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TRANSFORMED L1-ADI AND COMPACT ADI DIFFERENCE SCHEMES FOR TWO-DIMENSIONAL TIME-FRACTIONAL SUB-DIFFUSION EQUATIONS

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Abstract This paper proposes two novel alternating direction implicit (ADI) difference schemes for solving two-dimensional time-fractional sub-diffusion equations, taking into account the weak initial singularity of the solutions. To accurately capture the rapid evolution near the initial time, the Caputo time-fractional derivative is discretized using a transformed L1 method. Furthermore, central and compact finite difference methods are employed in spatial discretization to enhance computational efficiency. By using a discrete Grönwall inequality, the error estimates in H^1-norm of the two schemes are obtained. Numerical examples are presented to illustrate the effectiveness of the proposed schemes.

Keywords Time-fractional sub-diffusion equation, alternating direction implicit, compact ADI scheme, error estimation.

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1. Introduction

In the last few decades, fractional partial differential equations (PDEs) have gained considerable attention due to their applications in various fields of science and engineering, such as physics, medicine and biology; see, [4, 25, 26, 31, 32, 37, 42]. Since analytical solutions to fractional PDEs are often unavailable [8, 30], the development of accurate and efficient numerical methods is of great importance. Consequently, this field has attracted significant research attention. In this paper, we consider the effective numerical solution of the following two-dimensional time-fractional sub-diffusion equation

$$\partial_t^\alpha u(x,y,t) - \varepsilon\Delta u(x,y,t) = f(x,y,t), \qquad (x,y,t)\in\Omega\times(0,T], \tag{1.1}$$

$$u(x,y,0) = \phi(x,y), \qquad (x,y)\in\Omega, \tag{1.2}$$

$$u(x,y,t) = 0, \qquad (x,y)\in\partial\Omega,\ t\in(0,T], \tag{1.3}$$

where $\varepsilon > 0$ is a given constant, $\Omega = (0,l)\times(0,l)$ is a finite rectangular domain, $\partial\Omega$ denotes the boundary of Ω, f, ϕ, u are continuous functions and $u(x,y,0) = \phi(x,y)$. The Caputo fractional derivative $\partial_t^\alpha u$ is defined as

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,r)}{\partial r}\frac{1}{(t-r)^\alpha}dr, \quad 0<\alpha<1,$$

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here $\Gamma(\cdot)$ is the Gamma function.

There has been growing interest in studying effective numerical methods for solving the time-fractional sub-diffusions (1.1)-(1.3) in recent years [2, 10, 18, 27, 45, 52, 54]. On one hand, for linear problems, Sun [36] proposed the L1 difference scheme to solve the diffusion-wave equation and the slow diffusion equation when the fractional derivative order $\alpha(0 < \alpha < 1)$. They proved that the scheme is solvable, unconditionally stable and the convergence order is $\mathcal{O}(\tau^{3-\alpha} + h^2)$. Liu and Xu [20] established a method based on the L1 scheme in time and the Legendre spectral methods in space, for solving the time-fractional diffusion equation when $\alpha \in (0, 1)$. Yang and Zhang [43] used a new nonlinear finite-volume scheme to calculate the two-dimensional sub-diffusion equation on distorted meshes. On the other hand, for nonlinear problems, Jin [9] presented a general framework to analyze discretization errors, and verified it for the L1 scheme and convolution quadrature generated by backward difference formulas. Since the L1 scheme does not hold for any homogeneous equations with smooth data, they [11] also revisited the error estimates of the method, and established an $\mathcal{O}(\tau)$ convergence rate for both smooth and non-smooth initial data. Li et al. [15] proposed a linearized Galerkin finite element method to solve multi-dimensional fractional reaction-subdiffusion equations. They established unconditionally optimal error estimates in L^∞-norm. Pandey et al. [28] applied an operational matrix for fractional order differentiation and converted the two-dimensional time fractional order reaction–diffusion equation into a system of algebraic equations. Mohammad and Saadaoui [24] presented a new definition of fractional derivative based on a kernel function and established its application and theoretical results.

It is well-known that the solutions of the sub-diffusion equations are usually singular near the initial time $t = 0$, even for a smooth setting, see [12, 19, 35]. Therefore, the numerical discretization on the uniform mesh may lead to poor accuracy. Zhang et al. [49] constructed a compact difference method by approximating the Caputo derivative with non-uniform meshes, and the stability and H^1-norm convergence were proved. Wang et al. [39] analyzed an α-robust H^1-norm convergence based on the non-uniform meshes in time, employing an improved discrete fractional Grönwall inequality. However, both methods cannot avoid the singularity when $t \to 0$. Shi and Yang [34] proposed a time two-grid algorithm for solving the nonlinear generalized viscous Burgers' equation. Mohammad [23] utilized tight wavelet frames to present a framework to solve fractal-type fractional Riccate differential equations. For more numerical methods to solve the weak singularity at the initial time, we refer to [13, 38, 40, 44].

Recently, Li et al. [14, 33] used a change of variable $s = t^\alpha$ and then solved the s-fractional PDEs, by which they could avoid the singularity of the solution at the initial time. Qin et al. [29] proposed a transformed L1 scheme to avoid fast decay of the initial energy which possess an $\mathcal{O}(\tau^{2-\alpha})$ convergence rate. Later, Li et al. [16] conducted a linearized transformed L1 finite difference method for nonlinear coupled time-fractional Schrödinger equation and illustrated that the scheme was effective for small α. Although the transformed L1 schemes can deal with the singularity of the solution at initial time for time-fractional sub-diffusion equations, their implementations could be very expensive for multidimensional problems.

Due to their efficiency in solving multidimensional problems, the ADI techniques have been widely introduced to solve fractional diffusion equations [1, 3, 17, 21, 51]. There are also some works on the ADI methods for multidimensional fractional sub-diffusion equations. For example, Zhang and Sun [48] established two ADI schemes, in which the L1 approximation and backward Euler method are considered for the time stepping. Fairweather et al. [7] combined orthogonal spline collocation for the spatial discretization with ADI Crank-Nicolson method for the temporal integration to solve the fractional diffusion-wave equation. Zhou et al. [53] presented a non-uniform Crank-Nicolson ADI fast scheme to calculate three-dimensional nonlocal evolution equation in viscoelastic dynamics. Lyu and Vong [22] presented a weighted ADI scheme with the variable time steps method for solving two-dimensional diffusion-wave equations. They also study the stability and convergence of the scheme.

To further improve the accuracy in space, compact difference schemes have been widely developed for fractional PDEs [47]. For example, Du et al. [6] presented a compact difference scheme to solve the fractional diffusion-wave equation when $\alpha \in (1,2)$ and analyzed its unconditional stability and convergence in L^∞-norm. Based on the idea of weighted and shifted Grünwald difference operator, Wang and Vong [41] established a compact finite difference scheme for solving an equation with Riemann-Liouville fractional integral operator. For fractional sub-diffusion equations with Neumann boundary conditions, Cheng et al. [5] developed a compact ADI scheme and proved its convergence under the assumption of the weak singularity of solutions. For three-dimensional time-fractional equations, Zhai [46] proposed an implicit compact scheme by using fourth-order Padé approximation for spatial derivatives and the center difference approximation for the time derivative.

