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# EXTINCTION AND STATIONARY DISTRIBUTION OF A HEROIN EPIDEMIC MODEL WITH LÉVY JUMPS AND MARKOV SWITCHING*

**Kouling Li[1], Jinhai Guo[1] and Yongchang Wei[1,†]**

**Abstract** This paper endeavors to construct and embark on a rigorous investigation of a heroin epidemic model, subject to influences from both Lévy jumps and regime-switching. The primary objective is to conduct an in-depth analysis of the dynamical behaviors of this complex system. Firstly, we establish sufficient conditions for both the persistence in the mean and the extinction of the heroin epidemic model. Subsequently, under some certain conditions, we demonstrate the existence of a unique stationary distribution for this system. Finally, some numerical examples and figures are presented to illustrate and validate the theoretical results in an intuitive manner.

**Keywords** Heroin epidemic model, Lévy jumps, Markov switching, extinction, stationary distribution.

**MSC(2010)** 60H10, 34D05.

## 1. Introduction

Heroin is a highly addictive illicit drug that is currently classified as a Schedule I substance, signifying its lack of approved medical usage and substantial abuse potential [7]. For those addicted, withdrawal symptoms emerge upon discontinuation of use, with treatment protocols mirroring those for prescription opioids [37]. Repeated heroin use fosters tolerance among users, rendering them susceptible to overdose when their heart rate and breathing slow to hazardous levels. A particularly perilous scenario arises when individuals who have abstained from heroin for a while resume use, often returning to their previous dosage levels despite a decreased tolerance, thereby heightening their risk of overdose [7]. This emphasizes the enduring importance of rehabilitation and relapse prevention strategies for individuals addicted to heroin, which remain vital components in addressing drug addiction and facilitating recovery. To facilitate policymakers in effectively allocating resources for prevention and treatment purposes, several models have been developed specifically to address heroin addiction, with the majority of them being inspired by the White and Comiskey model [42], which utilizes ordinary differential equations (ODEs) and comprises three compartments: Susceptible individuals $S(t)$, drug users not in treatment $U_1(t)$, and drug users undergoing treatment $U_2(t)$. The literature contains several forms to model heroin addiction in a deterministic and stochastic continuous framework (see, e.g., [4, 8, 9, 11, 18, 27, 29, 30, 34, 35, 39, 40]). The classical heroin model with standard incidence

†The corresponding author.
[1]School of Information and Mathematics, Yangtze University, Jingzhou, China
*The second author supported by National Natural Science Foundation of China (No. 11871118).
Email: lkl1546162380@163.com(K. Li), xin3fei@21cn.com(J. Guo),
wyc897769778@yeah.net(Y. Wei)

is described by the following system:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \dfrac{\beta_1 S(t)U_1(t)}{N(t)} - \mu S(t), \\ \dfrac{\mathrm{d}U_1(t)}{\mathrm{d}t} = \dfrac{\beta_1 S(t)U_1(t)}{N(t)} - pU_1(t) + \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu + \delta_1)U_1(t), \\ \dfrac{\mathrm{d}U_2(t)}{\mathrm{d}t} = pU_1(t) - \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu + \delta_2)U_2(t), \end{cases} \tag{1.1}$$

here, at time $t$, the total population, denoted as $N(t) = S(t) + U_1(t) + U_2(t)$. The total population entering the susceptible per unit of time is represented by $\Lambda$. The probability of an individual transitioning to a drug user is $\beta_1$, while $\mu$ signifies the natural death rate across the entire population. The proportion of drug users undergoing treatment is denoted by $p$. $\beta_3$ represents the relapse rate of treated drug users back to untreated status. $\delta_1$ encompasses the drug-related deaths of untreated users and spontaneous recovery rates, whereas $\delta_2$ includes drug-related deaths of treated users and successful cure rates. The long-term behavior of system (1.1) is governed by the epidemic threshold $\mathfrak{R}_0 = \frac{\beta_1}{p+\mu+\delta_1}$ [42]. Specifically, if $\mathfrak{R}_0 > 1$, susceptible individuals and drug users will coexist indefinitely. Conversely, if $\mathfrak{R}_0 < 1$, drug users will eventually cease to exist within the local population. Notably, the rate at which susceptible individuals transition to drug users is the most influential factor. Hence, preventing drug use is considered more crucial than treating it in reducing its prevalence.

Because epidemic models are inevitably affected by stochastic factors [5, 13, 24, 31, 38, 46, 50, 51, 54], a stochastic version of the deterministic system (1.1)

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \dfrac{\beta_1 S(t)U_1(t)}{N(t)} - \mu S(t)\right]\mathrm{d}t + \alpha_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}U_1(t) = \left[\dfrac{\beta_1 S(t)U_1(t)}{N(t)} - pU_1(t) + \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu + \delta_1)U_1(t)\right]\mathrm{d}t \\ \qquad\qquad + \alpha_2 U_1(t)\mathrm{d}B_2(t), \\ \mathrm{d}U_2(t) = \left[pU_1(t) - \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu + \delta_2)U_2(t)\right]\mathrm{d}t + \alpha_3 U_2(t)\mathrm{d}B_3(t), \end{cases} \tag{1.2}$$

was considered by Liu et al. [29], where $B_i(t)$ represent independent standard Brownian motions with $B_i(0) = 0$, and $\alpha_i^2 > 0$, for $i = 1, 2, 3$ indicating the intensities of these Brownian motions. The authors demonstrated the uniqueness and positivity of the solution and established the conditions for the extinction of heroin addiction. In scenarios of persistence, they proved the existence of a stationary distribution. While incorporating white noise perturbation is indeed one way to introduce stochasticity into a deterministic model, it is not the only method. For instance, when the underlying population model may suddenly experience environmental shocks, such as massive diseases like avian influenza and SARS, earthquakes, hurricanes, and other disasters [2, 6, 12, 23, 26, 28, 41, 44, 52], Lévy jumps can be integrated into the models. Zhou et al. [52] established a threshold for a stochastic SIS model with Lévy jumps, which definitively determines disease extinction or persistence. They further demonstrated that Lévy noise can suppress disease outbreaks. Liu et al. [28] demonstrated that the persistence and extinction of two epidemic diseases in a stochastic delayed SIR model are critically influenced by time delay and Lévy noise, with the two diseases capable of coexisting under certain conditions. Li et al. [23] have investigated the asymptotic behavior of a heroin model incorporating Lévy jumps in relation to the equilibria of its deterministic counterpart system. El Fatini et al. [12] have explored the impact of Lévy noise perturbation on an epidemic model considering media coverage effects, analyzing the existence and uniqueness of global positive solutions and examining the dynamic behaviors near the disease-free and endemic equilibria of the corresponding deterministic system. Yang et al. [44] studied a stochastic SIR model with Lévy jumps, exploring solution uniqueness, a disease persistence threshold, a new ergodic distribution verification method. Liu et al. [26] established a stochastic SIAM epidemic model with Lévy jumps and delay. They proved the existence of a stationary distribution and derived sufficient conditions for the disease's persistence and extinction. Given the intricate nature of environmental factors, the inclusion of solely white noise may not yield predictions with sufficient precision. Consequently, to bolster the predictive accuracy of the deterministic system (1.1), Lévy noises are introduced on this foundation as follows:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \dfrac{\beta_1 S(t)U_1(t)}{N(t)} - \mu S(t)\right]\mathrm{d}t + \alpha_1 S(t)\mathrm{d}B_1(t) + \displaystyle\int_{\mathbb{U}} \chi_1(u)S(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \\ \mathrm{d}U_1(t) = \left[\dfrac{\beta_1 S(t)U_1(t)}{N(t)} - pU_1(t) + \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu+\delta_1)U_1(t)\right]\mathrm{d}t \\ \qquad\qquad + \alpha_2 U_1(t)\mathrm{d}B_2(t) + \displaystyle\int_{\mathbb{U}} \chi_2(u)U_1(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \\ \mathrm{d}U_2(t) = \left[pU_1(t) - \dfrac{\beta_3 U_1(t)U_2(t)}{N(t)} - (\mu+\delta_2)U_2(t)\right]\mathrm{d}t + \alpha_3 U_2(t)\mathrm{d}B_3(t) \\ \qquad\qquad + \displaystyle\int_{\mathbb{U}} \chi_3(u)U_2(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \end{cases} \tag{1.3}$$

where $\tilde{N}(\mathrm{d}u,\mathrm{d}t) = N(\mathrm{d}u,\mathrm{d}t) - \nu(\mathrm{d}u)\mathrm{d}t$ is a compensated Poisson random measure corresponding to a Poisson random measure $N(\mathrm{d}u,\mathrm{d}t)$ with characteristic measure $\nu(\mathrm{d}u)\,\mathrm{d}t$ on the product space $\mathbb{U}\times[0,\infty)$. Here, $\nu$ is a Lévy measure such that $\nu(\mathbb{U}) < \infty$.

Another type of environmental noise is the Markov chain, which is a valuable stochastic model frequently utilized in numerous applications [3, 15, 32, 36, 45]. It can be depicted as a transition between two or more environmental regimes, each characterized by distinct factors [10, 16, 17, 19, 21, 22, 25, 33, 43, 47–49, 53]. Using a Markov chain model, Lee et al. [21] forecasted a continuing increase in HIV/AIDS cases among both African American and Caucasian populations, with a pronounced racial disparity in the disease's burden. Zhang et al. [49] analyzed a stochastic SIS model with vaccination under regime switching. By constructing stochastic Lyapunov functions, they established sufficient conditions for the existence of a unique ergodic stationary distribution. Economou and Lopez-Herrero [10] developed computational methods for an environment-dependent SIS model, where infection and recovery rates are driven by a continuous-time Markov chain. Their approach quantifies the evolution of infectives under seasonal and environmental influences. Xu et al. [43] proposed a Markov chain model that predicts COVID-19 will persist long-term. Their model corrects the classical herd immunity formula, showing a much higher threshold is required. Jiang et al. [17] developed a Markov chain-switching heroin model to capture the impact of opium poppy harvesting cycles. They established the model's global positive solution and derived conditions for drug addiction extinction and persistence, demonstrating that increased noise intensities can accelerate extinction. Hridoy and Allen [16] investigated seasonal epidemic models using time-nonhomogeneous Markov chains and branching processes. They found that seasonal environments generally reduce disease emergence probability compared to constant environments, and identified conditions where seasonal variations in recovery rates can alter this pattern. The study offers new methods for optimizing the timing of disease control measures. By integrating the Markov chain into the modeling process, we can bolster the robustness and reliability of infectious disease models. Based on this premise, we incorporate the Markov chain into model (1.3) as follows:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda_{r_t} - \frac{\beta_{1,r_t}S(t)U_1(t)}{N(t)} - \mu_{r_t}S(t)\right]\mathrm{d}t + \alpha_{1,r_t}S(t)\mathrm{d}B_1(t) \\ \qquad\qquad + \int_{\mathbb{U}} \chi_{1,r_t}(u)S(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \\ \mathrm{d}U_1(t) = \left[\frac{\beta_{1,r_t}S(t)U_1(t)}{N(t)} - p_{r_t}U_1(t) + \frac{\beta_{3,r_t}U_1(t)U_2(t)}{N(t)} - (\mu_{r_t}+\delta_{1,r_t})U_1(t)\right]\mathrm{d}t \\ \qquad\qquad + \alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t) + \int_{\mathbb{U}} \chi_{2,r_t}(u)U_1(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \\ \mathrm{d}U_2(t) = \left[p_{r_t}U_1(t) - \frac{\beta_{3,r_t}U_1(t)U_2(t)}{N(t)} - (\mu_{r_t}+\delta_{2,r_t})U_2(t)\right]\mathrm{d}t + \alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t) \\ \qquad\qquad + \int_{\mathbb{U}} \chi_{3,r_t}(u)U_2(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \end{cases} \tag{1.4}$$

where $r_t$ is a right-continuous Markov chain.

In subsystem $j \in \mathbb{K} = \{1, 2, \cdots, k\}$, it obeys

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda_j - \frac{\beta_{1,j}S(t)U_1(t)}{N(t)} - \mu_j S(t)\right]\mathrm{d}t + \alpha_{1,j}S(t)\mathrm{d}B_1(t) \\ \qquad\qquad + \int_{\mathbb{U}} \chi_{1,j}(u)S(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \\ \mathrm{d}U_1(t) = \left[\frac{\beta_{1,j}S(t)U_1(t)}{N(t)} - p_j U_1(t) + \frac{\beta_{3,j}U_1(t)U_2(t)}{N(t)} - (\mu_j+\delta_{1,j})U_1(t)\right]\mathrm{d}t \\ \qquad\qquad + \alpha_{2,j}U_1(t)\mathrm{d}B_2(t) + \int_{\mathbb{U}} \chi_{2,j}(u)U_1(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \\ \mathrm{d}U_2(t) = \left[p_j U_1(t) - \frac{\beta_{3,j}U_1(t)U_2(t)}{N(t)} - (\mu_j+\delta_{2,j})U_2(t)\right]\mathrm{d}t + \alpha_{3,j}U_2(t)\mathrm{d}B_3(t) \\ \qquad\qquad + \int_{\mathbb{U}} \chi_{3,j}(u)U_2(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t). \end{cases} \tag{1.5}$$

Despite the realism demonstrated by epidemic models incorporating Lévy jumps and Markov chains [1, 14, 33], their stochastic analysis remains challenging and somewhat limited. To our knowledge, only sufficient conditions have been derived for the extinction and persistence of diseases in the presence of these three types of noises (e.g., as seen in (1.4)).

The content of this paper is organized as follows: Section 2 is dedicated to conducting a qualitative analysis of the stochastic epidemic model (1.4). Within Subsection 2.1, we utilize Lyapunov functions to establish a sufficient criterion for the persistence in the mean of the disease. Subsection 2.2 verifies the exponential extinction result under certain conditions. In Subsection 2.3, we derive the existence of a unique stationary distribution for system (1.4). Finally, Section 3 presents numerical examples to offer practical illustrations and explanations of our analytical findings.

