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# ON THE QUALITATIVE PROPERTIES OF THE SOLUTION OF A NON-LINEAR LANGEVIN EQUATION OF TWO FRACTAL DERIVATIVES

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**Abstract** In this paper, we study the qualitative properties of the solution of a non-linear fractal Langevin equation involving two distinct fractal orders. We prove the existence and uniqueness of solutions in the space $C[0,T]$. Additionally, we examine the continuous dependence of the solution on the parameters of the problem. Also, the Hyers–Ulam stability of the proposed problem will be studied. Moreover, the continuation of the problem will be proved. Finally, an example is provided to illustrate the applicability of the assumed conditions and to demonstrate the obtained results.

**Keywords** Langevin equation, fractal operators, fractal-fractional operators, continuous dependence, existence of solutions, Hyers-Ulam stability.

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## 1. Introduction

The Langevin equation for a Brownian particle with mass $m$ in a one-dimensional fluid bath is represented by

$$m\, v^{'}(t) + \lambda\, v(t) \;=\; F(t),$$

where $v(t) = x^{'}(t)$ is the velocity of the Brownian particle, $\lambda$ is a parameter describing friction between the particle and the bath, and the function $F(t)$ is a random force.

Let the Brownian particle has a unit mass, then its Langevin equation can be represented by [10]

$$x^{''}(t) + \lambda\, x^{'}(t) \;=\; F(t),$$

which can be written as

$$\frac{d}{dt}\,(\frac{d}{dt} + \lambda)\; x(t) \;=\; F(t).$$

Some works have studied the Langevin equation involving two fractional-order derivatives. In particular, some studies examined the model of the form [1–3, 5, 7, 18, 19, 24]

$${}^{C}D^{\alpha}\;(\;{}^{C}D^{\beta} + \lambda\;)\;x(t) \;=\; F(t),$$

where the equation has been analyzed under different non-linearities, boundary conditions, and fractional operators.

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Langevin equation of two fractal orders extends the classical Langevin model by incorporating two fractal derivatives, allowing it to describe systems with complex, scale-dependent behavior. This approach is useful for modeling physical processes in fractal or irregular media, where traditional calculus fails to capture memory effects and anomalous dynamics [14, 16, 17, 23].

Motivated by the works mentioned above, we aim to extend classical and fractional models by incorporating fractal memory effects and to study the initial value problem of the nonlinear delayed Langevin differential equation involving two distinct fractal order derivatives

$$D_\beta\ (D_\gamma + \lambda)\ x(t) \ = \ f(t, x(\phi(t))), \qquad t \in (0, T], \qquad x(0) = x_0 \tag{1.1}$$

where $D_\beta, D_\gamma$ are two different fractal derivatives of orders $\beta, \gamma \in (0, 1)$ [13, 21, 22], $\lambda$ is a positive parameter and $\phi$ is the delay function.

Our aim is to study a fractal integral equation (under certain assumptions) that is equivalent to the problem (1.1) in the functional space $C[0, T]$. We prove the existence of a continuous solution and investigate the continuous dependence of the solution on the parameter $\lambda$, the initial data $x_0$ and the functions $f, \phi$. Finally, we will establish the Hyers–Ulam stability of the problem.

Also, the continuation, as $\beta \to \gamma$ of the problem (1) to the problem

$$D_\gamma\ (D_\gamma + \lambda)\ x(t) \ = \ f(t, x(\phi(t))), \qquad t \in (0, T], \qquad x(0) = x_0 \tag{1.2}$$

and the continuation, as $\beta, \gamma \to 1$ of the problem (1) to the problem

$$\frac{d}{dt}\ (\frac{d}{dt} + \lambda)\ x(t) \ = \ f(t, x(\phi(t))), \qquad t \in (0, T] \qquad x(0) = x_0 \tag{1.3}$$

will be establish.

## 2. Existence of solution

Consider problem (1.1) under the following assumptions:
$(i)$ $f : [0, T] \times R \to R$ is measurable in $t \in [0, T]$ for any $x \in R$ and continuous in $x \in R$ for $t \in [0, T]$.
$(ii)$ There exist a bounded measurable function $a : [0, T] \to R$, $|a(t)| \le a$ and a positive constant $b$ such that

$$|f(t, x)| \ \le \ |a(t)| + b\ |x|. \tag{2.1}$$

$(iii)$ $\phi : [0, T] \to [0, T]$ is a continuous function and $\phi(t) \le t$.
$(iv)$ $\lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta} < 1$.

**Lemma 2.1.** *Problem (1.1) is equivalent to the fractal integral equation*

$$x(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds. \tag{2.2}$$

**Proof.** Let $x$ be a solution of problem (1.1). Then we have

$$D_\beta\ (D_\gamma + \lambda)\ x(t) \ = \ f(t, x(\phi(t))).$$

Let $y(t) = (D_\gamma + \lambda)\ x(t)$, then we obtain

$$D_\gamma\ x(t) \ = \ y(t) - \lambda\ x(t),$$

$$\frac{t^{1-\gamma}}{\gamma}\,\frac{dx(t)}{dt} \;=\; y(t)-\lambda\,x(t) \tag{2.3}$$

which implies that $y(0)=\lambda\,x(0)$.