The main goal of this paper is to construct effective and efficient numerical schemes, which can overcome the difficulties from the weak initial singularity of solutions of two-dimensional time-fractional sub-diffusion equations (1.1)-(1.3). We proposed two fully discrete schemes by using the transformed L1 approximation for the time integration with an ADI discretization and a compact ADI discretization in space. The new schemes combining the advantages of the transformed L1 scheme and the ADI technique, can avoid the singularity of the solution at an initial time, and allow for fast implementations. By using a new discrete Grönwall inequality, the error estimates in H^1-norm of the proposed schemes are established in the fully discrete sense, which are $\min\{2\alpha, 2-\alpha\}$ order in time and second (fourth) order in space for the ADI (compact ADI) scheme.

The key to ADI scheme and compact ADI scheme is to add the small term $\frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2 D_\tau^\alpha V_{i,j}^n$, where $V_{ij}^n = v(x_i,\ y_j,\ s_n)$. The added term makes sense since that after the change of variable, $D_\tau^\alpha V_{i,j}^n$ is bounded due to the obtained estimates (2.4). Moreover, the present convergence results are obtained by a discrete fractional inequality. Therefore, our results can be extended to the following semi-linear time-fractional problem

$$\partial_t^\alpha u(x,y,t) - \varepsilon\Delta u(x,y,t) = f(u),$$

if the nonlinear term is approximated by using some extrapolated methods. It should be pointed out that in previous references, there are also some ADI schemes, which are constructed under the assumptions of smooth solutions. In our opinion, smooth solutions are not typical cases of time fractional problems. And the related convergence results are usually obtained by the maximum principle. The way of the proof can not be extended to semi-linear problems.

The main contributions of the paper are as follows:

- A new ADI scheme is developed, taking the initial weak singularity of the solutions into consideration.
- The scheme is of order 4 in the spatial direction.
- The numerical scheme is very effective and can be easy to implement in actual applications.
- The stability and optimal error estimates of the schemes are presented.

The rest of the paper is organized as follows. In Section 2, we introduce the transformed L1-ADI and transformed L1 compact ADI schemes for the equation (1.1)-(1.3). In Section 3, the

H^1-norm convergence analysis of the transformed L1-ADI scheme is presented. The convergence analysis of the transformed L1 compact ADI scheme is established in Section 4. In Section 5, we present numerical experiments to validate our theoretical results. Finally, some conclusions are given in Section 6.

2. The numerical schemes

In this section, we will present two different numerical schemes for solving sub-diffusion equations. Let N, M_1 and M_2 be positive integers, we take the uniform time step size $\tau_s = T^\alpha/N(0 \leq \alpha \leq 1)$, $h_1 = l/M_1$ and $h_2 = l/M_2$ be spatial mesh sizes in x-direction and y-direction, respectively. Therefore, the space mesh is defined as $\Omega_h = \{(x_i, y_j)|\ x_i = ih_1,\ 0 \leq i \leq M_1,\ y_j = jh_2,\ 0 \leq j \leq M_2\}$ and the time mesh is defined as $\Omega_\tau = \{s_n|\ s_n = n\tau_s,\ 0 \leq n \leq N\}$. For any grid function $w = \{w_{ij}^n|\ 0 \leq i \leq M_1,\ 0 \leq j \leq M_2,\ 0 \leq n \leq N\}$, we denote

$$
\begin{aligned}
&w_{i+\frac{1}{2},j}^n = \frac{1}{2}(w_{i+1,j}^n + w_{i,j}^n), \quad w_{i,j+\frac{1}{2}}^n = \frac{1}{2}(w_{i,j+1}^n + w_{i,j}^n),\\
&\delta_x w_{i+\frac{1}{2},j}^n = \frac{1}{h_1}(w_{i+1,j}^n - w_{i,j}^n), \quad \delta_x^2 w_{i,j}^n = \frac{1}{h_1}(\delta_x w_{i+\frac{1}{2},j}^n - \delta_x w_{i-\frac{1}{2},j}^n),\\
&\mathcal{H}_x w_{i,j}^n = \begin{cases} \frac{1}{12}\left(w_{i-1,j}^n + 10w_{i,j}^n + w_{i+1,j}^n\right), & 1 \leq i \leq M_1 - 1,\ 0 \leq j \leq M_2,\\ w_{i,j}^n, & i = 0 \ \text{ or } \ M_1, \quad 0 \leq j \leq M_2. \end{cases}
\end{aligned}
$$

The difference operators $\delta_y w_{i,j+\frac{1}{2}}^n$, $\delta_y^2 w_{i,j}^n$ and $\mathcal{H}_y w_{i,j}^n$ are defined similarly.

To give out a transformed L1 approximation scheme, we start by transforming the sub-diffusion equation (1.1)-(1.3) into an equivalent form. By using the changes of variable $t = s^{1/\alpha}$ and denoting $v(x, y, s) = u(x, y, s^{1/\alpha})$, we have

$$
\begin{aligned}
\partial_t^\alpha u &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,y,r)}{\partial r} \frac{1}{(t-r)^\alpha} dr\\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^s \frac{\partial v(x,y,r)}{\partial r} \frac{1}{(s^{1/\alpha} - r^{1/\alpha})^\alpha} dr\\
&:= \partial_s^\alpha v.
\end{aligned}
$$

The equation (1.1)-(1.3) is then transformed into a problem solving for $v(x, y, s)$ such that

$$
\begin{aligned}
&\partial_s^\alpha v(x,y,s) = \varepsilon \Delta v + f(x,y,s^{1/\alpha}), && (x,y,s) \in \ \Omega \times (0, T^\alpha],\\
&v(x,y,0) = \phi(x,y), && (x,y) \in \ \Omega,\\
&v(x,y,s) = 0, && (x,y) \in \ \partial\Omega, s \in [0, T^\alpha].
\end{aligned}
\tag{2.1}
$$

