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THE MTH LEVEL FRACTIONAL DERIVATIVES WITH RESPECT TO ANOTHER FUNCTION

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Abstract In this paper, we study the integral equation

$$\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\Big(\psi(x) - \psi(t)\Big)^{\alpha-1} \phi(t)dt, \quad n-1 < \alpha \leq n, 0 < a < x < b.$$

We establish conditions ensuring the existence of a solution $\phi(x)$ in the space $L_1(a,b)$ and derive an explicit formula for this solution. We introduce a novel parametrization of the Hilfer fractional derivative with respect to another function ψ, which unifies and extends several classical operators. In this framework, we develop an extensive variant of Luchko's second level fractional derivative, termed the $\psi-$second level fractional derivative, encompassing the $\psi-$Riemann-Liouville, $\psi-$Caputo, and $\psi-$Hilfer derivatives as special cases. We analyze the relationships among these derivatives for various parameter values and within different function spaces, discussing their properties and significant results in fractional calculus. Finally, we propose a comprehensive generalization, the $\psi - m$th level fractional derivative, which facilitates the construction of fractional derivatives of any desired level. The main results focus on the $\psi-$second level derivative, as it unifies a wide range of established operators, simplifies the interpretation of results, and naturally supports further generalizations.

Keywords Fractional calculus, Riemann-Liouville fractional derivative, generalized fractional derivatives, $\psi-$Hilfer fractional derivative, $\psi-$Second level fractional derivative, $\psi - m$th level fractional derivatives.

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1. Introduction

One of the main purposes of this paper is to solve the integral equation (IE)

$$\varphi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\Big(\psi(x) - \psi(t)\Big)^{\alpha-1} \phi(t)dt, \quad n-1 < \alpha \leq n, 0 < a < x < b. \tag{1.1}$$

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The right-hand side defines an operator $\mathfrak{I}_{a^+}^{\alpha;\psi}$, referred to as the $\psi-$Riemann-Liouville fractional integral ($\psi-$RLFI), as defined in the next section (see Definition 2.4). We refer to (1.1) as an Abel IE with respect to another function, where $\varphi(x)$ is a given function and $\phi(x)$ is the unknown function. Our first objective is to establish necessary and sufficient conditions for the solvability of (1.1) in $L_1(a,b)$ in terms of the function $\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi$, within the space $\mathcal{AC}_{\psi+}^{n}[a,b]$ of φ (see (2.5)).

Equation (1.1) is closely related to the classical Abel IE, as discussed in [29] and Samko's book ([26], Ch. 1, Sec. 2, (2.1)), with a slight variation in the behavior of the solution. In particular, by setting $\psi(x) = x$ in (1.1), we retrieve the classical Abel IE. The Abel IE plays a crucial role in fractional calculus (FC) due to its unique solution, which exists under certain natural conditions on the function space $\mathfrak{I}_{a^+}^{\alpha}(L_1(a,b))$, and this solution is represented by the Riemann–Liouville fractional derivative (RLFD).

FC lacks a unique and universally accepted definition of the fractional derivative (FD). Various formulations exist, each offering different perspectives on fractional differentiation, and there is no single answer to what the FD of a given function should be. These definitions are not necessarily equivalent, which can pose challenges in obtaining consistent results. However, this diversity also enriches the structure of FC and presents challenges in finding connections among different FD formulations. Consequently, deriving general formulas that relate multiple definitions of FDs is of significant mathematical interest.

Three main families of FDs have been extensively studied: The RLFD, the Caputo fractional derivative (CFD), and the Hilfer fractional derivative (HFD). These derivatives are regarded as first-level FDs [16]. Beyond these, numerous other derivatives have been proposed as left-inverse operators to the Riemann–Liouville fractional integrals (RLFIs). As the field continues to evolve, researchers have introduced new fractional derivatives and integrals, each characterized by distinct kernel functions, further expanding the landscape of FC [1,6,13,16,24,26]. Recent contributions have introduced innovative FD formulations that deepen our understanding of fractional operators and their interrelationships [2–4,8,17–19,23,28].

From an applied perspective, individual FC formulations are valuable in specific contexts. However, from a theoretical standpoint, repeatedly proving fundamental properties for each fractional operator is inefficient. Mathematicians strive to establish broad frameworks that generalize results, allowing multiple operators to be studied as special cases. This approach facilitates mathematical analysis, enhances conceptual clarity, and extends the applicability of fractional operators.

To contribute to this unification effort, fractional operators defined with respect to another function have been introduced ([13], Ch. 2, Sec. 2.5; [26], Ch. 4, Sec. 18.2). These operators, often referred to as $\psi-$RLFIs and $\psi-$RLFDs, provide a unifying structure for various FD definitions. In 2017, Almeida [4] introduced the $\psi-$CFD and investigated its fundamental properties. Building on this work, Sousa and Oliveira [28] extended the framework in 2018 by incorporating Hilfer's concept, leading to the formulation of the $\psi-$HFD. These contributions have been widely recognized in the research community, as reflected by their extensive citations in both theoretical and applied contexts. Furthermore, the emergence of higher-order FDs levels, such as the second-level FD (2LFD) and the mth level FD introduced by Luchko in 2020 [16] and Bany-Ahmad et al. (2025) [2,3], highlights the need for a broader mathematical framework to systematically incorporate these operators and reveal their interconnections.

Motivated by these developments, our study aims to generalize and extend some existing theories by introducing a novel parametrization of the $\psi-$HFD. This leads to the definition of the $\psi-$second level fractional derivative ($\psi-$2LFD), which provides a flexible framework for fractional differentiation. By choosing ψ appropriately, the $\psi-$2LFD smoothly interpolates between classical first-level FDs and unifies numerous well-established derivatives, including the Hilfer, Caputo, Riemann-Liouville, Hadamard, Katugampola, Chen, Jumarie, Prabhakar, Erdélyi-Kober, and Weyl derivatives (see [28], Sec. 5).

Building on this foundation, we propose a further generalization: The $\psi - m$th level FDs for the mth compositions of the nth-order derivative. The $\psi - m$th level FD is not merely one of the potential left-inverse operators of the RLFI; its primary advantage lies in its incorporation of any level FDs, specifically the first level FDs, which are widely studied in literature. Several fundamental properties of these new operators are derived, reinforcing their theoretical significance. The growing body of research on FDs with respect to another function, including works such as [5,7,9,11,14,15,22,25,27,30], highlights the potential for novel applications and underscores the relevance of this framework.

The paper is structured as follows: Section 2 provides definitions for specific function spaces, including weighted spaces, along with the definitions of RLFI, RLFD, and HFD, as well as $\psi-$RLFI, $\psi-$RLFD, $\psi-$CFD, and $\psi-$HFD, supported by important lemmas. In Section 3, we establish the necessary and sufficient conditions for the solvability of Eq. (1.1) and present an explicit formula for the solution $\phi(x)$. Furthermore, we introduce an alternative parametrization of the ψ-HFD to bridge gaps in this area, facilitating the discovery of more comprehensive generalizations, and examine various properties of this fractional operator. Section 4 extends this framework by introducing a comprehensive version of the 2LFD, referred to as the ψ-2LFD, establishing connections with other FD formulations and exploring fundamental properties and lemmas. Expanding on the concepts introduced in Sections 3 and 4, Section 5 presents a significant generalization known as the $\psi - m$th level FDs, which incorporates any level of FDs. Finally, conclusions are provided in Section 6.

2. Preliminaries

This section delves into weighted spaces and fundamental concepts associated with FIs and FDs with respect to another function ψ. These concepts are essential for achieving the main objectives of the paper.

Let $\mathcal{C}[a,b]$, $\mathcal{AC}[a,b]$, $\mathcal{AC}^n[a,b]$, $\mathcal{C}^n[a,b]$, and $L_p(a,b)$, $p \geq 1$ be defined as the spaces of continuous functions, absolutely continuous, n-times absolutely continuous functions, n-times continuously differentiable functions on the finite interval $[a,b]$, and Lebesgue integrable functions on (a,b), respectively. For further information, please refer to citations [13,26].

The notation $\mathcal{AC}[a,b]$ denotes the space of functions that are absolutely continuous on the interval $[a,b]$, which is described as follows ([13], (1.1.5) and (1.1.6)):

$$\varphi \in \mathcal{AC}[a,b] \Leftrightarrow \exists \phi \in L_1(a,b) : \varphi(x) = c + \int_a^x \phi(t)dt, \quad x \in [a,b], \tag{2.1}$$

and therefore an absolutely continuous function $\varphi(x)$ has a summable derivative $\varphi'(x) = \phi(x)$ almost everywhere on $[a,b]$. Thus (2.1) yields $\phi(t) = \varphi'(t)$ and $c = \varphi(a)$.

Definition 2.1. ([13,26]) Let $\varphi \in L_1(a,b)$, $\alpha > 0$, where $\alpha \in \mathbb{R}$. Then the RLFIs ($\mathfrak{I}^\alpha_{a+}\varphi$ and $\mathfrak{I}^\alpha_{b-}\varphi$) of the function φ on $[a,b]$ with order α are defined by

$$\left(\mathfrak{I}^\alpha_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad x > a \tag{2.2}$$

and

$$\left(\mathfrak{I}_{b^-}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}\varphi(t)dt,\quad x<b, \tag{2.3}$$

where $\Gamma(\alpha)$ is the Gamma function. Moreover, the RLFDs of order α, $n-1<\alpha\leq n$ are defined by

$$\left(\mathfrak{D}_{a^+}^{\alpha}\varphi\right)(x)=\left(\frac{d}{dx}\right)^n\left(\mathfrak{I}_{a^+}^{n-\alpha}\varphi\right)(x),\ \text{and}\ \left(\mathfrak{D}_{b^-}^{\alpha}\varphi\right)(x)=\left(-\frac{d}{dx}\right)^n\left(\mathfrak{I}_{b^-}^{n-\alpha}\varphi\right)(x). \tag{2.4}$$

On the other hand, to promote more inclusive generalizations of spaces that align with our topic, we should recall the following definition: Let $\psi\in\mathcal{C}^n[a,b]$ such that $\psi'(x)>0$ on $[a,b]$ ([11], Definition 1.1 for left side). Then, we have

$$\mathcal{AC}_{\psi+}^n[a,b]=\left\{\varphi:[a,b]\to\mathbb{R}\text{ and }\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\varphi\in\mathcal{AC}[a,b]\right\} \tag{2.5}$$

and similarly for the other side

$$\mathcal{AC}_{\psi-}^n[a,b]=\left\{\varphi:[a,b]\to\mathbb{R}\text{ and }\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\varphi\in\mathcal{AC}[a,b]\right\}. \tag{2.6}$$

- For $n=1$, it is clear that $\mathcal{AC}_{\psi}^1[a,b]=\mathcal{AC}[a,b]$.
- For $\psi(x)=x$, we have $\mathcal{AC}_{\psi+}^n[a,b]=\mathcal{AC}^n[a,b]$.
- For $\psi(x)=lnx$, it follows that $\mathcal{AC}_{\psi+}^n[a,b]=\mathcal{AC}_{\delta}^n[a,b]$, where $\delta=x\frac{d}{dx}$, as presented in ([13], (1.1.13)).

Definition 2.2. The HFD can be expressed in two formulations as follows:

- D1. ([16]) The left-sided FD operator of order $0<\alpha\leq 1$ and type ν_1, where $0\leq\nu_1\leq 1-\alpha$, and the space ${}^H\mathcal{X}_{a^+}^1:=\left\{\varphi:\mathfrak{I}_{a^+}^{1-\alpha-\nu_1}\varphi\in\mathcal{AC}[a,b]\right\}$ such that ${}^H\mathfrak{D}_{a^+}^{\alpha,\nu_1}(\cdot):{}^H\mathcal{X}_{a^+}^1\to L_1(a,b)$ is defined by
$$\left({}^H\mathfrak{D}_{a^+}^{\alpha,\nu_1}\varphi\right)(x)=\left(\mathfrak{I}_{a^+}^{\nu_1}\frac{d}{dx}\mathfrak{I}_{a^+}^{1-\alpha-\nu_1}\varphi\right)(x). \tag{2.7}$$
- D2. ([26, 28]) Another parametrization of the left and right-sided HFDs of a function $\varphi\in\mathcal{C}^n(a,b)$ of order $n-1<\alpha<n$ and type $0\leq\beta_1\leq 1$ is defined by
$$\left({}^H\mathfrak{D}_{a^+}^{\alpha,\beta_1}\varphi\right)(x)=\left(\mathfrak{I}_{a^+}^{\beta_1(n-\alpha)}\left(\frac{d}{dx}\right)^n\mathfrak{I}_{a^+}^{(1-\beta_1)(n-\alpha)}\varphi\right)(x) \tag{2.8}$$
and
$$\left({}^H\mathfrak{D}_{b^-}^{\alpha,\beta_1}\varphi\right)(x)=\left(\mathfrak{I}_{b^-}^{\beta_1(n-\alpha)}\left(-\frac{d}{dx}\right)^n\mathfrak{I}_{b^-}^{(1-\beta_1)(n-\alpha)}\varphi\right)(x). \tag{2.9}$$

Moreover, by setting $\beta_1(1-\alpha)=\nu_1$ and $n=1$ in the formula 2.8, we obtain (2.7) as discussed in [16].

Definition 2.3. ([16]) Let $0 < \alpha \leq 1$, and the parameters ν_1, $\nu_2 \in \mathbb{R}$ satisfy the following conditions: $0 \leq \nu_1$, $0 \leq \nu_2$, $\alpha+\nu_1 \leq 1$, $\alpha+\nu_1+\nu_2 \leq 2$. Then the 2LFD $\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2)}(\cdot)\right)$ of order α, and type (ν_1,ν_2) on the space ${}^{2L}\mathcal{X}_{a^+}^{1} = \left\{\varphi : \mathfrak{I}_{a^+}^{2-\alpha-\nu_1-\nu_2}\varphi, \mathfrak{I}_{a^+}^{\nu_2}\left(\frac{d}{dx}\right)\mathfrak{I}_{a^+}^{2-\alpha-\nu_1-\nu_2}\varphi \in \mathcal{AC}[a,b]\right\}$, where ${}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2)}(\cdot) : {}^{2L}\mathcal{X}_{a^+}^{1} \longrightarrow L_1(a,b)$ is defined by

$$\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2)}\varphi\right)(x) = \left(\mathfrak{I}_{a^+}^{\nu_1}\frac{d}{dx}\mathfrak{I}_{a^+}^{\nu_2}\frac{d}{dx}\mathfrak{I}_{a^+}^{2-\alpha-\nu_1-\nu_2}\varphi\right)(x). \tag{2.10}$$

The formula (2.10) includes two pairs of first-order derivative compositions; hence, it is referred to as the 2LFD [16]. Therefore, the RLFD, CFD, and HFD represent first-level derivatives in this sense.

On the other hand, the following definitions and lemmas will be with respect to another function ψ. For more details, please refer to [13, 26].

Definition 2.4. Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $\varphi \in L_1(a,b)$, and $\psi \in \mathcal{C}^n[a,b]$ such that $\psi'(x) > 0$ on $[a,b]$.

- D1. ([26]) The left-sided FI operator of a function φ with respect to another function ψ on $[a,b]$ is defined by

$$\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\alpha-1}\varphi(t)dt \tag{2.11}$$

  and the right-sided FI operator is defined by

$$\mathfrak{I}_{b^-}^{\alpha;\psi}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)\Big(\psi(t)-\psi(x)\Big)^{\alpha-1}\varphi(t)dt. \tag{2.12}$$

- D2. ([13]) The left and right-sided $\psi-$RLFDs of a function φ are defined by

$$\left(\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x),$$
$$\text{and } \mathfrak{D}_{b^-}^{\alpha;\psi}\varphi(x) = \left(\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x). \tag{2.13}$$

- D3. ([4]) The left and right-sided $\psi-$CFDs of a function $\varphi \in \mathcal{C}^n[a,b]$ of order α are given by

$$\left({}^{C}\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x),$$
$$\text{and } \left({}^{C}\mathfrak{D}_{b^-}^{\alpha;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{b^-}^{n-\alpha;\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x). \tag{2.14}$$

- D4. ([28]) The ψ-HFDs $\left(\text{left-sided } {}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}(\cdot) \text{ and right-sided } {}^{H}\mathfrak{D}_{b^-}^{\alpha,\beta;\psi}(\cdot)\right)$ of a function $\varphi \in \mathcal{C}^n[a,b]$ of order α and type $0 \leq \beta_1 \leq 1$ are defined by

$$\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta_1;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{a^+}^{\beta_1(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}_{a^+}^{(1-\beta_1)(n-\alpha);\psi}\varphi\right)(x) \tag{2.15}$$

  and

$$\left({}^{H}\mathfrak{D}_{b^-}^{\alpha,\beta_1;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{b^-}^{\beta_1(n-\alpha);\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}_{b^-}^{(1-\beta_1)(n-\alpha);\psi}\varphi\right)(x). \tag{2.16}$$

Lemma 2.1. ([13,26]) *Let $\alpha > 0$, $\beta > 0$, and $\delta \in \mathbb{R}$ with $\delta > 0$.*

- *L1. Let $\varphi \in L_1(a,b)$. Then $\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\mathfrak{I}_{a^+}^{\beta;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{a^+}^{\alpha+\beta;\psi}\varphi\right)(x)$, and $\left(\mathfrak{I}_{b^-}^{\alpha;\psi}\mathfrak{I}_{b^-}^{\beta;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{b^-}^{\alpha+\beta;\psi}\varphi\right)(x)$.*
- *L2. If $\varphi(x) = \Big(\psi(x)-\psi(a)\Big)^{\delta-1}$, then*

$$\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \frac{\Gamma(\delta)\Big(\psi(x)-\psi(a)\Big)^{\delta+\alpha-1}}{\Gamma(\delta+\alpha)}, \text{ and } \left(\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \frac{\Gamma(\delta)\Big(\psi(x)-\psi(a)\Big)^{\delta-\alpha-1}}{\Gamma(\delta-\alpha)}.$$

3. Another parametrization of the $\psi-$Hilfer fractional derivative

Motivated by HFD's Definition 2.2 (D2) and $\psi-$HFD's Definition 2.4 (D4), we have discovered that these definitions can be utilized to develop a new parametrization of the $\psi-$HFD. The introduction of this alternative parametrization is essential for bridging gaps in this field and facilitating the discovery of more comprehensive generalizations, such as the $\psi-$2LFD, which will be discussed in the next section.