Multiplying both side of (2.3) by $\gamma\,t^{\gamma-1}$, we obtain

$$\frac{dx(t)}{dt} \;=\; \gamma\,t^{\gamma-1}\,y(t)-\lambda\,\gamma\,t^{\gamma-1}\,x(t).$$

Integrating both side and using $x(0)=x_0$, we obtain

$$x(t) \;=\; x_0+\int_0^t \gamma\,s^{\gamma-1}\,y(s)\,ds-\lambda\int_0^t \gamma\,s^{\gamma-1}\,x(s)\,ds. \tag{2.4}$$

Also, we have

$$\begin{aligned} D_\beta\,y(t) &= f(t,x(\phi(t))),\\ \frac{t^{1-\beta}}{\beta}\,\frac{dy(t)}{dt} &= f(t,x(\phi(t))) \end{aligned}$$

and

$$\frac{dy(t)}{dt} \;=\; \beta\,t^{\beta-1}\,f(t,x(\phi(t))).$$

Integrating both side and using $y(0)=\lambda\,x_0$, we obtain

$$y(t) \;=\; \lambda\,x_0+\int_0^t \beta\,s^{\beta-1}\,f(s,x(\phi(s)))\,ds.$$

Substituting into (2.4), we get the result

$$x(t) \;=\; x_0+\lambda\,x_0\,t^{\gamma}+\int_0^t \gamma\,s^{\gamma-1}\int_0^s \beta\,\theta^{\beta-1}\,f(\theta,x(\phi(\theta)))\,d\theta\,ds-\lambda\int_0^t \gamma\,s^{\gamma-1}\,x(s)\,ds.$$

Conversely, we have

$$\begin{aligned} \frac{d}{dt}\,x(t) &= \lambda\,x_0\,\gamma\,t^{\gamma-1}+\gamma\,t^{\gamma-1}\int_0^t \beta\,\theta^{\beta-1}\,f(\theta,x(\phi(\theta)))\,d\theta-\lambda\,\gamma\,t^{\gamma-1}\,x(t),\\ \frac{t^{1-\gamma}}{\gamma}\,\frac{d}{dt}\,x(t) &= \lambda\,x_0+\int_0^t \beta\,\theta^{\beta-1}\,f(\theta,x(\phi(\theta)))\,d\theta-\lambda\,x(t) \end{aligned}$$

and

$$(D_\gamma+\lambda)\,x(t) \;=\; \lambda\,x_0+\int_0^t \beta\,\theta^{\beta-1}\,f(\theta,x(\phi(\theta)))\,d\theta.$$

Differentiating, we obtain

$$\frac{d}{dt}\,(D_\gamma+\lambda)\,x(t) \;=\; \beta\,t^{\beta-1}\,f(t,x(\phi(t)))$$

and

$$D_\beta\,(D_\gamma+\lambda)\,x(t) \;=\; f(t,x(\phi(t))).$$

□

Now, by the same way, the following two lemmas can be easily proved.

**Lemma 2.2.** *Problem (1.2) is equivalent to the fractal integral equation*

$$x(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \gamma\ \theta^{\gamma-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds. \quad (2.5)$$

**Lemma 2.3.** *Problem (1.3) is equivalent to the integral equation*

$$x(t) \ = \ x_0 + \lambda\ x_0\ t + \int_0^t \int_0^s f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t x(s)\ ds. \quad (2.6)$$

**Theorem 2.1.** *Let the assumptions (i) − (iv) be satisfied, then problem (1.1) has at least one continuous solution $x \in C[0,T]$.*

**Proof.** Define the set

$$Q_r \ = \ \{x \in C[0,T] : ||x||_c \leq r\}, \quad \text{where} \ \ r = \frac{|x_0| + \lambda\ |x_0|\ T + a\ \frac{T^{\gamma+\beta}}{\gamma+\beta}}{1 - (\lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta})}$$

and define the operator $F$ by

$$Fx(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds.$$

Now, let $x \in Q_r$, then

$$\begin{aligned}
|Fx(t)| &= |\ x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds\ | \\
&\leq |x_0| + \lambda\ |x_0|\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |f(\theta, x(\phi(\theta)))|\ d\theta\ ds + \lambda \int_0^t \gamma\ s^{\gamma-1}\ |x(s)|\ ds \\
&\leq |x_0| + \lambda\ |x_0|\ T^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |a(\theta)|\ d\theta\ ds \\
&\quad + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ b\ |x(\phi(\theta))|\ d\theta\ ds \ + \ \lambda\ ||x||_c \int_0^t \gamma\ s^{\gamma-1}\ ds \\
&\leq |x_0| + \lambda\ |x_0|\ T + a \int_0^t \gamma\ s^{\gamma-1}\ s^\beta\ ds + b\ ||x||_c \int_0^t \gamma\ s^{\gamma-1}\ s^\beta\ ds + \lambda\ ||x||_c\ t^\gamma \\
&\leq |x_0| + \lambda\ |x_0|\ T + a\ \gamma\ \frac{t^{\gamma+\beta}}{\gamma+\beta} + b\ ||x||_c\ \gamma\ \frac{t^{\gamma+\beta}}{\gamma+\beta} + \lambda\ ||x||_c\ T^\gamma \\
&\leq |x_0| + \lambda\ |x_0|\ T + a\ \frac{T^{\gamma+\beta}}{\gamma+\beta} + b\ r\ \frac{T^{\gamma+\beta}}{\gamma+\beta} + \lambda\ r\ T,
\end{aligned}$$

then

$$||Fx||_c \ \leq \ |x_0| + \lambda\ |x_0|\ T + a\ \frac{T^{\gamma+\beta}}{\gamma+\beta} + b\ r\ \frac{T^{\gamma+\beta}}{\gamma+\beta} + \lambda\ r\ T \ = \ r.$$

This proves that the operator $F$ maps $Q_r$ into itself and the class $\{Fx\}$ is uniformly bounded in $Q_r$.