We now apply the L1 scheme to the fractional differential operator in (2.1) and arrive at [29]

$$
\begin{aligned}
\partial_s^\alpha v(x,y,s_n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{s_n} \frac{\partial v(x,y,r)}{\partial r} \frac{1}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha} dr\\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^n \frac{v(x,y,s_k) - v(x,y,s_{k-1})}{\tau_s} \int_{s_{k-1}}^{s_k} \frac{dr}{(s_n^{1/\alpha} - r^{1/\alpha})^\alpha} + Q^n\\
&= \sum_{k=1}^n a_{n,n-k}(v(x,y,s_k) - v(x,y,s_{k-1})) + Q^n,
\end{aligned}
\tag{2.2}
$$

where we have approximate $\frac{\partial v(x,y,r)}{\partial r}$ within the integral by $\frac{v(x,y,s_k)-v(x,y,s_{k-1})}{\tau_s}$, Q^n is the truncation error, $a_{n,n-k}$ is defined as

$$\begin{aligned} a_{n,n-k} &= \frac{1}{\tau_s\Gamma(1-\alpha)}\int_{s_{k-1}}^{s_k}\frac{dr}{(s_n^{1/\alpha}-r^{1/\alpha})^\alpha} \\ &= \frac{\alpha}{\tau_s\Gamma(1-\alpha)}\int_{s_{k-1}^{1/\alpha}}^{s_k^{1/\alpha}}\frac{dt}{(s_n^{1/\alpha}-t)^\alpha t^{1-\alpha}}dt \\ &= \frac{\alpha}{\tau_s\Gamma(1-\alpha)}\int_{s_{k-1}^{1/\alpha}/s_n^{1/\alpha}}^{s_k^{1/\alpha}/s_n^{1/\alpha}}\frac{ds}{(1-s)^\alpha s^{1-\alpha}} \\ &= \frac{\alpha}{\tau_s\Gamma(1-\alpha)}(B(s_k^{1/\alpha}/s_n^{1/\alpha},\alpha,1-\alpha)-B(s_{k-1}^{1/\alpha}/s_n^{1/\alpha},\alpha,1-\alpha)). \end{aligned}$$

Here we change the variable $r=t^\alpha$ and $s=t\cdot s_n^{1/\alpha}$ in the second and third equations.

Omitting the truction error, we can obtain

$$\begin{aligned} D_\tau^\alpha w^n :=& \sum_{k=1}^n a_{n,n-k}(v^k-v^{k-1}) \\ =& a_{n,0}w^n+\sum_{k=1}^{n-1}(a_{n,n-k}-a_{n,n-k-1})w^k-a_{n,n-1}w^0. \end{aligned} \tag{2.3}$$

As pointed out in [14, 29], problem (2.2) admits a unique solution, satisfying

$$\|\frac{\partial^l v}{\partial s^l}(x,y,s)\| \le C_0(1+s^{1/\alpha+1-l}), \quad \text{for } l=1,2, \tag{2.4}$$

where $\|\cdot\|$ denotes the usual L^2-norm.

2.1. A transformed L1 fully discrete ADI scheme

Considering (2.1) at the mesh point $(x_i,\ y_j,\ s_n)$, it holds that

$$D_s^\alpha v(x_i,\ y_j,\ s_n) = \varepsilon\Delta v(x_i,\ y_j,\ s_n)+f(x_i,\ y_j,\ s_n). \tag{2.5}$$

Applying the center difference scheme in the spatial direction, we have

$$\partial_s^\alpha v(x_i,\ y_j,\ s_n) = \varepsilon\delta_x^2 v(x_i,\ y_j,\ s_n)+\varepsilon\delta_y^2 v(x_i,\ y_j,\ s_n)+f(x_i,\ y_j,\ s_n)+R_{1,ij}^n,$$

where $|R_{1,ij}^n|\le C_1(h_1^2+h_2^2)$, $(x_i,\ y_j,\ s_n)\in\Omega_h\times\Omega_\tau$ with a positive constance C_1 independent of $h_1,\ h_2$ and τ_s.

Let $V_{ij}^n=v(x_i,y_j,s_n)$ and $f_{ij}^n=f(x_i,y_j,s_n^{1/\alpha})$ represent the discrete approximations of v and f at the grid point (x_i,y_j) and time s_n. Applying the scheme (2.2) yields

$$\partial_\tau^\alpha V_{i,j}^n = \varepsilon\delta_x^2 V_{i,j}^n+\varepsilon\delta_y^2 V_{i,j}^n+f_{i,j}^n+R_{1,ij}^n-Q_{ij}^n. \tag{2.6}$$

Adding $\frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\partial_\tau^\alpha V_{i,j}^n$ on both sides of the equation (2.6) obtains

$$\partial_\tau^\alpha V_{i,j}^n+\frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\partial_\tau^\alpha V_{i,j}^n = \varepsilon\delta_x^2 V_{i,j}^n+\varepsilon\delta_y^2 V_{i,j}^n+f_{i,j}^n+R_{2,ij}^n, \tag{2.7}$$

where the truncation error is

$$R_{2,ij}^n = R_{1,ij}^n - Q_{ij}^n + \frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\partial_\tau^\alpha V_{i,j}^n. \tag{2.8}$$

Replacing the grid function $V_{i,j}^n$ by the numerical solutions $v_{i,j}^n$ and ignoring the error terms, we obtain the discrete scheme

$$\left(\mathcal{I} + \frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\right) D_\tau^\alpha v_{i,j}^n - \varepsilon\Delta_h v_{i,j}^n = f_{i,j}^n, \tag{2.9}$$

where $\Delta_h = \delta_x^2 + \delta_y^2$, and $\mathcal{I}$ is the unit operator, which satisfies $\mathcal{I}v = v$. Using (2.3), we can rewrite the discrete scheme (2.9) as

$$\left(\mathcal{I} - r_n\delta_x^2\right)\left(\mathcal{I} - r_n\delta_y^2\right) v_{i,j}^n = \frac{1}{a_{n,0}}\left\{ f_{i,j}^n + \left(I + r_n^2\delta_x^2\delta_y^2\right)\left[\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})v_{i,j}^k + a_{n,n-1}v_{i,j}^0\right]\right\}, \tag{2.10}$$

where $r_n = \frac{\varepsilon}{a_{n,0}}$ and $f_{i,j}^n = f(x_i, y_j, s_n^{1/\alpha})$. The initial and boundary conditions are given as

$$v_{i,j}^0 = \phi(x_i, y_j), \qquad 0 \le i \le M_1, \quad 0 \le j \le M_2, \tag{2.11}$$

$$v_{i,j}^n = 0, \qquad (x_i, y_j) \text{ on } \partial\Omega, \quad 0 \le n \le N. \tag{2.12}$$

In order to solve the discrete system (2.10)-(2.12), we define intermediate variables $v_{i,j}^* = (\mathcal{I} - r_n\delta_y^2)v_{i,j}^n$. The numerical scheme (2.10)-(2.12) then takes two steps. First, we solve

$$\begin{cases} a_{n,0}\left(\mathcal{I} - r_n\delta_x^2\right) v_{i,j}^* = f_{i,j}^n + \left(\mathcal{I} + r_n^2\delta_x^2\delta_y^2\right)\left[\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})v_{i,j}^k + a_{n,n-1}v_{i,j}^0\right], \\ v_{0,j}^* = \left(\mathcal{I} - r_n\delta_y^2\right) v_{0,j}^n, v_{M_1,j}^* = (\mathcal{I} - r_n\delta_y^2)v_{M_1,j}^n, \end{cases} \tag{2.13}$$

for $j = 1, 2, ..., M_2 - 1$. Then we solve

$$\begin{cases} \left(\mathcal{I} - r_n\delta_y^2\right) u_{i,j}^n = v_{i,j}^*, \ 1 \le j \le M_2 - 1, \\ v_{i,0}^n = \mu(x_i, y_0, s_n^{1/\alpha}), \ v_{i,M_2}^n = \mu(x_i, y_{M_2}, s_n^{1/\alpha}), \end{cases} \tag{2.14}$$

for $i = 1, 2, ..., M_1 - 1$. Since the coefficient matrices of the systems in both (2.13) and (2.14) are strictly diagonally dominant, we conclude that the fully discrete scheme (2.10)-(2.12) is uniquely solvable.

