e}_{j,x} = 2r\bar{d}_j + 2s\bar{a}_j + p\bar{b}_j + q\bar{e}_j - \bar{b}_{j+1} - \bar{e}_{j+1}, \\ \bar{f}_{j,x} = 2\bar{a}_j + \bar{c}_{j+1} - p\bar{c}_j + 2\bar{d}_j - q\bar{f}_j + \bar{f}_{j+1}. \end{cases} \tag{3.20}$$

Taking the initial values $\bar{a}_0 = \bar{d}_0 = \frac{1}{2}\beta_0(t)$, and $\bar{b}_0 = \bar{c}_0 = \bar{e}_0 = \bar{f}_0 = k_0(t) = 0$, then from the above Eq. (3.20) we can get a set of values

$$\begin{aligned}
&\bar{a}_1 = -\frac{1}{2}xk_1(t) + \beta_1(t),\ \bar{b}_1 = r\beta_0(t),\ \bar{c}_1 = -\beta_0(t),\ \bar{d}_1 = -\frac{1}{2}xk_1(t) + \beta_1(t),\\
&\bar{e}_1 = s\beta_0(t),\ \bar{f}_1 = -\beta_0(t),\ \bar{b}_2 = \frac{\beta_0(t)}{\varepsilon - 1}(r_x - \varepsilon s_x + \varepsilon pr - qr + \varepsilon qs - \varepsilon ps) + 2\beta_1(t)r - k_1(t)xr,\\
&\bar{c}_2 = -\beta_0(t)q + k_1(t)x - 2\beta_1(t),\ \bar{e}_2 = \frac{\beta_0(t)}{\varepsilon - 1}(s_x - r_x + qr - pr + \varepsilon ps - qs) + 2\beta_1(t)s - k_1(t)xs,\\
&\bar{f}_2 = -\beta_0(t)p + k_1(t)x - 2\beta_1(t),\ \bar{a}_2 = \beta_0(t)r - \frac{1}{2}k_2(t)x + \beta_2(t),\ \bar{d}_2 = \beta_0(t)s - \frac{1}{2}k_2(t)x + \beta_2(t),\\
&\bar{a}_3 = \frac{\beta_0(t)}{\varepsilon - 1}(r_x - \varepsilon s_x + 2\varepsilon pr - 2qr + 2\varepsilon qs - 2\varepsilon ps) + 2\beta_1(t)(r - 2\partial^{-1}qr)\\
&\qquad - k_1(t)\partial^{-1}(2r + xr_x) - \frac{1}{2}k_3(t)x,\\
&\bar{d}_3 = \frac{\beta_0(t)}{\varepsilon - 1}(s_x - r_x + 2qr - 2pr + 2\varepsilon ps\\
&\qquad - 2qs) + 2\beta_1(t)(s - 2\partial^{-1}qs) - k_1(t)\partial^{-1}(2s + xs_x) - \frac{1}{2}k_3(t)x,\\
&\bar{c}_3 = \frac{\beta_0(t)}{\varepsilon - 1}(q_x - \varepsilon p_x + \varepsilon p^2 + q^2 - 2\varepsilon pq) + k_1(t)(1 + xq) + 2\beta_1(t)q + k_2(t)x - 2\beta_2(t) - 2\beta_0(t)r,\\
&\bar{f}_3 = \frac{\beta_0(t)}{\varepsilon - 1}(p_x - q_x - q^2 + 2pq - \varepsilon p^2) + k_1(t)xp - 2\beta_1(t)p + k_2(t)x - 2\beta_0(t)s - 2\beta_2(t),\\
&\cdots.
\end{aligned} \tag{3.21}$$

Note that

$$\begin{aligned}
\bar{V}_{2,+}^{[n]} &= \sum_{j=0}^{m}(\bar{a}_j g_1(m-j) + \bar{b}_j g_2(m-j) + \bar{c}_j g_3(m-j) + \bar{d}_j g_4(m-j)\\
&\quad + \bar{e}_j g_5(m-j) + \bar{f}_j g_6(m-j))\\
&=: \lambda^m \bar{V}_2 - \bar{V}_{2,-}^{[m]},
\end{aligned} \tag{3.22}$$

$$\lambda_{t,+}^{[m]} = \sum_{j\geq 0} k_j(t)\lambda^{m-j} - \sum_{j=m+1}^{\infty} k_j(t)\lambda^{m-j} = \lambda^m\lambda_t - \lambda_{t,-}^{[m]},$$

we then calculate that

$$
\begin{aligned}
-(\bar{V}_{2,+}^{[m]})_x + [\bar{U}, \bar{V}_{2,+}^{[m]}] + \frac{\partial \bar{U}}{\partial \lambda}\lambda_{t,+}^{[m]} =& (\bar{V}_{2,+}^{[m]})_x - [\bar{U}, \bar{V}_{2,-}^{[m]}] - \frac{\partial \bar{U}}{\partial \lambda}\lambda_{t,-}^{[m]} \\
=& (\bar{b}_{m+1} + \varepsilon \bar{e}_{m+1})g_2(0) + (\bar{b}_{m+1} + \bar{e}_{m+1})g_5(0) \\
& - (\bar{c}_{m+1} + \varepsilon \bar{f}_{m+1})g_3(0) - (\bar{c}_{m+1} + \bar{f}_{m+1})g_6(0).
\end{aligned} \tag{3.23}
$$

Assume

$$
\bar{V}_1^{[n]} = \bar{V}_{1,+}^{[n]} + \frac{1}{2}c_{n+1}g_1(0) + \frac{1}{2}f_{n+1}g_4(0), \tag{3.24}
$$

then the zero-curvature equation

$$
\frac{\partial \bar{U}}{\partial \bar{u}}\bar{u}_t - \bar{V}_{2,x}^{[m]} + [\bar{U}, \bar{V}_2^{[m]}] + \frac{\partial \bar{U}}{\partial \lambda}\lambda_t^{[m]} = 0 \tag{3.25}
$$

leads to the nonisospectral integrable couplings system of C-KdV equation hierarchy as follows:

$$
\bar{u}_{t_m} = \begin{pmatrix} q \\ r \\ p \\ s \end{pmatrix}_{t_m} = \begin{pmatrix} \bar{c}_{m+1,x} \\ -\bar{a}_{m+1,x} \\ \bar{f}_{m+1,x} \\ -\bar{d}_{m+1,x} \end{pmatrix} = \begin{pmatrix} \partial & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 \\ 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & -\partial \end{pmatrix} \begin{pmatrix} \bar{c}_{m+1} \\ \bar{a}_{m+1} \\ \bar{f}_{m+1} \\ \bar{d}_{m+1} \end{pmatrix} = J_1 \begin{pmatrix} \bar{c}_{m+1} \\ \bar{a}_{m+1} \\ \bar{f}_{m+1} \\ \bar{d}_{m+1} \end{pmatrix}. \tag{3.26}
$$