In this section, we present an alternative parametrization of the $\psi-$HFD and explore various properties of this fractional operator. To begin, we will reformulate Definition 2.2 (D1) in a more generalized form, enabling us to establish a new generalization of the $\psi-$HFD.

Furthermore, since theorems and our work will be stated in terms of the class $\mathcal{AC}^n$, we will redefine certain spaces that are well-suited for these topics as follows: Let $n-1 < \alpha \leq n$ and $0 \leq \nu_1 \leq n-\alpha$. The generalized spaces for the HFD, according to the definition provided by Y. Luchko in [16], are defined as follows:

$${}^H\mathcal{X}_{a^+}^1 := \left\{\varphi : \mathfrak{I}_{a^+}^{n-\alpha-\nu_1}\varphi \in \mathcal{AC}^n[a,b]\right\}, \text{ and } {}^H\mathcal{X}_{b^-}^1 := \left\{\varphi : \mathfrak{I}_{b^-}^{n-\alpha-\nu_1}\varphi \in \mathcal{AC}^n[a,b]\right\}.$$

These spaces makes sense in the space $\mathcal{AC}^n[a,b]$. For $n = 1$, we obtain the related space as indicated in Definition 2.2 (D1).

Definition 3.1. Let ${}^H\mathfrak{D}_{a^+}^{\alpha,\nu_1} : {}^H\mathcal{X}_{a^+}^1 \to L_1(a,b)$, where $n-1 < \alpha \leq n$ and $0 \leq \nu_1 \leq n-\alpha$. The left-sided HFD operator of order α and type ν_1 is defined as follows:

$$\left({}^H\mathfrak{D}_{a^+}^{\alpha,\nu_1}\varphi\right)(x) = \left(\mathfrak{I}_{a^+}^{\nu_1}\left(\frac{d}{dx}\right)^n \mathfrak{I}_{a^+}^{n-\alpha-\nu_1}\varphi\right)(x) \tag{3.1}$$

and the right-sided HFD operator ${}^H\mathfrak{D}_{b^-}^{\alpha,\nu_1} : {}^H\mathcal{X}_{b^-}^1 \to L_1(a,b)$ is defined as follows:

$$\left({}^H\mathfrak{D}_{b^-}^{\alpha,\nu_1}\varphi\right)(x) = \left(\mathfrak{I}_{b^-}^{\nu_1}\left(-\frac{d}{dx}\right)^n \mathfrak{I}_{b^-}^{n-\alpha-\nu_1}\varphi\right)(x). \tag{3.2}$$

- Let $\alpha = n$. Then by the assumption in Definition 3.1, we have that $\nu_1 = 0$, and thus we obtain the standard derivative.

Before we move forward, we need to prove some lemmas and theorems to facilitate our progress toward our goals in this section and beyond.

Lemma 3.1. *Let $\psi \in \mathcal{C}^n[a,b]$ such that $\psi'(x) > 0$ on $[a,b]$. If $\phi \in L_1(a,b)$, then both $\psi'\phi$ and $\frac{\phi}{\psi'}$ are in $L_1(a,b)$.*

Proof. For $\psi'\phi \in L_1(a,b)$: Since $\psi \in \mathcal{C}^n[a,b]$ and $\psi'(x) > 0$ for all $x \in [a,b]$, it follows that ψ' is continuous and thus bounded on $[a,b]$. Therefore, there exists a constant $M > 0$ such that $\psi'(x) \leq M$ for all $x \in [a,b]$. Consequently, we have $\int_a^b |\psi'(x)\phi(x)|\, dx \leq M \int_a^b |\phi(x)|\, dx < \infty$.

For $\frac{\phi}{\psi'} \in L_1(a,b)$: Let $\phi \in L_1(a,b)$. Since ψ' is continuous and strictly greater than zero, we define $m := \min_{x\in[a,b]} \psi'(x) > 0$, i.e., $\psi'(x) \geq m > 0$ for all $x \in [a,b]$. Thus,

$$\int_a^b \left| \frac{\phi(x)}{\psi'(x)} \right| dx \leq \frac{1}{m} \int_a^b |\phi(x)| dx < \infty.$$

□

Lemma 3.2. *Let $n-1 < \alpha \leq n$ and $\varphi \in L_1(a,b)$. Then we have $\left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{n;\psi} \varphi \right)(x) = \varphi(x)$.*

Proof. Let $\varphi \in L_1(a,b)$. Then the first order case, $n=1$:

$$\left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right) \mathfrak{I}_{a^+}^{1;\psi} \varphi \right)(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right) \int_a^x \psi'(t)\varphi(t) dt = \varphi(x).$$

Using semigroup property from Lemma 2.1 (L1), we have

$$\begin{aligned} \left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{n;\psi} \varphi \right)(x) &= \left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n-1} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right) \mathfrak{I}_{a^+}^{1;\psi} \mathfrak{I}_{a^+}^{n-1;\psi} \varphi \right)(x) \\ &= \left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{n-2} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right) \mathfrak{I}_{a^+}^{1;\psi} \mathfrak{I}_{a^+}^{n-2;\psi} \varphi \right)(x) \\ &= \dots \\ &= \varphi(x). \end{aligned}$$

□

Lemma 3.3. *Let $\alpha > 0$ and φ be continuous from the right at $x = a$ for $x \in [a,b]$. Then $\mathfrak{I}_{a^+}^{\alpha;\psi} \varphi(a) = \lim\limits_{x \to a^+} \mathfrak{I}_{a^+}^{\alpha;\psi} \varphi(x) = 0$.*

Proof. Since φ is continuous from the right at $x = a$, and using Lemma 2.1 (L2). Thus,

$$|\varphi(x)| < \mathcal{M}, \text{ for some positive constant } \mathcal{M}.$$

Therefore,

$$\left| \mathfrak{I}_{a^+}^{\alpha;\psi} \varphi(x) \right| < \left(\mathfrak{I}_{a^+}^{\alpha;\psi} \mathcal{M} \right)(x) = \mathcal{M} \frac{\big(\psi(x) - \psi(a)\big)^{\alpha}}{\Gamma(\alpha+1)}.$$

Since $\alpha > 0$, the right-hand side → 0 as $x \to a^+$, which completes the proof. □

Remark 3.1. We can obtain the same result as in Lemma 3.3 by setting $\gamma = 0$ in ([28], Lemma 4). Moreover, for $\psi(x) = x$, we obtain the result in ([10], Lemma 13).

Theorem 3.1. *Let $n-1 < \alpha \leq n$. Then the following rules are valid:*

- *T1. If $\varphi \in L_1(a,b)$, then*

$$\left(\mathfrak{D}_{a^+}^{\alpha;\psi} \mathfrak{I}_{a^+}^{\alpha;\psi} \varphi \right)(x) = \varphi(x). \tag{3.3}$$

- *T2. Let* $\varphi \in L_1(a,b)$*, and* $\varphi_{\psi+}^{[n-k]} := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-k}\varphi$*. Then*

$$\mathfrak{I}_{a^+}^{\alpha;\psi}\left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x) = \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\alpha-k}}{\Gamma(\alpha-k+1)}\varphi_{\psi+}^{[n-k]}(a). \tag{3.4}$$

- *T3. Let* $\varphi \in L_1(a,b)$ *and* $\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi \in \mathcal{AC}_{\psi+}^{n}[a,b]$*. Then*

$$\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\alpha-k}}{\Gamma(\alpha-k+1)}\varphi_{\psi+}^{[n-k]}\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi(a). \tag{3.5}$$

Moreover, if φ *is continuous from the right at* $x = a$ *for* $x \in [a,b]$*, then* $\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \varphi(x)$*.*

Proof.

- T1. Let $\varphi \in L_1(a,b)$. Using (2.13), Lemma 3.2, and semigroup property from Lemma 2.1 (L1), we have

$$\left(\mathfrak{D}_{a^+}^{\alpha;\psi}\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}_{a^+}^{n-\alpha;\psi}\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}_{a^+}^{n;\psi}\varphi\right)(x) = \varphi(x).$$

- T2. For $0 < \alpha \leq 1$, utilizing Definition 2.4, then ψ is invertible for all $x \in [a,b]$.

$$\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\alpha-1}\varphi(t)dt.$$

By making a change of variable: $\psi(t) = \psi(x) - \Big(\psi(u)\Big)^{\frac{1}{\alpha}}$ (see [21], or [20], Ch. 2, Thm. 2), we have

$$\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi(x) = \frac{1}{\Gamma(\alpha+1)}\int_{\psi^{-1}(0)}^{\psi^{-1}(\psi(x)-\psi(a))^{\alpha}}\psi'(u)\,\varphi\left(\psi^{-1}\left(\psi(x)-(\psi(u))^{\frac{1}{\alpha}}\right)\right)du. \tag{3.6}$$

Now, by taking $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)$ to both sides of (3.6), and applying the Leibniz integral rule, we obtain

$$\begin{aligned}&\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi(x)\\ &= \frac{(\psi(x)-\psi(a))^{\alpha-1}\varphi(a)}{\Gamma(\alpha)}\\ &\quad+\frac{1}{\Gamma(\alpha+1)}\int_{\psi^{-1}(0)}^{\psi^{-1}(\psi(x)-\psi(a))^{\alpha}}\psi'(u)\left(\frac{1}{\psi'(x)}\frac{\partial}{\partial x}\right)\varphi\left(\psi^{-1}\left(\psi(x)-(\psi(u))^{\frac{1}{\alpha}}\right)\right)du.\end{aligned}$$

Reversing the change of variable $\psi(x) - \Big(\psi(u)\Big)^{\frac{1}{\alpha}} = \psi(t)$, we obtain

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\mathfrak{I}_{a^+}^{\alpha,\psi}\varphi(x)$$

$$
\begin{aligned}
&=\frac{(\psi(x)-\psi(a))^{\alpha-1}\varphi(a)}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\int_a^x(\psi(x)-\psi(t))^{\alpha-1}\left(\frac{1}{\psi'(x)}\frac{\partial}{\partial x}\right)\varphi(t)dt\\
&=\frac{(\psi(x)-\psi(a))^{\alpha-1}\varphi(a)}{\Gamma(\alpha)}+\mathfrak{I}^{\alpha;\psi}_{a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\varphi(x). \qquad (3.7)
\end{aligned}
$$

Now, for $n-1<\alpha\leq n$, and taking $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)$ to both sides of (3.7), we have

$$
\begin{aligned}
\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\left[\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\mathfrak{I}^{\alpha,\psi}_{a^+}\varphi(x)\right]=&\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\left[\frac{(\psi(x)-\psi(a))^{\alpha-1}\varphi(a)}{\Gamma(\alpha)}\right]\\
&+\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\left[\mathfrak{I}^{\alpha;\psi}_{a+}\underbrace{\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\varphi(x)}_{:=\varphi^{[1]}_{\psi+}(x)}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^2\mathfrak{I}^{\alpha,\psi}_{a^+}\varphi(x)=&\frac{(\psi(x)-\psi(a))^{\alpha-2}\varphi(a)}{\Gamma(\alpha-1)}+\frac{(\psi(x)-\psi(a))^{\alpha-1}\varphi^{[1]}_{\psi+}(a)}{\Gamma(\alpha)}\\
&+\mathfrak{I}^{\alpha;\psi}_{a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^2\varphi(x).
\end{aligned}
$$

By repeated iterations, we obtain (T2).

- T3. By utilizing (2.13) and Lemma 2.1 (L1), along with (T2), we obtain

$$
\begin{aligned}
\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x)&=\mathfrak{I}^{\alpha;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\underbrace{\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi(x)}_{\text{say }=:\vartheta(x)}\\
&=\varphi(x)-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\alpha-k}}{\Gamma(\alpha-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi(a).
\end{aligned}
$$

Furthermore, if φ is continuous as $x\to a^+$, using Lemma 3.3, we have $\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x)=\varphi(x)$.

□

Remark 3.2. Referring to the Theorem 3.1, we have the following:

- R1. By replacing α with $n-\alpha$ in (3.4), it is clear that there is a relationship between $\psi-$RLFD and $\psi-$CFD as follows:

$$
\left({}^C\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x)=\left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x)-\sum_{k=0}^{n-1}\frac{\Big(\psi(x)-\psi(a)\Big)^{k-\alpha}}{\Gamma(k-\alpha+1)}\varphi^{[k]}_{\psi+}(a). \qquad (3.8)
$$

- R2. By setting $\psi(x)=x$ in (3.3), (3.4), (3.5), and (3.8), we obtain the results in ([26], (2.57)) ([20], Thm. 2), and ([13], (2.1.39), (2.4.6)), respectively. Furthermore, (3.8) is valid for $\varphi\in\mathcal{AC}^n_{\psi+}[a,b]$ as presented in ([11], (16)).

- R3. By setting $\psi(x) = ln(x)$ in (3.5), we obtain the result related to the Hadamard FD, as presented in ([13], (2.7.48)).

Now, one of the main purposes of this section is to solve the IE (1.1) and establish necessary and sufficient conditions for the solvability. Afterwards, we will define the basic domain of the $\psi-$RLFD as $\mathcal{X}^{0;\psi}_{a^+} := \mathfrak{I}^{\alpha;\psi}_{a^+}(L_1(a,b))$ which serves as the solution formula to the Abel IE with respect to another function. We will apply a similar strategy to other types of FDs and extend these spaces to larger function spaces.

The results obtained for the Abel IE with respect to another function (1.1), along with the definitions for the basic $\psi-$RLFD, $\psi-$CFD, $\psi-$HFD, and $\psi-$2LFD, are analogous to those presented in Samako's book ([26], Ch. 1, Sec. 2) (see also [29]) and in Y. Luchko's paper [16]. With some modifications and generalizations, we will define the basic spaces for $\psi-$RLFD, $\psi-$CFD, and $\psi-$HFD. These definitions align with the new parametrization of the $\psi-$HFD, enabling us to study various properties of these basic spaces and determine the assumptions that guarantee their existence.

In the following, we assume that $\psi \in \mathcal{C}^n[a,b]$ and $\psi'(x) > 0$ on $[a,b]$. Also, for brevity, we will prove the results only for the left FD, as the methods for the right FDs are analogous with appropriate adjustments.

Definition 3.2. Let $\mathcal{X}^{0;\psi}_{a^+}$ denote the space of functions φ represented by the left-sided FI of order α, where $\alpha > 0$, with respect to another function ψ of a summable function, defined as follows:

$$\mathcal{X}^{0;\psi}_{a^+} := \mathfrak{I}^{\alpha;\psi}_{a^+}(L_1(a,b)) = \left\{\varphi : \varphi = \mathfrak{I}^{\alpha;\psi}_{a^+}\phi, \quad \phi \in L_1(a,b)\right\} \tag{3.9}$$

similarly, the space of the right-sided FI with respect to another function ψ is defined by

$$\mathcal{X}^{0;\psi}_{b^-} := \mathfrak{I}^{\alpha;\psi}_{b^-}(L_1(a,b)) = \left\{\varphi : \varphi = \mathfrak{I}^{\alpha;\psi}_{b^-}\phi, \quad \phi \in L_1(a,b)\right\}. \tag{3.10}$$

Remark 3.3. Building on earlier discussions, we note the following:

- R1. Eq. (1.1) can be reformulated as follows:

$$\varphi(x) = \left(\mathfrak{I}^{\alpha;\psi}_{a^+}\phi\right)(x), \quad \phi \in L_1(a,b). \tag{3.11}$$

- R2. By setting $\psi(x) = ln(x)$ in Eq. (1.1), we obtain the Hadamard IE in ([12], Eq. (1.4)).
- R3. By setting $\psi(x) = x$ in (3.9), we obtain the definition in ([26], Definition 2.3).
- R4. By setting $\psi(x) = ln(x)$ in (3.9), we obtain the definition in ([13], (2.7.50)).

Let $\varphi \in \mathcal{X}^{0;\psi}_{a^+}$. In the following theorem, we aim to clarify the conditions under which the Eq. (1.1) is solvable.

Theorem 3.2. *In order that $\varphi \in \mathcal{X}^{0;\psi}_{a^+}$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, it is necessary and sufficient that*

$$\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b], \tag{3.12}$$

and that

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k \mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi(a) = 0, \;\; \textit{for } k = 0,1,2,\ldots,n-1. \tag{3.13}$$

If these conditions are satisfied, the Eq. (3.11) has a unique solution in $L_1(a,b)$ *given by*

$$\phi(x)=\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\frac{1}{\Gamma(n-\alpha)}\int_a^x\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-\alpha-1}\varphi(t)dt. \tag{3.14}$$

Moreover, in a comparable manner to the other side, let $\varphi\in\mathcal{X}^{0;\psi}_{b^-}$. *It is necessary and sufficient that*

$$\mathfrak{I}^{n-\alpha;\psi}_{b^-}\varphi\in\mathcal{AC}^{\mathrm{n}}_{\psi\text{-}}[a,b],\ \ \textit{and}\ \ \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k\mathfrak{I}^{n-\alpha;\psi}_{b^-}\varphi(b)=0,\ \ \textit{for } k=0,1,2,\dots,n-1. \tag{3.15}$$

Proof. Necessity: Let $\varphi\in\mathcal{X}^{0;\psi}_{a^+}$, and the Eq. (3.11) be solvable in $L_1(a,b)$. Applying the operator $\mathfrak{I}^{n-\alpha;\psi}_{a^+}$ to both sides of Eq. (3.11) and using the semigroup property from Lemma 2.1 (L1) with α replaced by $n-\alpha$ and $\beta=\alpha$, we obtain

$$\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)=\left(\mathfrak{I}^{n;\psi}_{a^+}\phi\right)(x),\quad \phi\in L_1(a,b). \tag{3.16}$$

Applying the operator $\mathfrak{D}^{k;\psi}_{a^+}$ to both sides of (3.16) for $k=0,1,2,\dots,n-1$ and utilizing Theorem 2.5 in [11], we find

$$\mathfrak{D}^{k;\psi}_{a^+}\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)=\left(\mathfrak{I}^{n-k;\psi}_{a^+}\phi\right)(x),\quad \phi\in L_1(a,b),\ k=0,1,2,\dots,n-1. \tag{3.17}$$

In particular, for $k=n-1$,

$$\mathfrak{D}^{n-1;\psi}_{a^+}\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)=\left(\mathfrak{I}^{1;\psi}_{a^+}\phi\right)(x),\quad \phi\in L_1(a,b), \tag{3.18}$$

and hence from (2.11) and (2.13) we get

$$\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)=\int_a^x\psi'(t)\phi(t)dt,\quad \phi\in L_1(a,b). \tag{3.19}$$

After differentiation we obtain

$$\phi(x)=\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x), \tag{3.20}$$

which in conjunction with (2.11) yields (3.14). Therefore, if Eq. (3.11) has a solution, this solution is necessarily given by Eq. (3.14), and thus it is unique.