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, and $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is increasing, right continuous and $\mathcal{F}_0$ contains all

$\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_n(t))$ be a $n$-dimensional independent standard Brownian motion, and $\{r_t\}_{t\geq 0}$ is a right-continuous Markov chain and taking the value $\mathbb{K} = \{1, 2, \cdots, k\}$ with generator $\Psi = (\psi_{jm})_{k\times k}$ given by

$$\mathbb{P}\{r_{t+\Delta t} = m | r_t = j\} = \begin{cases} \psi_{jm}\Delta t + o(\Delta t), & \text{if } m \neq j, \\ 1 + \psi_{jm}\Delta t + o(\Delta t), & \text{if } m = j, \end{cases}$$

as $\Delta t \to 0$, $\lim\limits_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$, $\psi_{jm} \geq 0$ is the transition rate from $j$ to $m$ for $m \neq j$ and $\psi_{jj} = -\sum\limits_{m\neq j} \psi_{jm}$ for all $j = 1, 2, \cdots, k$. Meanwhile, suppose that $r_t$ is irreducible [45], that is, it has a unique stationary distribution $\pi = \{\pi_1, \pi_2, \cdots, \pi_k\}$ that satisfies $\pi\Psi = 0$, $\sum\limits_{j=1}^{k} \pi_j = 1$ and $\pi_j > 0$, for all $j \in \mathbb{K}$. In this paper, suppose that the assumption $(A_0)$: $\psi_{jm} > 0$ for any $m \neq j$. Moreover, assume that $B(t)$, $N(u, t)$ and $r_t$ are mutually independent. Additionally, for convenience, denote $\check{a} = \max\{a_j\}$, $\hat{a} = \min\{a_j\}$, $j \in \mathbb{K}$, and $\mathbb{R}^n_+ = \{(z_1, \cdots, z_n) \in \mathbb{R}^n | z_i > 0, i = 1, \cdots, n\}$.

Let $X(t) = (Z(t), r_t)$ be a two component process, where $Z(t) \in \mathbb{R}^n$ represents the following hybrid system with Lévy jumps:

$$\mathrm{d}Z(t) = \mathscr{A}_{1,r_t}(Z(t))\mathrm{d}t + \mathscr{A}_{2,r_t}(Z(t))\mathrm{d}B(t) + \int_{\mathbb{U}} \mathscr{A}_{3,r_t}(Z(t-), u)\tilde{N}(\mathrm{d}u, \mathrm{d}t), \tag{1.6}$$

where $\mathscr{A}_1 : \mathbb{R}^n \times \mathbb{K} \to \mathbb{R}^n$, $\mathscr{A}_2 : \mathbb{R}^n \times \mathbb{K} \to \mathbb{R}^{n\times d}$ satisfying $\mathscr{A}_{2,j}(z)\mathscr{A}^T_{2,j}(z) = \mathcal{G}(j, z)$ and $\mathscr{A}_3 : \mathbb{R}^n \times \mathbb{U} \times \mathbb{K} \to \mathbb{R}^n$ are measurable functions. For any $\mathcal{H}(j, z) \in C^{1,2}(\mathbb{R}^n \times \mathbb{K}; \mathbb{R}_+)$, the generator $\mathcal{L}$ of the process $X(t) = (Z(t), r_t)$ is given as follows:

$$\begin{aligned}\mathcal{L}\mathcal{H}(j, z) = &\sum_{m=1}^{k} \psi_{jm}\mathcal{H}(m, z) + \mathcal{H}_z(j, z)\mathscr{A}_{1,j}(z) + \frac{1}{2}\sum_{i,l=1}^{n} \mathcal{G}_{il}(j, z)\frac{\partial^2 \mathcal{H}}{\partial z_i \partial z_l} \\ &+ \int_{\mathbb{U}} \left[\mathcal{H}(z + \mathscr{A}_{3,j}(z, u), j) - \mathcal{H}(z, j) - \mathcal{H}_z(z, j)\mathscr{A}_{3,j}(z, u)\right]\nu(\mathrm{d}u),\end{aligned} \tag{1.7}$$

where $\mathcal{H}_z = (\frac{\partial \mathcal{H}}{\partial z_1}, \frac{\partial \mathcal{H}}{\partial z_2}, \cdots, \frac{\partial \mathcal{H}}{\partial z_n})$.

## 2. Qualitative analysis

In order to facilitate the following discussions in this paper, some basic result is introduced as follows:

**Lemma 2.1.** [14,53] *Assume that there exists a bounded domain $\tilde{D} \in \mathbb{R}^n$ with regular boundary $\Gamma$ satisfying the following properties:*

*$(S_1)$ for all $j, m \in \mathbb{K}$, $m \neq j$, $\psi_{jm} > 0$;*

*$(S_2)$ for all $j \in \mathbb{K}$, the diffusion matrix $\mathcal{G}(z, j)$ is symmetric and there exists some constant $\theta_0 \in (0, 1]$, $\theta_0 \in \mathbb{R}^n$ such that*

$$\theta_0|\eta|^2 \leq \eta^T\mathcal{G}(z, j)\eta \leq \theta_0^{-1}|\eta|^2, \text{ for all } \eta \in \mathbb{R}^n;$$

*$(S_3)$ for all $j \in \mathbb{K}$, there exists a nonnegative function $\mathcal{H}(z, j) : \tilde{D}^c \to \mathbb{R}$ such that $\mathcal{H}(\cdot, j)$ is twice continuously differentiable and for some positive $L_0$,*

$$\mathcal{L}\mathcal{H}(z, j) \leq -L_0, \text{ for all } (z, j) \in \tilde{D}^c \times \mathbb{K}.$$

*Then $X(t) = (Z(t), r_t)$ of system (1.6) has a unique stationary measure $\lambda_*(\cdot,\cdot) = (\lambda_*(\cdot, j)) : j \in \mathbb{K})$ which is ergodic. And for all Borel measurable function $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{K} \to \mathbb{R}^n$ satisfying*

$$\sum_{j\in\mathbb{K}} \int_{\mathbb{R}^n} |f(Z,j)|\lambda_*(Z,j)\mathrm{d}Z < \infty,$$

*then*

$$\mathbb{P}\left(\frac{1}{t}\int_0^t f(Z(s), r_s)\mathrm{d}s \to \sum_{j\in\mathbb{K}} \int_{\mathbb{R}^n} f(Z,j)\lambda_*(Z,j)\mathrm{d}Z\right) = 1.$$

To discuss the dynamics of system (1.4), we first establish some fundamental lemmas that will be instrumental in our subsequent discussions.

**Lemma 2.2.** *Let $(S(t), U_1(t), U_2(t), r_t)$ be the solution of system (1.4), then $(S(t), U_1(t), U_2(t), j)$ to system (1.4) is stochastically ultimate bounded.*

**Proof.** Define a $C^2$-function $V_0(S, U_1, U_2) = e^{\theta_2 t}(1+S+U_1+U_2)^{\theta_1} = e^{\theta_2 t}(1+N)^{\theta_1}$,$\theta_1 \in (0,1)$, $\theta_2$ will be determined later. From the generalized Itô's formula, we have

$$\begin{aligned}\mathrm{d}V_0 =&\mathcal{L}V_0\mathrm{d}t + \theta_1 \left(1+N(t)\right)^{\theta_1-1} e^{\theta_2 t}\Big[\alpha_{1,r_t}S(t)\mathrm{d}B_1(t) + \alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t)\\ &+ \alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t)\Big] + \int_{\mathbb{U}} e^{\theta_2 t}\Big[(1+N(t-) + \chi_{1,r_t}(u)S(t-) + \chi_{2,r_t}(u)U_1(t-)\\ &+ \chi_{3,r_t}(u)U_2(t-))^{\theta_1} - (1+N(t-))^{\theta_1}\Big]\tilde{N}(\mathrm{d}u,\mathrm{d}t), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}\mathcal{L}V_0 =& e^{\theta_2 t}\Big[\theta_2(1+N)^{\theta_1} + \theta_1(1+N)^{\theta_1-1}(\Lambda_r - \mu_r S - (\mu_r+\delta_{1,r})U_1 - (\mu_r+\delta_{2,r})U_2)\\ &+ \frac{\theta_1(\theta_1-1)}{2}(\alpha_{1,r}^2S^2 + \alpha_{2,r}^2U_1^2 + \alpha_{3,r}^2U_2^2)(1+N)^{\theta_1-2} + M_t^0\Big],\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}M_t^0 =& \int_{\mathbb{U}} \Big[(1+S+U_1+U_2+\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2)^{\theta_1}\\ &- (1+S+U_1+U_2)^{\theta_1} - \theta_1(1+S+U_1+U_2)^{\theta_1-1}(\chi_{1,r}(u)S\\ &+ \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2)\Big]\nu(\mathrm{d}u).\end{aligned} \tag{2.3}$$

According to Taylor formula and $\theta_1 \in (0,1)$, it is easy to see

$$\begin{aligned}&M_t^0\\ =&\int_{\mathbb{U}} \Big[(1+N)^{\theta_1} + \theta_1(1+N)^{\theta_1-1}(\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2) + \frac{\theta_1(\theta_1-1)}{2}\\ &\times \Big(1+N+\theta_3\left(\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2\right)\Big)^{\theta_1-2}(\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2)^2\\ &- (1+N)^{\theta_1} - \theta_1(1+N)^{\theta_1-1}(\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2)\Big]\nu(\mathrm{d}u)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{U}} \Bigg[ \frac{\theta_1(\theta_1-1)}{2} \Big(1+N+\theta_2\Big(\chi_{1,r}(u)S+\chi_{2,r}(u)U_1+\chi_{3,r}(u)U_2\Big)\Big)^{\theta_1-2}(\chi_{1,r}(u)S \\
&\quad + \chi_{2,r}(u)U_1+\chi_{3,r}(u)U_2)^2\Bigg]\nu(\mathrm{d}u) \\
&\leq 0,
\end{aligned} \tag{2.4}
$$

where $\theta_3$ represents an arbitrary number that falls within the range $(0,1)$.

Combining (2.2), (2.4), and the basic inequality $S^2+U_1^2+U_2^2 \geq \frac{(S+U_1+U_2)^2}{3}$, we get

$$
\begin{aligned}
&\mathcal{L}V_0 \\
\leq& \Bigg[\theta_2(1+N)^{\theta_1}+\theta_1(1+N)^{\theta_1-1}(\check{\Lambda}-\hat{\mu}N)+\frac{\theta_1(\theta_1-1)}{2}(\hat{\alpha}_1^2S^2+\hat{\alpha}_2^2U_1^2+\hat{\alpha}_3^2U_2^2)(1+N)^{\theta_1-2}\Bigg]e^{\theta_2 t} \\
\leq& \theta_1(1+N)^{\theta_1-2}\Bigg[\frac{\theta_2}{\theta_1}(1+N)^2+(1+N)(\check{\Lambda}-\hat{\mu}N)+\frac{\theta_1-1}{6}(\hat{\alpha}_1^2\vee\hat{\alpha}_2^2\vee\hat{\alpha}_3^2)N^2\Bigg]e^{\theta_2 t} \\
=& \theta_1(1+N)^{\theta_1-2}\Bigg\{-\Bigg[\hat{\mu}-\frac{\theta_1-1}{6}(\hat{\alpha}_1^2\vee\hat{\alpha}_2^2\vee\hat{\alpha}_3^2)-\frac{\theta_2}{\theta_1}\Bigg]N^2+\frac{\theta_2}{\theta_1}+\check{\Lambda}+(\check{\Lambda}-\hat{\mu}-\frac{2\theta_2}{\theta_1})N\Bigg\}e^{\theta_2 t} \\
\leq& e^{\theta_2 t}G_1(\theta_1),
\end{aligned}
$$

where $\theta_2$ is a positive constant such that $\frac{\theta_2}{\theta_1} < \hat{\mu} - \frac{\theta_1-1}{6}(\hat{\alpha}_1^2\vee\hat{\alpha}_2^2\vee\hat{\alpha}_3^2)$ and $G_1(\theta_1)$ is a positive constant of $\theta_1$.

By integrating (2.1) from 0 to $t$ and then taking the expectation, we obtain

$$
\begin{aligned}
&\mathbb{E}V_0(S(t),U_1(t),U_2(t),r_t) \\
=& V_0(S(0),U_1(0),U_2(0),r_0)+\mathbb{E}\int_0^t \mathcal{L}V_0(S(s),U_1(s),U_2(s),r_s)\mathrm{d}s \\
\leq& (1+N(0))^{\theta_1}+\frac{1}{\theta_2}G_1(\theta_1)(e^{\theta_2 t}-1) \\
\leq& (1+N(0))^{\theta_1}+\frac{1}{\theta_2}G_1(\theta_1)e^{\theta_2 t},
\end{aligned}
$$

which implies

$$
\mathbb{E}(1+N)^{\theta_1} \leq e^{-\theta_2 t}(1+N(0))^{\theta_1}+\frac{1}{\theta_2}G_1(\theta_1) \leq (1+N(0))^{\theta_1}+\frac{1}{\theta_2}G_1(\theta_1)G_2.
$$

Moreover, let $\xi$ be a sufficiently large number such that $0 < \frac{G_2}{\xi} < 1$. By Chebyshev's inequality, it can be concluded that

$$
\mathbb{P}\Big\{(1+N)^{\theta_1} > \xi\Big\} \leq \frac{(1+N)^{\theta_1}}{\xi} \leq \frac{G_2}{\xi}\eta < 1.
$$

Therefore, $\limsup_{t\to\infty} \mathbb{P}\{(1+N)^{\theta_1} > \xi\} \leq \eta$. That is, the solutions of system (1.4) is stochastically ultimately bounded. □

**Remark 2.1.** From Lemma 2.2 and Lemma 2.1 in [50], it derive that for any solution $(S(t), U_1(t), U_2(t), r_t)$ of system (1.4),

$$
\lim_{t\to\infty} \frac{S(t)+U_1(t)+U_2(t)}{t} = 0, \ a.s.
$$

Throughout the rest of this paper, we always assume the following assumption holds:

$$(A_1): \quad |\chi_{i,j}(u)| \leq \omega_j < 1, \ \text{ for all } u \in \mathbb{U}, i = 1,2,3, \ j \in \mathbb{K}.$$

**Lemma 2.3.** *Let $(S(t), U_1(t), U_2(t), r_t)$ be the solution of system (1.4). If the assumption $(A_1)$ is satisfied, and further assume that $(A_2): \quad \hat{\mu} > \frac{1}{2}(\breve{\alpha_1}^2 \vee \breve{\alpha_2}^2 \vee \breve{\alpha_3}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})$, then*

$$\lim_{t\to\infty} \frac{\int_0^t \alpha_{1,r_s} S(s)\mathrm{d}B_1(s)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t \alpha_{2,r_s} U_1(s)\mathrm{d}B_2(s)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t \alpha_{3,r_s} U_2(s)\mathrm{d}B_3(s)}{t} = 0, \ a.s.$$