When $m = 1$, the first nonlinear equation is as follows:

$$
\begin{cases}
q_{t_1} = -\beta_0(t)q_x + k_1(t), \\
r_{t_1} = -\beta_0(t)r_x + \dfrac{1}{2}k_2(t), \\
p_{t_1} = -\beta_0(t)p_x + k_1(t), \\
s_{t_1} = -\beta_0(t)s_x + +\dfrac{1}{2}k_2(t).
\end{cases} \tag{3.27}
$$

When $n = 2$, the second nonlinear equation is as follows:

$$
\begin{cases}
q_{t_2} = \dfrac{\beta_0(t)}{\varepsilon - 1}(q_{xx} - \varepsilon p_{xx} + 2\varepsilon p p_x + 2q q_x - 2\varepsilon(pq)_x) + k_1(t)(xq + q_x) + 2\beta_1(t)q_x \\
\qquad -2\beta_0(t)r + k_2(t),\ \varepsilon \neq 1, \\
r_{t_2} = \dfrac{\beta_0(t)}{1 - \varepsilon}(r_{xx} - \varepsilon s_{xx} + 2\varepsilon p r_x + 2\varepsilon p_x r - 2q r_x - 2q_x r - 2\varepsilon p s_x - 2\varepsilon p_x s \\
\qquad +2\varepsilon q s_x + 2\varepsilon q_x s) - 2\beta_1(t)(r_x - 2qr) + k_1(t)(2r + x r_x) + \dfrac{1}{2}k_3(t),\ \varepsilon \neq 1, \\
p_{t_2} = \dfrac{\beta_0(t)}{\varepsilon - 1}(p_{xx} - q_{xx} - 2q q_2 - 2\varepsilon p p_x - 2(pq)_x) + k_1(t)(p + x p_1) - 2\beta_0(t)s_x \\
\qquad -2\beta_1(t)p_x + k_2(t),\ \varepsilon \neq 1, \\
s_{t_2} = \dfrac{\beta_0(t)}{1 - \varepsilon}(s_{xx} - r_{xx} - 2p r_x - 2p_x r + 2q r_x + 2q_x r + 2\varepsilon p s_x + 2\varepsilon p_x s - 2q s_x \\
\qquad -2q_x s) - 2\beta_1(t)(s_x - 2qs) + k_1(t)(2s + x s_x) + \dfrac{1}{2}k_3(t),\ \varepsilon \neq 1.
\end{cases} \tag{3.28}
$$

3.4. Bi-Hamiltonian structures of the nonisospectral C-KdV integrable couplings hierarchy (3.26)

In order to obtain the bi-Hamltonian structures of nonisospectral soliton hierarchy (3.26), we are able to use another approach, which is two sets of component-trace identity (see [42])

$$\frac{\delta}{\delta\bar{u}}\int\left(\left\langle W_1,\frac{\partial U_2}{\partial\lambda}\right\rangle+\left\langle W_2,\frac{\partial U_1}{\partial\lambda}\right\rangle\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left(\left\langle W_1,\frac{\partial U_2}{\partial\bar{u}}\right\rangle+\left\langle W_2,\frac{\partial U_1}{\partial\bar{u}}\right\rangle\right), \tag{3.29}$$

$$\frac{\delta}{\delta\bar{u}}\int\left(\left\langle W_1,\frac{\partial U_1}{\partial\lambda}\right\rangle+\varepsilon\left\langle W_2,\frac{\partial U_2}{\partial\lambda}\right\rangle\right)dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left(\left\langle W_1,\frac{\partial U_1}{\partial\bar{u}}\right\rangle+\varepsilon\left\langle W_2,\frac{\partial U_2}{\partial\bar{u}}\right\rangle\right). \tag{3.30}$$

We can simply write the spectral matrix $\bar{U}$ and $\bar{V}_2$ in the form of a block matrix

$$\bar{U}=\begin{pmatrix}U_1 & \varepsilon U_2\\ U_2 & U_1\end{pmatrix},\ \bar{V}_2=\begin{pmatrix}W_1 & \varepsilon W_2\\ W_2 & W_1\end{pmatrix},$$

where

$$\begin{aligned}
&U_1=\begin{pmatrix}\frac{q-\lambda}{2} & -r\\ 1 & \frac{-q+\lambda}{2}\end{pmatrix},\ U_2=\begin{pmatrix}\frac{p-\lambda}{2} & -s\\ 1 & \frac{-p+\lambda}{2}\end{pmatrix},\\
&W_1=\begin{pmatrix}\bar{a} & \bar{b}\\ \bar{c} & -\bar{a}\end{pmatrix},\ W_2=\begin{pmatrix}\bar{d} & \bar{e}\\ \bar{f} & -\bar{d}\end{pmatrix},\\
&\bar{a}=\sum_{j\geq 0}\bar{a}_j\lambda^{-j},\ \bar{b}=\sum_{j\geq 0}\bar{b}_j\lambda^{-j},\ \bar{c}=\sum_{j\geq 0}\bar{c}_j\lambda^{-j},\\
&\bar{d}=\sum_{j\geq 0}\bar{d}_j\lambda^{-j},\ \bar{e}=\sum_{j\geq 0}\bar{e}_j\lambda^{-j},\ \bar{f}=\sum_{j\geq 0}\bar{f}_j\lambda^{-j}.
\end{aligned} \tag{3.31}$$

Substituting (3.31) in to two sets of component-trace identity (3.29) and comparing powers of λ, we obtain

$$\begin{aligned}
&\frac{\delta}{\delta\bar{u}}\int(-\bar{a}_{m+1}-\bar{d}_{m+1})dx=(m-\gamma)\begin{pmatrix}\bar{d}_m\\ -\bar{d}_m-\bar{f}_m\\ \bar{a}_m\\ -\bar{a}_m-\bar{c}_m\end{pmatrix},\\
&\frac{\delta}{\delta\bar{u}}\int(-\bar{a}_{m+1}-\varepsilon\bar{d}_{m+1})dx=(m-\gamma)\begin{pmatrix}\bar{a}_m\\ -\bar{a}_m-\bar{c}_m\\ \varepsilon\bar{d}_m\\ -\varepsilon\bar{d}_m-\varepsilon\bar{f}_m\end{pmatrix}.
\end{aligned} \tag{3.32}$$