According to (2.11) and (2.13), relation (3.17) can be rewritten in the form

$$\begin{aligned}&\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)\\ &=\frac{1}{\Gamma(n-k)}\int_a^x\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-k-1}\phi(t)dt,\quad k=0,1,2,\dots,n-1.\end{aligned} \tag{3.21}$$

From (3.21) we see that $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)\in\mathcal{C}[a,b]$ for $k=0,1,2,\dots,n-2$, while (3.19) yields $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)\in\mathcal{AC}[a,b]$ in accordance with Lemma 3.1 and (2.1). By (2.5), this means that $\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]$ and hence (3.12) holds. Relation (3.13) follows from (3.21) by passing to the limit $x \to a^+$, and thus the necessity is proved.

Sufficiency: Assuming conditions (3.12) and (3.13) are satisfied, we have

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x) \in \mathcal{AC}[a,b],$$

and hence

$$\frac{d}{dx}\left[\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)\right] \in L_1(a,b),$$

and

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\left[\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)\right] \in L_1(a,b).$$

Therefore the function $\phi(x)$ given by (3.14) exists almost everywhere on $[a,b]$ and belongs to $L_1(a,b)$. Let us show that it is indeed a solution of (3.11). For this purpose we substitute (3.14) into the right-hand side of (3.11) and denote the result by $f(x)$, i.e.,

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\alpha-1}\left[\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^{n}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(t)\right]dt. \tag{3.22}$$

We shall show that $f(x) = \varphi(x)$, which proves the theorem. (3.22) is an equation of the type (3.11) with respect to $\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(t)$. It is certainly solvable since it is a merely a notation. So by (3.14) we have

$$\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x) = \Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x),$$

that is,

$$\frac{d}{dx}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x) = \frac{d}{dx}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x). \tag{3.23}$$

The functions $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)$ and $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)$ are absolutely continuous on $[a,b]$, the first by assumption and the second by virtue of (3.19) with $f(x)$ in the right-hand side. Hence

$$\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x) - \Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x) = c, \tag{3.24}$$

with some constant c. We have $\lim_{x\to a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x) = 0$ due to condition (3.13) and $\lim_{x\to a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x) = 0$ because (3.22) is a solvable equation. Hence $c = 0$, and thus

$$\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x) = \Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^{n-1}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x).$$

Similar arguments yield the relations

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-k}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-k}\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x),\qquad \text{for } k=1,2,\dots,n.$$

When $k=n$, we have

$$\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)=\left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}f\right)(x)$$

or

$$\frac{1}{\Gamma(n-\alpha)}\int_a^x\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-\alpha-1}[\varphi(t)-f(t)]\,dt=0.$$

The latter is equation of the form (3.11). The uniqueness of its solution leads to the relation $\varphi(x)-f(x)=0$. This completes the proof of theorem.

Similarly, the same procedure can be applied to the space $\mathcal{X}_{b^-}^{0;\psi}$ to obtain the desired result.

□

Remark 3.4. From earlier discussions, we note:

- R1. By setting $\psi(x)=x$ in Theorem 3.2, we find that $\mathcal{AC}_{\psi+}^n[a,b]=\mathcal{AC}^n[a,b]$. Consequently, this leads to the result presented in ([26], Thm. 2.3; see also Thm. 2.1).
- R2. By setting $\psi(x)=\ln x$ in Theorem 3.2, we obtain the result in ([12], Corollary 3.2), noting that the space $X(a,b)=L_1(a,b)$, and $\mathcal{AC}_0[a,b]=\mathcal{AC}_{\psi+}[a,b]$ for $\psi(x)=\ln x$.

Corollary 3.1. *On the space of functions $\mathcal{X}_{a^+}^{0;\psi}$, and utilizing Theorem 3.2, the unique fractional derivative $\mathfrak{D}_{a^+}^{\alpha;\psi}$, $n-1<\alpha\le n$, is given by*

$$\begin{aligned}\left(\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x)=&\left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)\\ =&\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\int_a^x\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-\alpha-1}\varphi(t)dt.\end{aligned}\tag{3.25}$$

Lemma 3.4. *Let $n-1<\alpha\le n$ and $\varphi\in\mathcal{X}_{a^+}^{0;\psi}$. Then we have $\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x)=\varphi(x)$.*

Proof. Let $\varphi\in\mathcal{X}_{a^+}^{0;\psi}$. Then $\varphi(x)=\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\phi\right)(x)$, $\phi\in L_1(a,b)$. Thus, using Theorem 3.1 (T1), we obtain

$$\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x)=\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\mathfrak{D}_{a^+}^{\alpha;\psi}\mathfrak{I}_{a^+}^{\alpha;\psi}\phi\right)(x)=\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\phi\right)(x)=\varphi(x).$$

□

Remark 3.5. Based on prior discussions, we outline:

- R1. By setting $\psi(x)=x$ in Corollary 3.1, Lemma 3.4, and taking into account Remark 3.3 (R3), we obtain the results presented in ([16], Thm. 3.2) and ([26], (2.58)), respectively.
- R2. The formula (3.25) can be used to define the FD $\mathfrak{D}_{a^+}^{\alpha;\psi}$ of the order $\alpha=n=1$ as the first order derivative with respect to another function ψ:

$$\left(\mathfrak{D}_{a^+}^{1;\psi}\varphi\right)(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\left(\mathfrak{I}_{a^+}^{0;\psi}\varphi\right)(x)=\frac{1}{\psi'(x)}\frac{d\varphi}{dx}=\frac{\mathfrak{D}_{a^+}^{1}\varphi}{\psi'(x)},\ \text{or}\ \left(\psi'(x)\mathfrak{D}_{a^+}^{1;\psi}\varphi\right)(x)=\frac{d\varphi}{dx}.\tag{3.26}$$

When $\psi(x)=x$ and $n=1$, we obtain the formula in ([16], (3.5)). Similarly, when $\psi(x)=\ln(x)$ and $n=1$, we derive the formula in ([13], (2.7.13)).

Remark 3.6 ([11], Thm. 2.3)**.** Let $\varphi \in L_1(a,b)$, $\varphi \in \mathcal{AC}^n_{\psi+}[a,b]$. Then the FD of φ with respect to ψ exists almost everywhere and given by

$$\begin{aligned}\left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x) =& \frac{1}{\Gamma(n-\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\varphi(t)dt \\ &+\sum_{k=0}^{n-1}\frac{\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k\varphi(a^+)}{\Gamma(k-\alpha+1)}(\psi(x)-\psi(a))^{k-\alpha}.\end{aligned}$$

Now, after studying various properties of the space $\mathcal{X}^{0;\psi}_{a^+}$, we can begin defining the domain for the $\psi-$RLFD and other FDs, as well as expanding the spaces accordingly to suit the specific type of FD. The results we obtained for the basic spaces and their extensions for these FDs are analogous to those presented in [16] for the first-level FDs.

The basic space for the left side of $\psi-$RLFD in (2.13) is defined by $\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi(x) : \mathcal{X}^{0;\psi}_{a^+} \to L_1(a,b)$, and similarly for the other side, where $\mathfrak{D}^{\alpha;\psi}_{b^-}\varphi(x) : \mathcal{X}^{0;\psi}_{b^-} \to L_1(a,b)$. The term "basic" refers to $\mathcal{X}^{0;\psi}$ of $\mathfrak{D}^{\alpha;\psi}$, such that $\mathcal{X}^{0;\psi}$ can mean $\mathcal{X}^{0;\psi}_{a^+}$ or $\mathcal{X}^{0;\psi}_{b^-}$, depending on the given FD.

It is evident from Remark 3.6 that (2.13) makes sense for a space of functions larger than $\mathcal{X}^{0;\psi}$ on the interval $[a,b]$, defined as follows:

$$^{RL}\mathcal{X}^{1;\psi}_{a^+} = \left\{\varphi : \mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]\right\}, \text{ and } ^{RL}\mathcal{X}^{1;\psi}_{b^-} = \left\{\varphi : \mathfrak{I}^{n-\alpha;\psi}_{b^-}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi\text{-}}[a,b]\right\}. \tag{3.27}$$

These spaces provide sufficient conditions for the existence of the derivatives in (2.13). Hence, if $\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]$, then by (2.5), (2.1), and Lemma 3.1, there exist $\phi(\cdot) \in L_1(a,b)$ such that

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi(a) + (\mathfrak{I}_{a^+}\phi)(x).$$

Thus,

$$\left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x) = \frac{\phi(x)}{\psi'(x)} \in L_1(a,b).$$

Remark 3.7. By setting $\psi(x) = x$ and $n = 1$ in the formula (3.27) for the space $^{RL}\mathcal{X}^{1;\psi}_{a^+}$, we obtain the space of the RLFD as presented in ([16], (3.6)).

Definition 3.3. The extension of the basic $\psi-$RLFD $\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi(x) : \mathcal{X}^{0;\psi}_{a^+} \to L_1(a,b)$ to the domain $\mathcal{X}^{1;\psi}_{a^+}$ is called the $\psi-$RLFD of order α, $n-1 < \alpha \le n$:

$$\left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x) = \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x), \quad \mathfrak{D}^{\alpha;\psi}_{a^+}\varphi(x) : \mathcal{X}^{1;\psi}_{a^+} \to L_1(a,b). \tag{3.28}$$

On the other hand, the basic spaces of $\psi-$CFDs are defined as follows:

$$^{C}\mathcal{X}^{0;\psi}_{a^+} = \left\{\varphi \in \mathcal{X}^{0;\psi}_{a^+} : \mathfrak{I}^{n-\alpha;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right\} \tag{3.29}$$

and

$$^{C}\mathcal{X}^{0;\psi}_{b^-} = \left\{\varphi \in \mathcal{X}^{0;\psi}_{b^-} : \mathfrak{I}^{n-\alpha;\psi}_{b^-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{n-\alpha;\psi}_{b^-}\varphi\right\}. \tag{3.30}$$

Thus, the basic spaces of $\psi-$CFDs in (2.14) are defined by ${}^C\mathfrak{D}^{\alpha;\psi} : {}^C\mathcal{X}^{0;\psi} \to L_1(a,b)$.

To streamline notation and the proof of certain results, we will employ the following notation:

$$\varphi^{[n]}_{\psi+}(x) := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi(x), \text{ and } \varphi^{[n]}_{\psi-}(x) := \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi(x). \tag{3.31}$$

Theorem 3.3. *The space* ${}^C\mathcal{X}^{0;\psi}_{a^+}$ *contains the functions* $\varphi \in \mathcal{AC}^n_{\psi+}[a,b]$ *that satisfy the condition*

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k \varphi(a) = 0,$$

$k = 0, 1, \dots, n-1$ *for all* $n \in \mathbb{N}$.

Proof. Let $\varphi \in {}^C\mathcal{X}^{0;\psi}_{a^+}$. By (3.29), we have $\varphi \in \mathcal{X}^{0;\psi}_{a^+}$ and

$$\mathfrak{I}^{n-\alpha;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi.$$

Thus, by (3.14), we have

$$\phi = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi. \text{ Hence, } \phi = \mathfrak{I}^{n-\alpha;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi.$$

Applying the operator $\mathfrak{I}^{\alpha;\psi}_{a^+}$ to the last relation, we find

$$\mathfrak{I}^{\alpha;\psi}_{a^+}\phi = \mathfrak{I}^{n;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi. \text{ Using (3.11), we have } \varphi = \mathfrak{I}^{n;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi.$$

Now, by applying $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}$ to both sides of the last relation and using Lemma 3.2, formulas (2.5) and (2.1), we obtain

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1} \varphi(x) = \mathfrak{I}^{1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi = \int_a^x \frac{d}{dt}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-1} \varphi(t)dt.$$

This completes the proof of theorem. $\square$

Theorem 3.4. *The basic* $\psi-$*CFD and the basic* $\psi-$*RLFD are identical when restricted to the domain* ${}^C\mathcal{X}^{0;\psi}$.

Proof. According to ([11], Lemma 2.3), these functions can be represented in the form

$$\begin{aligned}\varphi(x) = &\frac{1}{(n-1)!}\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-1}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \varphi(t)dt \\ &+ \underbrace{\sum_{k=0}^{n-1} \frac{\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k \varphi(a)}{k!}\Big(\psi(x)-\psi(a)\Big)^k}_{\text{equals zero by Theorem 3.3.}}.\end{aligned}$$

$$\text{Hence, } \varphi(x) = (\mathfrak{I}^{n;\psi}_{a^+}\varphi^{[n]}_{\psi+})(x),\ x \in [a,b], \text{ where } \varphi^{[n]}_{\psi+} := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \varphi \in L_1(a,b). \tag{3.32}$$

Using (3.32) with $n-1<\alpha\leq n$, then

$$\begin{aligned}\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\varphi\right)(x) &= \left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi^{[n]}_{\psi+}\right)(x)\\ &= \left(\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi^{[n]}_{\psi+}\right)(x)\\ &= \left(\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{n-\alpha;\psi}_{a^+}\mathfrak{I}^{n;\psi}_{a^+}\varphi^{[n]}_{\psi+}\right)(x)\\ &= \left(\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x). \qquad (3.33)\end{aligned}$$

With this, we finalize the proof of theorem. □

Remark 3.8. Building on earlier discussions, we note the following:

- R1. By setting $\psi(x)=x$ and $n=1$ in (3.29), we derive the result presented in ([16], (3.13)). Consequently, Theorem 3.3 aligns with the findings reported in [16].
- R2. The result of Theorem 3.4 can be obtained by utilizing Theorem 3.3 in relation (3.8).
- R3. By Theorem 3.4, it is clear that there is nothing new about the basic spaces of $\psi-$CFDs, as they are identical to the basic $\psi-$RLFDs when confined to the domain ${}^{C}\mathcal{X}^{0;\psi}$.

According to Theorem 3.3, the expression in (2.14) is valid for a space of functions larger than ${}^{C}\mathcal{X}^{0;\psi}$ on the interval $[a,b]$, as detailed below:

$${}^{C}\mathcal{X}^{1;\psi}_{a^+}=\left\{\varphi:\varphi\in\mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]\right\},\ \text{and}\ {}^{C}\mathcal{X}^{1;\psi}_{b^-}=\left\{\varphi:\varphi\in\mathcal{AC}^{\mathrm{n}}_{\psi\text{-}}[a,b]\right\}.$$

These spaces provide sufficient conditions for the existence of the derivatives in (2.14). Hence, if $\varphi\in\mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]$, then by (2.5) there exists $\phi(x)\in L_1(a,b)$ such that

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\varphi(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\varphi(a)+\mathfrak{I}_{a^+}\phi(x).$$

$$\text{Thus,}\ \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\varphi(x)=\frac{\phi(x)}{\psi'(x)}\in L_1(a,b).$$

Definition 3.4. The extension of the basic $\psi-$CFD ${}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi(x):{}^{C}\mathcal{X}^{0;\psi}_{a^+}\to L_1(a,b)$ to the domain ${}^{C}\mathcal{X}^{1;\psi}_{a^+}$ is called the $\psi-$CFD of order α, $n-1<\alpha\leq n$:

$$\left({}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x)=\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\varphi\right)(x),\quad {}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi(x):{}^{C}\mathcal{X}^{1;\psi}_{a^+}\to L_1(a,b).$$

Remark 3.9. For functions in ${}^{C}\mathcal{X}^{1;\psi}_{a^+}$, a clear connection exists between the $\psi-$RLFDs and $\psi-$CFDs, as noted in Remark 3.2 (R1 and R2).