Now, let $x \in Q_r$ and $|t_2 - t_1| < \delta$ such that $0 < t_1 < t_2 < T$, then

$$|Fx(t_2) - Fx(t_1)| = |\ x_0 + \lambda\ x_0\ t_2^\gamma + \int_0^{t_2} \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds$$

$$
\begin{aligned}
& -\lambda \int_0^{t_2} \gamma\ s^{\gamma-1}\ x(s)\ ds\ +\lambda \int_0^{t_1} \gamma\ s^{\gamma-1}\ x(s)\ ds \\
& -x_0 - \lambda\ x_0\ t_1^{\gamma} - \int_0^{t_1} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds\ | \\
\leq\ & \lambda\ |x_0|\ |t_2^{\gamma} - t_1^{\gamma}| + \int_{t_1}^{t_2} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ |f(\theta, x(\phi(\theta)))|\ d\theta\ ds \\
& +\lambda \int_{t_1}^{t_2} \gamma\ s^{\gamma-1}\ |x(s)|\ ds \\
\leq\ & \lambda\ |x_0|\ |t_2^{\gamma} - t_1^{\gamma}| + \int_{t_1}^{t_2} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ |a(\theta)|\ d\theta\ ds \\
& + \int_{t_1}^{t_2} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ b\ |x(\phi(\theta))|\ d\theta\ ds\ + \lambda\ ||x||_c \int_{t_1}^{t_2} \gamma\ s^{\gamma-1}\ ds \\
\leq\ & \lambda\ |x_0|\ |t_2^{\gamma} - t_1^{\gamma}| + a \int_{t_1}^{t_2} \gamma\ s^{\gamma-1}\ s^{\beta}\ ds + b\ ||x||_c \int_{t_1}^{t_2} \gamma\ s^{\gamma-1}\ s^{\beta}\ ds \\
& +\lambda\ ||x||_c \int_{t_1}^{t_2} \gamma\ s^{\gamma-1}\ ds \\
\leq\ & \lambda\ |x_0|\ |t_2^{\gamma} - t_1^{\gamma}| + a \int_{t_1}^{t_2} s^{\gamma+\beta-1}\ ds + b\ r \int_{t_1}^{t_2} s^{\gamma+\beta-1}\ ds\ +\ \lambda\ r \int_{t_1}^{t_2} s^{\gamma-1}\ ds,
\end{aligned}
$$

thus $\{Fx(t)\}$ is equi-continuous on $[0,T]$ and by Arzela-Ascoli Theorem [4] the class of function $\{Fx(t)\}$ is relatively compact [9]. Then the operator $F$ is compact [4, 12].

Let $\{x_n\} \subset Q_r$ such that $x_n \to x$, then

$$
Fx_n(t) \ = \ x_0 + \lambda\ x_0\ t^{\gamma} + \int_0^{t} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ f(\theta, x_n(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^{t} \gamma\ s^{\gamma-1}\ x_n(s)\ ds
$$

and

$$
\begin{aligned}
\lim_{n\to\infty} Fx_n(t) &= x_0 + \lambda\ x_0\ t^{\gamma} + \int_0^{t} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ \lim_{n\to\infty} f(\theta, x_n(\phi(\theta)))\ d\theta\ ds \\
&\quad -\lambda \int_0^{t} \gamma\ s^{\gamma-1}\ \lim_{n\to\infty} x_n(s)\ ds \\
&= x_0 + \lambda\ x_0\ t^{\gamma} + \int_0^{t} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ f(\theta, \lim_{n\to\infty} x_n(\phi(\theta)))\ d\theta\ ds \\
&\quad -\lambda \int_0^{t} \gamma\ s^{\gamma-1}\ x(s)\ ds \\
&= x_0 + \lambda\ x_0\ t^{\gamma} + \int_0^{t} \gamma\ s^{\gamma-1} \int_0^{s} \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^{t} \gamma\ s^{\gamma-1}\ x(s)\ ds \\
&= Fx(t),
\end{aligned}
$$

then $F$ is continuous [6].

By Schauder fixed point Theorem [8], there exist at least one solution $x \in Q_r \subset C[0,T]$ of the fractal integral equation (2.2) and we have $\frac{dx}{dt} \in L_1[0,T]$ which proves that $x \in AC[0,T]$.

Consequently, from the equivalent of problem (1.1) and equation (2.2), there exist at least one solution $x \in AC[0,T]$ of the problem (1.1). $\square$

## 3. Uniqueness of the solution

Now, consider the following assumption:
$(i)^*$ $f : [0,T] \times R \to R$ is measurable in $t \in [0,T]$ for every $x \in R$ and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq b\ |x-y|, \qquad b > 0 \tag{3.1}$$

and $f(t,0) = a(t)$ is a bounded measurable function.