In terms of initials of (3.21), one can find that $\gamma = 0$ via substituting $m = 0$ into the above Eq. (3.32). Hence, we have

$$\frac{\delta}{\delta\bar{u}}\int(\frac{-\bar{a}_{m+2}-\bar{d}_{m+2}}{m+1})dx = \begin{pmatrix} \bar{d}_{m+1} \\ -\bar{d}_{m+1}-\bar{f}_{m+1} \\ \bar{a}_{m+1} \\ -\bar{a}_{m+1}-\bar{c}_{m+1} \end{pmatrix} = \frac{\delta\widetilde{H}_{1_{m+1}}}{\delta\bar{u}},$$

$$\frac{\delta}{\delta\bar{u}}\int(\frac{-\bar{a}_{m+2}-\varepsilon\bar{d}_{m+2}}{m+1})dx = \begin{pmatrix} \bar{a}_{m+1} \\ -\bar{a}_{m+1}-\bar{c}_{m+1} \\ \varepsilon\bar{d}_{m+1} \\ -\varepsilon\bar{d}_{m+1}-\varepsilon\bar{f}_{m+1} \end{pmatrix} = \frac{\delta\widetilde{H}_{2_{m+1}}}{\delta\bar{u}},$$

where the Hamiltonian functioals are $\widetilde{H}_{1_{m+1}} = \int\frac{-\bar{a}_{m+2}-\bar{d}_{m+2}}{m+1}dx,\ \widetilde{H}_{2_{m+1}} = \int\frac{-\bar{a}_{m+2}-\varepsilon\bar{d}_{m+2}}{m+1}dx.$ For the first component, one has the Hamiltonian structures of the nonisospectral integrable couplings hierarchy (3.26) as follows:

$$\bar{u}_{t_m} = \begin{pmatrix} 0 & 0 & -\partial & -\partial \\ 0 & 0 & -\partial & 0 \\ -\partial & -\partial & 0 & 0 \\ -\partial & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{d}_m \\ -\bar{d}_m-\bar{f}_m \\ \bar{a}_m \\ -\bar{a}_m-\bar{c}_m \end{pmatrix} = \bar{J}_1\frac{\delta\bar{H}_{1_m}}{\delta\bar{u}}. \tag{3.33}$$

From the recurrence relationship (3.20) and (3.33), the recursion operator N is obtained

$$\begin{pmatrix} \bar{d}_{m+1} \\ -\bar{d}_{m+1}-\bar{f}_{m+1} \\ \bar{a}_{m+1} \\ -\bar{a}_{m+1}-\bar{c}_{m+1} \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{pmatrix}\begin{pmatrix} \bar{d}_m \\ -\bar{d}_m-\bar{f}_m \\ \bar{a}_m \\ -\bar{a}_m-\bar{c}_m \end{pmatrix} + \frac{x}{2}k_{m+1}(t)\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \frac{\partial^{-1}}{2}k_m(t)\begin{pmatrix} p \\ -p \\ q \\ -q \end{pmatrix},$$

where

$$\begin{aligned} N_{11} =& -\partial+\partial^{-1}q\partial-\varepsilon\partial^{-1}p\partial-\varepsilon s-\varepsilon\partial^{-1}s\partial+r+\partial^{-1}r\partial, \\ N_{12} =& r+\partial^{-1}r\partial-\varepsilon s-\varepsilon\partial^{-1}s\partial,\ N_{13} = \partial+s+\partial^{-1}s\partial-r-\partial^{-1}r\partial+\partial^{-1}p\partial-\partial^{-1}q\partial, \\ N_{14} =& s+\partial^{-1}s\partial-r-\partial^{-1}r\partial,\ N_{22} = \partial+\varepsilon s+\varepsilon\partial^{-1}s\partial-r-\partial^{-1}r\partial+q-\varepsilon p, \end{aligned}$$

$$\begin{aligned}
N_{21} =& 2\partial + \varepsilon s + \varepsilon\partial^{-1}s\partial - r - \partial^{-1}r\partial + \varepsilon\partial^{-1}p\partial - \partial^{-1}q\partial + 2 - 2\varepsilon + q - \varepsilon p,\\
N_{23} =& -2\partial + r + \partial^{-1}r\partial - s - \partial^{-1}s\partial + \partial^{-1}q\partial - \partial^{-1}p\partial + p - q,\\
N_{24} =& r + \partial^{-1}r\partial - s - \partial^{-1}s\partial - \partial + p - q,\ N_{32} = \varepsilon s + \varepsilon\partial^{-1}s\partial - \varepsilon r - \varepsilon\partial^{-1}r\partial,\\
N_{31} =& \varepsilon s + \varepsilon\partial^{-1}s\partial - \varepsilon r - \varepsilon\partial^{-1}r\partial + \varepsilon\partial^{-1}p\partial - \varepsilon\partial^{-1}q\partial + \varepsilon\partial,\\
N_{33} =& -\partial + r + \partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + \partial^{-1}q\partial - \varepsilon\partial^{-1}p\partial,\\
N_{41} =& -2\varepsilon\partial + \varepsilon r + \varepsilon\partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + \varepsilon\partial^{-1}q\partial - \varepsilon\partial^{-1}p\partial + \varepsilon p - \varepsilon q,\\
N_{43} =& 2\partial + \varepsilon\partial^{-1}p\partial - \partial^{-1}q\partial + \varepsilon\partial^{-1}s\partial + \varepsilon s - r - \partial^{-1}r\partial + q - \varepsilon p + 2 - 2\varepsilon,\\
N_{42} =& -\varepsilon\partial + \varepsilon r + \varepsilon\partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + \varepsilon p - \varepsilon q,\\
N_{34} =& r + \partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial,\ N_{44} = \partial + \varepsilon\partial^{-1}s\partial + \varepsilon s - r - \partial^{-1}r\partial + q - \varepsilon p.
\end{aligned}$$

Therefore, we derive the first component of the bi-Hamiltonian structures of the integrable hierarchy (3.26) as follows:

$$\bar{u}_m = \bar{J}_1\frac{\delta \bar{H}_{1_{m+1}}}{\delta \bar{u}} = \bar{J}_1 N\frac{\delta \bar{H}_{1_m}}{\delta \bar{u}} + \frac{x}{2}k_{m+1}(t)\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix} + \frac{\partial^{-1}}{2}k_m(t)\begin{pmatrix}p\\-p\\q\\-q\end{pmatrix}, \tag{3.34}$$

with the second Hamiltonian operator $\bar{J}_1 N$. For the second component, the bi-Hamiltonian structures of nonisospectral integrable couplings hierarchy (3.26) is obtained as follows:

$$\bar{u}_{t_m} = \begin{pmatrix}-\partial & -\partial & 0 & 0\\ -\partial & 0 & 0 & 0\\ 0 & 0 & -\frac{\partial}{\varepsilon} & -\frac{\partial}{\varepsilon}\\ 0 & 0 & -\frac{\partial}{\varepsilon} & 0\end{pmatrix}\begin{pmatrix}\bar{a}_m\\ -\bar{a}_m - \bar{c}_m\\ \varepsilon\bar{d}_m\\ -\varepsilon\bar{d}_m - \varepsilon\bar{f}_m\end{pmatrix} = \bar{J}_2\frac{\delta \bar{H}_{m_2}}{\delta \bar{u}}. \tag{3.35}$$