2.2. A fully discrete compact ADI scheme

Applying $\mathcal{H}_x\mathcal{H}_y$ to both side of (2.5) and using Lemma 2.3 in [50], we have

$$\mathcal{H}_x\mathcal{H}_y\partial_s^\alpha v(x_i,\ y_j,\ s_n) = \mathcal{H}_y\delta_x^2 v(x_i,\ y_j,\ s_n) + \mathcal{H}_x\delta_y^2 v(x_i,\ y_j,\ s_n) + \mathcal{H}_x\mathcal{H}_y f(x_i\ y_j,\ s_n) + R_{3,ij}^n,$$

where $|R_{3,ij}^n| \le C_2(h_1^4 + h_2^4)$ with a positive constant C_2 independent of $h_1,\ h_2$ and τ_s.

Let $V_{ij}^n = v(x_i,\ y_j,\ s_n)$ and $f_{ij}^n = f(v(x_i,\ y_j,\ s_n^{1/\alpha}))$. By using (2.2), we obtain

$$\mathcal{H}_x\mathcal{H}_y\partial_s^\alpha V_{ij}^n = \mathcal{H}_y\delta_x^2 V_{ij}^n + \mathcal{H}_x\delta_y^2 V_{ij}^n + \mathcal{H}_x\mathcal{H}_y f_{ij}^n + R_{3,ij}^n - \mathcal{H}_x\mathcal{H}_y Q_{ij}^n. \tag{2.15}$$

Submitting $\frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\partial_\tau^\alpha V_{i,j}^n$ in both sides of the equation (2.15), we have

$$\mathcal{H}_x\mathcal{H}_y\partial_\tau^\alpha V_{ij}^n + \frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\partial_\tau^\alpha V_{i,j}^n = \mathcal{H}_y\delta_x^2 V_{ij}^n + \mathcal{H}_x\delta_y^2 V_{ij}^n + \mathcal{H}_x\mathcal{H}_y f_{ij}^n + R_{4,ij}^n, \tag{2.16}$$

where $R_{4,ij}^n = R_{3,ij}^n - \mathcal{H}_x\mathcal{H}_y Q_{ij}^n + \frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2\partial_\tau^\alpha V_{i,j}^n$.

Replacing the grid function $V_{i,j}^n$ by the numerical solutions $v_{i,j}^n$ and ignoring the error terms, we obtain the discrete scheme

$$\mathcal{H}_x\mathcal{H}_y D_\tau^\alpha v_{ij}^n + \frac{\varepsilon^2}{a_{n,0}^2}\delta_x^2\delta_y^2 D_\tau^\alpha v_{i,j}^n = \mathcal{H}_y\delta_x^2 v_{ij}^n + \mathcal{H}_x\delta_y^2 v_{ij}^n + \mathcal{H}_x\mathcal{H}_y f_{ij}^n. \tag{2.17}$$

By (2.3), (2.17) can be rewritten as

$$\begin{aligned} & a_{n,0}\left(\mathcal{H}_x - r_n\delta_x^2\right)\left(\mathcal{H}_y - r_n\delta_y^2\right) v_{i,j}^n \\ = & \mathcal{H} f_{i,j}^n + \left(\mathcal{H} + r_n^2\delta_x^2\delta_y^2\right)\left[\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})v_{i,j}^k + a_{n,n-1}v_{i,j}^0\right], \end{aligned} \tag{2.18}$$

where $\mathcal{H} = \mathcal{H}_x\mathcal{H}_y$. Setting $v_{i,j}^* = (\mathcal{H}_y - r_n\delta_y^2)v_{ij}^n$, (2.18) is then transformed into solving the following systems

$$\begin{cases} a_{n,0}\left(\mathcal{H}_x - r_n\delta_x^2\right)v_{i,j}^* = \mathcal{H}_x\mathcal{H}_y f_{i,j}^n + \left(\mathcal{H}_x\mathcal{H}_y + r_n^2\delta_x^2\delta_y^2\right)\left[\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})v_{i,j}^k + a_{n,n-1}v_{i,j}^0\right], \\ \left(\mathcal{H}_y - r_n\delta_y^2\right)v_{i,j}^n = v_{i,j}^*, \quad 1 \le i \le M_1 - 1,\ 1 \le j \le M_2 - 1, \end{cases} \tag{2.19}$$

with initial and boundary conditions (2.11)-(2.12).

3. H^1-norm convergence analysis of ADI scheme

In this section, we present the fully discrete error analysis of the transformed L1-ADI scheme in H^1-norm. We first present some definitions that will be used in the rest of the paper. We define the space of grid functions on Ω_h,

$$V_h = \{v | v = \{v_{i,j} | (x_i, y_j) \in \Omega_h\} \ \text{ and } v_{i,j} = 0 \text{ if } \ (x_i, y_j) \in \partial\Omega_h.$$

For grid functions $w, v \in V_h$, we define the discrete inner product

$$\langle w, v\rangle_h = h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} w_{i,j}\cdot v_{i,j},$$

and denote $\|v\| = \sqrt{\langle v, v\rangle_h}$. Notations $\|\delta_x^2 v\|$, $\|\delta_y^2 v\|$, $\|\mathcal{H}_x v\|$, $\|\mathcal{H}_y v\|$ and $\|\mathcal{H} v\|$ are defined similarly. We also define

$$\|\delta_x v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |\delta_x v_{i-\frac{1}{2},j}|^2},$$

$$\|\delta_y \delta_x^2 v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |\delta_y \delta_x^2 v_{i,j-\frac{1}{2}}|^2},$$

$$\|\Delta_h v\| = \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |\Delta_h v_{i,j}|^2},$$

and $\|\delta_y v\|$, $\|\delta_x \delta_y^2 v\|$ are similarly defined. The following Sobolev norms are also needed:

$$\begin{aligned} \|\nabla_h v\| &= \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}, \\ \|v\|_{H^1} &= \sqrt{\|v\|^2 + \|\nabla_h v\|^2}, \\ \|v^n\|_A &= \sqrt{\|\nabla_h v^n\|^2 + r_n^2(\|\delta_x \delta_y^2 v^n\|^2 + \|\delta_y \delta_x^2 v^n\|^2)}. \end{aligned} \tag{3.1}$$

Moreover, we give the equivalence of the norm $\|\cdot\|_A$.

Theorem 3.1. *The norm $\|u\|_A$ exists if $u \in V_h$.*

Proof. First, we prove the positive definite. Since $\|\cdot\|^2 \geq 0$ and $r_n^2 \geq 0$, we can have $\|\cdot\|_A^2 \geq 0$. If $\|v\|_A = 0$, we can get $\|\nabla_h v\| = 0$, $\|\delta_x \delta_y^2 v^n\| = 0$, $\|\delta_y \delta_x^2 v^n\| = 0$. Then, we can have $v = \text{const}$. Since v satisfies $v_{i,j} = 0$ if $(x_i, y_j) \in \partial\Omega_h$, we can get $v = 0$.