A one-parameter family $\mathfrak{D}^{\alpha;\psi}$, $\alpha\geq 0$ of the linear operators, is classified as FDs if and only if it satisfies the Fundamental Theorem (FT) of FC [16]. We shall now define the subsequent space:

$${}^{FT}\mathcal{X}^{\psi}_{a^+}=\mathfrak{I}^{n-\alpha;\psi}_{a^+}\left(L_1(a,b)\right)=\left\{\forall\varphi\in{}^{FT}\mathcal{X}^{\psi}_{a^+},\exists\phi\in L_1(a,b):\varphi(x)=\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\phi\right)(x)\right\}. \qquad (3.34)$$

Lemma 3.5. *The $\psi-$CFD given by Definition 3.4 is a left-inverse operator to $\psi-$RLFI, i.e., the relation $\left({}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x)=\varphi(x)$, $x\in[a,b]$, is valid for any φ from the space ${}^{FT}\mathcal{X}^{\psi}_{a^+}$.*

Proof. Utilizing Lemma 3.2 and relation (3.34), we find

$$\begin{aligned}\left({}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x)&=\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x)\\&=\left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{n;\psi}_{a^+}\phi\right)(x)\\&=\varphi(x).\end{aligned}$$

□

Likewise, let $0\le\nu_1\le n-\alpha$. The basic spaces for the $\psi-$HFD are defined as follows:

$$^{H}\mathcal{X}^{0;\psi}_{a^+}=\left\{\varphi\in\mathcal{X}^{0;\psi}_{a^+}:\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1;\psi}_{a^+}\varphi(x)\right\}\tag{3.35}$$

and

$$^{H}\mathcal{X}^{0;\psi}_{b^-}=\left\{\varphi\in\mathcal{X}^{0;\psi}_{b^-}:\mathfrak{I}^{\nu_1;\psi}_{b^-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi(x)=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1;\psi}_{b^-}\varphi(x)\right\},\tag{3.36}$$

are the basic spaces of $\psi-$HFD for (3.39) and (3.40) which are defined on ${}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}:{}^{H}\mathcal{X}^{0;\psi}\to L_1(a,b)$. In a manner analogous to the space ${}^{C}\mathcal{X}^{0;\psi}_{a^+}$ for the $\psi-$CFD presented in Theorem 3.3, we consider the case where $n-\alpha=\nu_1$ in formula (3.29). The space ${}^{H}\mathcal{X}^{0;\psi}_{a^+}$ contains, in particular, the functions $\varphi\in\mathcal{AC}^n_{\psi+}[a,b]$ that satisfy the condition

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k\varphi(a)=0,\ k=0,1,\dots,n-1\text{ for all }n\in\mathbb{N}.\tag{3.37}$$

Moreover, let

$$^{H}\mathcal{B}^{0;\psi}_{a^+}=\left\{\varphi\in\mathcal{X}^{0;\psi}_{a^+}:\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi(x)=\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi(x)\right\}\tag{3.38}$$

be another basic space for the $\psi-$HFD. This formulation is valid, yielding results analogous to those established in Theorem 3.3, upon replacing α with $\alpha+\nu_1$ in formula (3.29), where $0<\alpha+\nu_1\le n$.

Next, the alternative parametrizations of the $\psi-$HFD are defined as follows:

Definition 3.5. Let $n-1<\alpha\le n$, $n\in\mathbb{N}$, $0\le\nu_1\le n-\alpha$, $\psi\in\mathcal{C}^n([a,b],\mathbb{R})$ such that ψ is increasing and $\psi'(x)\neq0$ for all $x\in[a,b]$. The left-sided $\psi-$HFD of order α and type ν_1 is defined by

$$\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x)=\left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x)\tag{3.39}$$

and the right-sided $\psi-$HFD is defined by

$$\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{b^-}\varphi\right)(x)=\left(\mathfrak{I}^{\nu_1;\psi}_{b^-}\Big(-\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{b^-}\varphi\right)(x).\tag{3.40}$$

If the domain of the $\psi-$HFD is ${}^{H}\mathcal{X}_{a^+}^{0;\psi}$ or ${}^{H}\mathcal{B}_{a^+}^{0;\psi}$, we shall refer to it as the basic $\psi-$HFD. In these spaces, we have the following theorem.

Theorem 3.5. *The basic $\psi-$HFD of order α and type ν_1 in (3.39) is identical to $\psi-$RLFD restricted to the domain ${}^{H}\mathcal{X}_{a^+}^{0;\psi}$, and coincides with the basic $\psi-$CFD restricted to the domain ${}^{H}\mathcal{B}_{a^+}^{0;\psi}$.*

Proof. Using relation (3.35) and Lemma 2.1 (L1), we find

$$\begin{aligned}\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x) &= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi\right)(x)\\ &= \left(\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}_{a^+}^{\nu_1;\psi}\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi\right)(x)\\ &= \left(\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}_{a^+}^{n-\alpha;\psi}\varphi\right)(x)\\ &= \left({}^{RL}\mathfrak{D}_{a^+}^{\alpha,\psi}\varphi\right)(x),\quad \varphi\in {}^{H}\mathcal{X}_{a^+}^{0;\psi}.\end{aligned}$$

Furthermore, in the basic space ${}^{H}\mathcal{B}_{a^+}^{0;\psi}$, and utilizing (3.38), we have

$$\begin{aligned}\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x) &= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi\right)(x)\\ &= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\,\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\varphi\right)(x)\\ &= \left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n\varphi\right)(x)\\ &= \left({}^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}\varphi\right)(x),\quad \varphi\in {}^{H}\mathcal{B}_{a^+}^{0;\psi}.\end{aligned}$$

□

Remark 3.10. By utilizing relation (3.37) in (3.8), we find that the $\psi-$RLFD, $\psi-$CFD, and $\psi-$HFD are identical when confined to the domains ${}^{H}\mathcal{X}_{a^+}^{0;\psi}$ or ${}^{H}\mathcal{B}_{a^+}^{0;\psi}$, i.e., $\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x) = \left({}^{C}\mathfrak{D}_{a^+}^{\alpha,\psi}\varphi\right)(x) = \left({}^{RL}\mathfrak{D}_{a^+}^{\alpha,\psi}\varphi\right)(x)$.

However, the domain of the basic $\psi-$HFD on the interval $[a,b]$ can be extended to larger spaces. In this case, the condition $\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k\varphi(a)=0$, $k=0,1,\ldots,n-1$ for all $n\in\mathbb{N}$, may not be satisfied of functions. Specifically, we have

$${}^{H}\mathcal{X}_{a^+}^{1;\psi}=\left\{\varphi:\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi\in\mathcal{AC}_{\psi+}^{n}[a,b]\right\},\ \text{and}\ {}^{H}\mathcal{X}_{b^-}^{1;\psi}=\left\{\varphi:\mathfrak{I}_{b^-}^{n-\alpha-\nu_1;\psi}\varphi\in\mathcal{AC}_{\psi-}^{n}[a,b]\right\}.$$

These spaces provide sufficient conditions for the existence of the derivatives in (3.39) and (3.40). Hence, if $\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi\in\mathcal{AC}_{\psi+}^{\mathrm{n}}[a,b]$, then by (2.5) there exist $\phi(\cdot)\in L_1(a,b)$ such that

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi(a)+\mathfrak{I}_{a^+}\phi(x).$$

Thus,

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\varphi(x)=\frac{\phi(x)}{\psi'(x)}\in L_1(a,b).$$

Definition 3.6. The extension of the basic $\psi-$HFD to the domains ${}^{H}\mathcal{X}^{1;\psi}_{a^+}$ or ${}^{H}\mathcal{X}^{1;\psi}_{b^-}$ is referred to as the $\psi-$HFD as defined by Definition 3.5. The space of the left-sided $\psi-$HFD is given by ${}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+} : {}^{H}\mathcal{X}^{1;\psi}_{a^+} \to L_1(a,b)$, and the space of the right-sided $\psi-$HFD is given by ${}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{b^-} : {}^{H}\mathcal{X}^{1;\psi}_{b^-} \to L_1(a,b)$.

Theorem 3.6. *The new parametrization of $\psi-$HFD given by (3.39) is a left-inverse operator to $\psi-$RLFI, i.e., the relation $\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\,\mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x) = \varphi(x)$, $x \in [a,b]$, is valid for any φ from the space ${}^{FT}\mathcal{X}^{\psi}_{a^+}$ given by (3.34).*

Proof.

$$\begin{aligned}\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\,\mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\,\mathfrak{I}^{\alpha;\psi}_{a^+}\,\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\,\mathfrak{I}^{n;\psi}_{a^+}\,\phi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{n;\psi}_{a^+}\,\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\,\phi\right)(x)\\ &= \left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\,\phi\right)(x)\\ &= \varphi(x),\ \varphi \in {}^{FT}\mathcal{X}^{\psi}_{a^+}.\end{aligned}$$

□

Remark 3.11. By setting $\psi(x) = x$ and $n = 1$ in (3.34), along with Lemma 3.5, (3.35), Definition 3.5, and Theorem 3.6, we derive the results presented in ([16], (3.21), (3.19), (3.23), (3.26), and (3.27)), respectively.

For $\xi_1 = \alpha + \nu_1$, where $n-1 < \xi_1 \leq n$, we have

$$\begin{aligned}\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi(x) &= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\left(\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x),\\ \text{and } \mathfrak{D}^{\xi_1;\psi}_{b^-}\varphi(x) &= \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\left(\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{b^-}\varphi\right)(x).\end{aligned} \tag{3.41}$$

Therefore, the $\psi-$HFD as defined above can be expressed as follows:

$$\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\,\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1-\alpha;\psi}_{a^+}\,\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi\right)(x) \tag{3.42}$$

and

$$\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{b^-}\varphi\right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{b^-}\,\mathfrak{D}^{\xi_1;\psi}_{b^-}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1-\alpha;\psi}_{b^-}\,\mathfrak{D}^{\xi_1;\psi}_{b^-}\varphi\right)(x). \tag{3.43}$$

Remark 3.12. By applying Definition 3.5, we derive the following results:

- R1. By setting $\nu_1 = \beta_1(n-\alpha)$ in (3.39) and (3.40), we obtain (2.15) and (2.16), respectively.
- R2. Let $\xi_1 = \alpha + \nu_1$. The parameters ξ_1, ν_1 satisfy: $n-1 < \xi_1 \leq n,\ 0 \leq n-\xi_1 < 1,\ \alpha \leq \xi_1,\ \ \nu_1 < \xi_1,\ \ n-\xi_1 < n-\nu_1,\ \ 0 \leq \nu_1 < 1.$
- R3. If $\alpha = n$ and under the condition $0 \leq \nu_1 \leq n-\alpha$, it follows that $\nu_1 = 0$. Thus,

$$\varphi^{[n]}_{\psi+}(x) = \lim_{\alpha\to n^-}\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi(x),\ \text{and}\ \varphi^{[n]}_{\psi-}(x) = \lim_{\alpha\to n^-}\mathfrak{D}^{\xi_1;\psi}_{b^-}\varphi(x).$$

Consequently, if $\alpha = n$, we have $\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x) = \varphi_{\psi+}^{[n]}(x)$, and $\left({}^{H}\mathfrak{D}_{b^-}^{\alpha,\nu_1;\psi}\varphi\right)(x) = \varphi_{\psi-}^{[n]}(x)$. Moreover, if $\alpha = n$ and $\psi(x) = x$, we obtain the usual derivative.

Theorem 3.7. *Let $n-1 < \alpha \le n$, $\alpha + \nu_1 \le n$, $\psi \in \mathcal{C}^{n+1}[a,b]$, and $\mathfrak{I}_{a^+}^{n-\xi_1;\psi}\varphi \in \mathcal{AC}_{\psi+}^{n+1}[a,b]$. Then we have*

$$\begin{aligned}&\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x)\\ =&\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(a)+\frac{1}{\Gamma(\xi_1-\alpha+1)}\int_a^x\Big(\psi(x)-\psi(t)\Big)^{\xi_1-\alpha}\frac{d}{dt}\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(t)dt\end{aligned}$$

and for $\mathfrak{I}_{b^-}^{n-\xi_1;\psi}\varphi \in \mathcal{AC}_{\psi-}^{n+1}[a,b]$, we obtain

$$\begin{aligned}&\left({}^{H}\mathfrak{D}_{b^-}^{\alpha,\nu_1;\psi}\varphi\right)(x)\\ =&\frac{\Big(\psi(b)-\psi(x)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\mathfrak{D}_{b^-}^{\xi_1;\psi}\varphi(b)-\frac{1}{\Gamma(\xi_1-\alpha+1)}\int_x^b\Big(\psi(t)-\psi(x)\Big)^{\xi_1-\alpha}\frac{d}{dt}\mathfrak{D}_{b^-}^{\xi_1;\psi}\varphi(t)dt.\end{aligned}$$

Proof. Using Definition 2.4 (D1) and integrating (3.42) by parts, we obtain

$$\begin{aligned}&\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x)\\ =&\left(\mathfrak{I}_{a^+}^{\xi_1-\alpha;\psi}\,\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi\right)(x)\\ =&\frac{1}{\Gamma(\xi_1-\alpha)}\int_a^x\underbrace{\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\xi_1-\alpha-1}}_{\text{dv}}\underbrace{\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(t)}_{\text{u}}\,dt\\ =&\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(a)+\frac{1}{\Gamma(\xi_1-\alpha+1)}\int_a^x\Big(\psi(x)-\psi(t)\Big)^{\xi_1-\alpha}\frac{d}{dt}\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(t)dt. \quad (3.44)\end{aligned}$$

Similarly, one obtains the other case. □

Particularly, let φ be continuous from the right at $x = a$ for $x \in [a,b]$ where $0 < n - \xi_1$. Then by Lemma 3.3, we have $\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(a) = \mathfrak{D}_{a^+}^{n;\psi}\mathfrak{I}_{a^+}^{n-\xi_1;\psi}\varphi(a) = 0$. Thus,

$$\left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x) = \frac{1}{\Gamma(\xi_1-\alpha+1)}\int_a^x\Big(\psi(x)-\psi(t)\Big)^{\xi_1-\alpha}\frac{d}{dt}\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi(t)dt.$$

Remark 3.13. By setting $\nu_1 = n - \alpha$ in Theorem 3.7, we derive the result for $\psi-$CFD as presented in ([4], Thm. 1).

Lemma 3.6. *Let $n-1 < \alpha \le n$ and $\varphi \in L_1(a,b)$.*

- *L1. For $\mathfrak{I}^{n-\xi_1;\psi}\varphi \in \mathcal{AC}_{\psi}^{\mathrm{n}}[a,b]$, we have*

$$\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\,{}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{a^+}^{\xi_1;\psi}\,\mathfrak{D}_{a^+}^{\xi_1;\psi}\varphi\right)(x),$$
$$\textit{and}\ \left(\mathfrak{I}_{b^-}^{\alpha;\psi}\,{}^{H}\mathfrak{D}_{b^-}^{\alpha,\nu_1;\psi}\varphi\right)(x) = \left(\mathfrak{I}_{b^-}^{\xi_1;\psi}\,\mathfrak{D}_{b^-}^{\xi_1;\psi}\varphi\right)(x).$$

- *L2. For* $\mathfrak{I}^{n-\nu_1;\psi}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi}[a,b]$*, we obtain*

$$\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\ \mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\ \mathfrak{D}^{\nu_1;\psi}_{a^+}\varphi\right)(x),$$
$$\text{and } \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{b^-}\ \mathfrak{I}^{\alpha;\psi}_{b^-}\varphi\right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{b^-}\ \mathfrak{D}^{\nu_1;\psi}_{b^-}\varphi\right)(x).$$

Proof. Let $\varphi \in L_1(a,b)$ and $\mathfrak{I}^{n-\xi_1;\psi}_{a^+}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]$. Then by (3.42) and Lemma 2.1 (L1), we have

$$\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\ {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\alpha;\psi}_{a^+}\mathfrak{I}^{\xi_1-\alpha;\psi}_{a^+}\ \mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi\right)(x).$$

Furthermore, for $\mathfrak{I}^{n-\nu_1;\psi}_{a^+}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]$, we have

$$\begin{aligned}\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\ \mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{n-\nu_1;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\mathfrak{D}^{\nu_1;\psi}_{a^+}\varphi\right)(x).\end{aligned}$$

Similarly, we can identify the right-sided version. ☐

Theorem 3.8. *Let* $n-1<\alpha\le n$*,* $n\in\mathbb{N}$*, and* $\varphi \in L_1(a,b)$*.*

- *T1. If* $\mathfrak{I}^{n-\xi_1;\psi}_{a^+}\varphi \in \mathcal{AC}^{n}_{\psi+}[a,b]$*, then*

$$\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\ {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x) = \varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}}{\Gamma(\xi_1-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_1;\psi}_{a^+}\varphi(a). \tag{3.45}$$

- *T2. If* $\mathfrak{I}^{n-\nu_1;\psi}_{a^+}\varphi \in \mathcal{AC}^{\mathrm{n}}_{\psi+}[a,b]$*, we have*

$$\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\ \mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x) = \varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\nu_1-k}}{\Gamma(\nu_1-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\nu_1;\psi}_{a^+}\varphi(a). \tag{3.46}$$

Proof. (T1) follows clearly by applying Lemma 3.6 (L1) and Theorem 3.1 (T3). Similarly, (T2) can be proven in the same manner using Lemma 3.6 (L2). ☐

Remark 3.14. By applying Theorem 3.8, we derive the following results:

- R1. By setting $\nu_1 = n-\alpha$ in Theorem 3.8 (T1), we derive the result for $\psi-$CFD as presented in ([4], Thm. 4).
- R2. If φ is continuous from the right at $x=a$ for $x\in[a,b]$, then by Lemma 3.3, we have
    1. $\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\ {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x) = \varphi(x)$.
    2. $\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\ \mathfrak{I}^{\alpha;\psi}_{a^+}\varphi\right)(x) = \varphi(x)$. Furthermore, for $\nu_1 = n-\alpha$, we obtain ([4], Thm. 5).

Remark 3.15. In Definition 3.5 of $\psi-$HFDs, the primary FDs $\psi-$RLFD and $\psi-$CFD are introduced from (3.39) and (3.40). By choosing $\psi(x)$ and varying parameters such as α, ν_1, a, and b, a diverse range of FDs can be derived from these formulas. As will be discussed below, we recommend referring to [4, 24, 28] for further details on these broad groups of FDs:

- For $\nu_1 = 0$ in Definition (3.5), we obtain the $\psi-$RLFDs as defined in Definition 2.4 (D2).

$$\left({}^{H}\mathfrak{D}^{\alpha,0;\psi}_{a^+}\varphi\right)(x) = \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x). \tag{3.47}$$

  Furthermore, by (3.47), we can derive the following results and additional insights (see [28], Sec. 5):

  - For $\psi(x) = x$, we have the RLFD $\left(\mathfrak{D}^{\alpha}_{a^+}\varphi\right)(x)$.
  - For $\psi(x) = x$, $a = 0$, we have the Riemann FD $\left({}^{R}\mathfrak{D}^{\alpha}_{+}\varphi\right)(x)$.
  - For $\psi(x) = x$, $a = c$, we obtain the Chen FD $\left(\mathfrak{D}^{\alpha}_{c}\varphi\right)(x)$.
  - For $\psi(x) = x$, $a = 0$ and $\varphi(x) = f(x) - f(0)$, we have the Jumarie FD $\left(\mathfrak{D}^{\alpha}_{x}f\right)(x)$.
  - For $\psi(x) = x$ and $\varphi(x) = \mathbb{E}^{-\xi_1}_{\rho,n-\alpha}\left[\omega(x-t)^{\rho}\right]f(x)$, we get the Prabhakar FD $\mathfrak{D}^{\omega,\rho}_{a^+,\xi_1,\alpha}\varphi(x)$, where ω, $\rho \in \mathbb{R}$, $\mathbb{E}$ is the three-parametric Mittag$-$Leffler function.
  - For $\psi(x) = ln(x)$, we have the Hadamard FD $\left({}^{H}\mathfrak{D}^{\alpha}_{a^+}\varphi\right)(x)$.
  - For $\psi(x) = x^{\rho}$. Multiplying both sides of (3.39) by ρ^{α}, we obtain the Katugampola FD as follows: $\rho^{\alpha}\,{}^{H}\mathfrak{D}^{\alpha,0;x^{\rho}}_{a^+}\varphi(x) = {}^{\rho}\mathfrak{D}^{\alpha}_{a^+}\varphi(x)$.