From the recurrence relationship (3.20) and (3.35), we have

$$\begin{pmatrix}\bar{a}_{m+1}\\ -\bar{a}_{m+1} - \bar{c}_{m+1}\\ \varepsilon\bar{d}_{m+1}\\ -\varepsilon\bar{d}_{m+1} - \varepsilon\bar{f}_{m+1}\end{pmatrix} = P\begin{pmatrix}\bar{a}_m\\ -\bar{a}_m - \bar{c}_m\\ \varepsilon\bar{d}_m\\ -\varepsilon\bar{d}_m - \varepsilon\bar{f}_m\end{pmatrix} + \frac{x}{2}k_{m+1}(t)\begin{pmatrix}-1\\1\\-\varepsilon\\ \varepsilon\end{pmatrix} + \frac{\partial^{-1}}{2}k_m(t)\begin{pmatrix}p\\-p\\ \varepsilon q\\-\varepsilon q\end{pmatrix}, \tag{3.36}$$

with the recursion operator P

$$P = \begin{pmatrix}P_{11} & P_{12} & P_{13} & P_{14}\\ P_{21} & P_{22} & P_{23} & P_{24}\\ P_{31} & P_{32} & P_{33} & P_{34}\\ P_{41} & P_{42} & P_{43} & P_{44}\end{pmatrix},$$

where

$$\begin{aligned}
P_{11} =& -\partial + \partial^{-1}q\partial - \varepsilon\partial^{-1}p\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + r + \partial^{-1}r\partial,\\
P_{12} =& r + \partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial,\ P_{13} = \partial + s + \partial^{-1}s\partial - r - \partial^{-1}r\partial + \partial^{-1}p\partial - \partial^{-1}q\partial,\\
P_{14} =& s + \partial^{-1}s\partial - r - \partial^{-1}r\partial,\ P_{22} = \partial + \varepsilon s + \varepsilon\partial^{-1}s\partial - r - \partial^{-1}r\partial + q - \varepsilon p,\\
P_{24} =& r + \partial^{-1}r\partial - s - \partial^{-1}s\partial - \partial + p - q,\\
P_{21} =& 2\partial + \varepsilon s + \varepsilon\partial^{-1}s\partial - r - \partial^{-1}r\partial + \varepsilon\partial^{-1}p\partial - \partial^{-1}q\partial - 2 - 2\varepsilon + q - \varepsilon p,\\
P_{23} =& -2\partial + r + \partial^{-1}r\partial - s - \partial^{-1}s\partial + \partial^{-1}q\partial - \partial^{-1}p\partial + p - q,\\
P_{31} =& \varepsilon s + \varepsilon\partial^{-1}s\partial - \varepsilon r - \varepsilon\partial^{-1}r\partial + \varepsilon\partial^{-1}p\partial - \varepsilon\partial^{-1}q\partial + \varepsilon\partial,\\
P_{32} =& \varepsilon s + \varepsilon\partial^{-1}s\partial - \varepsilon r - \varepsilon\partial^{-1}r\partial,\ P_{34} = r + \partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial,\\
P_{33} =& -\partial + r + \partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + \partial^{-1}q\partial - \varepsilon\partial^{-1}p\partial,\\
P_{41} =& -2\varepsilon\partial + \varepsilon r + \varepsilon\partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + \varepsilon\partial^{-1}q\partial - \varepsilon\partial^{-1}p\partial + \varepsilon p - \varepsilon q,\\
P_{43} =& 2\partial + \varepsilon\partial^{-1}p\partial - \partial^{-1}q\partial + \varepsilon\partial^{-1}s\partial + \varepsilon s - r - \partial^{-1}r\partial + q - \varepsilon p + 2 - 2\varepsilon,\\
P_{42} =& -\varepsilon\partial + \varepsilon r + \varepsilon\partial^{-1}r\partial - \varepsilon s - \varepsilon\partial^{-1}s\partial + \varepsilon p - \varepsilon q,\\
P_{44} =& \partial + \varepsilon\partial^{-1}s\partial + \varepsilon s - r - \partial^{-1}r\partial + q - \varepsilon p.
\end{aligned}$$

Therefore, we derive the second component of the bi-Hamiltonian structures of the integrable hierarchy (3.26) as follows:

$$\bar{u}_m = \bar{J}_2\frac{\delta \bar{H}_{2m+1}}{\delta \bar{u}} = \bar{J}_2 P\frac{\delta \bar{H}_{2m}}{\delta \bar{u}} + \frac{x}{2}k_{m+1}(t)\begin{pmatrix} -1 \\ 1 \\ -\varepsilon \\ \varepsilon \end{pmatrix} + \frac{\partial^{-1}}{2}k_m(t)\begin{pmatrix} p \\ -p \\ \varepsilon q \\ -\varepsilon q \end{pmatrix}, \tag{3.37}$$

with the second Hamiltonian operator $\bar{J}_2 P$.

4. Infinte conservation laws of C-KdV equation with the variable coefficient (2.16)

In this section, we further investigate the properties of the equations derived from the isospectral and nonisospectral integrable hierarchies (2.14), including the infinte conservation laws of certain nonlinear evolution equations.

Now, we want to find infinitely many conserved quantities of the C-KdV equation with the variable coefficient (2.16). It is easy to get its lax pair

$$\begin{cases} \varphi_x = U\varphi,\ U = \begin{pmatrix} \dfrac{q-\lambda}{2} & -r \\ 1 & \dfrac{-q+\lambda}{2} \end{pmatrix}, \\ \varphi_t = W\varphi,\ W = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \end{cases} \tag{4.1}$$

where $A=\frac{1}{2}\alpha_0(t)\lambda^2+\alpha_1(t)\lambda+\alpha_0(t)r+\alpha_2(t),\ B=\alpha_0(t)r\lambda-\alpha_0(t)r_x+\alpha_0(t)rq+2\alpha_0(t)r,\ C=-\alpha_0(t)\lambda-\alpha_0(t)q-2\alpha_1(t)$. The lax pair can be wirtten as

$$\begin{cases}\varphi_{1,x}=\dfrac{1}{2}(q-\lambda)\varphi_1-r\varphi_2,\\ \varphi_{2,x}=\varphi_1+\dfrac{1}{2}(\lambda-q)\varphi_2,\end{cases}\quad\begin{cases}\varphi_{1,t}=A\varphi_1+B\varphi_2,\\ \varphi_{2,t}=C\varphi_1-A\varphi_2.\end{cases}\tag{4.2}$$