Then, we prove the homogeneity. Let $v = \lambda u$, we have

$$\begin{aligned} \|\lambda u\|_A &= \sqrt{\|\nabla_h \lambda u^n\|^2 + r_n^2(\|\delta_x \delta_y^2 \lambda u^n\|^2 + \|\delta_y \delta_x^2 \lambda u^n\|^2)} \\ &= \sqrt{\lambda^2 \|\nabla_h u^n\|^2 + r_n^2 \lambda^2 (\|\delta_x \delta_y^2 u^n\|^2 + \|\delta_y \delta_x^2 u^n\|^2)} \\ &= |\lambda| \|u\|_A. \end{aligned}$$

Finally, we prove the triangle inequality. We now prove $\|u+v\|_A^2 \leq (\|u\|_A + \|v\|_A)^2$, for $\forall u, v \in V_h$.

By using the definition of the discrete inner product and the holder inequality, we can get

$$\begin{aligned} \|w + v\|^2 &= h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (w_{i,j} + v_{i,j})^2 \\ &\leq h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (w_{i,j}^2 + 2|w_{i,j}||v_{i,j}^2| + v_{i,j}^2) \\ &= \|w\|^2 + \|v\|^2 + 2\sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |w_{i,j}|^2} \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} |v_{i,j}|^2} \\ &= \|w\|^2 + 2\|w\|\|v\| + \|v\|^2 \\ &= (\|w\| + \|v\|)^2. \end{aligned} \tag{3.2}$$

By using (3.2), we have

$$
\begin{aligned}
\|u+v\|_A^2 =&\|\nabla_h(v+u)\|^2 + r_n^2\|\delta_x\delta_y^2(u+v)\|^2 + r_n^2\|\delta_y\delta_x^2(u+v)\|^2\\
\leq&(\|\nabla_h u\| + \|\nabla_h v\|)^2 + r_n^2(\|\delta_x\delta_y^2 u\| + \|\delta_x\delta_y^2 v\|)^2 + r_n^2(\|\delta_y\delta_x^2 u\| + \|\delta_y\delta_x^2 v\|)^2\\
=&\|u\|_A^2 + \|v\|_A^2 + 2(\|\nabla_h u\|\|\nabla_h v\| + r_n^2\|\delta_x\delta_y^2 u\|\|\delta_x\delta_y^2 v\| + r_n^2\|\delta_y\delta_x^2 u\|\|\delta_y\delta_x^2 v\|)\\
\leq&\|u\|_A^2 + \|v\|_A^2 + 2\sqrt{\|\nabla_h u\|^2 + r_n^2\|\delta_x\delta_y^2 u\| + r_n^2\|\delta_y\delta_x^2 u\|}\\
&\times\sqrt{\|\nabla_h v\|^2 + r_n^2\|\delta_x\delta_y^2 v\| + r_n^2\|\delta_y\delta_x^2 v\|}\\
=&\|u\|_A^2 + \|v\|_A^2 + 2\|u\|_A\|v\|_A\\
=&(\|u\|_A + \|v\|_A)^2.
\end{aligned}
$$

Thus, we end the proof. □

We present below some properties of the coefficients $a_{n,n-k}$ useful to the analysis later.

Lemma 3.1. ([29], Lemma 3.1) *For $n \geq 1$, it holds that*

$$0 \leq a_{n,n-1} \leq a_{n,n-2} \leq \ldots \leq a_{n,0}.$$

Lemma 3.2. *For $n \geq 2$, we have*

$$a_{n,0} \geq \frac{\alpha N}{T^\alpha \Gamma(2-\alpha) n^{1-\alpha}}.$$

Proof. Since

$$a_{n,n-j} = \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \int_{t_{j-1}/t_n}^{t_j/t_n} \frac{dz}{(1-z)^\alpha z^{1-\alpha}},$$

we see that

$$
\begin{aligned}
a_{n,0} &= \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \int_{t_{n-1}/t_n}^{1} \frac{dz}{(1-z)^\alpha z^{1-\alpha}}\\
&= \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \int_{(\frac{n-1}{n})^{1/\alpha}}^{1} \frac{dz}{(1-z)^\alpha z^{1-\alpha}}\\
&\geq \frac{\alpha}{\tau_s \Gamma(1-\alpha)} \int_{(\frac{n-1}{n})^{1/\alpha}}^{1} \frac{dz}{(1-z)^\alpha}\\
&= \frac{\alpha}{\tau_s \Gamma(2-\alpha)} \left(1 - \left(\frac{n-1}{n}\right)^{1/\alpha}\right)^{1-\alpha}\\
&\geq \frac{\alpha}{\tau_s \Gamma(2-\alpha)} \left(1 - \frac{n-1}{n}\right)^{1-\alpha}\\
&= \frac{\alpha N}{T^\alpha \Gamma(2-\alpha) n^{1-\alpha}}.
\end{aligned}
$$

□

For grid functions, the following lemma reflects the relationship between their inner product and their A-norm.

Lemma 3.3. *For any grid functions $u^n, v^n \in V_h$, we have*

$$|\langle u^n + r_n^2 \delta_x^2 \delta_y^2 u^n, \Delta_h v^n \rangle_h| \le \|u^n\|_A \|v^n\|_A, \tag{3.3}$$

where (3.3) *holds when $u^n = v^n$.*

Proof. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\langle u^n + r_n^2 \delta_x^2 \delta_y^2 u^n, \delta_x^2 v^n \rangle_h| &\le |\langle \delta_x u^n, \delta_x v^n \rangle_h| + r_n^2 |\langle \delta_y \delta_x^2 u^n, \delta_y \delta_x^2 v^n \rangle_h| \\ &\le \|\delta_x u^n\| \|\delta_x v^n\| + r_n^2 \|\delta_y \delta_x^2 u^n\| \|\delta_y \delta_x^2 v^n\|.\end{aligned}$$

A similar proof gives

$$|\langle u^n + r_n^2 \delta_x^2 \delta_y^2 u^n, \delta_y^2 v^n \rangle_h| \le \|\delta_y u^n\| \|\delta_y v^n\| + r_n^2 \|\delta_x \delta_y^2 u^n\| \|\delta_x \delta_y^2 v^n\|.$$