**Remark 3.1.** From the assumption $(i)^*$, we obtain that $f$ satisfies the assumptions $(i)$ and $(ii)$.

**Theorem 3.1.** *Let the assumptions $(i)^*$, $(iii)$ and $(iv)$ be satisfied, then the solution $x \in C[0,T]$ of the problem (1.1) is unique.*

**Proof.** As the assumptions of Theorem 2.1 be satisfied, and the solution of the integral equation (2.2) exists. Let $x$ and $\bar{x}$ be two solutions of equation (2.2), then

$$\begin{aligned}
|x(t) - \bar{x}(t)| = &|\ x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds \\
&-x_0 - \lambda\ x_0\ t^\gamma - \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, \bar{x}(\phi(\theta)))\ d\theta\ ds \\
&+\lambda \int_0^t \gamma\ s^{\gamma-1}\ \bar{x}(s)\ ds\ | \\
\leq& \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |\ f(\theta, x(\phi(\theta))) - f(\theta, \bar{x}(\phi(\theta)))\ |\ d\theta\ ds \\
&+\lambda \int_0^t \gamma\ s^{\gamma-1}\ |\ \bar{x}(s) - x(s)\ |\ ds \\
\leq& \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ b\ |\ x(\phi(\theta)) - \bar{x}(\phi(\theta))\ |\ d\theta\ ds\ +\ \lambda\ ||\bar{x} - x||_c \int_0^t \gamma\ s^{\gamma-1}\ ds \\
\leq& b\ ||x - \bar{x}||_c \int_0^t \gamma\ s^{\gamma-1} s^\beta\ ds\ +\ \lambda\ ||\bar{x} - x||_c\ t^\gamma \\
\leq& b\ ||x - \bar{x}||_c\ \gamma\ \frac{t^{\gamma+\beta}}{\gamma+\beta}\ +\ \lambda\ T^\gamma\ ||\bar{x} - x||_c \\
\leq& b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ ||x - \bar{x}||_c\ +\ \lambda\ T\ ||\bar{x} - x||_c,
\end{aligned}$$

then

$$||x - \bar{x}||_c\ \leq\ (\ \lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ )\ ||x - \bar{x}||_c$$

and

$$[1 - (\ \lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ )]\ ||x - \bar{x}||_c\ \leq\ 1.$$

Since $(\ \lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ ) < 1$, then

$$||x - \bar{x}||_c = 0$$

which implies that $x = \bar{x}$ and the solution $x \in C[0,T]$ of the fractal integral equation (2.2) is unique. Consequently, the solution of problem (1.1) is unique. □

# 4. Continuous dependence

**Definition 4.1.** [4] The solution $x \in C[0,T]$ of problem (1.1) depends continuously on the parameter $\lambda$, the initial value $x_0$ and on the function $f$, if $\forall \epsilon > 0,\ \exists \delta > 0$ such that

$$\max\{\ |\lambda - \lambda^*|,\ \ |x_0 - x_0^*|,\ \ |f - f^*|\ \} < \delta$$

implies

$$||x - x^*||_c < \epsilon,$$

where $x^*$ is the unique solution of equation

$$x^*(t) = x_0^* + \lambda^*\ x_0^*\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f^*(\theta, x^*(\phi(\theta)))\ d\theta\ ds - \lambda^* \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds. \quad (4.1)$$

**Theorem 4.1.** *Let the assumptions of Theorem 3.1 be satisfied. Then the unique solution of problem (1.1) depends continuously on the parameter $\lambda$, the initial condition $x_0$ and on the function $f$.*

**Proof.** Let $x, x^*$ be two solutions of (2.2) and (4.1) respectively, then

$$\begin{aligned}
|x(t) - x^*(t)| = &\ |\ x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds \\
&-\lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds\ + \lambda^* \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds \\
&-x_0^* - \lambda^*\ x_0^*\ t^\gamma - \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f^*(\theta, x^*(\phi(\theta)))\ d\theta\ ds\ | \\
= &\ |\ x_0 - x_0^*\ +\ \lambda\ x_0\ t^\gamma - \lambda\ x_0^*\ t^\gamma\ +\ \lambda\ x_0^*\ t^\gamma - \lambda^*\ x_0^*\ t^\gamma \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds \\
&- \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x^*(\phi(\theta)))\ d\theta\ ds \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x^*(\phi(\theta)))\ d\theta\ ds \\
&- \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f^*(\theta, x^*(\phi(\theta)))\ d\theta\ ds \\
&-\lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds\ +\ \lambda \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds \\
&-\lambda \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds\ +\ \lambda^* \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds\ | \\
\leq &\ |x_0 - x_0^*|\ +\ \lambda\ |x_0 - x_0^*|\ t^\gamma\ +\ |\lambda - \lambda^*|\ x_0^*\ t^\gamma \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |\ f(\theta, x(\phi(\theta))) - f(\theta, x^*(\phi(\theta)))\ |\ d\theta\ ds \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |\ f(\theta, x^*(\phi(\theta))) - f^*(\theta, x^*(\phi(\theta)))\ |\ d\theta\ ds \\
&+\lambda \int_0^t \gamma\ s^{\gamma-1}\ |\ x^*(s) - x(s)\ |\ ds\ +\ |\ \lambda^* - \lambda\ | \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds\ |
\end{aligned}$$