Since

$$\frac{\partial}{\partial t}\frac{\partial}{\partial x}\log\varphi_1=\frac{\partial}{\partial x}\frac{\partial}{\partial t}\log\varphi_1,\tag{4.3}$$

it follows that

$$\frac{\partial}{\partial t}(\frac{1}{2}(q-\lambda)-r\kappa)=\frac{\partial}{\partial x}(A+B\kappa).\tag{4.4}$$

Assume

$$\kappa=\frac{\varphi_2}{\varphi_1}=\kappa_0+\frac{\kappa_1}{\lambda}+\frac{\kappa_2}{\lambda^2}+\frac{\kappa_3}{\lambda^3}+\cdots,\tag{4.5}$$

then we have

$$\kappa_x=\frac{\varphi_{2,x}}{\varphi_1}-\frac{\varphi_{1,x}\varphi_2}{\varphi_1^2}=(\lambda-q)\kappa+r\kappa^2+1.\tag{4.6}$$

Substituting (4.5) into (4.6) and comparing the powers of λ^{-j}, we get

$$\begin{aligned}&\kappa_0=0,\ \lambda\kappa_1+1=0,\ \kappa_1=-1,\\ &q+\kappa_2=0,\ \kappa_2=-q,\\ &2r\kappa_1^2-q\kappa_2+\kappa_3=\kappa_{2,x},\ \kappa_3=q_x+q^2-2r,\\ &2r\kappa_1\kappa_2-q\kappa_3+\kappa_4=\kappa_{3,x},\ \kappa_4=q_{xx}+3qq_x-4qr-2r_x+q^3,\\ &\vdots\\ &\kappa_{n,x}=\kappa_{n+1}-q\kappa_n+r(\sum_{l+k=n}\kappa_l\kappa_k),\ \kappa_{n+1}=\kappa_{n,x}+q\kappa_n-r(\sum_{l+k=n}\kappa_l\kappa_k).\end{aligned}\tag{4.7}$$

If we change φ_1 to φ_2 in (4.3), we can obtain

$$\frac{\partial}{\partial t}(\frac{1}{2}(\lambda-q)+\frac{\varphi_1}{\varphi_2})=\frac{\partial}{\partial x}(C\frac{\varphi_1}{\varphi_2}-A).$$

Set

$$\nu=\frac{\varphi_1}{\varphi_2}=\frac{\nu_1}{\lambda}+\frac{\nu_2}{\lambda^2}+\frac{\nu_3}{\lambda^3}+\cdots,$$

then

$$\nu_x=\frac{\varphi_{1,x}\varphi_2-\varphi_1\varphi_{2,x}}{\varphi_2^2}=(q-\lambda)\nu-\nu^2-r.$$

Using the same method as above, we get

$$\begin{aligned}&\nu_1+r=0,\ \nu=-r,\ q\nu_1-\nu_2=\nu_{1,x},\ \nu_2=-qr+r_x,-2\nu_1^2+q\nu_2-\nu_3=\nu_{2,x},\\ &\nu_3=-2r^2-q^2r+2qr_x-r_{xx}+q_xr,\ -2\nu_1\nu_2+q\nu_3-\nu_4=\nu_{3,x},\\ &\nu_4=r_{xxx}-q_{xx}r+3qq_xr-3q_xr_x-3qr_{xx}+3q^2r_x-2qr^2+2rr_x+2r^2q-q^3r,\\ &\vdots\\ &\nu_{n,x}=q\nu_n-\nu_{n+1}-\sum_{l+k=n}\nu_k\nu_l,\ \nu_{n+1}=q\nu_n-\nu_{n,x}-\sum_{l+k=n}\nu_k\nu_l.\end{aligned}$$

We find that the conservation laws are the same for ν and κ. Next we start from κ to find the conservation law of the nonlinear evolution equation (2.16). From (4.4), the conserved density is $\frac{q}{2} - r\kappa$, and the corresponding conserved flow is $A + B\kappa$. For different λ^{-j} there are different conserved densities corresponding to different conserved flows. For example, taking λ^{-1} gives rise to

$$\frac{\partial}{\partial t}(r) = \frac{\partial}{\partial x}(\alpha_0(t)r_x - 2\alpha_0(t)qr - 2\alpha_0(t)r),$$

and taking λ^{-2} gives

$$\frac{\partial}{\partial t}(qr) = \frac{\partial}{\partial x}(\alpha_0(t)rq_x + \alpha_0(t)qr_x - 2\alpha_0(t)r^2 - 2\alpha_0(t)qr).$$

In this way we can find a higher conserved density $\frac{1}{2}q - r\kappa_3$, $\frac{1}{2}q - r\kappa_4$, $\frac{1}{2}q - r\kappa_5$, $\cdots$ and their corresponding conserved flows.

5. Lie point symmetries and invariant solutions

In recent years, Lie symmetry analysis method has been widely applied to differential equations [4], Lie algebras [14, 28], and other mathematical fields. It is one of the most effective methods for deriving explicit solutions to nonlinear differential equations. Zhao considered the optimal system, similarity reductions, invariant solutions, and conservation laws of the Heisenberg equation [55]. Furthermore, Seadawy, Rizvi et al. applied Lie symmetry analysis to obtain conservation laws as well as exact solutions for Chen-Lee-Liu equation and Selkov- Schnakenberg system, among others [1, 3, 35, 35–37]. In this section, we consider the Lie point symmetries and invariant solutions of Eqs. (2.18) and (2.21).

5.1. Lie point symmetries of Burgers equation (2.21)

First, for Eq. (2.21) $u_t - u_{xx} - 2uu_x = 0$, we consider a one-parameter Lie group of transformations

$$\begin{aligned}
x^* &= X_1(x, t, u, \varepsilon) = x + \varepsilon\xi_1(x, t, u) + O(\varepsilon^2),\\
t^* &= X_2(x, t, u, \varepsilon) = t + \varepsilon\xi_2(x, t, u) + O(\varepsilon^2),\\
u^* &= U(x, t, u, \varepsilon) = x + \varepsilon\xi_1(x, t, u) + O(\varepsilon^2),
\end{aligned}$$

with a continuous parameter ε. It satisfies $x^* = x$, $t^* = t$, $u^* = u$ when $\varepsilon = e$, without loss of generality $\varepsilon = 0$ corresponds to the identity element e. The corresponding generator of Lie group is

$$X = \xi_1(x, t, u)\frac{\partial}{\partial x} + \xi_2(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u}.$$