*and*

$$\lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{U}} \chi_{1,r_s}(u) S(s-)\tilde{N}(\mathrm{d}u, \mathrm{d}s)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{U}} \chi_{2,r_s}(u) U_1(s-)\tilde{N}(\mathrm{d}u, \mathrm{d}s)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{U}} \chi_{3,r_s}(u) U_2(s-)\tilde{N}(\mathrm{d}u, \mathrm{d}s)}{t} = 0, \ a.s.$$

**Proof.** Define a $C^2$-function $V_1(S, U_1, U_2) = (1 + S + U_1 + U_2)^2 = (1 + N)^2$, similar to the proof of Lemma 2.2, we derive

$$\begin{aligned}\mathcal{L}V_1 \leq & -\left[\hat{\mu} - \frac{1}{2}(\breve{\alpha_1}^2 \vee \breve{\alpha_2}^2 \vee \breve{\alpha_3}^2) - \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right]N^2 + \left((\check{\Lambda} - \hat{\mu}) + \breve{\omega}^2\nu(\mathbb{U})\right)N \\ & + \check{\Lambda} + \breve{\omega}^2\nu(\mathbb{U}).\end{aligned}$$

From Lemma 2.2 in [50] and the assumption $(A_2)$, we get

$$\lim_{t\to\infty} \frac{\int_0^t \alpha_{1,r_s} S(s)\mathrm{d}B_1(s)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t \alpha_{2,r_s} U_1(s)\mathrm{d}B_2(s)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t \alpha_{3,r_s} U_2(s)\mathrm{d}B_3(s)}{t} = 0, \ a.s.,$$

and

$$\lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{U}} \chi_{1,r_s}(u) S(s-)\tilde{N}(\mathrm{d}u, \mathrm{d}s)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{U}} \chi_{2,r_s}(u) U_1(s-)\tilde{N}(\mathrm{d}u, \mathrm{d}s)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{U}} \chi_{3,r_s}(u) U_2(s-)\tilde{N}(\mathrm{d}u, \mathrm{d}s)}{t} = 0, \ a.s.$$

□

Next, we will investigate the globally unique positive solution of system (1.4).

**Theorem 2.1.** *If the assumption $(A_1)$ is satisfied, then there is a unique solution $(S(t), U_1(t), U_2(t), r_t)$ to system (1.4) on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+ \times \mathbb{K}$ with probability one, namely, $(S(t), U_1(t), U_2(t), r_t) \in \mathbb{R}^3_+ \times \mathbb{K}$ for all $t > 0$, a.s.*

**Proof.** The proof is simple and it is omitted here. □

**Remark 2.2.** From Theorem 2.1, we secure the existence of unique, globally positive solutions for system (1.4), which aligns perfectly with their biological significance.

## 2.1. Persistence in the mean of drug users

In practical scenarios, the persistence of a disease holds greater relevance and significance. Moving forward, we will derive the conditions for the persistence of heroin drug abusers. To facilitate this analysis, we first present the following assumption:

$$(A_3): \quad K_{2,j} = 2\sqrt{\frac{\check{\delta}_j \Lambda_j}{2}} + 3\sqrt[3]{\Lambda_j^2 \beta_{1,j}} - \mu_j - \check{\delta}_j - 2\Lambda_j - \check{\alpha}_j^2 - \left[\frac{1}{1-\omega_j} - 1 + \omega_j\right]\nu(\mathbb{U}) > 0, \quad \text{for all } j \in \mathbb{K}, \tag{2.5}$$

here, $\check{\delta}_j = \max\{\delta_{1,j},\, \delta_{2,j}\}$, $\check{\alpha}_j^2 = \max\{\alpha_{1,j}^2,\, \alpha_{2,j}^2,\, \alpha_{3,j}^2\}$ and define

$$\Re_s^1 = \frac{\sum_{j=1}^{k} \pi_j \left[\frac{3\sqrt{11}}{5}\left(K_{2,j}\mu_j\right)^{\frac{1}{2}} + \mu_j\right]}{\sum_{j=1}^{k} \pi_j \left(\Lambda_j + \beta_{1,j} + K_{1,j} + \frac{\alpha_{1,j}^2}{2}\right)}, \tag{2.6}$$

where $K_{1,j} = \frac{\omega_j^2}{2(1-\omega_j)^2}\nu(\mathbb{U})$.

**Theorem 2.2.** *Let $(S(t), U_1(t), U_2(t), r_t)$ be the solution of system (1.4). If the assumptions $(A_1)$-$(A_3)$ are satisfied, and suppose further that $\Re_s^1 > 1$, then system (1.4) will be persistent in the mean.*

**Proof.** Define a $C^2$-function as follows:

$$V_3 = \frac{1}{N} + 2\ln N + S - \ln S + U_1 + \frac{99}{100}U_2.$$

Applying the generalized Itô's formula to $V_3$, we obtain

$$\begin{aligned}
\mathrm{d}V_3 =& \mathcal{L}V_3 \mathrm{d}t - \frac{1}{N^2(t)}[\alpha_{1,r_t}S(t)\mathrm{d}B_1(t) + \alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t) + \alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t)] \\
&+ \frac{2}{N(t)}[\alpha_{1,r_t}S(t)\mathrm{d}B_1(t) + \alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t) + \alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t)] \\
&+ \left(1 - \frac{1}{S(t)}\right)\alpha_{1,r_t}S(t)\mathrm{d}B_1(t) + \alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t) + \frac{99}{100}\alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t) \\
&+ \int_{\mathbb{U}}\left[\frac{1}{S(t-)(1+\chi_{1,r_t}(u)) + U_1(t-)(1+\chi_{2,r_t}(u)) + U_2(t-)(1+\chi_{3,r_t}(u))}\right. \\
&\left. - \frac{1}{N(t-)}\right]\tilde{N}(\mathrm{d}u,\mathrm{d}t) + 2\int_{\mathbb{U}}[\ln(N(t-) + \chi_{1,r_t}(u)S(t-) + \chi_{2,r_t}(u)U_1(t-) \\
&+ \chi_{3,r_t}(u)U_2(t-)) - \ln N(t-)]\tilde{N}(\mathrm{d}u,\mathrm{d}t) + \int_{\mathbb{U}}[\chi_{1,r_t}(u)S(t-) \\
&- \ln(1+\chi_{1,r_t}(u))]\tilde{N}(\mathrm{d}u,\mathrm{d}t) + \int_{\mathbb{U}}[\chi_{2,r_t}(u)U_1(t-)]\tilde{N}(\mathrm{d}u,\mathrm{d}t) \\
&+ \frac{99}{100}\int_{\mathbb{U}}[\chi_{3,r_t}(u)U_2(t-)]\tilde{N}(\mathrm{d}u,\mathrm{d}t),
\end{aligned} \tag{2.7}$$

where

$$\mathcal{L}V_3 = -\frac{1}{N^2}\left[\Lambda_r - \mu_r S - (\mu_r + \delta_{1,r})U_1 - (\mu_r + \delta_{2,r})U_2\right] + \frac{2}{N}\left[\Lambda_r - \mu_r S - (\mu_r + \delta_{1,r})U_1\right.$$

$$
\begin{aligned}
&- (\mu_r + \delta_{2,r})U_2\Big] + \Lambda_r - \frac{\beta_{1,r}SU_1}{N} - \mu_r S - \frac{\Lambda_r}{S} + \frac{\beta_{1,r}U_1}{N} + \mu_r + \frac{\beta_{1,r}SU_1}{N} - p_r U_1 \\
&+ \frac{\beta_{3,r}U_1U_2}{N} - (\mu_r + \delta_{1,r})U_1 + \frac{99}{100}\left(p_r U_1 - \frac{\beta_{3,r}U_1U_2}{N} - (\mu_r + \delta_{2,r})U_2\right) + \frac{\alpha_{1,r}^2}{2} \\
&+ \frac{\alpha_{1,r}^2 S^2 + \alpha_{2,r}^2 U_1^2 + \alpha_{3,r}^2 U_2^2}{N^3} - \frac{\alpha_{1,r}^2 S^2 + \alpha_{2,r}^2 U_1^2 + \alpha_{3,r}^2 U_2^2}{N^2} \\
&+ \int_{\mathbb{U}} \frac{\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2}{(S + U_1 + U_2)^2}\nu(\mathrm{d}u) \\
&+ \int_{\mathbb{U}} \left[\frac{1}{S(1+\chi_{1,r}(u)) + U_1(1+\chi_{2,r}(u)) + U_2(1+\chi_{3,r}(u))} - \frac{1}{N}\right]\nu(\mathrm{d}u) \\
&+ \int_{\mathbb{U}} [\chi_{1,r}(u) - \ln(1+\chi_{1,r}(u))]\nu(\mathrm{d}u) \\
&+ 2\int_{\mathbb{U}} \Big[\ln(N + \chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2) - \ln N \\
&- \frac{1}{N}(\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2)\Big]\nu(\mathrm{d}u).
\end{aligned}
\tag{2.8}
$$

Taking account to the inequalities $\frac{S^2+U_1^2+U_2^2}{N^2} \leq 1$, we get

$$
\frac{\alpha_{1,r}^2 S^2 + \alpha_{2,r}^2 U_1^2 + \alpha_{3,r}^2 U_2^2}{N^3} \leq \frac{\check{\alpha}_r^2(S^2 + U_1^2 + U_2^2)}{N^3} \leq \frac{\check{\alpha}_r^2}{N}. \tag{2.9}
$$

Meanwhile, let $\hat{\chi}_r(u) = \min\{\chi_{1,r}(u), \chi_{2,r}(u), \chi_{3,r}(u)\}$ and $\check{\chi}_r(u) = \max\{\chi_{1,r}(u), \chi_{2,r}(u), \chi_{3,r}(u)\}$, we have

$$
\begin{aligned}
&\int_{\mathbb{U}} \left[\frac{1}{S(1+\chi_{1,r}(u)) + U_1(1+\chi_{2,r}(u)) + U_2(1+\chi_{3,r}(u))} - \frac{1}{N}\right]\nu(\mathrm{d}u) \\
&+ \int_{\mathbb{U}} \left[\frac{\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2}{N^2}\right]\nu(\mathrm{d}u) \\
\leq &\frac{1}{N}\int_{\mathbb{U}} \left[\frac{1}{1+\hat{\chi}_r(u)} - 1 + \check{\chi}_r(u)\right]\nu(\mathrm{d}u).
\end{aligned}
\tag{2.10}
$$

Moreover, from the assumption $(A_1)$, we derive

$$
\begin{aligned}
&\int_{\mathbb{U}} \Big[\ln(N + \chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2) - \ln N \\
&- \frac{1}{N}(\chi_{1,r}(u)S + \chi_{2,r}(u)U_1 + \chi_{3,r}(u)U_2)\Big]\nu(\mathrm{d}u) \leq 0.
\end{aligned}
\tag{2.11}
$$

Combining (2.8)-(2.11), it can be concluded that

$$
\begin{aligned}
\mathcal{L}V_3 \leq &- \frac{\Lambda_r}{N^2} + \frac{\mu_r}{N} + \frac{\delta_{1,r}U_1}{N^2} + \frac{\delta_{2,r}U_2}{N^2} + \frac{2\Lambda_r}{N} - 2\mu_r + \frac{\check{\alpha}_r^2}{N} + \Lambda_r - \frac{99}{100}\mu_r N - \frac{\Lambda_r}{S} \\
&+ \frac{\beta_{3,r}U_1U_2}{100N} + \mu_r + \beta_{1,r} - \frac{\beta_{1,r}S}{N} + \frac{\int_{\mathbb{U}}[\frac{1}{1+\hat{\chi}_r(u)} - 1 + \check{\chi}_r(u)]\nu(\mathrm{d}u)}{N} \\
&+ \int_{\mathbb{U}} [\chi_{1,r}(u) - \ln(1+\chi_{1,r}(u))]\nu(\mathrm{d}u) + \frac{\alpha_{1,r}^2}{2}.
\end{aligned}
\tag{2.12}
$$

According to Taylor formula and $(A_1)$, we have

$$\begin{aligned}
&\int_{\mathbb{U}}[\chi_{i,r}(u)-\ln(1+\chi_{i,r}(u))]\nu(\mathrm{d}u)\\
\leq&\int_{\mathbb{U}}\left[\chi_{i,r}(u)-\chi_{i,r}(u)+\frac{\chi_{i,r}^2(u)}{2(1+\theta_4\chi_{i,r}(u))^2}\right]\nu(\mathrm{d}u)\\
=&\int_{\mathbb{U}}\left[\frac{\chi_{i,r}^2(u)}{2(1+\theta_4\chi_{i,r}(u))^2}\right]\nu(\mathrm{d}u)\\
\leq&\frac{\omega_r^2}{2(1-\omega_r)^2}\nu(\mathbb{U}),\ i=1,2,3,
\end{aligned}\tag{2.13}$$

where $\theta_4\in(0,1)$ is an arbitrary number.

From (2.13), we obtain

$$\begin{aligned}
\mathcal{L}V_3\leq&\frac{\mu_r+\check{\delta}_r+2\Lambda_r+\check{\alpha}_r^2+\int_{\mathbb{U}}[\frac{1}{1+\hat{\chi}_r(u)}-1+\check{\chi}_r(u)]\nu(\mathrm{d}u)}{N}-2\sqrt{\frac{\check{\delta}_rS}{N^2}\cdot\frac{\Lambda_r}{2S}}\\
&-3\sqrt[3]{\frac{\Lambda_r}{N^2}\cdot\frac{\beta_{1,r}S}{N}\cdot\frac{\Lambda_r}{S}}+\Lambda_r-\frac{99}{100}\mu_rN+\frac{\beta_{3,r}U_1U_2}{100N}+\beta_{1,r}+K_{1,r}+\frac{\alpha_{1,r}^2}{2},
\end{aligned}\tag{2.14}$$

where $K_{1,r}$ is defined as (2.6).