It then follows from the Cauchy-Schwartz inequality that

$$\begin{aligned}|\langle u^n + r_n^2 \delta_x^2 \delta_y^2 u^n, \Delta_h v^n \rangle_h &\le |\langle u^n + r_n^2 \delta_x^2 \delta_y^2 u^n, \delta_x^2 v^n \rangle_h| + |\langle u^n + r_n^2 \delta_x^2 \delta_y^2 u^n, \delta_y^2 v^n \rangle_h| \\ &\le \|\delta_x u^n\| \, \|\delta_x v^n\| + r_n^2 \left\|\delta_y \delta_x^2 u^n\right\| \left\|\delta_y \delta_x^2 v^n\right\| \\ &\quad + \|\delta_y u^n\| \, \|\delta_y v^n\| + r_n^2 \left\|\delta_x \delta_y^2 u^n\right\| \left\|\delta_x \delta_y^2 v^n\right\| \\ &\le \left(\|\nabla_h u^n\|^2 + r_n^2 \left(\left\|\delta_x \delta_y^2 u^n\right\|^2 + \left\|\delta_y \delta_x^2 u^n\right\|^2\right)\right)^{\frac{1}{2}} \\ &\quad \times \left(\|\nabla_h v^n\|^2 + r_n^2 \left(\left\|\delta_x \delta_y^2 v^n\right\|^2 + \left\|\delta_y \delta_x^2 v^n\right\|^2\right)\right)^{\frac{1}{2}} \\ &= \|u^n\|_A \, \|v^n\|_A \, .\end{aligned}$$

We complete the proof. □

The following Grönwall inequalities, taken from Theorem 3.1 in [29], are the key ingredients in proving the stability and the convergence of the proposed schemes under the H^1-norm.

Lemma 3.4. (Discrete fractional Grönwall inequality). *For any finite time $s_N = T^\alpha > 0$ and a given non-negative sequence $\{\lambda_i\}_{i=0}^{N-1}$, assume that there exists a positive constant λ^*, independent of the time step τ_s, such that $\lambda^* \ge \sum_{i=1}^{N-1} \lambda_i$. Suppose that the grid functions $\{\omega^n | n \ge 0\}$ satisfy*

$$D_\tau^\alpha (\omega^n)^2 \le \sum_{i=1}^{n} \lambda_{n-i} \left(\omega^i\right)^2 + \omega^n \left(Q^n + \zeta\right), \quad n \ge 1,$$

where $\{Q^n |\ n \ge 1\}$ are as defined in (2.2). *Then, there exist positive constants C_1^* and τ_s^* such that, when $\tau_s \le \tau_s^*$,*

$$\omega^j \le 2E_\alpha \left(2\lambda^* s_j\right) \left[\omega^0 + C_1^* \left(\tau_s^{2-\alpha} + \zeta\right)\right], \quad 1 \le j \le N,$$

where $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function.

Now, we are in the position to present the following convergence result of the transformed L1-ADI scheme. We first present the stability of the numerical scheme.

Theorem 3.2. *Let* $\{v_{i,j}^n \mid (x_i, y_j) \in \Omega_h, 1 \le n \le N\}$ *be the solution of the difference scheme* (2.10)-(2.12)*. Then, there exist positive constants* C_1^* *and* τ_1 *such that, when* $\tau_s \le \tau_1$*, it holds that*

$$\|v^n\|_A^2 \le \|v^0\|_A^2 + C_1^* \max_{1\le l\le N} \|f^l\|^2, \quad 1 \le n \le N,$$

where v_0 *means the solution at the initial time and* ε *is a constant.*

Proof. Taking the inner product of (2.7) with $-\frac{2}{a_{n,0}} h_1 h_2 \Delta_h v_{ij}^n$ and summing i, j from 1 to $M_1 - 1$ and 1 to $M_2 - 1$, we have

$$\begin{aligned}
&-2h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (v_{ij}^n + r_n^2 \delta_x^2 \delta_y^2 v_{ij}^n) \Delta_h v_{ij}^n + 2\frac{\varepsilon}{a_{n,0}} \|\Delta_h v^n\|^2 \\
=&\sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \left(-2h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (v_{ij}^k + r_n^2 \delta_x^2 \delta_y^2 v_{ij}^k) \Delta_h v_{ij}^n \right) \\
&+2\frac{a_{n,n-1}}{a_{n,0}} h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} (v_{ij}^0 + r_n^2 \delta_x^2 \delta_y^2 v_{ij}^0) \Delta_h v_{ij}^n - \frac{2}{a_{n,0}} h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} f_{ij}^n \Delta_h v_{ij}^n.
\end{aligned}$$

Applying Lemma 3.3, we have

$$\begin{aligned}
2\|v^n\|_A^2 + 2\frac{\varepsilon}{a_{n,0}} \|\Delta_h v^n\|^2 \le& \sum_{k=1}^{n-1} 2\frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} \|v^k\|_A \|v^n\|_A + 2\frac{a_{n,n-1}}{a_{n,0}} \|v^0\|_A \|v^n\|_A \\
&-\frac{2}{a_{n,0}} h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} f_{ij}^n \Delta_h v_{ij}^n.
\end{aligned}$$

By using Cauchy-Schwartz inequality, we can get

$$\begin{aligned}
2\|v^n\|_A^2 + 2\frac{\varepsilon}{a_{n,0}} \|\Delta_h v^n\|^2 \le& \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} (\|v^k\|_A^2 + \|v^n\|_A^2) + \frac{a_{n,n-1}}{a_{n,0}} (\|v^0\|_A^2 + \|v^n\|_A^2) \\
&-\frac{2}{a_{n,0}} h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} f_{ij}^n \Delta_h v_{ij}^n \\
\le& \sum_{k=1}^{n-1} \frac{a_{n,n-k-1} - a_{n,n-k}}{a_{n,0}} (\|v^k\|_A^2 + \|v^n\|_A^2) + \frac{a_{n,n-1}}{a_{n,0}} (\|v^0\|_A^2 + \|v^n\|_A^2) \\
&+\frac{2\varepsilon}{a_{n,0}} \|\Delta_h v^n\|^2 + \frac{1}{2\varepsilon a_{n,0}} \|f^n\|^2,
\end{aligned}$$

where the following inequality is used

$$-\frac{2}{a_{n,0}} h_1h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} f_{ij}^n \Delta_h v_{ij}^n \le \frac{2\varepsilon}{a_{n,0}} \|\Delta_h v^n\|^2 + \frac{1}{2\varepsilon a_{n,0}} \|f^n\|^2.$$

After simply computing, we can obtain

$$2a_{n,0}\|v^n\|_A^2 \le \sum_{k=1}^{n-1} (a_{n,n-k-1} - a_{n,n-k})(\|v^k\|_A^2 + \|v^n\|_A^2) + a_{n,n-1}(\|v^0\|_A^2 + \|v^n\|_A^2) + \frac{1}{2\varepsilon} \|f^n\|^2.$$

Together with the following equality

$$\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})+a_{n,n-1}=a_{n,0},$$

it further implies

$$a_{n,0}\|v^n\|_A^2\le\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})\|v^k\|_A^2+a_{n,n-1}\|v^0\|_A^2+\frac{1}{2\varepsilon}\|f^n\|^2.$$

Based on the definition of the discrete fractional operator (2.3), we can get

$$D_\tau^\alpha\|v^n\|_A^2\le\frac{1}{2\varepsilon}\|f^n\|^2. \tag{3.4}$$

Applying the discrete fractional Grönwall inequality, there exists positive constants C_1^* and τ_1 such that when $\tau_s\le\tau_1$ (3.4) further implies that

$$\|v^n\|_A^2\le\|v^0\|_A^2+C_1^*\max_{1\le l\le N}\|f^l\|^2.$$

Thus, the proof is finished. □

Next, the analysis of the convergence results is given.