Thus, the corresponding $2th$ extend infinitesimal generators are given by

$$X^{(2)} = \xi_1(x, t, u)\frac{\partial}{\partial x} + \xi_2(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u} + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_t^{(1)}\frac{\partial}{\partial u_t} + \eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}},$$

where

$$\begin{aligned}
&\eta_x^{(1)} = D_x\eta - (D_x\xi_1)u_x - (D_x\xi_2)u_t, \quad \eta_t^{(1)} = D_t\eta - (D_t\xi_1)u_x - (D_t\xi_2)u_t,\\
&\eta_{xx}^{(2)} = D_x\eta_x^{(1)} - (D_x\xi_1)u_{xx} - (D_x\xi_2)u_{xt},
\end{aligned} \tag{5.1}$$

and the $D_x,\ D_t$ are the total derivative operators

$$D_x = \frac{D}{Dx} = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots + u_{xi_1i_2\cdots i_n}\frac{\partial}{\partial u_{i_1i_2\cdots i_n}},$$
$$D_t = \frac{D}{Dt} = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots + u_{ti_1i_2\cdots i_n}\frac{\partial}{\partial u_{i_1i_2\cdots i_n}},\ i_j = x, t.$$

The invariance condition of Eq. (2.21) here is

$$\eta^{(2)}_{xx} + 2u_x\eta + 2u\eta^{(1)}_x - \eta^{(1)}_t = 0, \tag{5.2}$$

when

$$u_t = 2uu_x + u_{xx},$$

where $\eta^{(2)}_{xx},\ \eta^{(1)}_x,\ \eta^{(1)}_t$ are given by Eq. (5.1). From the Eq. (5.2) we obtain the overdetermined system of the partial differential equations

$$(\xi_1)_x = \frac{1}{2}(\xi_2)_t,\ (\xi_1)_t = -(\xi_2)_t u - 2\eta,\ (\xi_1)_u = 0,\ (\xi_2)_{ttt} = 0,\ (\xi_2)_x = 0,$$
$$(\xi_2)_u = 0,\ \eta_t = -\frac{1}{2}(\xi_2)_{tt}u,\ \eta_x = \frac{1}{4}(\xi_2)_{tt},\ \eta_u = -\frac{1}{2}(\xi_2)_t.$$

Solving the overdetermined system, we have

$$\xi_1 = \frac{1}{2}\alpha x + \frac{1}{2}\beta x - 2\delta t + \epsilon, \quad \xi_2 = \frac{1}{2}\alpha t^2 + \beta t + \gamma, \quad \eta = -\frac{1}{2}\alpha tu - \frac{1}{2}\beta u - \frac{1}{4}\alpha x + \delta,$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon$ are five arbitrary parameters. Hence a nontrivial five-parameter Lie group of transformations acting on $(x,t,u)-$space is admitted by the Burgers equation (2.21) with infinitesimal generators given by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ X_3 = -2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ X_4 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u},\\ &X_5 = \frac{1}{2}tx\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - (\frac{1}{4}x + \frac{1}{2}tu)\frac{\partial}{\partial u}. \end{aligned} \tag{5.3}$$

The commutator table for the Lie algebra arising from the infinitesimal generators (5.3) are

By solving the initial value problems for the first order system of ODE's

$$\begin{aligned} &\frac{dx^*}{d\varepsilon} = \xi_1(x^*, t^*, u^*),\ x^*|_{\varepsilon=0} = x,\\ &\frac{dt^*}{d\varepsilon} = \xi_2(x^*, t^*, u^*),\ t^*|_{\varepsilon=0} = t,\\ &\frac{du^*}{d\varepsilon} = \eta(x^*, t^*, u^*),\ u^*|_{\varepsilon=0} = u, \end{aligned}$$

we obtain the corresponding one-parameter Lie group of transformations

$$\begin{aligned} G_1 &: (x,t,u) \longrightarrow (x, t+\varepsilon, u),\\ G_2 &: (x,t,u) \longrightarrow (x+\varepsilon, t, u),\\ G_3 &: (x,t,u) \longrightarrow (x - 2t\varepsilon, t, u+\varepsilon),\\ G_4 &: (x,t,u) \longrightarrow (e^{\frac{1}{2}\varepsilon}x, e^{\varepsilon}t, u+\varepsilon),\\ G_5 &: (x,t,u) \longrightarrow (\frac{2x}{2-\varepsilon t}, \frac{2t}{2-\varepsilon t}, u+\varepsilon). \end{aligned} \tag{5.4}$$

Table 1. The commutator table of Lie algebra.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_2$	X_1	X_4
X_2	0	0	0	$\frac{1}{2}X_2$	$-4X_3$
X_3	$-X_2$	0	0	$-\frac{1}{2}X_3$	0
X_4	$-X_1$	$-\frac{1}{2}X_2$	X_3	0	X_5
X_5	$-X_4$	$4X_3$	0	$-X_5$	0

Therefore, we can conclude that if function $u(x,t)$ is a solution of Burgers equation (2.21), then the following functions are also solutions

$$\begin{aligned}
&G_1(\varepsilon)\cdot u(x,t) = u(x, t-\varepsilon),\\
&G_2(\varepsilon)\cdot u(x,t) = u(x-\varepsilon, t),\\
&G_3(\varepsilon)\cdot u(x,t) = u(x+2t\varepsilon, t)+\varepsilon,\\
&G_4(\varepsilon)\cdot u(x,t) = e^{-\frac{1}{2}\varepsilon}u(e^{-\frac{1}{2}\varepsilon}x, e^{-\varepsilon}t),\\
&G_5(\varepsilon)\cdot u(x,t) = u((1-\frac{1}{2}\varepsilon)tx, \frac{2t}{2+\varepsilon t})+\varepsilon.
\end{aligned}$$

5.2. Invariant solutions of Burgers equuation (2.21)

Next, we find the invariant solutions of Eq. (2.21) through different infinitesimal generators corresponding to the one-parameter Lie transformation group.

5.2.1. Solutions through infinitesimal generator X_3 (parameter δ)

The corresponding one-parameter Lie group of transformations G_3 obtained at Eq. (5.4)

$$x^* = x - 2t\varepsilon,\ t^* = t,\ u^* = u + \varepsilon.$$

Invariant solutions $u = \theta(x,t)$ of Burgers equation (2.21) corresponding X_3 satisfy

$$X(u-\theta_1(x,t)) = 0,\ at\ u-\theta_1(x,t) = 0,$$

i.e.,

$$1 + 2t\frac{\partial\theta_1(x,t)}{\partial x} = 0. \tag{5.5}$$

The solution of (5.5) is obtained by solving the characteristic equations

$$\frac{dx}{2t} = \frac{dt}{0} = \frac{du}{1}, \tag{5.6}$$

which yield two invariants

$$\zeta_1 = t^2,\ f = u + \frac{x}{2t}.$$

Thus, the solutions to (5.5) are defined by the invariant form

$$u + \frac{x}{2t} = f(\zeta_1),$$

or, solving for u,

$$u = \theta_1(x,t) = f(\zeta_1) - \frac{x}{2t}. \tag{5.7}$$

Substitution of (5.7) into Burgers equation (2.21) leads to $f(\zeta)$ satisfying

$$2\zeta_1 f^{'} + f = 0,$$

so that invariant solutions of Burgers equation (2.21) resulting fron X_3 are

$$u = \theta_1(x,t) = \frac{C_1}{t} - \frac{x}{2t},$$

where C_1 is arbitrary constant.