Meanwhile, according to assumption $(A_1)$, it implies $\int_{\mathbb{U}}[\frac{1}{1+\hat{\chi}_r(u)}-1+\check{\chi}_r(u)]\nu(\mathrm{d}u)\leq\left[\frac{1}{1-\omega_r}-1+\omega_r\right]\nu(\mathbb{U})$, then (2.14) can be further expressed as

$$\mathcal{L}V_3\leq-\frac{3\sqrt{11}}{5}\sqrt{K_{2,r}\mu_r}+\Lambda_r+\beta_{1,r}+\frac{1}{100}\beta_{3,r}U_1+K_{1,r}+\frac{\alpha_{1,r}^2}{2}-\mu_r,\tag{2.15}$$

where $K_{2,r}$ is defined as the assumption $(A_3)$.

Hence, together (2.7) with (2.15), we have

$$\begin{aligned}
&V_3(S(t),U_1(t),U_2(t))\\
\leq&V_3(S(0),U_1(0),U_2(0))+\frac{1}{100}\int_0^t\beta_{3,r_s}U_1(s)\mathrm{d}s+M_t^1+M_t^2+M_t^3+M_t^4\\
&+\int_0^t\left[-\frac{3\sqrt{11}}{5}\left(K_{2,r_s}\mu_{r_s}\right)^{\frac{1}{2}}-\mu_{r_s}+\Lambda_{r_s}+\beta_{1,r_s}+K_{1,r_s}+\frac{\alpha_{1,r_s}^2}{2}\right]\mathrm{d}s,
\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}
M_t^1=&\int_0^t\left[-\frac{2\alpha_{1,r_s}S(s)}{N^2(s)}+\frac{\alpha_{1,r_s}S(s)}{N(s)}+\left(1-\frac{1}{S(s)}\right)\alpha_{1,r_s}S(s)\right]\mathrm{d}B_1(s),\\
M_t^2=&\int_0^t\left[-\frac{2\alpha_{1,r_s}U_1(s)}{N^2(s)}+\frac{\alpha_{1,r_s}U_1(s)}{N(s)}+\alpha_{2,r_s}U_1(s)\right]\mathrm{d}B_2(s),\\
M_t^3=&\int_0^t\left[-\frac{2\alpha_{1,r_s}U_2(s)}{N^2(s)}+\frac{a\alpha_{1,r_s}U_2(s)}{N(s)}+\frac{99}{100}\alpha_{3,r_s}U_2(s)\right]\mathrm{d}B_3(s),
\end{aligned}$$

and

$$M_t^4=2\int_0^t\int_{\mathbb{U}}\left[\frac{1}{S(s-)(1+\chi_{1,r_s}(u))+U_1(s-)(1+\chi_{2,r_s}(u))+U_2(s-)(1+\chi_{3,r_s}(u))}\right.$$

$$- \frac{1}{S(s-)+U_1(s-)+U_2(s-)} + \Big[ \ln[N(s-) + \chi_{1,r_s}(u)S(s-) + \chi_{2,r_s}(u)U_1(s-) + \chi_{3,r_s}(u)U_2(s-)] - \ln N(s-)\Big] + \big[\chi_{1,r_s}(u)S(s-) - \ln(1+\chi_{1,r_s}(u))\big] + \big[\chi_{2,r_s}(u)U_1(s-)\big] + \frac{99}{100}\big[\chi_{3,r_s}(u)U_2(s-)\big]\Bigg]\tilde{N}(\mathrm{d}u,\mathrm{d}s).$$

Dividing both sides of (2.16) by $t$ simultaneously, it derives

$$\begin{aligned}&\frac{V_3(S(t),U_1(t),U_2(t))}{t}\\ \leq&\frac{V_3(S(0),U_1(0),U_2(0))}{t} + \frac{1}{100t}\int_0^t \beta_{3,r_s}U_1(s)\mathrm{d}s + \frac{M_t^1}{t} + \frac{M_t^2}{t} + \frac{M_t^3}{t} + \frac{M_t^4}{t}\\ &+ \frac{1}{t}\int_0^t\left[-\frac{3\sqrt{11}}{5}(K_{2,r_s}\mu_{r_s})^{\frac12} - \mu_{r_s} + \Lambda_{r_s} + \beta_{1,r_s} + K_{1,r_s} + \frac{\alpha_{1,r_s}^2}{2}\right]\mathrm{d}s.\end{aligned}$$

According to Lemma 2.3, we have $\lim_{t\to\infty}\frac{M_t^1}{t} = \lim_{t\to\infty}\frac{M_t^2}{t} = \lim_{t\to\infty}\frac{M_t^3}{t} = \lim_{t\to\infty}\frac{M_t^4}{t} = 0$, a.s. Consequently, by the Birkhoff Ergodic theory, we obtain

$$\begin{aligned}&\liminf_{t\to\infty}\frac{\int_0^t U_1(s)\mathrm{d}s}{t}\\ \geq&\frac{100}{\breve{\beta}_3}\left[\sum_{j=1}^k \pi_j\left(\frac{3\sqrt{11}}{5}(K_{2,j}\mu_j)^{\frac12} + \mu_j\right) - \sum_{j=1}^k \pi_j\big(\Lambda_j + \beta_{1,j} + K_{1,j} + \frac{\alpha_{1,j}^2}{2}\big)\right]\\ \geq&\frac{100}{\breve{\beta}_3}\sum_{j=1}^k \pi_j L_1(\mathfrak{R}_s^1 - 1) > 0, \quad a.s.,\end{aligned} \tag{2.17}$$

where $L_1 = \Lambda_j + \beta_{1,j} + K_{1,j} + \frac{\alpha_{1,j}^2}{2}$.

Note that $\frac{\beta_{3,r}U_1U_2}{N} < \beta_{3,r}U_1$ and $\frac{\beta_{3,r}U_1U_2}{N} < \beta_{3,r}U_2$, from (2.15), it also can be concluded that

$$\mathcal{L}V_3 \leq -\frac{3\sqrt{11}}{5}\sqrt{K_{2,r}\mu_r} + \Lambda_r + \beta_{1,r} + \frac{1}{100}\beta_{3,r}U_2 + K_{1,r} + \frac{\alpha_{1,r}^2}{2} - \mu_r.$$

Similar to the proof of (2.17), it is easy to see that

$$\liminf_{t\to\infty}\frac{\int_0^t U_2(s)\mathrm{d}s}{t} \geq \frac{100}{\breve{\beta}_3}\sum_{j=1}^k \pi_j L_1(\mathfrak{R}_s^1 - 1) > 0, \quad a.s. \tag{2.18}$$

Moreover, from the generalized Itô's formula to $V_4 = U_1 + U_2$, we have

$$\begin{aligned}\mathrm{d}V_4 \leq&\Big(\beta_{1,r_t}S(t) - (\mu_{r_t}+\delta_{1,r_t})U_1(t) - (\mu_{r_t}+\delta_{2,r_t})U_2(t)\Big)\mathrm{d}t + \alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t)\\ &+ \int_{\mathbb{U}}[\chi_{2,r_t}(u)U_1(t-)]\,\tilde{N}(\mathrm{d}u,\mathrm{d}t) + \alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t)\\ &+ \int_{\mathbb{U}}[\chi_{3,r_t}(u)U_2(t-)]\,\tilde{N}(\mathrm{d}u,\mathrm{d}t).\end{aligned} \tag{2.19}$$

Integrating (2.19) from 0 to $t$, and then dividing both sides by $t$, we obtain

$$\begin{aligned}\frac{\check{\beta}_1}{t}\int_0^t S(s)\mathrm{d}s \geq & \frac{U_1(t)+U_2(t)}{t} - \frac{U_1(0)+U_2(0)}{t} + \frac{1}{t}\int_0^t (\mu_{r_s}+\delta_{1,r_s})U_1(s)\mathrm{d}s \\ & + \frac{1}{t}\int_0^t (\mu_{r_s}+\delta_{2,r_s})U_2(s)\mathrm{d}s - \frac{M_t^5}{t} - \frac{M_t^6}{t} - \frac{M_t^7}{t},\end{aligned}$$

where $M_t^5 = \int_0^t \alpha_{2,r_s}U_1(s)\mathrm{d}B_2(s)$, $M_t^6 = \int_0^t \alpha_{3,r_s}U_3(s)\mathrm{d}B_3(s)$ and $M_t^7 = \int_0^t\int_{\mathbb{U}}[\chi_{2,r_s}(u)U_1(s-)] + [\chi_{3,r_s}(u)U_2(s-)]\tilde{N}(\mathrm{d}u,\mathrm{d}s)$.

From Lemma 2.3, we have $\lim_{t\to\infty}\frac{M_t^5}{t} = \lim_{t\to\infty}\frac{M_t^6}{t} = \lim_{t\to\infty}\frac{M_t^7}{t} = 0$, a.s. Consequently, by the Birkhoff Ergodic theory, (2.17) and (2.18), we get

$$\begin{aligned}\liminf_{t\to\infty}\frac{\int_0^t S(s)\mathrm{d}s}{t} \geq & \frac{\hat{\mu}+\hat{\delta}_1}{\check{\beta}_1}\liminf_{t\to\infty}\frac{\int_0^t U_1(s)\mathrm{d}s}{t} + \frac{\hat{\mu}+\hat{\delta}_2}{\check{\beta}_1}\liminf_{t\to\infty}\frac{\int_0^t U_2(s)\mathrm{d}s}{t} \\ > & \left(\frac{2\hat{\mu}+\hat{\delta}_1+\hat{\delta}_2}{\check{\beta}_1}\right)\frac{100}{\check{\beta}_3}\sum_{j=1}^k \pi_j L_1(\mathfrak{R}_s^1 - 1) \\ > & 0, \ a.s.\end{aligned}$$

□

**Remark 2.3.** From Theorem 2.2, it follows under the assumptions $(A_1)$-$(A_3)$ and $\mathfrak{R}_s^1 > 1$, system (1.4) will be persistent in the mean, that is, $\liminf_{t\to\infty}\frac{\int_0^t S(s)\mathrm{d}s}{t} > 0$, $\liminf_{t\to\infty}\frac{\int_0^t U_1(s)\mathrm{d}s}{t} > 0$, and $\liminf_{t\to\infty}\frac{\int_0^t U_2(s)\mathrm{d}s}{t} > 0$, a.s.

## 2.2. Extinction of drug users

The extinction of diseases serve as crucial benchmarks with significant implications in real world. This section delves into this aspect by analyzing system (1.4). To simplify our calculations, let us introduce the following notation:

$$\mathfrak{R}_e^0 = \frac{\sum_{j=1}^k \pi_j\beta_{1,j}}{\sum_{j=1}^k \pi_j\left[\mu_j + \hat{\delta}_j + \frac{\hat{\alpha}_j^2}{4}\right]}, \tag{2.20}$$

where $\hat{\delta}_j = \min\{\delta_{1,j},\delta_{2,j}\}$ and $\hat{\alpha}_j^2 = \min\{\alpha_{2,j}^2,\alpha_{3,j}^2\}$.

**Theorem 2.3.** *Let $(S(t),U_1(t),U_2(t),r_t)$ be the solution of system (1.4). If the assumptions $(A_1)$ and $(A_2)$ are satisfied, and further assume that $\mathfrak{R}_e^0 < 1$, then the heroin drug abusers will become extinct with an exponential probability of 1, that is, $\limsup_{t\to\infty}\frac{\ln U_1(t)}{t} = 0, \limsup_{t\to\infty}\frac{\ln U_2(t)}{t} = 0$, a.s. Moreover, the distribution of $S(t)$ converges weakly to the measure $\lambda(\cdot,j)$ with density*

$$\varrho_1(x,j) = A_0 x^{-2-\frac{2a}{\tilde{\alpha}_{1,j}^2}}\tilde{\alpha}_{1,j}^{-2+\frac{2a}{\tilde{\alpha}_{1,j}^2}} e^{\frac{2\check{\Lambda}+ax}{\tilde{\alpha}_{1,j}^2 x}},$$

*where $\check{\chi}_i(u) = \max\{\chi_{i,j}(u),\ j\in\mathbb{K}\}$, $\hat{\chi}_i(u) = \min\{\chi_{i,j}(u),\ j\in\mathbb{K}\}$, $i = 1,2,3$, $\tilde{\alpha}_{1,j} = \alpha_{1,j} + \int_{\mathbb{U}}[\check{\chi}_1(u) - \ln(1+\check{\chi}_1(u))]\nu(\mathrm{d}u)$, $a = \hat{\mu} + \int_{\mathbb{U}}(\hat{\chi}_1(u) - \check{\chi}_1(u))\nu(\mathrm{d}u)$, and $A_0$ is determined by the normalization condition $\int_0^\infty \varrho_1(x,j)\mathrm{d}x = 1$.*

**Proof.** Consider the first equation of system (1.4) and $\tilde{N}(\mathrm{d}u,\mathrm{d}t) = N(\mathrm{d}u,\mathrm{d}t) - \mathrm{d}t\nu(\mathrm{d}u)$, we obtain

$$\begin{aligned}\mathrm{d}S(t) = &\left[\Lambda_{r_t} - \frac{\beta_{1,r_t}S(t)U_1(t)}{N(t)} - \mu_{r_t}S(t) - S(t)\int_{\mathbb{U}}\chi_{1,r_t}(u)\nu(\mathrm{d}u)\right]\mathrm{d}t\\ &+ \alpha_{1,r_t}S(t)\mathrm{d}B_1(t) + \int_{\mathbb{U}}\chi_{1,r_t}(u)S(t-)N(\mathrm{d}u,\mathrm{d}t).\end{aligned}$$

By comparison theorem in [2], we get

$$\begin{aligned}\mathrm{d}S(t) \le &\left[\check{\Lambda} - \hat{\mu}S(t) - S(t)\int_{\mathbb{U}}\hat{\chi}_1(u)\nu(\mathrm{d}u)\right]\mathrm{d}t + \alpha_{1,r_t}S(t)\mathrm{d}B_1(t)\\ &+ \int_{\mathbb{U}}\check{\chi}_1(u)S(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t)\\ = &\left[\check{\Lambda} - \left(\hat{\mu} + \int_{\mathbb{U}}(\hat{\chi}_1(u) - \check{\chi}_1(u))\nu(\mathrm{d}u)\right)S(t)\right]\mathrm{d}t + \alpha_{1,r_t}S(t)\mathrm{d}B_1(t)\\ &+ \int_{\mathbb{U}}\check{\chi}_1(u)S(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t). \end{aligned} \tag{2.21}$$

Let $(\overrightarrow{S}(t), j)$ be the global positive solution by the following system with initial value $(\overrightarrow{S}(0), j)$

$$\mathrm{d}\overrightarrow{S}(t) = [\check{\Lambda} - a\overrightarrow{S}(t)]\mathrm{d}t + \alpha_{1,j}\overrightarrow{S}(t)\mathrm{d}B_1(t) + \int_{\mathbb{U}}\check{\chi}_1(u)\overrightarrow{S}(t-)\tilde{N}(\mathrm{d}u,\mathrm{d}t), \tag{2.22}$$

where $a = \hat{\mu} + \int_{\mathbb{U}}(\hat{\chi}_1(u) - \check{\chi}_1(u))\nu(\mathrm{d}u)$.