  However, for the right side of (3.40), we have the following:

  - For $\psi(x) = x$, $a = -\infty$ with $\nu_1 = 0$, we obtain the Liouville FD ${}^{L}\mathfrak{D}^{\alpha}_{+}\varphi(x)$.

- For $\nu_1 = n - \alpha$ in Definition 3.5, we obtain the $\psi-$CFDs as defined in Definition 2.4 (D3).

$$\left({}^{H}\mathfrak{D}^{\alpha,n-\alpha;\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x) = \left({}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x). \tag{3.48}$$

  Moreover, from (3.48), we have the following:

  - For $\psi(x) = x$, we obtain the CFD $\left({}^{C}\mathfrak{D}^{\alpha}_{a^+}\varphi\right)(x)$.
  - For $\psi(x) = x^{\sigma}$ and $\varphi(x) = x^{\sigma(\eta+\alpha)}f(x)$, we get the Erdélyi–Kober as follows: $x^{-\sigma\eta}\,{}^{H}\mathfrak{D}^{\alpha,1;\psi}_{a^+}\varphi(x) = {}^{EK}\mathfrak{D}^{\alpha}_{a^+,\sigma,\eta}f(x)$, where $\eta \in \mathbb{R}$, $\sigma > 0$.
  - For $\psi(x) = ln(x)$, we have the Caputo$-$Hadamard FD ${}^{CH}\mathfrak{D}^{\alpha}_{a^+}\varphi(x)$.
  - For $\psi(x) = x^{\rho}$. Multiplying both sides of (3.39) by ρ^{α}, then we obtain the Caputo-Katugampola (Caputo-type) FD as follows: $\rho^{\alpha}\,{}^{H}\mathfrak{D}^{\alpha,n-\alpha;x^{\rho}}_{a^+}\varphi(x) = {}^{CK}\mathfrak{D}^{\alpha,\rho}_{a^+}\varphi(x)$.

  However, for the right side of (3.40), we have the following:

  - For $\psi(x) = x$, $a = -\infty$ with $\nu_1 = n - \alpha$, we obtain the Liouville–Caputo ${}^{LC}\mathfrak{D}^{\alpha}_{+}\varphi(x)$.

  Furthermore, there are numerous other types of FDs that can be derived from the new parameterization of the $\psi-$HFD. However, we prefer using the parameterization in Definition 3.5 because it allows for a straightforward generalization, as we will discuss in the subsequent subsections.

4. The $\psi-$second level fractional derivative

Inspired by the definitions in $\psi-$HFD Definition 3.5 and 2LFD Definition 2.3, we have discovered that these definitions can be utilized to characterize a broader version of 2LFD, namely $\psi-$2LFD, for a function φ with respect to another function ψ. The importance of this new concept lies in bridging the gaps between other well-known concepts such as $\psi-$RLFD, $\psi-$CFD, and

$\psi-$HFD. Furthermore, it plays a crucial role in discovering connections between these concepts and studying their characteristics simultaneously using the same FD operator. This approach ultimately leads to significant results and greater generalization in the field of FC.

To begin, we will reformulate Definition 2.3 in a more general form to better align with our objective. Then, we will redefine some spaces that are suitable for this type of FD.

Similarly, in the context of Hilfer spaces discussed in Section 3, the generalized spaces for the 2LFD of order $n-1<\alpha\le n$ and type (ν_1,ν_2), where $\nu_1,\nu_2\in\mathbb{R}$, $\alpha+\nu_1\le n$ and $\alpha+\nu_1+\nu_2\le 2n$, are defined according to the Definition 2.3 as follows:

$$^{2L}\mathcal{X}^1_{a^+}=\left\{\varphi:\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2}_{a^+}\varphi,\mathfrak{I}^{\nu_2}_{a^+}\left(\frac{d}{dx}\right)^n\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2}_{a^+}\varphi\in\mathcal{AC}^n[a,b]\right\}$$

and

$$^{2L}\mathcal{X}^1_{b^-}=\left\{\varphi:\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2}_{b^-}\varphi,\mathfrak{I}^{\nu_2}_{b^-}\left(-\frac{d}{dx}\right)^n\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2}_{b^-}\varphi\in\mathcal{AC}^n[a,b]\right\}.$$

- For $n=1$, we obtain the related space in Definition 2.3. This generalization will be necessary to further develop the concept of the 2LFD.

Definition 4.1. Let the parameters ν_1, $\nu_2\in\mathbb{R}$ satisfy the following conditions: $0\le\nu_1$, $0\le\nu_2$, $\alpha+\nu_1\le n$, $\alpha+\nu_1+\nu_2\le 2n$. Then the 2LFD of order α, $n-1<\alpha\le n$ and type (ν_1,ν_2), such that $^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2)}_{a^+}:{}^{2L}\mathcal{X}^1_{a^+}\longrightarrow L_1(a,b)$ is defined by

$$\left(^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2)}_{a^+}\varphi\right)(x)=\left(\mathfrak{I}^{\nu_1}_{a^+}\left(\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_2}_{a^+}\left(\frac{d}{dx}\right)^n\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2}_{a^+}\varphi\right)(x)\tag{4.1}$$

similarly, we can obtain the right-sided $^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2)}_{b^-}:{}^{2L}\mathcal{X}^1_{b^-}\longrightarrow L_1(a,b)$ as follows:

$$\left(^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2)}_{b^-}\varphi\right)(x)=\left(\mathfrak{I}^{\nu_1}_{b^-}\left(\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_2}_{b^-}\left(\frac{d}{dx}\right)^n\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2}_{b^-}\varphi\right)(x).\tag{4.2}$$

Remark 4.1. When $\nu_2=0$ in (4.1) and (4.2), its results will be in (3.1) and (3.2), respectively. Additionally, for $n=1$ in (4.1) and (3.1), we obtain (2.10) and (2.7), respectively.

Similar to the approach taken for the $\psi-$HFD in Section 3, the basic spaces for the $\psi-$2LFD with $n-1<\alpha\le n$ are defined as follows:

$$\begin{aligned}^{2L}\mathcal{X}^{0;\psi}_{a^+}=&\Bigg\{\varphi\in\mathcal{X}^{0;\psi}_{a^+}:\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1;\psi}_{a^+}\varphi\text{ and}\\&\mathfrak{I}^{\nu_1+\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1+\nu_2;\psi}_{a^+}\varphi\Bigg\}\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}^{2L}\mathcal{X}^{0;\psi}_{b^-}=&\Bigg\{\varphi\in\mathcal{X}^{0;\psi}_{b^-}:\mathfrak{I}^{\nu_1;\psi}_{b^-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1;\psi}_{b^-}\varphi\text{ and}\\&\mathfrak{I}^{\nu_1+\nu_2;\psi}_{b^-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1+\nu_2;\psi}_{b^-}\varphi\Bigg\}.\end{aligned}\tag{4.4}$$

Furthermore, let ${}^{2L}\mathcal{B}^{0;\psi}_{a^+}$ be another basic space for the $\psi-$2LFD defined as follows:

$$ {}^{2L}\mathcal{B}^{0;\psi}_{a^+} = \left\{ \varphi \in \mathcal{X}^{0;\psi}_{a^+} : \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+} \varphi(x) = \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \varphi(x) \right\}. \tag{4.5} $$

Theorem 4.1. *The spaces* ${}^{2L}\mathcal{X}^{0;\psi}_{a^+}$ *and* ${}^{2L}\mathcal{B}^{0;\psi}_{a^+}$ *contain the functions* $\varphi \in \mathcal{AC}^n_{\psi+}[a,b]$ *that satisfy the condition* $\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^k \varphi(a) = 0$, $k = 0, 1, \ldots, n-1$ *for all* $n \in \mathbb{N}$. *Similarly, this holds for the other side.*

Proof. Following the approach used for the space ${}^{C}\mathcal{X}^{0;\psi}_{a^+}$ in Theorem 3.3, we can complete the proof by setting $n - \alpha = \nu_1$, $\nu_1 + \nu_2$, or $2n - \alpha - \nu_1 - \nu_2$ in the proof of Theorem 3.3. □

Indeed, the swapping between the derivatives and integrals in (4.3), (4.4), and (4.5) is valid by setting $n - \alpha$ in the relation (3.33) with the same assumptions used in the previous proof.

Inspired by (4.1) and Definition 3.5, we will introduce an advanced form of the ψ-2LFD in the following definition. This definition will serve as the foundation for our generalization in the next section, namely the $\psi - m$th level FD.

Definition 4.2. Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and the parameters $\nu_1, \nu_2 \in \mathbb{R}$, $0 \leq \nu_1$, $0 \leq \nu_2$, $\alpha + \nu_1 \leq n$, $\alpha + \nu_1 + \nu_2 \leq 2n$, and $\psi \in \mathcal{C}^n([a,b], \mathbb{R})$ such that ψ is increasing, $\psi'(x) \neq 0$ for all $x \in [a,b]$. The left-sided $\psi-$2LFD of order α and type (ν_1, ν_2) is defined by

$$ \left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+} \varphi \right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{a^+} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{\nu_2;\psi}_{a^+} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+} \varphi \right)(x) \tag{4.6} $$

and the right-sided $\psi-$2LFD is defined by

$$ \left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-} \varphi \right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{b^-} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{\nu_2;\psi}_{b^-} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{b^-} \varphi \right)(x). \tag{4.7} $$

Note that, if $\xi_2 = \alpha + \nu_1 + \nu_2 - n$, $n-1 < \xi_2 \leq n$, then

$$ \left(\mathfrak{D}^{\xi_2;\psi}_{a^+} \varphi \right)(x) = \left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+} \varphi \right)(x) = \left(\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{n-\xi_2;\psi}_{a^+} \varphi \right)(x) \tag{4.8} $$

and

$$ \left(\mathfrak{D}^{\xi_2;\psi}_{b^-} \varphi \right)(x) = \left(\left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{b^-} \varphi \right)(x) = \left(\left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}^{n-\xi_2;\psi}_{b^-} \varphi \right)(x). \tag{4.9} $$

Thus, the $\psi-$2LFD, as defined above, can be expressed as follows:

$$ \left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+} \varphi \right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{a^+} \varphi^{[n]}_{\psi+} \mathfrak{I}^{\nu_2;\psi}_{a^+} \mathfrak{D}^{\xi_2;\psi}_{a^+} \varphi \right)(x), $$

and

$$ \left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-} \varphi \right)(x) = \left(\mathfrak{I}^{\nu_1;\psi}_{b^-} \varphi^{[n]}_{\psi-} \mathfrak{I}^{\nu_2;\psi}_{b^-} \mathfrak{D}^{\xi_2;\psi}_{b^-} \varphi \right)(x). $$

However, if the domain of the $\psi-$2LFD is ${}^{2L}\mathcal{X}^{0;\psi}_{a^+}$ or ${}^{2L}\mathcal{B}^{0;\psi}_{a^+}$, we will refer to it as the basic $\psi-$2LFD. In these spaces, we have the following lemma.

Theorem 4.2. *The basic $\psi-$2LFD of order α and type (ν_1,ν_2) is identical to the basic $\psi-$RLFD restricted to the domain ${}^{2L}\mathcal{X}^{0;\psi}_{a^+}$ and coincides with the basic $\psi-$CFD restricted to the domain ${}^{2L}\mathcal{B}^{0;\psi}_{a^+}$.*

Proof. On the space ${}^{2L}\mathcal{X}^{0;\psi}_{a^+}$, we have

$$\begin{aligned}\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_1+\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x), \quad \varphi\in {}^{2L}\mathcal{X}^{0;\psi}_{a^+}.\end{aligned}$$

Moreover, on the space ${}^{2L}\mathcal{B}^{0;\psi}_{a^+}$, we have

$$\begin{aligned}\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x)\\ &= \left(\mathfrak{I}^{n-\alpha;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x)\\ &= \left({}^{C}\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x), \quad \varphi\in {}^{2L}\mathcal{B}^{0;\psi}_{a^+}.\end{aligned}$$

□

Remark 4.2. From prior discussions, we outline the following points:

- R1. According to (3.8) and Theorem 4.1, the results of Theorem 4.2 are identical when confined to the domains ${}^{2L}\mathcal{X}^{0;\psi}_{a^+}$ or ${}^{2L}\mathcal{B}^{0;\psi}_{a^+}$, i.e., $\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left({}^{C}\mathfrak{D}^{\alpha,\psi}_{a^+}\varphi\right)(x) = \left({}^{RL}\mathfrak{D}^{\alpha,\psi}_{a^+}\varphi\right)(x)$, which also equals $\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x)$ as noted in Remark 3.10, and thus it is nothing new.
- R2. By setting $\psi(x)=x$ and $n=1$ in (4.3), Theorem 4.1, and Theorem 4.2, we obtain the related results discussed in ([16], Sec. 3.4).

Otherwise, the domain of the basic $\psi-$2LFD ${}^{2L}\mathcal{X}^{0;\psi}$ on the interval $[a,b]$ can be expanded to the larger space of functions as follows:

$${}^{2L}\mathcal{X}^{1;\psi}_{a^+} = \left\{\varphi : \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi, \mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi \in \mathcal{AC}^n_{\psi+}[a,b]\right\} \tag{4.10}$$

and

$${}^{2L}\mathcal{X}^{1;\psi}_{b^-} = \left\{\varphi : \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{b^-}\varphi, \mathfrak{I}^{\nu_2;\psi}_{b^-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{b^-}\varphi \in \mathcal{AC}^n_{\psi-}[a,b]\right\}. \tag{4.11}$$

These spaces provide sufficient conditions for the existence of the derivatives in (4.6) and (4.7). Hence, if $\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\varphi \in \mathcal{AC}_{\psi+}^{\mathrm{n}}[a,b]$, then by (2.5), there exist $\phi(x) \in L_1(a,b)$ such that

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\varphi(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n-1}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\varphi(a) + \mathfrak{I}_{a^+}\phi(x).$$

Thus,

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\varphi(x) = \frac{\phi(x)}{\psi'(x)} \in L_1(a,b).$$

Furthermore, if $\mathfrak{I}_{a^+}^{\nu_2;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\varphi \in \mathcal{AC}_{\psi+}^{n}[a,b]$, then

$$\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{\nu_2;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\varphi = \frac{\phi(x)}{\psi'(x)} \in L_1(a,b).$$

Remark 4.3. Building on earlier discussions, we note the following:

- R1. Setting $n = 1$ and $\psi(x) = x$ in (4.10), we obtain the related spaces ([16], Sec. 3.4).
- R2. Note that, if $\xi_2 = \alpha + \nu_1 + \nu_2 - n$, $n - 1 < \xi_2 \leq n$, then

$$^{2L}\mathcal{X}_{a^+}^{1;\psi} = \left\{\varphi : \mathfrak{I}_{a^+}^{n-\xi_2;\psi}\varphi, \mathfrak{I}_{a^+}^{\nu_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi \in \mathcal{AC}_{\psi+}^{n}[a,b]\right\}.$$

Definition 4.3. The extension of the basic $\psi-$2LFD to the domains $^{2L}\mathcal{X}_{a^+}^{1;\psi}$ or $^{2L}\mathcal{X}_{b^-}^{1;\psi}$ is referred to as the $\psi-$2LFD as defined by Definition 4.2. The space of the left-sided $\psi-$2LFD is given by $^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi} : {}^{2L}\mathcal{X}_{a^+}^{1;\psi} \to L_1(a,b)$, and the space of the right-sided $\psi-$2LFD is given by $^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi} : {}^{2L}\mathcal{X}_{b^-}^{1;\psi} \to L_1(a,b)$.

In order to demonstrate and verify the usage of the term "fractional derivative", we articulate and validate the FT of FC for the $\psi-$2LFD as outlined in the following theorem.

Theorem 4.3. *On the space $^{FT}\mathcal{X}_{a^+}^{\psi}$ defined by (3.34). The new parametrization of the $\psi-$2LFD given by (4.6) is a left-inverse operator to the $\psi-$RLFI for $x \in [a,b]$, i.e., the relation*

$$\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\,\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \varphi(x), \textit{ holds for any } \varphi \textit{ from the space } {}^{FT}\mathcal{X}_{a^+}^{\psi}.$$

Proof. The proof is somewhat analogous to that employed in the proof of Theorem 3.6.

$$\begin{aligned}
\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\,\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) &= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{\nu_2;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\,\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{\nu_2;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\,\mathfrak{I}_{a^+}^{n;\psi}\phi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{\nu_2;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{n;\psi}\mathfrak{I}_{a^+}^{2n-\alpha-\nu_1-\nu_2;\psi}\phi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{n;\psi}\mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi}\phi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{n-\alpha;\psi}\phi\right)(x)\\
&= \varphi(x).
\end{aligned}$$

□

Theorem 4.4. *Let $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$. Consider the following conditions: $\nu_2 = 0$, $\nu_2 \geq n$, or $\alpha + \nu_1 + \nu_2 \leq n$, along with $\nu_2 = \nu_1$ when φ is continuous from the right at $x = a$ on the interval $x \in [a, b]$. Under these conditions, we observe that the $\psi-$2LFD reduces to the $\psi-$HFD, as indicated in formula (3.39).*

Proof. For $\nu_2 = 0$, we have

$$\begin{aligned}\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,0);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{0;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x)\\ &= \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x).\end{aligned}$$

Now, for $\nu_2 > n$, we have

$$\begin{aligned}\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{\nu_2-n;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1+\nu_2-n;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha-(\nu_1+\nu_2-n);\psi}_{a^+}\varphi\right)(x)\\ &= \left({}^{H}\mathfrak{D}^{\alpha,\nu_1+\nu_2-n;\psi}_{a^+}\varphi\right)(x).\end{aligned}$$

Specifically, for $\nu_2 = n$, we have $\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x)$.