5.2.2. Solutions through infinitesimal generator $X_1 + X_2$

For the generator $X_1 + X_2$, the characteristic equation is written as

$$\frac{dt}{1} = \frac{dx}{1} = \frac{du}{0}.$$

The solution of the characteristic equations yield two invariants of $X_1 + X_2$

$$\zeta_2 = -x + t, \ g = u.$$

Thus, the solution of Eq. (2.21) is given by the invariant form

$$u = g(-x+t) = g(\zeta_2). \tag{5.8}$$

Substituting (5.8) into (2.21), we can obtain the ordinary differential equation

$$g^{'} + 2gg^{'} - g^{''} = 0. \tag{5.9}$$

Solving this reduced equation (5.9), we obtain the closed-form solution solution as follows

$$u(x,t) = g(-x+t)\frac{-C_1 + 2\tan(\frac{C_2+t-x}{C_1})}{2C_1},$$

where $C_1,\ C_2$ are arbitrary constants.

5.3. Lie point symmetries of C-KdV equation (2.18)

In what follows, we consider the Lie point symmetries and invariant solutions of partial differential equations system (2.18)

$$u_t - u_{xx} - 2uu_x - 2v_x = 0,$$
$$v_t + v_{xx} - 2vu_x - 2v_x u = 0,$$

which admits the infinitesimal generators

$$X = \xi_1(x,t,u,v)\frac{\partial}{\partial x} + \xi_2(x,t,u,v)\frac{\partial}{\partial t} + \eta^1(x,t,u,v)\frac{\partial}{\partial u} + \eta^2(x,t,u,v)\frac{\partial}{\partial v}.$$

Thus the second extend infinitesimal generator corresponding to Eq. (2.18) takes the form

$$X^{(2)} = X + \eta_t^{(1)1}\frac{\partial}{\partial u_t} + \eta_t^{(1)2}\frac{\partial}{\partial v_t} + \eta_x^{(1)1}\frac{\partial}{\partial u_x} + \eta_x^{(1)2}\frac{\partial}{\partial v_x} + \eta_{xx}^{(2)1}\frac{\partial}{\partial u_{xx}} + \eta_{xx}^{(2)2}\frac{\partial}{\partial v_{xx}},$$

where

$$\begin{aligned}
&\eta_t^{(1)1} = D_t(\eta^1) - D_t(\xi_1)u_x - D_t(\xi_2)u_t,\ \eta_t^{(1)2} = D_t(\eta^2) - D_t(\xi_1)v_x - D_t(\xi_2)v_t,\\
&\eta_x^{(1)1} = D_x(\eta^1) - D_x(\xi_1)u_x - D_x(\xi_2)u_t,\ \eta_x^{(1)2} = D_x(\eta^2) - D_x(\xi_1)v_x - D_x(\xi_2)v_t,\\
&\eta_{xx}^{(2)1} = D_x(\eta_x^{(1)1}) - D_x(\xi_1)u_{xx} - D_x(\xi_2)u_{xt},\ \eta_{xx}^{(2)2} = D_x(\eta_x^{(1)2}) - D_x(\xi_1)v_{xx} - D_x(\xi_2)v_{xt}.
\end{aligned}$$

For PDE system (2.18), the invariance condition is

$$\begin{aligned}
&\eta_{xx}^{(2)1} + 2\eta_x^{(1)2} + 2u\eta_x^{(1)1} + u_x\eta^1 - \eta_t^{(1)1} = 0,\\
&\eta_t^{(1)2} + \eta_{xx}^{(2)2} - 2u_x\eta^2 - 2v\eta_x^{(1)1} + 2v_x\eta^1 - 2u\eta_x^{(1)2} = 0,
\end{aligned}$$

when

$$u_t = u_{xx} + 2uu_x + 2v_x,\ v_t = -v_{xx} + 2vu_x + 2v_xu.$$

From the invariance condition, we obtain the overdermined system as follows:

$$\begin{aligned}
&(\xi_1)_{tt} = 0,\ (\xi_1)_x = \frac{1}{2}(\xi_2)_t,\ (\xi_1)_u = 0,\ (\xi_1)_v = 0,\ (\xi_2)_{tt} = 0,\ (\xi_2)_x = 0,\\
&(\xi_2)_u = 0,\ (\xi_2)_v = 0, \eta^1 = -\frac{1}{2}(\xi_2)_t u - \frac{1}{2}(\xi_1)_t,\ \eta^2 = -(\xi_2)_t v.
\end{aligned} \tag{5.10}$$

The system (5.10) gives rise to

$$\xi_1 = \frac{1}{2}\varrho x + \varsigma t + \tau,\ \xi_2 = \varrho t + \sigma,\ \eta^1 = -\frac{1}{2}\varrho u - \frac{1}{2}\sigma,\ \eta^2 = -\varrho v, \tag{5.11}$$

where ϱ, σ, ς, τ are arbitrary constants. Hence a nontrivial four-parameter Lie group of transformations acting on $(x,t,u,v)-$space is admitted by the Eq. (5.11) with infinitesimal generators given by

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial u},\ X_4 = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{5.12}$$

The commutator table for the Lie algebra arising from the infinitesimal generators (5.12) is Consider infinitesimal generator $X_i,\ i = 1,2,3,4$. By solving the following system of ODEs with initial value problems

$$\begin{aligned}
&\frac{dx^*}{d\varepsilon} = \xi_1(x^*,t^*,u^*,v^*),\ x^*|_{\varepsilon=0} = x,\ \frac{dt^*}{d\varepsilon} = \xi_2(x^*,t^*,u^*,v^*),\ t^*|_{\varepsilon=0} = t,\\
&\frac{du^*}{d\varepsilon} = \eta^1(x^*,t^*,u^*,v^*),\ u^*|_{\varepsilon=0} = u,\ \frac{dv^*}{d\varepsilon} = \eta^1(x^*,t^*,u^*,v^*),\ v^*|_{\varepsilon=0} = v,
\end{aligned}$$