Theorem 3.3. *Assume that the problem* (2.1) *has a unique solution satisfies* (2.4) *and the exact solution $V(x,y,t)$ is smooth in the domain $\Omega\times[0,T^\alpha]$. Let $\{v_{i,j}^n\mid(x_i,y_j)\in\Omega_h,1\le n\le N\}$ be the solution of the difference scheme* (2.10)-(2.12). *Then there exist constants C and τ_2 such that, when $\tau_s\le\tau_2$, the error $e^n=V^n-v^n$ satisfies*

$$\|e^n\|_{H^1}\le C(\tau_s^{\min\{2\alpha,2-\alpha\}}+h_1^2+h_2^2),\quad 1\le n\le N,$$

where C is independent of τ_s, h_1 and h_2.

Proof. Subtracting (2.9) from (2.7) and considering the equation (2.8), we can show

$$(I+r_n^2\delta_x^2\delta_y^2)D_\tau^\alpha e_{ij}^n-\varepsilon\Delta_h e_{ij}^n=R_{2,ij}^n. \tag{3.5}$$

Rewrite $R_{2,ij}^n$ as

$$R_{2,ij}^n=R_{1,ij}^n-(\mathcal{I}+r_n^2\delta_x^2\delta_y^2)Q_{ij}^n+r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V_{ij}^n.$$

Thus, (3.5) can be written as

$$(I+r_n^2\delta_x^2\delta_y^2)D_\tau^\alpha e_{ij}^n-\varepsilon\Delta_h e_{ij}^n=R_{1,ij}^n-(\mathcal{I}+r_n^2\delta_x^2\delta_y^2)Q_{ij}^n+r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V_{ij}^n. \tag{3.6}$$

Taking the inner product of (3.6) with $-\Delta_h e^n$, we have

$$\begin{aligned}&-a_{n,0}\langle(I+r_n^2\delta_x^2\delta_y^2)e^n,\Delta_h e^n\rangle_h+\varepsilon\|\Delta_h e^n\|^2\\&=-a_{n,n-1}\langle(I+r_n^2\delta_x^2\delta_y^2)e^0,\Delta_h e^n\rangle_h-\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})\langle(I+r_n^2\delta_x^2\delta_y^2)e^k,\Delta_h e^n\rangle_h\\&\quad+\langle(I+r_n^2\delta_x^2\delta_y^2)Q^n,\Delta_h e^n\rangle_h-\langle(R_1^n+r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V^n),\Delta_h e^n\rangle_h.\end{aligned}$$

From Lemma 3.1, we know that

$$a_{n,n-k-1}-a_{n,n-k}>0 \text{ and } a_{n,n-1}\geq 0.$$

It then follows from Lemma 3.3 and partial integral formula that

$$\begin{aligned}a_{n,0}\|e^n\|_A^2\leq a_{n,n-1}\|e^0\|_A\|e^n\|_A+\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})\|e^k\|_A\|e^n\|_A\\ +\|Q^n\|_A\|e^n\|_A+\langle\nabla_h(R_1^n+r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V^n),\nabla_h e^n\rangle_h.\end{aligned}$$

Using Cauchy-Schwartz inequality, we have

$$\begin{aligned}a_{n,0}\|e^n\|_A^2\leq a_{n,n-1}\|e^0\|_A^2+\frac{1}{4}a_{n,n-1}\|e^n\|_A^2+\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})\|e^k\|_A^2\\ +\frac{1}{4}\sum_{k=1}^{n-1}(a_{n,n-k-1}-a_{n,n-k})\|e^n\|_A^2+\|Q^n\|_A\|e^n\|_A+\|\nabla_h R_1^n\|\|e^n\|_A\\ +\|\nabla_h r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V^n\|\|e^n\|_A.\end{aligned}$$

It then follows that

$$D_\tau^\alpha\|e^n\|_A^2\leq\frac{1}{4}a_{n,0}\|e^n\|_A^2+\|Q^n\|_A\|e^n\|_A+\|\nabla_h R_1^n\|\|e^n\|_A+\|\nabla_h r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V^n\|\|e^n\|_A. \tag{3.7}$$

Furthermore, for the last two term in (3.7). Since $R1_{i,j}^n$ can be written as

$$R_{1,ij}^n=-\frac{h_1^2}{12}\frac{\partial^4 V_{ij}^n}{\partial x^4}-\frac{h_1^4}{360}\frac{\partial^6 V_{ij}^n}{\partial x^6}-\frac{h_2^2}{12}\frac{\partial^4 V_{ij}^n}{\partial y^4}-\frac{h_2^4}{360}\frac{\partial^6 V_{i,j}^n}{\partial y^6}+\mathcal{O}(h_1^6+h_2^6).$$

Then, we have

$$\begin{aligned}\delta_x R_{1,ij}^n&=\frac{1}{h_1}\left[R_{1,i+\frac{1}{2}\ j}^n-R_{i,i-\frac{1}{2}\ j}^n\right]\\ &=\frac{1}{h_1}\left[h_1\frac{\partial R_{1,ij}^n}{\partial x}+\mathcal{O}(h_1^3)\right]\\ &=-\frac{h_1^2}{12}\frac{\partial^5 V_{ij}^n}{\partial x^5}-\frac{h_2^2}{12}\frac{\partial^5 V_{ij}^n}{\partial x\partial y^4}+\mathcal{O}(h_1^4+h_1^2+h_2^4)\\ &=\mathcal{O}(h_1^2+h_2^2).\end{aligned}$$

Similarly, we can obtain $\delta_y R_{1,ij}^n=\mathcal{O}(h_1^2+h_2^2)$. Since $\|\nabla_h R_1\|=\sqrt{\|\delta_x R_1\|^2+\|\delta_y R_1\|^2}$, we have

$$\|\nabla_h R_1\|\leq C_3(h_1^2+h_2^2). \tag{3.8}$$

From Lemma 3.2, we have

$$\begin{aligned}\frac{1}{a_{n,0}}&\leq\frac{T^\alpha\Gamma(2-\alpha)n^{1-\alpha}}{\alpha N}\\ &=\frac{\Gamma(2-\alpha)T^\alpha}{\alpha}\cdot\frac{n^{1-\alpha}}{N}\end{aligned}$$

$$= \frac{\Gamma(2-\alpha)T^\alpha}{\alpha} \cdot \frac{n}{N} \cdot \frac{1}{n^\alpha}.$$