Additionally, for $\alpha + \nu_1 + \nu_2 \leq n$, we have

$$\begin{aligned}\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x)\\ &= \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x).\end{aligned}$$

Furthermore, if $\nu_2 = \nu_1$ and φ is continuous from the right at $x = a$, then utilizing Theorem 3.1 (T2) and Lemma 3.3, we have

$$\begin{aligned}&\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_1);\psi}_{a^+}\varphi\right)(x)\\ &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \underbrace{\mathfrak{I}^{2n-\alpha-2\nu_1;\psi}_{a^+}\varphi}_{\text{say } =: \vartheta(x)}\right)(x)\end{aligned}$$

$$
\begin{aligned}
&=\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\left[\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_1;\psi}_{a^+}\vartheta(x)-\underbrace{\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\nu_1-k}}{\Gamma(\nu_1-k+1)}\varphi^{[n-k]}_{\psi+}\vartheta(a)}_{\text{equals zero by Lemma 3.3}}\right]\\
&=\left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\varphi\right)(x)\\
&=\left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x).
\end{aligned}
$$

□

Remark 4.4. Building on earlier discussions, we note the following:

- R1. By setting $\psi(x) = x$ and $n = 1$ in Theorem 4.3 and Theorem 4.4, we obtain the results discussed in ([16], Thm. 3.3, Remark 3.9, respectively).
- R2. Based on Theorem 4.4 and utilizing Remark 3.15, we establish a connection between $\psi-$2LFD and $\psi-$HFD, $\psi-$RLFD, and $\psi-$CFD. Consequently, many well-known FDs can be obtained by selecting different forms of $\psi(x)$, such as 2LFD, HFD, RLFD, and CFD. Thus, the significance and novelty of this new concept emerge clearly from these connections, particularly in studying their characteristics simultaneously using the same FD operator.

Remark 4.5. According to Definition 4.2, we observe the following:

- R1. By adjusting the ν_2 condition in Definition 4.2 to $\nu_2 \geq n - \nu_1$ instead of $\nu_2 \geq 0$, although this is less comprehensive, and by setting $\xi_2 = \alpha + \nu_1 + \nu_2 - n$, we observe that $n - 1 < \xi_2 \leq n$. This condition is suitable for certain weighted spaces, such as $\mathcal{C}_{\xi_2;\psi}[a, b]$ (see [28]), and ensures that ξ_2 is always positive. Therefore, we will assume that all the conditions of Definition 4.2 for $\psi-$2LFD are fulfilled in all subsequent lemmas and theorems, with the assumption that $\nu_2 \geq n - \nu_1$, unless explicitly stated otherwise.
- R2. Taking into account Remark 4.5 (R1), the parameters $\xi_2 = \alpha+\nu_1+\nu_2-n$, $\xi_1 = \alpha+\nu_1$, and ν_1, ν_2 have the following properties:

$n-1 < \xi_2 \leq n$	$n - \xi_2 < 2n - \nu_1 - \nu_2$
$\xi_2 \leq \nu_1 + \nu_2$	$\xi_2 < \alpha + \nu_1 + \nu_2$
$\xi_2 \geq \alpha > n - 1$	$\xi_1 \leq \xi_2$ if $\nu_2 \geq n$
$0 \leq \nu_1 < 1$	$n \leq \nu_1 + \nu_2 < 2n$

- R3. Referring to Theorem 4.4, we observe that the conditions, except for $\nu_2 \geq n$, are not compatible with the modification made in Remark 4.5 (R1), Due to the contradictions arising from this modification. Nevertheless, we will subsequently demonstrate a theorem regarding the relationship between $\psi-$2LFD and $\psi-$HFD to accommodate these adjustments.
- R4. The case $\nu_2 = n$ is an important example that we will use to connect our results with the well-known findings regarding the amended condition, as mentioned in (R1). Therefore, for $\nu_2 = n$, we note the following:

- It is consistent with the requirement $\nu_2 \geq n - \nu_1$ and complies with the restrictions outlined in Definition 4.2 for $\psi-$2LFD.
- The parameters ξ_2 and ξ_1 are equivalent.
- By setting $\nu_1 = \beta_1(n-\alpha)$, $\nu_1 = n-\alpha$, and $\nu_1 = 0$, and considering Theorem 4.4 in conjunction with Remark 3.12 (R1), along with relations (3.48) and (3.47), we observe that the $\psi-$2LFD reduces to $\psi-$HFD, as indicated in formula (2.15), as well as to $\psi-$CFD and $\psi-$RLFD, respectively.

Lemma 4.1. *Let* $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$ *and* $\nu_2 \geq n - \nu_1$. *Then we have*

$$\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1-\alpha;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x) \tag{4.12}$$

and for $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{b^-}$, *we have*

$$\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1-\alpha;\psi}_{b^-}\mathfrak{D}^{\xi_1;\psi}_{b^-}\mathfrak{I}^{\xi_2;\psi}_{b^-}\mathfrak{D}^{\xi_2;\psi}_{b^-}\varphi\right)(x). \tag{4.13}$$

Proof. Let $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$. Taking into account Remark 4.5 (R1), we find that

$$\begin{aligned}
\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_2;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{2n-\alpha-\nu_1-\nu_2;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{\nu_2+\nu_1+\alpha-n+n-\alpha-\nu_1;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\mathfrak{I}^{\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{n-\alpha-\nu_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\mathfrak{I}^{\xi_1-\alpha;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x).
\end{aligned}$$

□

Theorem 4.5. *Let* $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$ *and* $\nu_2 \geq n - \nu_1$. *The relationship between* $\psi-$*2LFD and* $\psi-$*HFD is as follows:*

$$\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x).$$

Proof. This follows directly from relations (3.42) and (4.12). □

Theorem 4.6. *Let* $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$ *and* $\nu_2 \geq n - \nu_1$. *Then we have*

$$\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi\right)(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-\alpha-k}}{\Gamma(\xi_2-\alpha-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)$$

and for $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{b^-}$, *we have*

$$\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-}\varphi\right)(x) = \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{b^-}\varphi\right)(x) - \sum_{k=1}^{n}\frac{\Big(\psi(b)-\psi(x)\Big)^{\xi_2-\alpha-k}}{\Gamma(\xi_2-\alpha-k+1)}\varphi^{[n-k]}_{\psi-}\mathfrak{I}^{n-\xi_2;\psi}_{b^-}\varphi(b).$$

Proof. Given that $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$, it follows that $\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi \in \mathcal{AC}^n_{\psi+}[a,b]$. Therefore, by applying Theorem 4.5, relation (3.5), and Lemma 2.1 (L2), we have

$$\begin{aligned}\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\\
&= {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\left(\varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)\right)\\
&= {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi(x) - \sum_{k=1}^{n}\frac{{}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)\\
&= {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi(x) - \sum_{k=1}^{n}\frac{\mathfrak{I}^{\xi_1-\alpha;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)\\
&= {}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-\alpha-k}}{\Gamma(\xi_2-\alpha-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a).\end{aligned}$$

□

Remark 4.6. As discussed earlier, we note the following:

- R1. By setting $\nu_2 = n$ and utilizing Remark 4.5 (R4) along with relations (3.3), (3.42), and (4.12), we find that $\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1-\alpha;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi\right)(x) = \left({}^{H}\mathfrak{D}^{\alpha,\nu_1;\psi}_{a^+}\right)(x)$. Similarly, by applying Lemma 3.3 to Theorem 4.6, we can obtain the same result if φ is continuous from the right at $x = a$ for $x \in [a,b]$.
- R2. Utilizing Lemma 4.1 and the semigroup property from Lemma 2.1 (L1), we obtain

$$\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\,{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) = \left(\mathfrak{I}^{\xi_1;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x). \tag{4.14}$$

Theorem 4.7. *Let* $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$, $\mathfrak{I}^{n-\xi_1;\psi}_{a^+}\varphi \in \mathcal{AC}^{n+1}_{\psi+}[a,b]$, $\nu_2 \geq n-\nu_1$, *and* $\psi \in \mathcal{C}^{n+1}[a,b]$. *Then we have*

$$\begin{aligned}&\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x)\\
&= \frac{1}{\Gamma(\xi_1-\alpha+1)}\int_a^x\Big(\psi(x)-\psi(t)\Big)^{\xi_1-\alpha}\frac{d}{dt}\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi(t)dt\\
&\quad+\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\mathfrak{D}^{\xi_1;\psi}_{a^+}\varphi(a) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-\alpha-k}}{\Gamma(\xi_2-\alpha-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)\end{aligned}$$

and for $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{b^-}$, *and* $\mathfrak{I}^{n-\xi_1;\psi}_{b^-}\varphi \in \mathcal{AC}^{n+1}_{\psi-}[a,b]$, *we have*

$$\begin{aligned}&\left({}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-}\varphi\right)(x)\\
&= \frac{-1}{\Gamma(\xi_1-\alpha+1)}\int_x^b\Big(\psi(t)-\psi(x)\Big)^{\xi_1-\alpha}\frac{d}{dt}\mathfrak{D}^{\xi_1;\psi}_{b^-}\varphi(t)dt\\
&\quad+\frac{\Big(\psi(b)-\psi(x)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\mathfrak{D}^{\xi_1;\psi}_{b^-}\varphi(b) - \sum_{k=1}^{n}\frac{\Big(\psi(b)-\psi(x)\Big)^{\xi_2-\alpha-k}}{\Gamma(\xi_2-\alpha-k+1)}\varphi^{[n-k]}_{\psi-}\mathfrak{I}^{n-\xi_2;\psi}_{b^-}\varphi(b).\end{aligned}$$

Proof. The proof can be obtained by employing Theorem 4.6 and Theorem 3.7. ☐

Theorem 4.8. *Let* $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$ *and* $\nu_2 \geq n - \nu_1$. *Then we find that*

$$\begin{aligned}\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\,{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) =&\varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)\\ &-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}}{\Gamma(\xi_1-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n+\xi_2-\xi_1;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi(a).\end{aligned} \tag{4.15}$$

Proof. By Theorem 3.1 (T3) and Lemma 4.1, we have

$$\begin{aligned}&\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\,{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x)\\ =&\left(\mathfrak{I}^{\xi_1;\psi}_{a^+}\mathfrak{D}^{\xi_1;\psi}_{a^+}\underbrace{\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi}_{=:\vartheta(x)}\right)(x)\\ =&\vartheta(x)-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}}{\Gamma(\xi_1-k+1)}\vartheta^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_1;\psi}_{a^+}\vartheta(a)\\ =&\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi(x)-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}}{\Gamma(\xi_1-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_1;\psi}_{a^+}\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi(a)\\ =&\varphi(x)-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a)\\ &-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}}{\Gamma(\xi_1-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n+\xi_2-\xi_1;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi(a).\end{aligned}$$

☐

Theorem 4.9. *Let* $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$ *and* $\nu_2 \geq n$. *Then we have*

$$\begin{aligned}\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\,{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x) &= \left(\mathfrak{I}^{\xi_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\\ &= \varphi(x)-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi^{[n-k]}_{\psi+}\mathfrak{I}^{n-\xi_2;\psi}_{a^+}\varphi(a).\end{aligned}$$

Proof. Let $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}_{a^+}$. Using (3.5), we obtain

$$\begin{aligned}&\left(\mathfrak{I}^{\alpha;\psi}_{a^+}\,{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right)(x)\\ =&\left(\mathfrak{I}^{\alpha+\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{\nu_2;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\\ =&\left(\mathfrak{I}^{\alpha+\nu_1;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}^{n;\psi}_{a^+}\mathfrak{I}^{\nu_2-n;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right)(x)\end{aligned}$$

$$
\begin{aligned}
&= \left(\mathfrak{I}_{a^+}^{\alpha+\nu_1;\psi}\mathfrak{I}_{a^+}^{\nu_2-n;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{\xi_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi\right)(x)\\
&= \varphi(x) - \sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi_{\psi+}^{[n-k]}\mathfrak{I}_{a^+}^{n-\xi_2;\psi}\varphi(a).
\end{aligned}
$$

□

Theorem 4.10. *Let $\varphi \in {}^{2L}\mathcal{X}^{1;\psi}$ on the space $\mathcal{AC}_{\psi+}^{n+m}[a,b]$, where $m, n \in \mathbb{N}$, and suppose that $\nu_2 \geq 0$. Then for all $k \in \mathbb{N}$ we have*

$$
\left(\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\right)^k\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi\right)(x) = \frac{\Big(\psi(x)-\psi(a)\Big)^{k\alpha}\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi(c)}{\Gamma(k\alpha+1)}
$$

and

$$
\left(\left(\mathfrak{I}_{b^-}^{\alpha;\psi}\right)^k\left({}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi\right)(x) = \frac{(\psi(b)-\psi(x))^{k\alpha}\left({}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi(d)}{\Gamma(k\alpha+1)},
$$

for some $c \in (a,x)$ and $d \in (x,b)$, ensured by the mean value theorem for integrals.

Proof. By using Lemma 2.1 (L1), we obtain $\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\right)^k = \underbrace{\mathfrak{I}_{a^+}^{\alpha;\psi}\cdots\mathfrak{I}_{a^+}^{\alpha;\psi}}_{\text{k-times}} = \mathfrak{I}_{a^+}^{k\alpha;\psi}$.

Therefore,

$$
\begin{aligned}
&\left(\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\right)^k\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{k\alpha;\psi}\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi\right)(x)\\
&= \frac{1}{\Gamma(k\alpha)}\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{k\alpha-1}\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi(t)dt\\
&= \frac{\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi(c)}{\Gamma(k\alpha)}\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{k\alpha-1}dt\\
&= \frac{\Big(\psi(x)-\psi(a)\Big)^{k\alpha}\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\right)^m\varphi(c)}{\Gamma(k\alpha+1)}, \qquad \text{for some } c \in (a,x).
\end{aligned}
$$

□

Remark 4.7. Building on earlier discussions, we note the following:

- R1. Referring to Theorem 4.10, by setting $\nu_2 = 0$ and considering Theorem 4.4 along with Remark 3.12 (R1) and Remark 3.15, we obtain the following results: If $\nu_1 = \beta_1(n-\alpha)$, we obtain the result in ([28], Thm. 8). Additionally, if $\nu_1 = n-\alpha$, we derive the result in ([4], Thm. 7).
- R2. Referring to Theorem 4.9, by setting $\nu_2 = n$ and considering Remark 4.5 (R4), we obtain the result presented in Lemma 3.6 (L1). Furthermore, if $\nu_1 = \beta_1(n-\alpha)$, we arrive at the result in ([28], Thm. 5). If $\nu_1 = n-\alpha$, we obtain the result in ([4], Thm. 4). Additionally, if $\nu_1 = 0$ then $\xi_2 = \alpha$, leading to relation (3.5).

- R3. In Theorem 4.8, if φ is continuous from the right at $x = a$ for $x \in [a, b]$, then by Lemma 3.3, we have $\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\, {}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x) = \varphi(x)$.