Table 2. The commutator table of Lie algebra.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_2	X_1
X_2	0	0	0	$\frac{1}{2}X_2$
X_3	$-X_2$	0	0	$-\frac{1}{2}X_3$
X_4	$-X_1$	$-\frac{1}{2}X_2$	$\frac{1}{2}X_3$	0

we obtain the corresponding one-parameter Lie group of transformations, respectively

$$\begin{aligned}
&P_1 : (x,t,u,v) \longrightarrow (x, t+\varepsilon, u, v),\\
&P_2 : (x,t,u,v) \longrightarrow (x+\varepsilon, t, u, v),\\
&P_3 : (x,t,u,v) \longrightarrow (x+\varepsilon t, t, u-\frac{1}{2}\varepsilon, v),\\
&P_4 : (x,t,u,v) \longrightarrow (e^{\frac{1}{2}\varepsilon}x, e^{\varepsilon}t, e^{-\frac{1}{2}\varepsilon}u, e^{-\varepsilon}v).
\end{aligned}$$

Therefore, we conclude that if $u(x,t)$, $v(x,t)$ is a solution of Eq. (2.18), then the following functions are also solutions

$$\begin{aligned}
&p_1(\varepsilon)\cdot u(x,t) = u(x, t-\varepsilon),\ p_1\cdot v(x,t) = v(x,t-\varepsilon),\\
&p_2(\varepsilon)\cdot u(x,t) = u(x-\varepsilon, t),\ p_2(\varepsilon)\cdot v(x,t) = v(x-\varepsilon,t),\\
&p_3(\varepsilon)\cdot u(x,t) = u(x-\varepsilon t, t) - \frac{1}{2}\varepsilon,\ p_3(\varepsilon)\cdot v(x,t) = v(x-\varepsilon t,t),\\
&p_4(\varepsilon)\cdot u(x,t) = e^{-\frac{1}{2}\varepsilon}u(e^{-\frac{1}{2}\varepsilon}x, e^{-\varepsilon}t),\ p_4(\varepsilon)\cdot v(x,t) = e^{-\varepsilon}v(e^{-\frac{1}{2}\varepsilon}x, e^{-\varepsilon}t).
\end{aligned}$$

5.4. Invariant solutions of C-KdV equation (2.18)

5.4.1. Solutions through X_4

For the infinitesimal generator $X_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial u} - \frac{u}{2}\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$, the characteristic equation is written as

$$\frac{2dx}{x} = \frac{dt}{t} = \frac{2du}{-u} = \frac{du}{-v}$$

which have three invariants

$$\chi = \frac{x^2}{t},\ f = xu,\ g = tv,$$

that is

$$u = \frac{1}{x}f(\chi),\ v = \frac{1}{t}g(\chi). \tag{5.13}$$

Substituting (5.13) into (2.18), we get the ordinary differential equation

$$\begin{aligned}
&(2\chi-\chi^2)f' + 2f^2 - 2f - 4\chi^2 f'' - 4\chi f f' - 4\chi^2 g' = 0,\\
&(2\chi-\chi^2)g' - \chi g + 2fg + 4\chi^2 g'' - 4\chi f' g - 4\chi f g' = 0.
\end{aligned}$$

Solving the above ordinary differential equation, we obtain the solution of Eq. (2.18)

$$u(x,t) = -\frac{x}{2t},\ v(x,t) = \frac{C_1}{t}.$$

where C_1 is a arbitiary constant.

5.4.2. Solutions through X_3

The infinitesimal generator $X_3 = t\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial u}$ gives rise to the group-invariant solution of the form

$$u = \frac{1}{t}f(\frac{1}{2}t^2) - \frac{x}{2t},\ v = \frac{1}{t}g(\frac{1}{2}t^2). \tag{5.14}$$

Substituting (5.14) into (2.18), we get the ordinary differential equation

$$f^{'} = 0,\ g^{'} = 0.$$

Solving the above ordinary differential equation, we find that the Eq. (2.18) has the closed-form solutions as follows

$$u = C_1\frac{1}{t} - \frac{x}{2t},\ v = C_2\frac{1}{t},$$

where $C_1,\ C_2$ are arbitiary constants.

5.4.3. Solutions through $X_1 + X_3$

The infinitesimal generator $X_1 + X_3 = \frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial u}$ gives rise to the group-invariant solution of the form

$$u = f(x - \frac{1}{2}t^2) - \frac{1}{2}t,\ v = g(x - \frac{1}{2}t^2). \tag{5.15}$$

Substituting (5.15) into (2.18), we derive the ODES

$$\begin{aligned} &g^{''} - 2f^{'}g - 2fg^{'} = 0, \\ &f^{'} + 2ff^{'} + 2g^{'} + \frac{1}{2} = 0. \end{aligned} \tag{5.16}$$

Thus, we obtain the solutions of PDE system (2.18) and ODES (5.16)

$$u(x,t) = -\frac{1}{2}t,\ v(x,t) = -\frac{1}{4}x + \frac{1}{8}t^2 + C_1,$$

where C_1 is a arbitrary constant.

6. Conclusions and discussions

In this paper, we have introduced a novel isospectral-nonisospectral C-KdV hierarchy by generalizing the classical C-KdV spectral problem. Through reductions of these hierarchies, various generalized forms of classical equations are obtained, such as the variable coefficients isospectral and nonisospectral C-KdV equation, the variable coefficients generalized Burgers equation, and so on. The trace identity is employed to derive the Hamiltonian structures of the isospectral plus nonisospectral C-KdV hierarchy. Inspired by our previous research on Z_2^{ε} integrable couplings, the 2×2 matrix Lie algebra A_1 is extended to the 4×4 matrix Lie algebra g, which is applied to construct the integrable couplings of isospectral and nonisospectral C-KdV hierarchy. However, we find that the trace identity cannot generate the Hamiltonian structure of the above integrable couplings system. Therefore, the bi-Hamiltonian structure of the integrable couplings is obtained by applying the quadratic-form identity. This implies that these soliton hierarchies are integrable in the sense of Liouville. Furthermore, the nonlinear evolution equations derived from isospectral plus nonisospectral hierarchy have many good properties, such as infinite conservation laws, Lie point symmetries, invariant solutions, Darboux transformation etc. As applications, we obtain infinite conservation laws for the nonisospectral C-KdV equation, Lie point symmetries, and invariant solutions for Burgers equation and partial differential equation system. Many interesting aspects of integrable couplings still warrant further investigation. For instance, how the $\bar{\partial}$-dressing method can be utilized to construct solutions to the integrable couplings hierarchy remains an open question. Future research will focus on exploring these open questions to deepen the understanding of integrable couplings and their applications in mathematical physics.

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