$$\begin{aligned}
&\leq \delta \; + \lambda \; \delta \; t^{\gamma} \; + \delta \; x_0^* \; t^{\gamma} \; + \int_0^t \gamma \; s^{\gamma-1} \int_0^s \beta \; \theta^{\beta-1} \; b \; | \; x(\phi(\theta)) - x^*(\phi(\theta)) \; | \; d\theta \; ds \\
&\quad + \delta \int_0^t \gamma \; s^{\gamma-1} \int_0^s \beta \; \theta^{\beta-1} d\theta \; ds + \lambda \; ||x - x^*||_c \int_0^t \gamma \; s^{\gamma-1} ds + \delta \; ||x^*||_c \int_0^t \gamma \; s^{\gamma-1} ds \\
&\leq \delta \; + \; \delta \; \lambda \; T^{\gamma} \; + \; \delta \; x_0^* \; T^{\gamma} \; + \; b \; ||x - x^*||_c \int_0^t \gamma \; s^{\gamma-1} \; s^{\beta} \; ds \; + \; \delta \int_0^t \gamma \; s^{\gamma-1} \; s^{\beta} ds \\
&\quad + \lambda \; ||x - x^*||_c \; t^{\gamma} \; + \; \delta \; ||x^*||_c \; t^{\gamma} \\
&\leq \delta \; + \; \delta \; \lambda \; T \; + \; \delta \; x_0^* \; T \; + \; b \; ||x - x^*||_c \; \gamma \; \frac{t^{\gamma+\beta}}{\gamma+\beta} \; + \; \delta \; \gamma \; \frac{t^{\gamma+\beta}}{\gamma+\beta} \\
&\quad + \lambda \; ||x - x^*||_c \; T^{\gamma} \; + \; \delta \; ||x^*||_c \; T^{\gamma} \\
&\leq \delta \; ( \; 1 + \lambda \; T + x_0^* \; T \; ) \; + \; b \; \frac{T^{\gamma+\beta}}{\gamma+\beta} \; ||x - x^*||_c \; + \; \delta \; \frac{T^{\gamma+\beta}}{\gamma+\beta} \\
&\quad + \lambda \; T \; ||x - x^*||_c \; + \; \delta \; T \; ||x^*||_c,
\end{aligned}$$

then

$$||x - x^*||_c \; \leq \; \delta \; (1 + \lambda \; T + x_0^* \; T) \; + b \; \frac{T^{\gamma+\beta}}{\gamma+\beta} \; ||x - x^*||_c \; + \delta \; \frac{T^{\gamma+\beta}}{\gamma+\beta} \; + \lambda \; T \; ||x - x^*||_c \; + \delta \; T \; ||x^*||_c$$

and

$$[ \; 1 - (\lambda \; T + b \; \frac{T^{\gamma+\beta}}{\gamma+\beta}) \; ] \; ||x - x^*||_c \; \leq \; \delta \; (1 + \lambda \; T + x_0^* \; T) \; + \; \delta \; T \; ||x^*||_c \; + \; \delta \; \frac{T^{\gamma+\beta}}{\gamma+\beta},$$

thus

$$||x - x^*||_c \; \leq \; \frac{\delta \; (1 + \lambda \; T + x_0^* \; T) \; + \; \delta \; T \; ||x^*||_c \; + \; \delta \; \frac{T^{\gamma+\beta}}{\gamma+\beta}}{1 - (\lambda \; T + b \; \frac{T^{\gamma+\beta}}{\gamma+\beta})} \; = \; \epsilon.$$

□

**Definition 4.2.** The solution $x \in C[0, T]$ of problem (1.1) depends continuously on the delay function $\phi$, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$|\phi(t) - \phi^*(t)| < \delta, \qquad t \in [0, T]$$

implies

$$||x - x^*||_c < \epsilon,$$

where $x^*$ is the unique solution of equation

$$x^*(t) = x_0 + \lambda \; x_0 \; t^{\gamma} + \int_0^t \gamma \; s^{\gamma-1} \int_0^s \beta \; \theta^{\beta-1} \; f(\theta, x^*(\phi^*(\theta))) \; d\theta \; ds - \lambda \int_0^t \gamma \; s^{\gamma-1} \; x^*(s) \; ds. \quad (4.2)$$

**Theorem 4.2.** *Let the assumptions of Theorem 3.1 be satisfied. Then the unique solution of problem (1.1) depends continuously on the delay function $\phi$.*

**Proof.** Let $x, x^*$ be two solutions of (2.2) and (4.2) respectively, then

$$|x(t) - x^*(t)| = | \; x_0 + \lambda \; x_0 \; t^{\gamma} + \int_0^t \gamma \; s^{\gamma-1} \int_0^s \beta \; \theta^{\beta-1} \; f(\theta, x(\phi(\theta))) \; d\theta \; ds$$