Theorem 4.11. *Let* $\varphi \in {}^{2L}\mathcal{X}_{a^+}^{1;\psi}$, $\nu_2 \geq n - \nu_1$, *and* $\psi \in \mathcal{C}^n([a, b], \mathbb{R})$. *Consider the function* $\varphi(x) = \Big(\psi(x) - \psi(a)\Big)^{\delta-1}$, *where* $\delta > 0$. *Then*

$$\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x) = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}\Big(\psi(x) - \psi(a)\Big)^{\delta-\alpha-1} = \left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x).$$

Proof. Using Lemma 2.1 (L2) and Remark 4.7 (R1), we obtain

$$\begin{aligned}\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x) &= \mathfrak{I}_{a^+}^{\xi_1-\alpha;\psi}\,\mathfrak{D}_{a^+}^{\xi_1;\psi}\mathfrak{I}_{a^+}^{\xi_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\Big(\psi(x) - \psi(a)\Big)^{\delta-1} \\ &= \mathfrak{I}_{a^+}^{\xi_1-\alpha;\psi}\,\mathfrak{D}_{a^+}^{\xi_1;\psi}\Big(\psi(x) - \psi(a)\Big)^{\delta-1} \\ &= \frac{\Gamma(\delta)}{\Gamma(\delta-\xi_1)}\mathfrak{I}_{a^+}^{\xi_1-\alpha;\psi}\Big(\psi(x) - \psi(a)\Big)^{\delta-\xi_1-1} \\ &= \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)}\Big(\psi(x) - \psi(a)\Big)^{\delta-\alpha-1} \\ &= \left({}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\varphi\right)(x).\end{aligned}$$

Similarly, the same procedure can be applied to $\vartheta(x) = \Big(\psi(b) - \psi(x)\Big)^{\delta-1}$ to derive the result. □

Remark 4.8. Referring to Theorem 4.11, we have the following:

- R1. By setting $\delta = k + 1$ where $k \in \mathbb{N}$, we obtain

$$\begin{aligned}{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x) - \psi(a)\Big)^{k} &= \frac{k!}{\Gamma(k-\alpha+1)}\Big(\psi(x) - \psi(a)\Big)^{k-\alpha} \\ &= {}^{H}\mathfrak{D}_{a^+}^{\alpha,\nu_1;\psi}\Big(\psi(x) - \psi(a)\Big)^{k}\end{aligned}$$

and

$$\begin{aligned}{}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(b) - \psi(x)\Big)^{k} &= \frac{k!}{\Gamma(k-\alpha+1)}\Big(\psi(b) - \psi(x)\Big)^{k-\alpha} \\ &= {}^{H}\mathfrak{D}_{b^-}^{\alpha,\nu_1;\psi}\Big(\psi(b) - \psi(x)\Big)^{k}.\end{aligned}$$

- R2. When $\delta = \alpha - k + 1$, $k \in \mathbb{N}$, we observe that

$$\begin{aligned}{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x) - \psi(a)\Big)^{\alpha-k} &= \frac{\Gamma(\alpha-k+1)}{\Gamma(1-k)}\Big(\psi(x) - \psi(a)\Big)^{-k} \\ &= 0,\ \text{since } \Gamma(1-k) = \pm\infty,\ k \in \mathbb{N}.\end{aligned}$$

Additionally, for $k = 1, 2, \ldots, [\alpha] + 1$, and using Lemma 2.1 (L2), we have

$$\mathfrak{D}_{a^+}^{\alpha;\psi}\Big(\psi(x) - \psi(a)\Big)^{\alpha-k} = 0,\ \text{and}\ \mathfrak{D}_{b^-}^{\alpha;\psi}\Big(\psi(b) - \psi(x)\Big)^{\alpha-k} = 0. \tag{4.16}$$

Hence, from (4.16) and Lemma 4.1, we obtain

$$ {}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}=0,\ \text{and}\ {}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}=0 \tag{4.17}$$

similarly, we have

$$ {}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(b)-\psi(x)\Big)^{\xi_1-k}=0,\ \text{and}\ {}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(b)-\psi(x)\Big)^{\xi_2-k}=0.$$

- R3. The function φ is continuous from the right at $x=a$ for $x\in[a,b]$ and for any $\delta>0$. Thus, we can achieve the result of Theorem 4.11 by utilizing Remark 4.6 (R1). Moreover, by setting $\nu_1=\beta_1(n-\alpha)$ and considering Remark 3.12 (R1), we obtain the result presented in ([28], Lemma 5 and Remark 2). Additionally, for $\nu_1=n-\alpha$ and using Remark 3.15, we derive the results in ([4], Lemma 1 and relation (2)).

Theorem 4.12. *Let $\varphi,\vartheta\in{}^{2L}\mathcal{X}_{a^+}^{1;\psi}$ and $\nu_2\geq n-\nu_1$. Then*

$$\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)=\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x)$$

if and only if

$$\varphi(x)=\vartheta(x)+\sum_{k=1}^{n}c_k\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}+\sum_{k=1}^{n}d_k\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k},$$

where

$$c_k=\sum_{k=1}^{n}\frac{(\varphi)_{\psi+}^{[n-k]}\mathfrak{I}_{a^+}^{n+\xi_2-\xi_1;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}(\varphi-\vartheta)(a)}{\Gamma(\xi_1-k+1)},\ d_k=\sum_{k=1}^{n}\frac{(\varphi)_{\psi+}^{[n-k]}\mathfrak{I}_{a^+}^{n-\xi_2;\psi}(\varphi-\vartheta)(a)}{\Gamma(\xi_2-k+1)}.$$

Furthermore, on the right-hand side, let $\varphi,\vartheta\in{}^{2L}\mathcal{X}_{b^-}^{1;\psi}$. Then $\left({}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)=\left({}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x)$ if and only if

$$\varphi(x)=\vartheta(x)+\sum_{k=1}^{n}e_k\Big(\psi(b)-\psi(x)\Big)^{\xi_1-k}+\sum_{k=1}^{n}r_k\Big(\psi(b)-\psi(x)\Big)^{\xi_2-k},$$

where

$$e_k=\sum_{k=1}^{n}\frac{(\varphi)_{\psi-}^{[n-k]}\mathfrak{I}_{b^-}^{n+\xi_2-\xi_1;\psi}\mathfrak{D}_{b^-}^{\xi_2;\psi}(\varphi-\vartheta)(b)}{\Gamma(\xi_1-k+1)},\ r_k=\sum_{k=1}^{n}\frac{(\varphi)_{\psi-}^{[n-k]}\mathfrak{I}_{b^-}^{n-\xi_2;\psi}(\varphi-\vartheta)(b)}{\Gamma(\xi_2-k+1)}.$$

Proof. ($\Rightarrow$) Let $\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)=\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x)$, which implies that

$${}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\big(\varphi(x)-\vartheta(x)\big)=0.$$

By applying the left integral operator to both sides of this equivalence and utilizing Theorem 4.8, we obtain

$$\left(\mathfrak{I}_{a^+}^{\alpha;\psi}\ {}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\left(\varphi-\vartheta\right)\right)(x)=0.$$

This implies that

$$\begin{aligned}&(\varphi-\vartheta)(x)-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}}{\Gamma(\xi_1-k+1)}\varphi_{\psi+}^{[n-k]}\mathfrak{J}_{a^+}^{n+\xi_2-\xi_1;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}(\varphi-\vartheta)(a)\\&-\sum_{k=1}^{n}\frac{\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}{\Gamma(\xi_2-k+1)}\varphi_{\psi+}^{[n-k]}\mathfrak{J}_{a^+}^{n-\xi_2;\psi}(\varphi-\vartheta)(a)=0.\end{aligned}$$

Therefore, we can conclude that

$$\varphi(x)=\vartheta(x)+\sum_{k=1}^{n}c_k\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}+\sum_{k=1}^{n}d_k\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}. \tag{4.18}$$

($\Leftarrow$) To demonstrate the opposite, let (4.18) be given. By applying the derivative operator ${}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}(\cdot)$ to both sides of (4.18), we obtain

$$\begin{aligned}&\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)\\=&\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x)\\&+\underbrace{\sum_{k=1}^{n}c_k\,{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x)-\psi(a)\Big)^{\xi_1-k}+\sum_{k=1}^{n}d_k\,{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x)-\psi(a)\Big)^{\xi_2-k}}_{\text{equals zero by (4.17)}}\\=&\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x).\end{aligned}$$

□

In particular, for $n=1$, we obtain

$$\begin{aligned}&\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)\\=&\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x)\Leftrightarrow\varphi(x)\\=&\vartheta(x)+c\Big(\psi(x)-\psi(a)\Big)^{\xi_1-1}+d\Big(\psi(x)-\psi(a)\Big)^{\xi_2-1}\end{aligned}$$

where

$$c=\frac{\mathfrak{J}_{a^+}^{1+\xi_2-\xi_1;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}(\varphi-\vartheta)(a)}{\Gamma(\xi_1)},\ d=\frac{\mathfrak{J}_{a^+}^{1-\xi_2;\psi}(\varphi-\vartheta)(a)}{\Gamma(\xi_2)}.$$

Remark 4.9. Referring to Theorem 4.12, by setting $\nu_2=n$ and considering Remark 4.5 (R4), we obtain the following results: If $\nu_1=\beta_1(n-\alpha)$, we obtain the result in ([28], Thm. 6). Furthermore, if $\nu_1=n-\alpha$, we arrive at the result in ([4], Thm. 6).

Lemma 4.2. *Let* $\varphi\in{}^{2L}\mathcal{X}_{a^+}^{1;\psi}$ *and* $\vartheta\in{}^{2L}\mathcal{X}_{b^-}^{1;\psi}$*, and suppose that* $0\le\nu_2$*,* $0<\lambda$*,* $0<\beta$*, and* $0\le\mu$*. Then for* $\varphi(x)=\Big(\psi(x)-\psi(a)\Big)^{\beta-1}\mathbb{E}_{\mu,\beta}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\mu}\right]$*, and* $\vartheta(x)=\Big(\psi(b)-\psi(x)\Big)^{\beta-1}\mathbb{E}_{\mu,\beta}\left[\lambda\Big(\psi(b)-\psi(x)\Big)^{\mu}\right]$*, we have*

$$\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)=\Big(\psi(x)-\psi(a)\Big)^{\beta-\alpha-1}\mathbb{E}_{\mu,\beta-\alpha}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\mu}\right]$$

and

$$\left({}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\vartheta\right)(x) = \Big(\psi(b)-\psi(x)\Big)^{\beta-\alpha-1}\mathbb{E}_{\mu,\beta-\alpha}\left[\lambda\Big(\psi(b)-\psi(x)\Big)^{\mu}\right],$$

where $\mathbb{E}_\alpha(\cdot)$ is the Mittag-Leffler function with one parameter, and $\mathbb{E}_{\alpha,\beta}(\cdot)$ is referred to as a Mittag-Leffler type function (see [13]*, (1.8.1) and (1.8.17)).*

Proof. By utilizing the definition of a Mittag-Leffler type function and Theorem 4.11, we can deduce that

$$\begin{aligned}
&\left({}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right)(x)\\
=&{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\left(\sum_{k=0}^{\infty}\frac{\lambda^k\Big(\psi(x)-\psi(a)\Big)^{\mu k+\beta-1}}{\Gamma(\mu k+\beta)}\right)\\
=&\sum_{k=0}^{\infty}\frac{\lambda^k\,{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\Big(\psi(x)-\psi(a)\Big)^{\mu k+\beta-1}}{\Gamma(\mu k+\beta)}\\
=&\sum_{k=0}^{\infty}\frac{\lambda^k\Big(\psi(x)-\psi(a)\Big)^{\mu k+\beta-\alpha-1}}{\Gamma(\mu k+\beta-\alpha)}\\
=&\Big(\psi(x)-\psi(a)\Big)^{\beta-\alpha-1}\sum_{k=0}^{\infty}\frac{\lambda^k\Big(\psi(x)-\psi(a)\Big)^{\mu k}}{\Gamma(\mu k+\beta-\alpha)}\\
=&\Big(\psi(x)-\psi(a)\Big)^{\beta-\alpha-1}\mathbb{E}_{\mu,\beta-\alpha}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\mu}\right].
\end{aligned}$$

The result for the right side of the lemma can be proven similarly. □

Remark 4.10. Specifically, when $\beta=1$ and $\mu=\alpha$, and based on the property $\mathbb{E}_{\alpha,1}(\cdot)=\mathbb{E}_\alpha(\cdot)$ as referenced in [13], we have

$$\begin{aligned}
&{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\left(\mathbb{E}_\alpha\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\alpha}\right]\right)\\
=&\Big(\psi(x)-\psi(a)\Big)^{-\alpha}\mathbb{E}_{\alpha,1-\alpha}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\alpha}\right]\\
=&\Big(\psi(x)-\psi(a)\Big)^{-\alpha}\left(\sum_{k=0}^{\infty}\frac{\lambda^k\Big(\psi(x)-\psi(a)\Big)^{\alpha k}}{\Gamma(\alpha k-\alpha+1)}\right)\\
=&\Big(\psi(x)-\psi(a)\Big)^{-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}+\sum_{k=1}^{\infty}\frac{\lambda^k\Big(\psi(x)-\psi(a)\Big)^{\alpha k}}{\Gamma(\alpha k-\alpha+1)}\right)\\
=&\Big(\psi(x)-\psi(a)\Big)^{-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}+\lambda\Big(\psi(x)-\psi(a)\Big)^{\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k\Big(\psi(x)-\psi(a)\Big)^{\alpha k}}{\Gamma(\alpha k+1)}\right)\\
=&\frac{\Big(\psi(x)-\psi(a)\Big)^{-\alpha}}{\Gamma(1-\alpha)}+\lambda\mathbb{E}_{\alpha,1}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\alpha}\right]
\end{aligned}$$

$$= \frac{\Big(\psi(x)-\psi(a)\Big)^{-\alpha}}{\Gamma(1-\alpha)} + \lambda \mathbb{E}_{\alpha}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\alpha}\right].$$

Moreover, for $\beta = 1+\alpha$, and by utilizing Lemma 4.2, we have

$$ {}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\left(\Big(\psi(x)-\psi(a)\Big)^{\alpha}\mathbb{E}_{\mu,1+\alpha}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\mu}\right]\right) = \mathbb{E}_{\mu}\left[\lambda\Big(\psi(x)-\psi(a)\Big)^{\mu}\right].$$

Theorem 4.13. *On the weighted space $\mathcal{C}_{\gamma,\psi}$ of functions φ on $(a,b]$, as presented in [28], the $\psi-$2LFDs are bounded operators. For $n-1<\gamma\le n$ and $\nu_2 \ge n-\nu_1$, we have*

$$\left\|{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right\|_{\mathcal{C}_{\gamma,\psi}} \le \mathcal{K}\|\varphi\|_{\mathcal{C}^{2n}_{\gamma,\psi}}, \text{ and } \left\|{}^{2L}\mathfrak{D}_{b^-}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right\|_{\mathcal{C}_{\gamma,\psi}} \le \mathcal{K}\|\varphi\|_{\mathcal{C}^{2n}_{\gamma,\psi}},$$

where

$$\mathcal{K} = \frac{\Big(\psi(b)-\psi(a)\Big)^{2n-\alpha}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)\Gamma(n-\xi_2+1)}.$$

Proof. Using relations (4.12), (3.41), and (4.8), we obtain

$$\begin{aligned}
&\left\|{}^{2L}\mathfrak{D}_{a^+}^{\alpha,(\nu_1,\nu_2);\psi}\varphi\right\|_{\mathcal{C}_{\gamma,\psi}} \\
&= \left\|\mathfrak{I}_{a^+}^{\xi_1-\alpha;\psi}\mathfrak{D}_{a^+}^{\xi_1;\psi}\mathfrak{I}_{a^+}^{\xi_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi\right\|_{\mathcal{C}_{\gamma,\psi}} \\
&= \max_{x\in[a,b]}\left|\Big(\psi(x)-\psi(a)\Big)^{\gamma}\mathfrak{I}_{a^+}^{\xi_1-\alpha;\psi}\mathfrak{D}_{a^+}^{\xi_1;\psi}\mathfrak{I}_{a^+}^{\xi_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi(x)\right| \\
&\le \left\|\mathfrak{D}_{a^+}^{\xi_1;\psi}\mathfrak{I}_{a^+}^{\xi_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi\right\|_{\mathcal{C}_{\gamma;\psi}} \cdot \frac{1}{\Gamma(\xi_1-\alpha)}\max_{x\in[a,b]}\left|\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\xi_1-\alpha-1}dt\right| \\
&\le \frac{\Big(\psi(b)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\left\|\mathfrak{D}_{a^+}^{\xi_1;\psi}\mathfrak{I}_{a^+}^{\xi_2;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi\right\|_{\mathcal{C}_{\gamma;\psi}} \\
&\le \frac{\Big(\psi(b)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)} \cdot \max_{x\in[a,b]}\left|\Big(\psi(x)-\psi(a)\Big)^{\gamma}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{\xi_2+n-\xi_1;\psi}\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi(x)\right| \qquad (4.19) \\
&\le \frac{\left\|\mathfrak{D}_{a^+}^{\xi_2;\psi}\varphi\right\|_{\mathcal{C}^{n}_{\gamma;\psi}}\Big(\psi(b)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)\Gamma(\xi_2+n-\xi_1)} \cdot \max_{x\in[a,b]}\left|\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\xi_2+n-\xi_1-1}dt\right| \\
&\le \frac{\Big(\psi(b)-\psi(a)\Big)^{\xi_2-\alpha+n}}{\Gamma(\xi_1-\alpha+1)\Gamma(\xi_2+n-\xi_1+1)}\left\|\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{n-\xi_2;\psi}\varphi\right\|_{\mathcal{C}^{n}_{\gamma,\psi}} \\
&\le \frac{\Big(\psi(b)-\psi(a)\Big)^{\xi_2-\alpha+n}}{\Gamma(\xi_1-\alpha+1)\Gamma(\xi_2+n-\xi_1+1)} \cdot \max_{x\in[a,b]}\left|\begin{aligned}&\Big(\psi(x)-\psi(a)\Big)^{\gamma}\left(\tfrac{1}{\psi'(x)}\tfrac{d}{dx}\right)^{2n}\tfrac{1}{\Gamma(n-\xi_2)} \\ &\times\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-\xi_2-1}\varphi(t)dt\end{aligned}\right| \\
&\le \frac{\|\varphi\|_{\mathcal{C}^{2n}_{\gamma;\psi}}\Big(\psi(b)-\psi(a)\Big)^{\xi_2-\alpha+n}}{\Gamma(\xi_1-\alpha+1)\Gamma(\xi_2+n-\xi_1+1)\Gamma(n-\xi_2)} \cdot \max_{x\in[a,b]}\left|\int_a^x \psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-\xi_2-1}dt\right|
\end{aligned}$$

$$\leq \frac{\Big(\psi(b)-\psi(a)\Big)^{2n-\alpha}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)\Gamma(n-\xi_2+1)}\|\varphi\|_{\mathcal{C}^{2n}_{\gamma,\psi}}$$
$$= \mathcal{K}\|\varphi\|_{\mathcal{C}^{2n}_{\gamma,\psi}}.$$

□

Remark 4.11. Based on prior discussions, we outline:

- R1. If $\nu_2 \geq n$, then $\xi_1 \leq \xi_2$ by Remark 4.5 (R2). Therefore, we can conclude that the $\psi-$2LFDs are bounded operators, given by

$$\left\|{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi\right\|_{\mathcal{C}_{\gamma,\psi}} \leq \mathcal{S}\|\varphi\|_{\mathcal{C}^n_{\gamma,\psi}}, \text{ and } \left\|{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-}\varphi\right\|_{\mathcal{C}_{\gamma,\psi}} \leq \mathcal{S}\|\varphi\|_{\mathcal{C}^n_{\gamma,\psi}},$$

  where

$$\mathcal{S} = \frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}{\Gamma(\nu_1+1)\Gamma(\nu_2-n+1)\Gamma(n-\xi_2+1)}.$$