Moreover, By the similar arguments mentioned in [20], the auxiliary system (2.22) has a unique stationary measure $\lambda(\cdot, j)$ and $\lambda(\cdot, j)$ has probability density $\varrho_1(x,j) = A_0 x^{-2-\frac{2a}{\tilde{\alpha}_{1,j}^2}} \tilde{\alpha}_{1,j}^{-2+\frac{2a}{\tilde{\alpha}_{1,j}^2}} e^{\frac{2\check{\Lambda}+ax}{\tilde{\alpha}_{1,j}^2 x}}$, and $\tilde{\alpha}_{1,j} = \alpha_{1,j} + \int_{\mathbb{U}}[\check{\chi}_1(u) - \ln(1+\check{\chi}_1(u))]\nu(\mathrm{d}u)$, where $\int_0^\infty \varrho_1(x,j)\mathrm{d}x = 1$. From the ergodicity of $\overrightarrow{S}(t)$, we further get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \overrightarrow{S}(s)\mathrm{d}s = \sum_{j=1}^k\int_0^\infty \overrightarrow{S}\lambda(\mathrm{d}\overrightarrow{S}, j) = \sum_{j=1}^k\int_0^\infty \overrightarrow{S}\varrho_1(\overrightarrow{S}, j)\mathrm{d}\overrightarrow{S},\ a.s.,$$

which implies

$$\mathbb{E}\frac{1}{t}\int_0^t \overrightarrow{S}(s)\mathrm{d}s \to \sum_{j=1}^k\int_0^\infty \overrightarrow{S}\varrho_1(\overrightarrow{S}, j)\mathrm{d}\overrightarrow{S}.$$

Moreover, taking expectation of (2.22), we verify

$$\frac{\mathbb{E}\overrightarrow{S}(t)}{t} = \check{\Lambda} - \frac{a}{t}\mathbb{E}\int_0^t \overrightarrow{S}(s)\mathrm{d}s.$$

Hence, let $t \to \infty$, we get

$$\sum_{j=1}^k\int_0^\infty \overrightarrow{S}\varrho_1(\overrightarrow{S}, j)\mathrm{d}\overrightarrow{S} = \mathbb{E}\overrightarrow{S}(t) = \frac{\check{\Lambda}}{a}.$$

Therefore, combining (2.21) and (2.22), which implies $S(t) \leq \overrightarrow{S}(t)$ and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \left|S(s)-\frac{\check{\Lambda}}{a}\right|\mathrm{d}s=\sum_{j=1}^{k}\int_0^\infty \left|S-\frac{\check{\Lambda}}{a}\right|\lambda(\mathrm{d}S,j)=\sum_{j=1}^{k}\int_0^\infty \left|S-\frac{\check{\Lambda}}{a}\right|\varrho_1(S,j)\mathrm{d}S,$$

where $\lambda(x,j)$ is a probability measure and it further derives that the distribution of $S(t)$ weakly converges to $\lambda(S,j)$ with density $\varrho_1(S,j)$.

To estimate $U_1$ and $U_2$, applying the generalized Itô's formula to $\ln V_4$, where $V_4 = U_1 + U_2$, we obtain

$$\begin{aligned}\mathrm{d}\ln V_4 = {} & \mathcal{L}\ln V_4\mathrm{d}t+\frac{1}{U_1(t)+U_2(t)}\left[\alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t)+\alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t)\right]\\ &+\int_{\mathbb{U}}\ln\left(1+\frac{\chi_{2,r_t}(u)U_1(t-)+\chi_{3,r_t}(u)U_2(t-)}{U_1(t-)+U_2(t-)}\right)\tilde{N}(\mathrm{d}u,\mathrm{d}t),\end{aligned}\tag{2.23}$$

where

$$\begin{aligned}\mathcal{L}\ln V_4 = {} & \frac{1}{U_1+U_2}\left[\frac{\beta_{1,r}SU_1}{N}-p_rU_1+\frac{\beta_{3,r}U_1U_2}{N}-\big(\mu_r+\delta_{1,r}\big)U_1+p_rU_1\right.\\ &\left.-\frac{\beta_{3,r}U_1U_2}{N}-\big(\mu_r+\delta_{2,r}\big)U_2\right]-\frac{1}{2(U_1+U_2)^2}(\alpha_{2,r}^2U_1^2+\alpha_{3,r}^2U_2^2)\\ &+\int_{\mathbb{U}}\left[\ln\left(1+\frac{\chi_{2,r}(u)U_1+\chi_{3,r}(u)U_2}{U_1+U_2}\right)-\frac{1}{U_1+U_2}\big(\chi_{2,r}(u)U_1\right.\\ &\left.+\chi_{3,r}(u)U_2\big)\right]\nu(\mathrm{d}u).\end{aligned}\tag{2.24}$$

Note that for any $x > 0$, $\ln x - x \leq -1$, we have

$$\int_{\mathbb{U}}-\frac{\chi_{2,r}(u)U_1+\chi_{3,r}(u)U_2}{U_1+U_2}+\ln\left(1+\frac{\chi_{2,r}(u)U_1+\chi_{3,r}(u)U_2}{U_1+U_2}\right)\nu(\mathrm{d}u)\leq 0.\tag{2.25}$$

Meanwhile, the basic inequality $(x+y)^2 \leq 2(x^2+y^2)$ and $\hat{\alpha}_j^2 = \min\{\alpha_{2,j}^2, \alpha_{3,j}^2\}$ imply

$$\frac{1}{2(U_1+U_2)^2}(\alpha_{2,r}^2U_1^2+\alpha_{3,r}^2U_2^2)\geq\frac{1}{4(U_1^2+U_2^2)}\cdot\hat{\alpha}_r^2(U_1^2+U_2^2).\tag{2.26}$$

From (2.24)-(2.26), we get

$$\mathcal{L}\ln V_4 \leq \beta_{1,r}-\mu_r-\hat{\delta}_r-\frac{\hat{\alpha}_r^2}{4},\tag{2.27}$$

where $\hat{\delta}_j$ is defined as (2.20). Moreover, combining (2.23) and (2.27), we verify

$$\begin{aligned}\mathrm{d}\ln V_4 \leq {} & \left[\beta_{1,r_t}-\mu_{r_t}-\hat{\delta}_{r_t}-\frac{\hat{\alpha}_{r_t}^2}{4}\right]\mathrm{d}t\\ &+\frac{1}{U_1(t)+U_2(t)}\left[\alpha_{2,r_t}U_1(t)\mathrm{d}B_2(t)+\alpha_{3,r_t}U_2(t)\mathrm{d}B_3(t)\right]\\ &+\int_{\mathbb{U}}\ln\left(1+\frac{\chi_{2,r_t}(u)U_1(t-)+\chi_{3,r_t}(u)U_2(t-)}{U_1(t-)+U_2(t-)}\right)\tilde{N}(\mathrm{d}u,\mathrm{d}t).\end{aligned}\tag{2.28}$$

Integrating (2.28) from 0 to $t$, and then dividing both sides by $t$, we derive

$$\frac{\ln V_4(U_1(t),U_2(t))}{t} \leq \frac{\ln V_4(U_1(0),U_2(0))}{t} + \frac{M_t^8}{t} + \frac{M_t^9}{t} + \frac{M_t^{10}}{t} + \frac{1}{t}\int_0^t \left[\beta_{1,r_s} - \mu_{r_s} - \hat{\delta}_{r_s} - \frac{\hat{\alpha}_{r_s}^2}{4}\right], \tag{2.29}$$

where

$$M_t^8 = \int_0^t \frac{\alpha_{2_{r_s}} U_1(s)}{U_1(s)+U_2(s)}\mathrm{d}B_2(s),\ \ M_t^9 = \int_0^t \frac{\alpha_{3_{r_s}} U_2(s)}{U_1(s)+U_2(s)}\mathrm{d}B_3(s),$$
$$\text{and}\ \ M_t^{10} = \int_0^t\int_{\mathbb{U}} \ln\Big(1+\frac{\chi_{2,r_s}(u)U_1(s-)+\chi_{3,r_s}(u)U_2(s-)}{U_1(t-)+U_2(t-)}\Big)\tilde{N}(\mathrm{d}u,\mathrm{d}s).$$

According to Lemma 2.3, it derives that $\lim\limits_{t\to\infty}\frac{M_t^8}{t} = \lim\limits_{t\to\infty}\frac{M_t^9}{t} = \lim\limits_{t\to\infty}\frac{M_t^{10}}{t} = 0$, a.s. Hence, based on the Birkhoff Ergodic theorem and taking the upper limit on both sides of (2.29), we get

$$\limsup_{t\to\infty}\frac{\ln V_4(U_1(t),U_2(t))}{t} \leq \sum_{j=1}^k \pi_j\beta_{1,j} - \sum_{j=1}^k \pi_j\left[\mu_j + \hat{\delta}_j + \frac{\hat{\alpha}_j^2}{4}\right] = \sum_{j=1}^k \pi_j L_2(\mathfrak{R}_e^0 - 1),\ a.s.,$$

where $L_2 = \mu_j + \hat{\delta}_j + \frac{\hat{\alpha}_j^2}{4}$. Note that $\mathfrak{R}_e^0 < 1$, then we obtain

$$\limsup_{t\to\infty}\frac{\ln V_4(U_1(t),U_2(t))}{t} < 0,\ a.s.,$$

it gives $\limsup\limits_{t\to\infty}\frac{\ln U_1(t)}{t} < 0$, and $\limsup\limits_{t\to\infty}\frac{\ln U_2(t)}{t} < 0$, a.s. □

**Remark 2.4.** Theorem 2.3 asserts that under the conditions where $\mathfrak{R}_e^0 < 1$ and the assumptions $(A_1)$-$(A_2)$ are fulfilled, the heroin addiction will ultimately become extinct with an exponential probability of 1. Furthermore, this theorem sheds light on the fact that the intensity of the Brownian motion and the Markov chain $\{r_t\}_{t\geq 0}$ significantly influence the extinction process of heroin drug abusers.

## 2.3. Stationary distribution

Whether it has a stationary distribution is very important in the study of infectious disease models. This section will verify that system (1.4) has a unique stationary distribution.

**Theorem 2.4.** *If the assumptions $(A_1)$-$(A_3)$ are satisfied, and suppose further that $\mathfrak{R}_s^1 > 1$ and $\min\{2\mu_j + \alpha_{1,j}^2, \mu_j + \delta_{1,j}, \mu_j + \delta_{2,j}\} > \nu(\mathbb{U})$, for all $j \in \mathbb{K}$, then the solution $(S(t), U_1(t), U_2(t), r_t)$ of system (1.4) is ergodic and it exists a unique stationary distribution in $\mathbb{R}_+^3 \times \mathbb{K}$.*

**Proof.** Let $\kappa$ be a sufficiently small number with $0 < \kappa < 1$, define a bounded set as follow:

$$\tilde{D}_\kappa = \left\{(S,U_1,U_2) \in \mathbb{R}_+^3 : \kappa \leq S \leq \frac{1}{\kappa}, \kappa \leq U_1 \leq \frac{1}{\kappa}, \kappa^2 \leq U_2 \leq \frac{1}{\kappa^2}\right\}.$$

According to the assumption $(A_0)$, it is easy to see that the condition $(S_1)$ in Lemma 2.1 is held. Let $\mathscr{A}_{2,j}(z) = diag(\alpha_{1,j}, \alpha_{2,j}, \alpha_{3,j})z$ with $z = (S, U_1, U_2)$, and then $\mathcal{G}_j(z) = \mathscr{A}_{2,j}(z)\mathscr{A}_{2,j}^T(z) = diag(\alpha_{1,j}^2, \alpha_{2,j}^2, \alpha_{3,j}^2)|z|^2$ is positive definite, which implies the condition $(S_2)$ in Lemma 2.1 is satisfied. To this end, we only need to verify the condition $(S_3)$ in Lemma 2.1.