$$
\begin{aligned}
&-\lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds\ +\ \lambda \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds \\
&-x_0 - \lambda\ x_0\ t^\gamma - \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x^*(\phi^*(\theta)))\ d\theta\ ds\ | \\
=|&\ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds\ -\ \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds \\
&- \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi^*(\theta)))\ d\theta\ ds \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x(\phi^*(\theta)))\ d\theta\ ds \\
&- \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ f(\theta, x^*(\phi^*(\theta)))\ d\theta\ ds\ +\ \lambda \int_0^t \gamma\ s^{\gamma-1}\ x^*(s)\ ds\ | \\
\leq & \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |\ f(\theta, x(\phi(\theta))) - f(\theta, x(\phi^*(\theta)))\ |\ d\theta\ ds \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ |\ f(\theta, x(\phi^*(\theta))) - f(\theta, x^*(\phi^*(\theta)))\ |\ d\theta\ ds \\
&+\lambda \int_0^t \gamma\ s^{\gamma-1}\ |x^*(s) - x(s)|\ ds \\
\leq & \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ b\ |\ x(\phi(\theta)) - x(\phi^*(\theta))\ |\ d\theta\ ds \\
&+ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ b\ |\ x(\phi^*(\theta)) - x^*(\phi^*(\theta))\ |\ d\theta\ ds \\
&+\lambda\ ||x^* - x||_c \int_0^t \gamma\ s^{\gamma-1}\ ds \\
\leq &\ b\ \epsilon_\delta \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ d\theta\ ds\ +\ b\ ||x - x^*||_c\ \int_0^t \gamma\ s^{\gamma-1} \int_0^s \beta\ \theta^{\beta-1}\ d\theta\ ds \\
&+\lambda\ ||x^* - x||_c \int_0^t \gamma\ s^{\gamma-1}\ ds \\
\leq &\ b\ \epsilon_\delta \int_0^t \gamma\ s^{\gamma-1}\ s^\beta ds\ + b\ ||x - x^*||_c\ \int_0^t \gamma\ s^{\gamma-1}\ s^\beta ds\ + \lambda\ ||x^* - x||_c \int_0^t \gamma\ s^{\gamma-1} ds \\
\leq &\ b\ \epsilon_\delta\ \gamma\ \frac{t^{\gamma+\beta}}{\gamma+\beta}\ +\ b\ ||x - x^*||_c\ \gamma\ \frac{t^{\gamma+\beta}}{\gamma+\beta}\ +\ \lambda\ ||x^* - x||_c\ t^\gamma \\
\leq &\ b\ \epsilon_\delta\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ +\ b\ ||x - x^*||_c\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ +\ \lambda\ ||x^* - x||_c\ T^\gamma \\
\leq &\ b\ \epsilon_\delta\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ +\ b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ ||x - x^*||_c\ +\ \lambda\ T\ ||x - x^*||_c,
\end{aligned}
$$

then

$$||x - x^*||_c\ \leq\ b\ \epsilon_\delta\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ +\ b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ ||x - x^*||_c\ +\ \lambda\ T\ ||x - x^*||_c$$

and

$$[\ 1 - (\lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta})\ ]\ ||x - x^*||_c\ \leq\ b\ \epsilon_\delta\ \frac{T^{\gamma+\beta}}{\gamma+\beta},$$

thus

$$||x - x^*||_c \;\leq\; \frac{b\ \epsilon_\delta\ \frac{T^{\gamma+\beta}}{\gamma+\beta}}{1 - (\lambda\ T + b\ \frac{T^{\gamma+\beta}}{\gamma+\beta})} \;=\; \epsilon.$$

□

## 5. Hyers-Ulam stability

**Definition 5.1.** [11,15,20,21] Let the solution $x \in C[0,T]$ of the problem (1.1) be exists, then problem (1.1) is Hyers-Ulam stable, if $\forall \epsilon > 0, \exists \delta > 0$ such that for any $\delta$-approximate solution $x_s$ of problem (1.1) satisfying

$$|\ D_\beta\ (D_\gamma + \lambda)\ x_s(t)\ -\ f(t, x_s(\phi(t)))\ | < \delta,$$

implies

$$||x - x_s||_c < \epsilon.$$

**Theorem 5.1.** *Let the assumptions of Theorem 3.1 be satisfied. Then problem (1.1) is Hyers-Ulam stable.*

**Proof.** We have

$$-\,\delta\ <\ D_\beta\ (D_\gamma + \lambda)\ x_s(t)\ -\ f(t, x_s(\phi(t)))\ <\ \delta,$$
$$-\,\delta\ <\ \frac{t^{1-\beta}}{\beta}\ \frac{d}{dt}\ (D_\gamma + \lambda)\ x_s(t)\ -\ f(t, x_s(\phi(t)))\ <\ \delta.$$

Multiplying by $\beta\ t^{\beta-1}$, we obtain

$$-\delta\ \beta\ t^{\beta-1}\ <\ \frac{d}{dt}\ (D_\gamma + \lambda)\ x_s(t)\ -\ \beta\ t^{\beta-1}\ f(t, x_s(\phi(t)))\ <\ \delta\ \beta\ t^{\beta-1}.$$

Integrating and using $x_s(0) = x(0)$, then we get

$$-\,\delta\ t^\beta\ <\ (D_\gamma + \lambda)\ x_s(t) - \lambda\ x(0) - \int_0^t \beta\ \theta^{\beta-1}\ f(\theta, x_s(\phi(\theta)))\ d\theta\ <\ \delta\ t^\beta,$$
$$-\,\delta\ t^\beta\ <\ \frac{t^{1-\gamma}}{\gamma}\ \frac{dx_s(t)}{dt} + \lambda\ x_s(t) - \lambda\ x_0 - \int_0^t \beta\ \theta^{\beta-1}\ f(\theta, x_s(\phi(\theta)))\ d\theta\ <\ \delta\ t^\beta.$$

Multiplying by $\gamma\ t^{\gamma-1}$, we obtain

$$-\delta\ \gamma\ t^{\beta+\gamma-1}\ <\ A\ <\ \delta\ \gamma\ t^{\beta+\gamma-1},$$

where

$$A\ =\ \frac{dx_s(t)}{dt}\ +\ \lambda\ \gamma\ t^{\gamma-1}\ x_s(t)\ -\ \lambda\ x_0\ \gamma\ t^{\gamma-1}\ -\ \gamma\ t^{\gamma-1} \int_0^t \beta\ \theta^{\beta-1}\ f(\theta, x_s(\phi(\theta)))\ d\theta.$$