  This result is derived using a technique similar to that in the proof of Lemma 4.13, with some minor differences, particularly in (4.19), which becomes:

$$\leq \frac{\Big(\psi(b)-\psi(a)\Big)^{\xi_1-\alpha}}{\Gamma(\xi_1-\alpha+1)}\cdot \max_{x\in[a,b]}\left|\Big(\psi(x)-\psi(a)\Big)^{\gamma}\mathfrak{I}^{\xi_2-\xi_1;\psi}_{a^+}\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi(x)\right|$$
$$\leq \frac{\left\|\mathfrak{D}^{\xi_2;\psi}_{a^+}\varphi\right\|_{\mathcal{C}_{\gamma;\psi}}\Big(\psi(b)-\psi(a)\Big)^{\xi_2-\alpha}}{\Gamma(\xi_1-\alpha+1)\Gamma(\xi_2-\xi_1+1)}$$
$$\leq \cdots.$$

- R2. For $\gamma = 0$ and using the fact $\mathcal{C}_{0,\psi}[a,b] = \mathcal{C}[a,b]$. Then $\left|{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi(a)\right| = \left|{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-}\varphi(b)\right| = 0$, since

$$\left|{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{a^+}\varphi(x)\right| \leq \frac{\Big(\psi(x)-\psi(a)\Big)^{2n-\alpha}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)\Gamma(n-\xi_2+1)}\|\varphi\|_{\mathcal{C}^{2n}_{\gamma,\psi}},$$

  and

$$\left|{}^{2L}\mathfrak{D}^{\alpha,(\nu_1,\nu_2);\psi}_{b^-}\varphi(x)\right| \leq \frac{\Big(\psi(b)-\psi(x)\Big)^{2n-\alpha}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1)\Gamma(n-\xi_2+1)}\|\varphi\|_{\mathcal{C}^{2n}_{\gamma,\psi}}.$$

5. The $\psi - m$th level fractional derivative

In this section, we expand the construction from the previous section to encompass m compositions of the $nth-$order derivatives and corresponding RLFIs with respect to another function ψ. This expansion yields $\psi - m$th level FDs of order α, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and type $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$, where "$m$" represents the number of compositions of the $nth-$order derivative. Hence, if we have two pairs of compositions ($m = 2$) of the $nth-$order derivatives, it is called the 2LFD. In this sense, the RLFD, the CFD, and the HFD are the 1st level FDs with

$m = 1$. For convenience, we will use the following notation for the summation of parameters (ν), such that $\nu_1, \nu_2, \dots, \nu_m \in \mathbb{R}$, with $0 \leq \nu_k$ for $k = 1, 2, \dots, m$:

$$s_k := \sum_{i=1}^{k} \nu_i, \;\; k = 1, 2, \dots, m.$$

To clarify, if we consider the fourth level ($m = 4$), we will have four relations of s_k: $s_1 = \nu_1$; $s_2 = \nu_1 + \nu_2$; $s_3 = \nu_1 + \nu_2 + \nu_3$; $s_4 = \nu_1 + \nu_2 + \nu_3 + \nu_4$.

The appropriate spaces of functions for the basic $\psi - m$th level FDs of the $nth-$order derivative are specified as follows:

$$ {}^{mL}\mathcal{X}_{a^+}^{0;\psi} = \left\{ \varphi \in \mathcal{X}_{a^+}^{0;\psi} : \mathfrak{I}_{a^+}^{s_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \varphi = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{s_k;\psi} \varphi, \quad k = 1, 2, \cdots, m \right\} \tag{5.1}$$

and

$$ {}^{mL}\mathcal{X}_{b^-}^{0;\psi} = \left\{ \varphi \in \mathcal{X}_{b^-}^{0;\psi} : \mathfrak{I}_{b^-}^{s_k;\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \varphi = \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{b^-}^{s_k;\psi} \varphi, \quad k = 1, 2, \cdots, m \right\}. \tag{5.2}$$

Furthermore, let ${}^{mL}\mathcal{B}_{a^+}^{0;\psi}$ be another basic space for the $\psi-$2LFD defined as follows:

$$ \begin{aligned} {}^{mL}\mathcal{B}_{a^+}^{0;\psi} = \Bigg\{ \varphi \in \mathcal{X}_{a^+}^{0;\psi} : & \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{mn-\alpha-s_k;\psi} \varphi \\ & = \mathfrak{I}_{a^+}^{mn-\alpha-s_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \varphi, \quad k = 1, 2, \dots, m \Bigg\}. \end{aligned}$$

Similarly, we can define the right-sided ${}^{mL}\mathcal{B}_{b^-}^{0;\psi}$.

Theorem 5.1. *The spaces* ${}^{mL}\mathcal{X}_{a^+}^{0;\psi}$ *and* ${}^{mL}\mathcal{B}_{a^+}^{0;\psi}$ *contain the functions* $\varphi \in \mathcal{AC}_{\psi+}^n[a, b]$ *that satisfy the condition* $\left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^k \varphi(a) = 0$, $k = 0, 1, \dots, n-1$ *for all* $n \in \mathbb{N}$. *Similarly, this holds for the other side.*

Proof. Similar to the case of the space ${}^{C}\mathcal{X}_{a^+}^{0;\psi}$ for the $\psi-$CFD, as shown in Theorem 3.3, we complete the proof by replacing $n - \alpha$ with s_k or $mn - \alpha - s_k$. □

Inspired by definitions 3.5 and 4.2, we can express a comprehensive generalization of the m parameters as follows:

Definition 5.1. Let $n - 1 < \alpha \leq n$ with $n \in \mathbb{N}$ and $\nu_k \in \mathbb{R}$, where $0 \leq \nu_k$ and $\alpha + s_k \leq kn$ for $k = 1, 2, \dots, m$. Let $\psi \in \mathcal{C}^n([a, b], \mathbb{R})$ be an increasing function such that $\psi'(x) \neq 0$ for all $x \in [a, b]$. The left-sided $\psi - m$th level FD of order α and type (ν) is defined by

$$\left({}^{mL}\mathfrak{D}_{a^+}^{\alpha,(\nu);\psi} \varphi \right)(x) = \left(\left(\prod_{k=1}^{m} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{mn-\alpha-s_m;\psi} \varphi \right)(x) \tag{5.3}$$

and the right-sided $\psi - mth$ level FD is defined by

$$\left({}^{mL}\mathfrak{D}_{b^-}^{\alpha,(\nu);\psi} \varphi \right)(x) = \left(\left(\prod_{k=1}^{m} \mathfrak{I}_{b^-}^{\nu_k;\psi} \left(\frac{-1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{b^-}^{mn-\alpha-s_m;\psi} \varphi \right)(x). \tag{5.4}$$

If the domain of the $\psi - mth$ level FD is ${}^{mL}\mathcal{X}^{0;\psi}_{a^+}$ or ${}^{mL}\mathcal{B}^{0;\psi}_{a^+}$, we will refer to it as the basic $\psi - mth$ level FD. In these spaces, we have the following theorem.

Theorem 5.2. *The basic $\psi - mth$ level FD of order α and type type $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ of the relation (5.3) is identical to the basic $\psi-$RLFD restricted to the domain ${}^{mL}\mathcal{X}^{0;\psi}_{a^+}$, and coincides with the basic $\psi-$CFD restricted to the domain ${}^{mL}\mathcal{B}^{0;\psi}_{a^+}$. Similarly, we can apply this concept to the right side of (5.4) restricting it to the domains ${}^{mL}\mathcal{X}^{0;\psi}_{b^-}$, and ${}^{mL}\mathcal{B}^{0;\psi}_{b^-}$, respectively.*

Proof. On the space $\varphi \in {}^{mL}\mathcal{X}^{0;\psi}_{a^+}$, we have

$$\begin{aligned}
&\left({}^{mL}\mathfrak{D}^{\alpha,(\nu);\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_1;\psi}_{a^+}\right)\left(\left(\prod_{k=2}^{m} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \mathfrak{I}^{\nu_1+\nu_2;\psi}_{a^+}\right)\left(\left(\prod_{k=3}^{m} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\varphi\right)(x)\\
&= \dots\\
&= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{mn}\mathfrak{I}^{s_m;\psi}_{a^+}\,\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{mn}\mathfrak{I}^{mn-\alpha;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\mathfrak{D}^{\alpha;\psi}_{a^+}\varphi\right)(x), \quad \varphi \in {}^{mL}\mathcal{X}^{0;\psi}_{a^+}.
\end{aligned}$$

Furthermore, on the space ${}^{mL}\mathcal{B}^{0;\psi}_{a^+}$, we have

$$\begin{aligned}
&\left({}^{mL}\mathfrak{D}^{\alpha,(\nu);\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m-1} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{\nu_m;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m-1} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{\nu_m;\psi}_{a^+}\mathfrak{I}^{mn-\alpha-s_m;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m-1} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{(m-1)n-\alpha-s_{m-1};\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m-2} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{\nu_{m-1};\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\mathfrak{I}^{(m-1)n-\alpha-s_{m-1};\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x)\\
&= \left(\left(\prod_{k=1}^{m-2} \mathfrak{I}^{\nu_k;\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\right)\mathfrak{I}^{\nu_{m-1};\psi}_{a^+}\mathfrak{I}^{(m-1)n-\alpha-s_{m-1};\psi}_{a^+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n\varphi\right)(x)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(\prod_{k=1}^{m-2} \mathfrak{I}_{a^+}^{\nu_k;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\right)\mathfrak{I}_{a^+}^{(m-2)n-\alpha-s_{m-2};\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{2n}\varphi\right)(x)\\
&= \dots\\
&= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\mathfrak{I}_{a^+}^{n-\alpha-s_1;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{(m-1)n}\varphi\right)(x)\\
&= \left(\mathfrak{I}_{a^+}^{mn-\alpha;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{mn}\varphi\right)(x)\\
&= \left({}^{C}\mathfrak{D}_{a^+}^{\alpha;\psi}\varphi\right)(x), \quad \varphi \in {}^{mL}\mathcal{B}_{a^+}^{0;\psi}.
\end{aligned}
$$

□

Also, here the results on the spaces ${}^{mL}\mathcal{X}_{a^+}^{0;\psi}$ and ${}^{mL}\mathcal{B}_{a^+}^{0;\psi}$ are identical according to Theorem 5.1 and (3.8). Anyway, the domain of the basic $\psi-m$th level FDs on the interval $[a,b]$ can be expanded to a broader space of functions, analogous to the extension seen with the $\psi-$2LFD as follows:

$$
\begin{aligned}
{}^{mL}\mathcal{X}_{a^+}^{1;\psi} = \Bigg\{\varphi : \mathfrak{I}_{a^+}^{mn-\alpha-s_m;\psi}\varphi,&\\
\left(\prod_{k=i}^{m} \mathfrak{I}_{a^+}^{\nu_k;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\right)\mathfrak{I}_{a^+}^{mn-\alpha-s_m;\psi}\varphi \in \mathcal{AC}_{\psi+}^{n}[a,b], \quad i = 2,\dots,m\Bigg\}& \qquad (5.5)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{mL}\mathcal{X}_{b^-}^{1;\psi} = \Bigg\{\varphi : \mathfrak{I}_{b^-}^{mn-\alpha-s_m;\psi}\varphi,&\\
\left(\prod_{k=i}^{m} \mathfrak{I}_{b^-}^{\nu_k;\psi}\left(\frac{-1}{\psi'(x)}\frac{d}{dx}\right)^{n}\right)\mathfrak{I}_{b^-}^{mn-\alpha-s_m;\psi}\varphi \in \mathcal{AC}_{\psi-}^{n}[a,b], \quad i = 2,\dots,m\Bigg\}.& \qquad (5.6)
\end{aligned}
$$

These spaces provide sufficient conditions for the existence of the derivatives in (5.3) and (5.4).

Remark 5.1. For $m = 2$, the relations (5.1), (5.2), (5.5), (5.6), and Definition 5.1, reduce to the $\psi-$2LFD given by (4.3), (4.4), (4.10), (4.11), and Definition 4.2, respectively. Additionally, for $\psi(x) = x$ and $n = 1$, we obtain all the related spaces presented in [16].

Definition 5.2. The extension of the basic $\psi-mth$ level FD to the domains ${}^{mL}\mathcal{X}_{a^+}^{1;\psi}$ or ${}^{mL}\mathcal{X}_{b^-}^{1;\psi}$ is referred to as the $\psi-mth$ level FD as defined by Definition 5.1. The space of the left-sided $\psi-mth$ level FD is given by ${}^{mL}\mathfrak{D}_{a^+}^{\alpha,(\nu);\psi} : {}^{mL}\mathcal{X}_{a^+}^{1;\psi} \to L_1(a,b)$, and the space of the right-sided $\psi-mth$ level FD is given by ${}^{mL}\mathfrak{D}_{b^-}^{\alpha,(\nu);\psi} : {}^{mL}\mathcal{X}_{b^-}^{1;\psi} \to L_1(a,b)$.

Theorem 5.3. *On the space ${}^{FT}\mathcal{X}_{a^+}^{\psi}$ defined by (3.34). The $\psi-mth$ level FD given by (5.3) is a left-inverse operator to the $\psi-$RLFI for $x \in [a,b]$, i.e., the relation*

$$\left({}^{mL}\mathfrak{D}_{a^+}^{\alpha,(\nu);\psi}\,\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x) = \varphi(x), \ \textit{holds for any } \varphi \textit{ in the space } {}^{FT}\mathcal{X}_{a^+}^{\psi}.$$

Proof. The proof of this theorem follows similarly to what has been done in Theorem 4.3.

$$\left({}^{mL}\mathfrak{D}_{a^+}^{\alpha,(\nu);\psi}\,\mathfrak{I}_{a^+}^{\alpha;\psi}\varphi\right)(x)$$

$$
\begin{aligned}
&= \left(\left(\prod_{k=1}^{m} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{mn-\alpha-s_m;\psi} \, \mathfrak{I}_{a^+}^{\alpha;\psi} \varphi \right)(x) \\
&= \left(\left(\prod_{k=1}^{m} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{mn-\alpha-s_m;\psi} \, \mathfrak{I}_{a^+}^{n;\psi} \phi \right)(x) \\
&= \left(\left(\prod_{k=1}^{m-1} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{\nu_m;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{n;\psi} \, \mathfrak{I}_{a^+}^{mn-n+n-\alpha-s_m;\psi} \phi \right)(x) \\
&= \left(\left(\prod_{k=1}^{m-1} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{mn-n+n-\alpha-s_{m-1};\psi} \phi \right)(x) \\
&= \left(\left(\prod_{k=1}^{m-2} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{\nu_{m-1};\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{n;\psi} \, \mathfrak{I}_{a^+}^{mn-2n+n-\alpha-s_{m-1};\psi} \phi \right)(x) \\
&= \left(\left(\prod_{k=1}^{m-2} \mathfrak{I}_{a^+}^{\nu_k;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \right) \mathfrak{I}_{a^+}^{mn-2n+n-\alpha-s_{m-2};\psi} \phi \right)(x) \\
&= \dots \\
&= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathfrak{I}_{a^+}^{n;\psi} \mathfrak{I}_{a^+}^{mn-mn+n-\alpha-s_1;\psi} \phi \right)(x) \\
&= \left(\mathfrak{I}_{a^+}^{\nu_1;\psi} \mathfrak{I}_{a^+}^{n-\alpha-\nu_1;\psi} \phi \right)(x) \\
&= \left(\mathfrak{I}_{a^+}^{n-\alpha;\psi} \phi \right)(x) \\
&= \varphi(x), \ \varphi \in {}^{FT}\mathcal{X}_{a^+}^{\psi}.
\end{aligned}
$$

□

Remark 5.2. Based on the preceding discussions, we note the following:

- R1. By modifying the restriction $\alpha + s_k \leq kn$ for $k = 1, 2, \dots, m$ in Definition 5.1 to $kn - 1 < \alpha + s_k \leq kn$ (i.e, from $s_k \geq 0$ to $s_k \geq (k-1)n$) and defining $\xi_k = \alpha + s_k - (k-1)n$, we observe that $n - 1 < \xi_k \leq n$ and $\alpha \leq \xi_k$. This adjustment is relevant and very useful for certain weighted spaces, such as $\mathcal{C}_{\xi_k;\psi}$ (see [28]), ensuring that ξ_k remains positive and facilitates straightforward generalization to the mth-level FD or any desired level.
- R2. If $n \leq \nu_k$ for $k = 2, 3, \dots, m$, this implies that $\xi_{k-1} \leq \xi_k$ holds for any level of FD we consider.
- R3. By setting $m = 2$ in (R1) of this remark, we obtain the results presented in Remark 4.5.

Remark 5.3. Building on earlier discussions and Definition 5.1, it is evident that most results obtained in Sections 3 and 4 can be straightforwardly generalized to the mth-level FD or any level we desire.

6. Conclusion and future work

In this study, we generalized the Abel IE to an Abel IE with respect to another function ψ, as detailed in Eq. (1.1), and established necessary and sufficient conditions for its solvability. We introduced novel FDs, beginning with the $\psi-$HFD, followed by the $\psi-$2LFD, culminating in the comprehensive generalization termed the $\psi - m$th level FD. Through appropriate selection of parameters ν_k for $k = 1, 2, \ldots, m$ and the function ψ, these operators unify various definitions, encompassing a class of FDs such as the $\psi-$2LFD, $\psi-$HFD, $\psi-$RLFD, $\psi-$CFD, and others. We facilitated seamless transitions among different FDs using a single formula. We derived several significant properties of these new operators, focusing particularly on the $\psi-$2LFD due to its clarity, ease of application, and potential for generalization to any desired level. Additionally, we established connections to well-known results in the literature, as outlined in various remarks in Sections 3, 4, and 5. A promising direction for future research is the application of these novel FDs to various differential equations, along with investigating the existence and uniqueness of Cauchy-type problems using the mth level FDs in the weighted space $\mathcal{C}_{\xi_k,\psi}$. Additionally, we aim to adapt these novel FDs to align more closely with the representations discussed in [2]. There is also potential for generalization through variable-order FDs $\alpha(x)$. These and other directions will be explored in future work.

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