Define a $C^2$-function as follow:

$$\begin{aligned}\bar{V}(S, U_1, U_2, j) =& L_3\left(\frac{2}{N} + S - \ln S + \ln N + U_1 + \frac{99}{100}U_2 - l(j)\right) - \ln S - \ln U_1 \\ & - \ln U_2 + \frac{1}{\gamma}\left(S + U_1 + U_2\right)^{\gamma} \\ =& L_3 V_5 + V_6 + V_7,\ j \in \mathbb{K},\end{aligned}$$

here, $V_5 = \frac{2}{N} + \ln N + S - \ln S + U_1 + \frac{99}{100}U_2 - l(j)$, $V_6 = -\ln S - \ln U_1 - \ln U_2$, $V_7 = \frac{1}{\gamma}(S + U_1 + U_2)^{\gamma}$, $2 < \gamma < b_1$, and $L_3 > 0$ such that $-L_3\sum_{j=1}^{k}\pi_j L_4(\mathfrak{R}_s^1 - 1) + L_5 < -2$, where $b_1 = \min\left\{\frac{2\mu_j + \alpha_{1,j}^2 + b_2}{\alpha_{1,j}^2 + b_2 + \frac{2}{3}\mu_j}, \frac{\mu_j + \delta_{1,j} - 2b_2}{\alpha_{2,j}^2} + 1, \frac{\mu_j + \delta_{2,j}^2 - 2b_2}{\alpha_{3,j}^2} + 1, 3\right\}$, $b_2 = \breve{\omega}^2(1 + \breve{\omega})^{\gamma-2}\nu(\mathbb{U})$, $L_4 = \Lambda_j + \beta_{1,j} + K_{1,j} + \frac{\alpha_{1,j}^2}{2}$, $\mathfrak{R}_s^1$ is defined as (2.6), $l = (l(1), \cdots, l(k))^T$, $l(j)$ and $L_5$ will be determined in the proof. Obviously, $\bar{V}$ is continuous and goes to $\infty$ as $(S, U_1, U_2, j)$ goes to the boundary of $\mathbb{R}_+^3 \times \mathbb{K}$, it is easy to prove that $\bar{V}$ has a minimum value of $M_0$. Therefore, we can further define a nonnegative $C^2$-function $\tilde{V}$ as follow:

$$\tilde{V}(S, U_1, U_2, j) = L_3 V_5 + V_6 + V_7 - M_0,\ j \in \mathbb{K}.$$

Note that $V_5 = V_3 - l(j)$, where $V_3$ is defined in Theorem 2.2. From the generalized Itô's formula and (2.15), we have

$$\begin{aligned}\mathcal{L}V_5 \leq& -\frac{3\sqrt{11}}{5}\sqrt{K_{2,j}\mu_j} + \Lambda_j + \beta_{1,j} + \frac{1}{100}\beta_{1,j}U_1 + K_{1,j} + \frac{\alpha_{1,j}^2}{2} - \mu_j - \sum_{m=1}^{k}\psi_{jm}l(m) \\ =& -\left(\aleph_j + \sum_{m=1}^{k}\psi_{jm}l(m)\right) + \frac{1}{100}\beta_{3,j}U_1,\end{aligned} \tag{2.30}$$

where $\aleph_j = \frac{3\sqrt{11}}{5}\sqrt{K_{2,j}\mu_j} + \mu_j - \left(\Lambda_j + \beta_{1,j} + K_{1,j} + \frac{\alpha_{1,j}^2}{2}\right)$, $K_{1,j}$ is defined as (2.6) and $K_{2,j}$ is defined as the assumption $(A_3)$.

Define a vector $\aleph = (\aleph_1, \cdots, \aleph_k)^T$, then for $\aleph_j$ and the irreducible property of the generator matrix $\Psi$, there exists a solution of the following Poisson system expressed as $l = (l(1), \cdots, l(k))^T$ (see [19], Lemma 2.3), which satisfies

$$\Psi l + \aleph = \sum_{j=1}^{k}\pi_j\aleph_j\vec{1},$$

where $\vec{1} = (1, \cdots, 1)^T$, thus we have for all $j \in \mathbb{K}$

$$\sum_{m=1}^{k}\psi_{jm}l(m) + \aleph_j = \sum_{j=1}^{k}\pi_j\aleph_j. \tag{2.31}$$

From (2.30) and (2.31), it can be conclude that

$$\mathcal{L}V_5 \leq -\sum_{j=1}^{k}\pi_j L_4(\Re_s^1 - 1) + \frac{1}{100}\beta_{3,j}U_1. \tag{2.32}$$

Meanwhile,

$$\begin{aligned}
\mathcal{L}V_6 \leq & -\frac{\Lambda_j}{S} + \mu_j + p_j + \mu_j + \delta_{1,j} - \frac{p_j U_1}{U_2} + \beta_{3,j} + \mu_j + \delta_{2,j} + \frac{\alpha_{1,j}^2 + \alpha_{2,j}^2 + \alpha_{3,j}^2}{2} \\
& + \int_{\mathbb{U}} \left[\chi_{1,j}(u) - \ln(1+\chi_{1,j}(u))\right]\nu(\mathrm{d}u) + \int_{\mathbb{U}} \left[\chi_{2,j}(u) - \ln(1+\chi_{2,j}(u))\right]\nu(\mathrm{d}u) \\
& + \int_{\mathbb{U}} \left[\chi_{3,j}(u) - \ln(1+\chi_{3,j}(u))\right]\nu(\mathrm{d}u) \\
= & -\frac{\Lambda_j}{S} + \mu_j + p_j + \mu_j + \delta_{1,j} - \frac{p_j U_1}{U_2} + \beta_{3,j} + \mu_j + \delta_{2,j} + \frac{\alpha_{1,j}^2 + \alpha_{2,j}^2 + \alpha_{3,j}^2}{2} \\
& + \frac{3\breve{\omega}^2}{2(1-\breve{\omega})^2}\nu(\mathbb{U}).
\end{aligned} \tag{2.33}$$

Besides,

$$\begin{aligned}
\mathcal{L}V_7 = & (S+U_1+U_2)^{\gamma-1}\left(\Lambda_j - \mu_j S - (\mu_j+\delta_{1,j})U_1 - (\mu_j+\delta_{2,j})U_2\right) \\
& + \frac{\gamma-1}{2}(S+U_1+U_2)^{\gamma-2}\left(\alpha_{1,j}^2 S^2 + \alpha_{2,j}^2 U_1^2 + \alpha_{3,j}^2 U_2^2\right) \\
& + \int_{\mathbb{U}}\Bigg[\frac{1}{\gamma}(S+U_1+U_2+\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2)^\gamma - \frac{1}{\gamma}(S+U_1+U_2)^\gamma \\
& - (S+U_1+U_2)^{\gamma-1}(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2)\Bigg]\nu(\mathrm{d}u).
\end{aligned} \tag{2.34}$$

By Taylor formula and $(A_1)$, we obtain

$$\begin{aligned}
& \int_{\mathbb{U}}\Bigg[\frac{1}{\gamma}\left(S+U_1+U_2+\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2\right)^\gamma - \frac{1}{\gamma}\left(S+U_1+U_2\right)^\gamma \\
& - \left(S+U_1+U_2\right)^{\gamma-1}\left(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2\right)\Bigg]\nu(\mathrm{d}u) \\
= & \frac{1}{\gamma}\int_{\mathbb{U}}\Big[N^\gamma + \gamma N^{\gamma-1}(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2) \\
& + \frac{\gamma(\gamma-1)}{2}(N+\theta_5(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2))^{\gamma-2}(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1 \\
& + \chi_{3,j}(u)U_2)^2 - N^\gamma - \gamma N^{\gamma-1}(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2)\Big]\nu(\mathrm{d}u) \\
= & \frac{1}{\gamma}\int_{\mathbb{U}}\Big[\frac{\gamma(\gamma-1)}{2}(N+\theta_5(\chi_{1,j}(u)S+\chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2))^{\gamma-2}(\chi_{1,j}(u)S \\
& + \chi_{2,j}(u)U_1+\chi_{3,j}(u)U_2)^2\Big]\nu(\mathrm{d}u) \\
\leq & \frac{\gamma-1}{2}(N+\breve{\omega}N)^{\gamma-2}(\breve{\omega}N)^2\nu(\mathbb{U}),
\end{aligned} \tag{2.35}$$

where $\theta_5 \in (0,1)$ is an arbitrary number. Together (2.34) with (2.35), we derive

$$\begin{aligned}\mathcal{L}V_7 \leq& \Lambda_j N^{\gamma-1} - \mu_j S^\gamma - (\mu_j+\delta_{1,j})U_1^\gamma - (\mu_j+\delta_{2,j})U_2^\gamma \\ &+ \frac{\gamma-1}{2}N^{\gamma-2}\Big[(\alpha_{1,j}^2 S^2 + \alpha_{2,j}^2 U_1^2 + \alpha_{3,j}^2 U_2^2) + b_2 N^2\Big] \\ \leq& -\frac{1}{3}\gamma\mu_j S^\gamma - \frac{(\mu_j+\delta_{1,j})}{2}U_1^\gamma - \frac{(\mu_j+\delta_{2,j})}{2}U_2^\gamma + L_6, \end{aligned} \tag{2.36}$$

where

$$\begin{aligned}L_6 = \sup_{(S,U_1,U_2)\in\mathbb{R}_+^3}\Big\{& -\mu_j(1-\frac{1}{3}\gamma)S^\gamma - \frac{\mu_j+\delta_{1,j}}{2}U_1^\gamma - \frac{\mu_j+\delta_{2,j}}{2}U_2^\gamma + \Lambda_j N^{\gamma-1} \\ &+ \frac{\gamma-1}{2}N^{\gamma-2}\Big[\Big(\alpha_{1,j}^2 S^2 + \alpha_{2,j}^2 U_1^2 + \alpha_{3,j}^2 U_2^2\Big) + b_2 N^2\Big]\Big\}.\end{aligned}$$

Moreover, combining (2.32), (2.33) and (2.36), it is easy to verify

$$\begin{aligned}\mathcal{L}\tilde{V} \leq& - L_3\sum_{j=1}^{k}\pi_j L_4(\mathfrak{R}_s^1 - 1) + L_3\frac{1}{100}\beta_{3,j}U_1 - \frac{\Lambda_j}{S} - p_j\frac{U_1}{U_2} - \frac{\gamma\mu_j}{3}S^\gamma \\ &- \frac{\mu_j+\delta_{1,j}}{2}U_1^\gamma - \frac{\mu_j+\delta_{2,j}}{2}U_2^\gamma + L_5,\end{aligned}$$

where

$$L_5 = \mu_j + p_j + \mu_j + \delta_{1,j} + \beta_{3,j} + \mu_j + \delta_{2,j} + \frac{\alpha_{1,j}^2+\alpha_{2,j}^2+\alpha_{3,j}^2}{2} + \frac{3\breve{\omega}^2}{2(1-\breve{\omega})^2} + L_6.$$

Next, let $\kappa$ be sufficiently small in the set $\mathbb{R}_+^3\backslash\tilde{D}_\kappa \times \mathbb{K}$ satisfying

$$L_3\frac{1}{100}\beta_{3,j}\kappa < 1, \tag{2.37}$$

$$-\frac{\Lambda_j}{\kappa} + L_7 \leq -1, \tag{2.38}$$

$$-\frac{p_j}{\kappa} + L_7 \leq -1, \tag{2.39}$$

$$-\frac{\mu_j+\delta_{2,j}}{2\kappa^\gamma} + L_7 \leq -1, \tag{2.40}$$

$$-\frac{\mu_j+\delta_{1,j}}{2\kappa^\gamma} + L_7 \leq -1, \tag{2.41}$$

$$-\frac{\gamma\mu_j}{3\kappa^\gamma} + L_7 \leq -1, \tag{2.42}$$

where $L_7 = L_3\frac{1}{100}\beta_{3,j}U_1 + L_5$. For convenience, we divide $\mathbb{R}_+^3\backslash\tilde{D}_\kappa$ into six domains:

$$\begin{aligned}&\tilde{D}_1 = \{(S,U_1,U_2)\in\mathbb{R}_+^3 : 0 < U_1 < \kappa\},\ \tilde{D}_2 = \{(S,U_1,U_2)\in\mathbb{R}_+^3 : 0 < S < \kappa\}, \\ &\tilde{D}_3 = \{(S,U_1,U_2)\in\mathbb{R}_+^3 : 0 < U_2 < \kappa^2, U_1 \geq \kappa\}, \\ &\tilde{D}_4 = \left\{(S,U_1,U_2)\in\mathbb{R}_+^3 : U_2 > \frac{1}{\kappa^2}\right\},\end{aligned}$$

$$\tilde{D}_5 = \left\{ (S, U_1, U_2) \in \mathbb{R}^3_+ : U_1 > \frac{1}{\kappa} \right\},\ \tilde{D}_6 = \left\{ (S, U_1, U_2) \in \mathbb{R}^3_+ : S > \frac{1}{\kappa} \right\}.$$

Clearly, we derive from (2.37)-(2.42) that for any $(S, U_1, U_2, r) \in \mathbb{R}^3_+ \backslash \tilde{D}^c_\kappa \times \mathbb{K}$, $\mathcal{L}\tilde{V}(S, U_1, U_2, j) < -1$. By Lemma 2.1, it follows that system (1.4) is ergodic and it has a unique stationary distribution. □

**Remark 2.5.** Theorem 2.4 states that under the conditions, including assumptions $(A_1)$-$(A_3)$, $\mathfrak{R}^1_s > 1$, and for all $j \in \mathbb{K}$, the inequality $\min\{2\mu_j + \alpha^2_{1,j}, \mu_j + \delta_{1,j}, \mu_j + \delta_{2,j}\} > \nu(\mathbb{U})$ holds. Under these conditions, system (1.4) exhibits ergodicity and possesses a unique stationary distribution, suggesting that the disease will persist and spread within the population. Additionally, this theorem highlights the crucial role of the coefficients of Lévy jumps and the Markov chain $\{r_t\}_{t\geq 0}$ in influencing the persistence of heroin drug abusers.

## 3. Numerical simulations

By employing the positive preserving truncated Euler-Maruyama method, as outlined in [31,46], we derive a numerical method for simulating system (1.4) in state $j$, as detailed below:

$$\begin{cases} S(n+1) = S(n) + \left[\Lambda_j - \beta_{1,j}\breve{\pi}_0\left(\dfrac{S(n)U_1(n)}{N(n)}\right) - \mu_j\breve{\pi}_0\left(S(n)\right)\right]\Delta t + \alpha_{1,j}\breve{\pi}_0(S(n)\xi_1(n))\sqrt{\Delta t} \\ \qquad + \dfrac{\alpha^2_{1,j}}{2}\breve{\pi}_0(S(n))(\xi^2_1(n) - 1)\Delta t + \chi_{1,j}(y^*)\breve{\pi}_0(S(n))\Delta\mathbb{L}(n), \\ U_1(n+1) = U_1(n) + \left[\beta_{1,j}\breve{\pi}_0\left(\dfrac{S(n)U_1(n)}{N(n)}\right) - p_j\breve{\pi}_0(U_1(n)) + \beta_{3,j}\breve{\pi}_0\left(\dfrac{U_1(n)U_2(n)}{N(n)}\right)\right. \\ \qquad \left. - (\mu_j + \delta_{1,j})\breve{\pi}_0(U_1(n))\right]\Delta t + \alpha_{2,j}\breve{\pi}_0(U_1(n)\xi_2(n))\sqrt{\Delta t} \\ \qquad + \dfrac{\alpha^2_{2,j}}{2}\breve{\pi}_0(U_1(n))(\xi^2_2(n) - 1)\Delta t + \chi_{2,j}(y^*)\breve{\pi}_0(U_1(n))\Delta\mathbb{L}(n), \\ U_2(n+1) = U_2(n) + \left[p_j\breve{\pi}_0(U_1(n)) - \beta_{3,j}\breve{\pi}_0\left(\dfrac{U_1(n)U_2(n)}{N(n)}\right) - (\mu_j + \delta_{2,j})\breve{\pi}_0(U_2(n))\right]\Delta t \\ \qquad + \alpha_{3,j}\breve{\pi}_0(U_2(n)\xi_3(n))\sqrt{\Delta t} + \dfrac{\alpha^2_{3,j}}{2}\breve{\pi}_0(U_2(n))(\xi^2_3(n) - 1)\Delta t \\ \qquad + \chi_{3,j}(y^*)\breve{\pi}_0(U_2(n))\Delta\mathbb{L}(n), \end{cases} \tag{3.1}$$

where $\breve{\pi}_0(x) = 0 \vee x$, $N(n) = S(n) + U_1(n) + U_2(n)$, $y^* \in \mathbb{U}$, $M^* \in \mathbb{K}$, $n = 0, 1, 2, \cdots, M^*$, $\Delta t = \frac{T}{N^*}$ denotes the size of time step on $[0, T]$, $\xi_i(n)(i = 1, 2, 3)$ are independent Gaussian random variable, following $\mathcal{N}(0, 1)$, and $\Delta\mathbb{L}(n) \triangleq \mathbb{L}(t_{n+1}) - \mathbb{L}(t_n)$ obeys the Poisson distribution with intensity $\nu$ and $\mathbb{U} = (0, \infty)$.