Integrating, we get

$$|\ x_s(t)\ -\ x_0\ +\ \lambda \int_0^t \gamma\ \theta^{\gamma-1}\ x_s(\theta)\ d\theta\ -\ \lambda\ x_0\ t^\gamma$$

$$
\begin{aligned}
&- \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ f(\tau, x_s(\phi(\tau)))\ d\tau\ d\theta\ | \\
&< \delta\ \frac{\gamma}{\beta+\gamma}\ t^{\beta+\gamma} \\
&< \delta\ \frac{T^{\beta+\gamma}}{\beta+\gamma}.
\end{aligned}
$$

Let $\delta\ \frac{T^{\beta+\gamma}}{\beta+\gamma} = \delta^*$, then we obtain that

$$
|\ x_s(t) - x_0 - \lambda\, x_0\, t^\gamma - \int_0^t \gamma\, \theta^{\gamma-1} \int_0^\theta \beta\, \tau^{\beta-1}\, f(\tau, x_s(\phi(\tau)))\, d\tau\, d\theta + \lambda \int_0^t \gamma\, \theta^{\gamma-1}\, x_s(\theta)\, d\theta\ |\ <\ \delta^*.
$$

Now,

$$
\begin{aligned}
|x(t) - x_s(t)| &= |\ x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ f(\tau, x(\phi(\tau)))\ d\tau\ d\theta \\
&\quad -\lambda \int_0^t \gamma\ \theta^{\gamma-1}\ x(\theta)\ d\theta\ -\ x_s(t)\ | \\
&= |\ x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ f(\tau, x(\phi(\tau)))\ d\tau\ d\theta \\
&\quad -\lambda \int_0^t \gamma\ \theta^{\gamma-1}\ x(\theta)\ d\theta\ +\ \lambda \int_0^t \gamma\ \theta^{\gamma-1}\ x_s(\theta)\ d\theta \\
&\quad -x_0 - \lambda\ x_0\ t^\gamma - \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ f(\tau, x_s(\phi(\tau)))\ d\tau\ d\theta \\
&\quad +x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ f(\tau, x_s(\phi(\tau)))\ d\tau\ d\theta \\
&\quad -\lambda \int_0^t \gamma\ \theta^{\gamma-1}\ x_s(\theta)\ d\theta\ -\ x_s(t)\ | \\
&\le \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ |\ f(\tau, x(\phi(\tau))) - f(\tau, x_s(\phi(\tau)))\ |\ d\tau\ d\theta \\
&\quad +\lambda \int_0^t \gamma\ \theta^{\gamma-1}\ |x_s(\theta) - x(\theta)|\ d\theta \\
&\quad +|\ x_s(t) - x_0 - \lambda\ x_0\ t^\gamma - \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ f(\tau, x_s(\phi(\tau)))\ d\tau\ d\theta \\
&\quad +\lambda \int_0^t \gamma\ \theta^{\gamma-1}\ x_s(\theta)\ d\theta\ | \\
&\le \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ b\ |x(\phi(\tau)) - x_s(\phi(\tau))|\ d\tau\ d\theta \\
&\quad +\lambda\ ||x_s - x||_c \int_0^t \gamma\ \theta^{\gamma-1}\ d\theta\ +\ \delta^* \\
&\le b\ ||x - x_s||_c \int_0^t \gamma\ \theta^{\gamma-1} \int_0^\theta \beta\ \tau^{\beta-1}\ d\tau\ d\theta\ +\ \lambda\ ||x_s - x||_c\ t^\gamma\ +\ \delta^* \\
&\le b\ ||x - x_s||_c \int_0^t \gamma\ \theta^{\gamma-1}\ \theta^\beta\ d\theta\ +\ \lambda\ T^\gamma\ ||x_s - x||_c\ +\ \delta^*
\end{aligned}
$$

$$\leq b\ \gamma\ \frac{t^{\gamma+\beta}}{\gamma+\beta}\ ||x-x_s||_c\ +\ \lambda\ T\ ||x-x_s||_c\ +\ \delta^*$$
$$\leq b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ ||x-x_s||_c\ +\ \lambda\ T\ ||x-x_s||_c\ +\ \delta^*,$$

then

$$||x-x_s||_c\ \leq\ (\ \lambda\ T+b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ )\ ||x-x_s||_c\ +\ \delta^*$$

and

$$[1-(\ \lambda\ T+b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ )]\ ||x-x_s||_c\ \leq\ \delta^*,$$

thus

$$||x-x_s||_c\ \leq\ \frac{\delta^*}{1-(\ \lambda\ T+b\ \frac{T^{\gamma+\beta}}{\gamma+\beta}\ )}\ =\ \epsilon.$$

□

## 6. Continuation to classical problem

**Theorem 6.1.** *Let the assumptions of Theorem 3.1 be satisfied. Then problem (1.1) is continue to the problem (1.2) as* $\beta \to \gamma$.