**Example 3.1.** (Persistence in the mean) Consider the parameters in Table 1 and the initial value $(S(0), U_1(0), U_2(0), r_0) = (1.5, 0.8, 0.05, 1) \in \mathfrak{R}^3_+ \times \mathbb{K}$, $\nu(\mathbb{U}) = 0.05$. Let $r_t$ be the irreducible Markov chain with $\mathbb{K} = \{1, 2\}$, and the generator is $\Psi = \begin{pmatrix} -60 & 60 \\ 40 & -40 \end{pmatrix}$, and denote a step size $\Delta = 0.01$, by the one-step transition probability matrix $P = e^{\Delta\Phi}$, it has $P = \begin{pmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{pmatrix}$. The

**Table 1.** The values of the parameters.

| | $\Lambda$ | $\beta_1$ | $\mu$ | $\delta_1$ | $\delta_2$ | $\beta_3$ | $p$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| state1 | 0.264 | 0.4 | 0.224 | 0.034 | 0.12 | 0.06 | 0.024 | 0.004 | 0.003 | 0.004 |
| state2 | 0.274 | 0.5 | 0.234 | 0.044 | 0.22 | 0.16 | 0.034 | 0.014 | 0.013 | 0.014 |

irreducible property implies the unique stationary distribution of $r_t$ is $\pi = (\pi_1, \pi_2) = (0.4, 0.6)$ (see Figure 1(a)). To consider the influence of Lévy noises, define the following parameters, respectively:

($\bar{\mathbf{C}}_\mathbf{1}$) $\chi_{1,1}(u) = -0.36$, $\chi_{2,1}(u) = -0.38$, $\chi_{3,1}(u) = -0.34$, $\omega_1 = 0.38$, $\chi_{1,2}(u) = -0.26$, $\chi_{2,2}(u) = -0.28$, $\chi_{3,2}(u) = -0.24$, $\omega_2 = 0.28$.

By calculation, we obtain $\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha_1}^2 \vee \breve{\alpha_2}^2 \vee \breve{\alpha_3}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.220292 > 0$, $K_{1,1} = \frac{\omega_1^2}{2(1-\omega_1)^2}\nu(\mathbb{U}) = 0.009391$, $K_{1,2} = \frac{\omega_2^2}{2(1-\omega_2)^2}\nu(\mathbb{U}) = 0.003781$, $K_{2,1} = 2\sqrt{\frac{\check{\delta}_1\Lambda_1}{2}} + 3\sqrt[3]{\Lambda_1^2\beta_{1,1}} - \mu_1 - \check{\delta}_1 - 2\Lambda_1 - \breve{\alpha}_1^2 - \left[\frac{1}{1-\omega_1} - 1 + \omega_1\right]\nu(\mathbb{U}) = 0.279447 > 0$, $K_{2,2} = 2\sqrt{\frac{\check{\delta}_2\Lambda_2}{2}} + 3\sqrt[3]{\Lambda_2^2\beta_{1,2}} - \mu_2 - \check{\delta}_2 - 2\Lambda_2 - \breve{\alpha}_2^2 - \left[\frac{1}{1-\omega_2} - 1 + \omega_2\right]\nu(\mathbb{U}) = 0.295648 > 0$, and $\mathfrak{R}_s^1 = \frac{\sum_{j=1}^{k}\pi_j\left[\frac{3\sqrt{11}}{5}(K_{2,j}\mu_j)^{\frac{1}{2}} + \mu_j\right]}{\sum_{j=1}^{k}\pi_j\left(\Lambda_j + \beta_{1,j} + K_{1,j} + \frac{\alpha_{1,j}^2}{2}\right)} = 1.010826 > 1$.

($\bar{\mathbf{C}}_\mathbf{2}$) $\chi_{1,1}(u) = 0.06$, $\chi_{2,1}(u) = 0.09$, $\chi_{3,1}(u) = 0.09$, $\omega_1 = 0.09$, $\chi_{1,2}(u) = 0.16$, $\chi_{2,2}(u) = 0.19$, $\chi_{3,2}(u) = 0.19$, $\omega_2 = 0.19$.

By calculation, we obtain $\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha_1}^2 \vee \breve{\alpha_2}^2 \vee \breve{\alpha_3}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.223$, $K_{1,1} = 0.000245$, $K_{1,2} = 0.001376$, $K_{2,1} = 0.319647$, $K_{2,2} = 0.307864$, and $\mathfrak{R}_s^1 = 1.045022$.

($\bar{\mathbf{C}}_\mathbf{3}$) $\chi_{1,1}(u) = 0.28$, $\chi_{2,1}(u) = 0.29$, $\chi_{3,1}(u) = 0.24$, $\omega_1 = 0.29$, $\chi_{1,2}(u) = 0.38$, $\chi_{2,2}(u) = 0.39$, $\chi_{3,2}(u) = 0.34$, $\omega_2 = 0.39$.

By calculation, we obtain $\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha_1}^2 \vee \breve{\alpha_2}^2 \vee \breve{\alpha_3}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.2201$, $K_{1,1} = 0.004171$, $K_{1,2} = 0.010219$, $K_{2,1} = 0.29417$, $K_{2,2} = 0.2776250$, and $\mathfrak{R}_s^1 = 1.002568$.

Based on the parameters presented in Table 1 and conditions ($\bar{\mathbf{C}}_\mathbf{1}$) to ($\bar{\mathbf{C}}_\mathbf{3}$), it is evident that the assumptions (A1)-(A3) are satisfied. According to Theorem 2.2, this validates that system (1.4) exhibits persistence in the mean, as clearly illustrated in Figures 1(b), 1(c) and 1(d). Furthermore, this analysis suggests that the coefficients of Lévy noises have a significant impact on the dynamical behaviors of system (1.4).

**Example 3.2.** (Extinction) Consider the parameters as Table 2 and the initial value of system

**Table 2.** The values of the parameters.

| | $\Lambda$ | $\beta_1$ | $\mu$ | $\delta_1$ | $\delta_2$ | $\beta_3$ | $p$ | $\chi_1(u)$ | $\chi_2(u)$ | $\chi_3(u)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| state1 | 0.97 | 0.27285 | 0.19232 | 0.017 | 0.027 | 0.57 | 0.022 | 0.17 | 0.27 | 0.37 |
| state2 | 0.87 | 0.17285 | 0.19222 | 0.007 | 0.017 | 0.47 | 0.012 | 0.07 | 0.17 | 0.27 |

(1.4) is given by $(S(0), U_1(0), U_2(0), r_0) = (10, 2, 0.2, 1) \in \mathfrak{R}_+^3 \times \mathbb{K}$, $\nu(\mathbb{U}) = 0.6$. The irreducible Markov chain $r_t$ with the generator $\Psi = \begin{pmatrix} -70 & 70 \\ 30 & -30 \end{pmatrix}$, and the unique stationary distribution $\pi = (0.3, 0.7)$ (see Figure 2(a)). To consider the influence of Brownian motion, define the

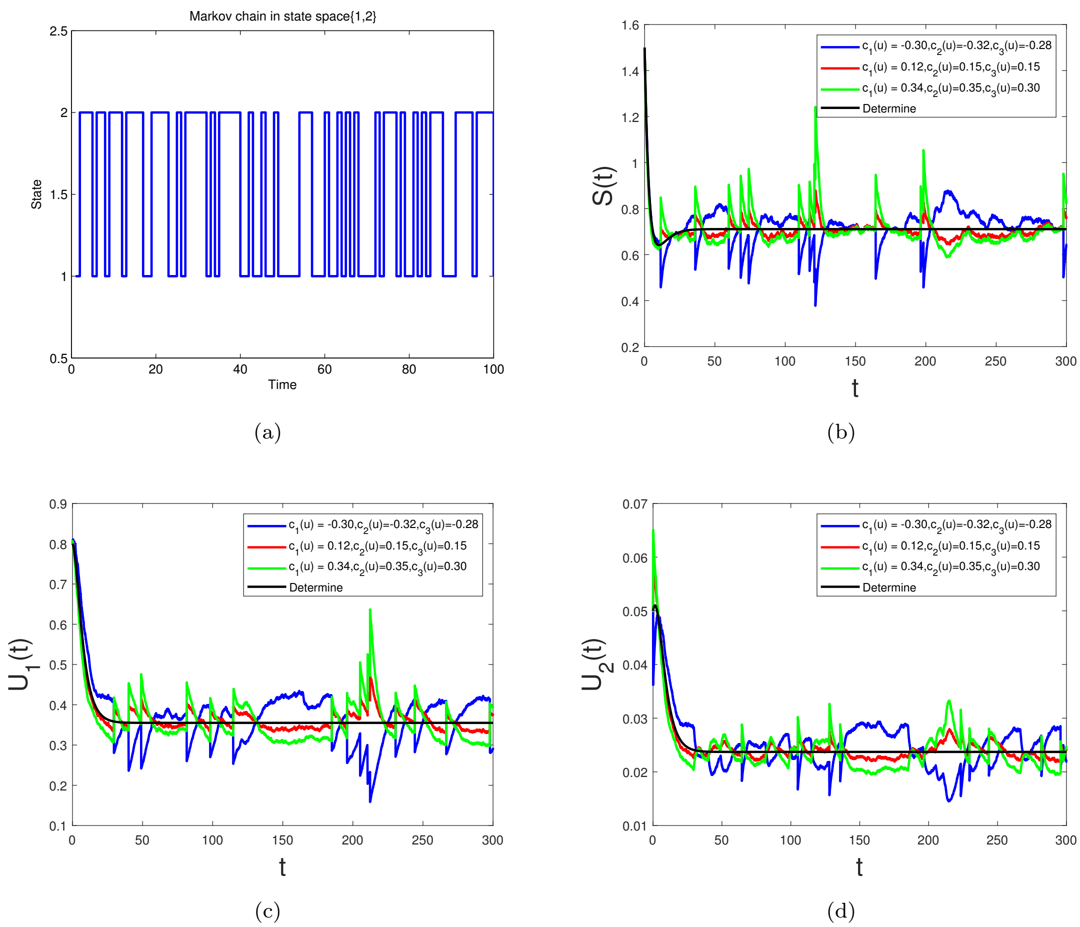


**Figure 1.** The path of the Markov chain and the variations in the trajectories of $S(t)$, $U_1(t)$, and $U_2(t)$ within systems (1.1) and (1.4) under varying intensities of Lévy noise, respectively, with the parameters outlined in Table 1.

following parameters, respectively:

$(\mathbf{C_1})$ $\alpha_{1,1} = 0.12$, $\alpha_{2,1} = 0.13$, $\alpha_{3,1} = 0.14$, $\alpha_{1,2} = 0.02$, $\alpha_{2,2} = 0.03$, $\alpha_{3,2} = 0.04$.

From Table 1, we obtain that $\breve{\omega} = 0.37$, $\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha}_1{}^2 \vee \breve{\alpha}_2{}^2 \vee \breve{\alpha}_3{}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.14135 > 0$, and $\mathfrak{R}_e^0 = \dfrac{\sum_{j=1}^{k} \pi_j \beta_{1,j}}{\sum_{j=1}^{k} \pi_j \left[\mu_j + \hat{\delta}_j + \frac{\hat{\alpha}_j^2}{4}\right]} = 0.999877 < 1$,

$(\mathbf{C_2})$ $\alpha_{1,1} = 0.32$, $\alpha_{2,1} = 0.33$, $\alpha_{3,1} = 0.34$, $\alpha_{1,2} = 0.22$, $\alpha_{2,2} = 0.23$, $\alpha_{3,2} = 0.24$.

By calculation, it has $\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha}_1{}^2 \vee \breve{\alpha}_2{}^2 \vee \breve{\alpha}_3{}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.09335$, and $\mathfrak{R}_e^0 = 0.931038$,

$(\mathbf{C_3})$ $\alpha_{1,1} = 0.5$, $\alpha_{2,1} = 0.51$, $\alpha_{3,1} = 0.52$, $\alpha_{1,2} = 0.40$, $\alpha_{2,2} = 0.41$, $\alpha_{3,2} = 0.42$.

By calculation, it get $\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha}_1{}^2 \vee \breve{\alpha}_2{}^2 \vee \breve{\alpha}_3{}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.01595$, and $\mathfrak{R}_e^0 = 0.81638$.

According to the parameters in Table 2 and $(\mathbf{C_1})$-$(\mathbf{C_3})$, it implies the assumptions $(A_1)$ and $(A_2)$ are satisfied. From Theorem 2.3, the heroin drug abusers $U_1(t)$ and $U_2(t)$ eventually become extinct with an exponential probability of 1, and the susceptible individuals $S(t)$ converges towards a stable constant value, as depicted in Figures 2(b), 2(c) and 2(d). This observation underscores that the intensity of noise exerts a pronounced influence on heroin users, with greater noise intensities leading to stronger effects.

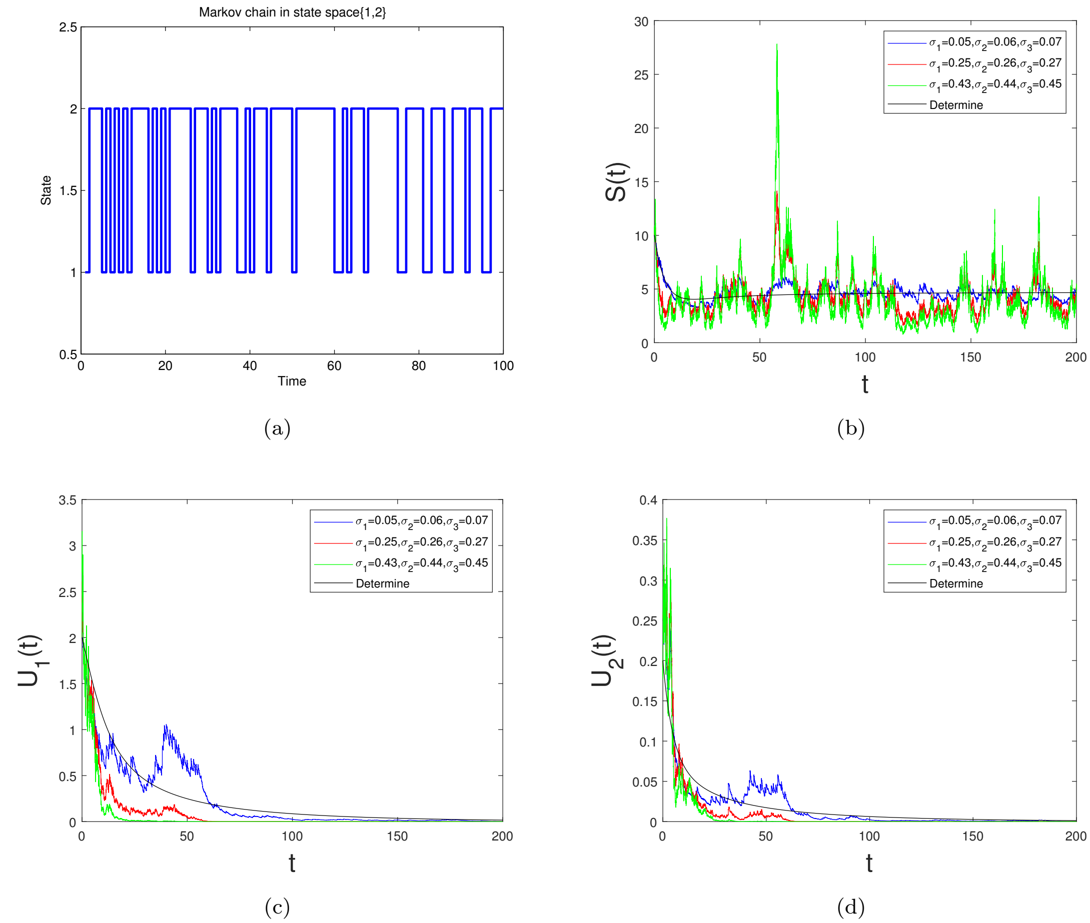


**Figure 2.** The path of the Markov chain and the variations in the trajectories of $S(t)$, $U_1(t)$, and $U_2(t)$ within systems (1.1) and (1.4) under varying intensities of white noise, respectively, with the parameters outlined in Table 2.

**Example 3.3.** (Distribution) Consider the parameters in Table 3, and the initial value of

**Table 3.** The values of the parameters.

| | $\Lambda$ | $\beta_1$ | $\mu$ | $\delta_1$ | $\delta_2$ | $\beta_3$ | $p$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\chi_1(u)$ | $\chi_2(u)$ | $\chi_3(u)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| state1 | 0.28 | 0.4 | 0.2555 | 0.01 | 0.05 | 0.35 | 0.01 | 0.1 | 0.09 | 0.08 | 0.1 | 0.09 | 0.08 |
| state2 | 0.4 | 0.6 | 0.322 | 0.24 | 0.25 | 0.45 | 0.02 | 0.12 | 0.15 | 0.2 | 0.12 | 0.15 | 0.2 |

system (1.4) is given by $(S(0), U_1(0), U_2(0), r_0) = (4, 0.5, 0.1, 1) \in \mathfrak{R}^3_+ \times \mathbb{K}$, $\omega_1 = 0.1$, $\omega_2 = 0.2$ and $\nu(\mathbb{U}) = 0.1$. Meanwhile, the choice of Markov chain $r_t$ is same as Example 3.2. Moreover, by calculation, we get

$$\hat{\mu} - \left(\frac{1}{2}(\breve{\alpha}_1{}^2 \vee \breve{\alpha}_2{}^2 \vee \breve{\alpha}_3{}^2) + \frac{1}{2}\breve{\omega}^2\nu(\mathbb{U})\right) = 0.25 > 0,$$

$$K_{2,1}=2\sqrt{\frac{\check{\delta}_1\Lambda_1}{2}}+3\sqrt[3]{\Lambda_1^2\beta_{1,1}}-\mu_1-\check{\delta}_1-2\Lambda_1-\check{\alpha}_1^2-\left[\frac{1}{1-\omega_1}-1+\omega_1\right]\nu(\mathbb{U})=0.216769,$$

$$\mathfrak{R}_s^{11}=\frac{\frac{3\sqrt{11}}{5}\left(K_{2,1}\mu_1\right)^{\frac{1}{2}}+\mu_1}{\Lambda_1+\beta_{1,1}+K_{1,1}+\frac{\alpha_{1,1}^2}{2}}=1.055719>1,$$

$$K_{2,2}=2\sqrt{\frac{\check{\delta}_2\Lambda_2}{2}}+3\sqrt[3]{\Lambda_2^2\beta_{1,2}}-\mu_2-\check{\delta}_2-2\Lambda_2-\check{\alpha}_2^2-\left[\frac{1}{1-\omega_2}-1+\omega_2\right]\nu(\mathbb{U})=0.363871,$$

$$\mathfrak{R}_s^{12}=\frac{\frac{3\sqrt{11}}{5}\left(K_{2,2}\mu_1\right)^{\frac{1}{2}}+\mu_2}{\Lambda_2+\beta_{1,2}+K_{1,2}+\frac{\alpha_{1,2}^2}{2}}=0.992908<1, \text{ and}$$

$$\mathfrak{R}_s^{1}=\frac{\sum\limits_{j=1}^{k}\pi_j\left[\frac{3\sqrt{11}}{5}\left(K_{2,j}\mu_j\right)^{\frac{1}{2}}+\mu_j\right]}{\sum\limits_{j=1}^{k}\pi_j\left(\Lambda_j+\beta_{1,j}+K_{1,j}+\frac{\alpha_{1,j}^2}{2}\right)}=1.029296>1,$$

which implies the conditions of Theorem 2.4 are ensured to be met. The simulation results for the untreated heroin user $U_1(t)$ in system (1.4) are presented in Figure 3. Notably, $\mathfrak{R}_s^{11}>1$ implies that $U_1(t)$ within subsystem 1 demonstrates stochastic persistence, as illustrated in Figure 3(a). Conversely, $\mathfrak{R}_s^{12}<1$ indicates that $U_1(t)$ in subsystem 2 tends towards extinction, as shown in Figure 3(c). When the Markov chain plays a role in stochasticity, it contributes to the persistence of $U_1(t)$ within the hybrid system, as evidenced in Figure 3(e). Thus, it can be inferred that the Markov chain $\{r_t\}_{t\geq 0}$ has both passive and positive influences on disease transmission.

# 4. Conclusions

External influences on drug abuse, particularly those affecting survival and transmission mechanisms, indicate that a stochastic heroin epidemic model with Lévy noises and Markov regime-switching offers superior realism compared to deterministic counterparts. By embedding these stochastic elements into the heroin epidemic model, this research advances comprehension of addiction progression stages and refines parameter estimation for targeted intervention strategies. Theoretical contributions include proving the existence and uniqueness of the solution for system (1.4) and identifying sufficient conditions for the extinction and persistence in the mean of heroin users. Additionally, the ergodic stationary distribution of system (1.4) is verified through Lyapunov stability analysis, providing insights into long-term system behavior.

Compared to existing literature, the key innovations of this paper are as follows:

1. Distinct from the work by White and Comiskey [42], this research integrates Markov chain dynamics and Lévy noises components into heroin epidemic model (1.1). This novel framework advances understanding of addiction progression stages and enables precise parameter calibration for epidemic containment. By utilizing appropriate Lyapunov functions, we establish:
(a) For $\mathfrak{R}_e^0<1$, mathematical extinction of both untreated ($U_1(t)$) and treated ($U_2(t)$) user populations in system (1.4);
(b) For $\mathfrak{R}_s^1>1$, mean persistence and ergodic stationary distribution emergence in system (1.4).
2. The system delineated in equation (1.4) represents an expansion of the system outlined

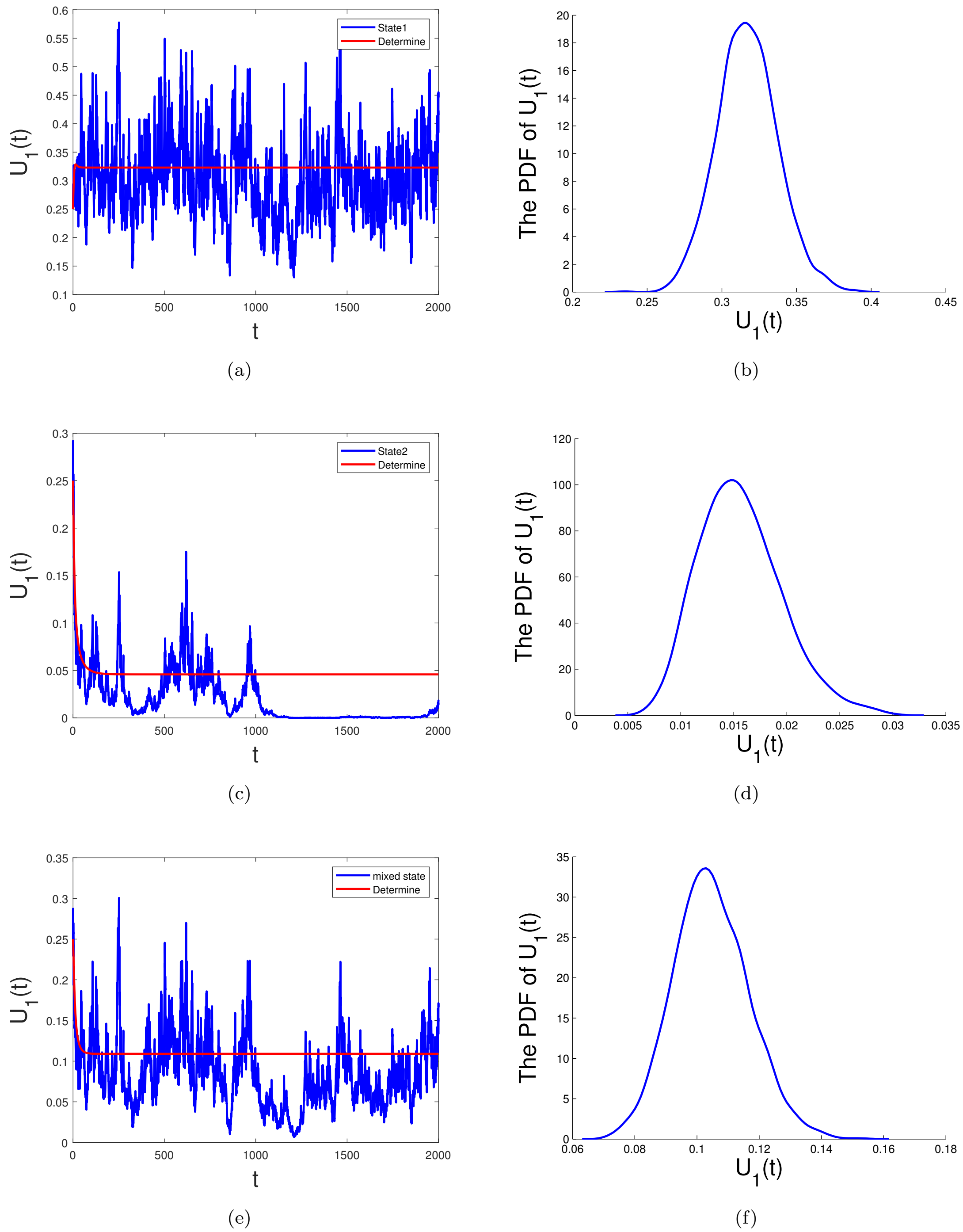


**Figure 3.** The trajectory and the probability density function of $U_1$ in state 1, state 2, and mixed states in system (1.1) and system (1.4) respectively.

in equation (1.2), incorporating considerations of Lévy noises and regime-switching. In contrast to the work by Liu et al. [29], we formulate the essential conditions for both persistence in the mean and the unique stationary distribution of system (1.4) at the identical critical threshold $\mathfrak{R}_s^1$. Theorems 2.2 and 2.3 unveil a remarkable insight: Lévy noises and the Markov chain can control the persistence and extinction of the disease, indicating their significant role in the spread of the epidemic (see Figures 1 and 2).

3. As an advanced extension of system (1.1), system (1.4) integrates Lévy noise excitations and Markov chain transitions, necessitating more intricate theoretical and numerical investigations than those focused solely on Lévy noises in Li et al. [23]. A novel insight emerges: The Markov chain, when active internally, can reconcile subsystem-specific stochastic persistence (Subsystem 1) and extinction (Subsystem 2) to maintain overall persistence in the hybrid system. This underscores the Markov chain's pivotal function in reducing heroin user extinction rates and shaping epidemic spread trajectories, with empirical support from Figures 3(a), 3(c) and 3(e).
4. Unlike Jiang et al.'s [17] bilinear incidence model, this study pioneers the incorporation of Markov chain regime-switching and Lévy noise elements within a standard incidence heroin epidemic framework. This innovative combination introduces unprecedented complexity by replacing the bilinear incidence function with its standard counterpart, thereby elevating both theoretical challenges and practical relevance in epidemic modeling. The enhanced complexity demands sophisticated mathematical tools to analyze persistence, extinction and stationary distribution dynamics.

## Conflicts of interest

This work does not have any conflicts of interest.

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