**Proof.** Let $x \in AC[0,T]$ be the solution of problem (1.1), then

$$\lim_{\beta\to\gamma} D_\beta\ (D_\gamma+\lambda)\ x(t) = f(t, x(\phi(t))),$$
$$\lim_{\beta\to\gamma} \frac{t^{1-\beta}}{\beta}\ \frac{d}{dt}\ (D_\gamma+\lambda)\ x(t) = f(t, x(\phi(t))),$$

then

$$D_\gamma\ (D_\gamma+\lambda)\ x(t)\ =\ f(t, x(\phi(t))).$$

And from (2.2), we get

$$\lim_{\beta\to\gamma} x(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \lim_{\beta\to\gamma}\ \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds,$$
$$x(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \gamma\ \theta^{\gamma-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds$$

which is the solution of problem (1.2). □

**Theorem 6.2.** *Let the assumptions of Theorem 3.1 be satisfied. Then problem (1.1) is continue to the problem (1.3) as* $\beta, \gamma \to 1$.

**Proof.** Let $x \in AC[0,T]$ be the solution of problem (1.1), then

$$D_\beta\ (D_\gamma+\lambda)\ x(t) = f(t, x(\phi(t))),$$
$$\lim_{\beta\to 1} \frac{t^{1-\beta}}{\beta}\ \frac{d}{dt}\ (D_\gamma+\lambda)\ x(t) = f(t, x(\phi(t))),$$
$$\frac{d}{dt}\ (D_\gamma+\lambda)\ x(t) = f(t, x(\phi(t)))$$

integrating both side, we get

$$(D_\gamma + \lambda)\ x(t) - \lambda\ x_0 = \int_0^t f(s, x(\phi(s)))\ ds,$$
$$\frac{t^{1-\gamma}}{\gamma}\ \frac{dx(t)}{dt} + \lambda\ x(t) = \lambda\ x_0 + \int_0^t f(s, x(\phi(s)))\ ds,$$
$$\lim_{\gamma\to 1}\ \frac{t^{1-\gamma}}{\gamma}\ \frac{dx(t)}{dt} + \lambda\ x(t) = \lambda\ x_0 + \int_0^t f(s, x(\phi(s)))\ ds,$$

then

$$(\frac{d}{dt} + \lambda)\ x(t)\ =\ \lambda\ x_0 + \int_0^t f(s, x(\phi(s)))\ ds$$

differentiating both side, we get

$$\frac{d}{dt}\ (\frac{d}{dt} + \lambda)\ x(t)\ =\ f(t, x(\phi(t))).$$

And from (2.2), we get

$$\lim_{\beta\to 1} x(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s \lim_{\beta\to 1}\ \beta\ \theta^{\beta-1}\ f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds,$$
$$x(t) = x_0 + \lambda\ x_0\ t^\gamma + \int_0^t \gamma\ s^{\gamma-1} \int_0^s f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t \gamma\ s^{\gamma-1}\ x(s)\ ds,$$
$$\lim_{\gamma\to 1} x(t) = x_0 + \lambda\ x_0\ \lim_{\gamma\to 1}\ t^\gamma + \int_0^t \lim_{\gamma\to 1}\ \gamma\ s^{\gamma-1} \int_0^s f(\theta, x(\phi(\theta)))\ d\theta\ ds$$
$$-\lambda \int_0^t \lim_{\gamma\to 1}\ \gamma\ s^{\gamma-1}\ x(s)\ ds,$$
$$x(t) = x_0 + \lambda\ x_0\ t + \int_0^t \int_0^s f(\theta, x(\phi(\theta)))\ d\theta\ ds - \lambda \int_0^t x(s)\ ds$$

which is the solution of problem (1.3). □

## 7. Example

Here, we present a simple example that illustrates the applicability of our results.

Take $T = 1$, then we have

$$D_\beta\ (D_\gamma + \lambda)\ x(t)\ =\ f(t, x(\phi(t))), \quad t \in (0, 1], \quad x(0) = x_0. \tag{7.1}$$

For the numerical illustration, we choose the following parameters: $\beta = 0.7,\ \gamma = 0.8,\ \lambda = 0.5$ and let $x_0 = 2$.

Also, we choose the function $f$ as:

$$f(t, x) = 0.1\ \sin(t) + 0.3\ x,$$

where $|a(t)| = |f(t, 0)| = 0.1\ |\sin(t)| \leq 0.1 = a$ and $b = 0.3$, and the delay function $\phi$ as: $\phi(t) = \frac{1}{2}\ t,$ where $\phi(t) \leq t$, then we get

$$f(t, x(\phi(t)))\ =\ 0.1\ \sin(t)\ +\ 0.3\ x(\frac{t}{2}).$$

Now, all the selected parameters and functions fulfill the assumed conditions.

Consequently, the numerical example takes the following form:

$$D_\beta\ (D_\gamma + 0.5)\ x(t)\ =\ 0.1\ \sin(t)\ +\ 0.3\ x(\frac{t}{2}),\quad t \in (0,1],\quad x(0) = 2.$$

## 8. Conclusions

This paper addressed the analysis of a non-linear fractal Langevin equation involving two fractal orders. We demonstrated the existence and uniqueness of solutions, and showed that this solution depends continuously on the parameter $\lambda$, the initial data $x_0$, and the functions $f, \phi$. Furthermore, we established the Hyers–Ulam stability of the problem, confirming the reliability of the model under small perturbations. In addition, we proved the continuation of the problem. Finally, a numerical example was presented to illustrate the applicability of the assumed conditions and to demonstrate the validity of the obtained results.

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