Since n is a positive integer, $0 < \alpha < 1$ and $n \le N$, we can get $\frac{n}{N} \le 1$, $\frac{1}{n^\alpha} < 1$. Thus, the above equation can be calculated as

$$\frac{1}{a_{n,0}} \le \frac{\gamma(2-\alpha)T^\alpha}{\alpha}.$$

From the definition of τ_s, we obtain that $O(\tau_s) = O(N^{-1})$. Note that, when $\bar{C}$ is a constant independent of N, $1/a_{n,0} \le \bar{C}\tau_s$ when n is far from N and $1/a_{n,0} \le C\tau_s^\alpha$ when n is nearby N. Therefore, for $1 \le n \le N$, we can obtain that $1/a_{n,0} \le C\tau_s^\alpha$. According to the definition of r_n and (2.4), we have

$$\|\nabla_h r_n^2 \delta_x^2 \delta_y^2 \partial_s^\alpha V^n\| = r_n^2 \|\nabla_h \delta_x^2 \delta_y^2 \partial_s^\alpha V^n\| \le C_4 \tau_s^{2\alpha}. \tag{3.9}$$

From (3.8) and (3.9), we find

$$D_\tau^\alpha \|e^n\|_A^2 \le \frac{1}{4} a_{n,0} \|e^n\|_A^2 + \left(\|Q^n\|_A + C_4\tau_s^{2\alpha} + C_3(h_1^2+h_2^2)\right) \|e^n\|_A.$$

Applying Lemma 4, there exists a constant τ_2, when $\tau_s \le \tau_2$, we have

$$\|e^n\|_A \le 2E_\alpha(2\lambda^* r_n) \left[\|e^0\|_A + C^*\left(\tau_s^{\min\{2-\alpha,2\alpha\}} + h_1^2 + h_2^2\right)\right],$$

where $C^* = \max\{C_3,\ C_4\}$. Since $\|\cdot\|_{H^1} \le \|\cdot\|_A$ and $\|e^0\|_A = 0$, and consequently,

$$\|e^n\|_{H^1} \le 2E_\alpha(2\lambda^* r_n) C^* \left(\tau_s^{\min\{2-\alpha,2\alpha\}} + h_1^2 + h_2^2\right).$$

This completes the proof. □

4. Convergence analysis of compact ADI scheme

We shall now derive an error estimate for the compact ADI scheme based on H^1-norm. For the reader's convenience, we let $\mathcal{H} = \mathcal{H}_x\mathcal{H}_y$ and $\Lambda_h = \mathcal{H}_y\delta_x^2 + \mathcal{H}_x\delta_y^2$, and we introduce two norms [50]

$$\|v^n\|_{\mathcal{H}_x} = \sqrt{\|\delta_x v\|^2 - \frac{h_1^2}{12}\|\delta_x^2 v\|^2}, \quad \|v^n\|_{\mathcal{H}_y} = \sqrt{\|\delta_y v\|^2 - \frac{h_2^2}{12}\|\delta_y^2 v\|^2}.$$

Since $\frac{2}{3}\|\delta_x v\|^2 \le \|v\|_{\mathcal{H}_x}^2 \le \|\delta_x v\|^2$, the two norms defined above are reasonable. The following results on the the inner product and the A-norm will be useful later.

Lemma 4.1. *For any grid function $v^n \in V_h$, it holds that*

$$\begin{aligned} -\langle \mathcal{H}_x v^n, \delta_x^2 v^n\rangle_h &= \|v^n\|_{\mathcal{H}_x}^2, \\ -\langle \mathcal{H}_y v^n, \delta_y^2 v^n\rangle_h &= \|v^n\|_{\mathcal{H}_y}^2. \end{aligned} \tag{4.1}$$

Proof. It follows from the definition of the operator $\mathcal{H}_x$ that

$$-\langle \mathcal{H}_x v^n, \delta_x^2 v^n\rangle = -\langle v^n, \delta_x^2 v^n\rangle_h - \frac{h_1^2}{12}\langle \delta_x^2 v^n, \delta_x^2 v^n\rangle_h$$

$$
\begin{aligned}
&= \langle \delta_x v^n, \delta_x v^n \rangle_h - \frac{h_1^2}{12} \langle \delta_x^2 v^n, \delta_x^2 v^n \rangle_h \\
&= \|\delta_x v^n\|^2 - \frac{h_1^2}{12} \|\delta_x^2 v^n\|^2 \\
&= \|v^n\|_{\mathcal{H}_x}^2 .
\end{aligned}
$$

The second equality (4.1) can be obtained similarly. □

Lemma 4.2. *For any $u^n, v^n \in V_h$, it holds that*

$$
\langle (\mathcal{H} + r_n^2 \delta_x^2 \delta_y^2) u^n, \Lambda_h v^n \rangle_h | \leq \frac{64}{27} \|u\|_A \|v\|_A .
$$

Proof. Similar to the proof of Lemma 3.3, by the definition of the operator Λ_h, we have

$$
|\langle \mathcal{H} u^n,\ \Lambda_h v^n \rangle_h| \leq |\langle \mathcal{H} u^n,\ \mathcal{H}_y \delta_x^2 v^n \rangle_h| + |\langle \mathcal{H} u^n,\ \mathcal{H}_x \delta_y^2 v^n \rangle_h| .
$$

Then the definitions of operators $\mathcal{H}$ and Λ_h lead to

$$
\begin{aligned}
|\langle \mathcal{H} u^n\ ,\ \mathcal{H}_y \delta_x^2 v^n \rangle_h| &= |\langle u^n + \frac{h_1^2}{12} \delta_x^2 u^n + \frac{h_2^2}{12} \delta_y^2 u^n + \frac{h_1^2 h_2^2}{144} \delta_x^2 \delta_y^2 u^n,\ \delta_x^2 v^n + \frac{h_2^2}{12} \delta_x^2 \delta_y^2 v^n \rangle_h| \\
&\leq |\langle u^n + \frac{h_1^2}{12} \delta_x^2 u^n + \frac{h_2^2}{12} \delta_y^2 u^n + \frac{h_1^2 h_2^2}{144} \delta_x^2 \delta_y^2 u^n,\ \delta_x^2 v^n \rangle_h| \\
&\quad + |\langle u^n + \frac{h_1^2}{12} \delta_x^2 u^n + \frac{h_2^2}{12} \delta_y^2 u^n + \frac{h_1^2 h_2^2}{144} \delta_x^2 \delta_y^2 u^n,\ \frac{h_2^2}{12} \delta_x^2 \delta_y^2 v^n \rangle_h| \\
&:= T_1 + T_2 .
\end{aligned} \tag{4.2}
$$

Using discrete Green formula and Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
T_1 &\leq |\langle \delta_x u^n, \delta_x v^n \rangle_h| + \frac{h_1^2}{12} |\langle \delta_x^2 u^n, \delta_x^2 v^n \rangle_h| + \frac{h_2^2}{12} |\langle \delta_x \delta_y^2 u^n, \delta_x v^n \rangle_h| + \frac{h_1^2 h_2^2}{144} |\langle \delta_x^2 \delta_y^2 u^n, \delta_x^2 v^n \rangle_h| \\
&\leq \|\delta_x u^n\| \|\delta_x v^n\| + \frac{h_1^2}{12} \|\delta_x^2 u^n\| \|\delta_x^2 v^n\| + \frac{h_2^2}{12} \|\delta_x \delta_y^2 u^n\| \|\delta_x v^n\| + \frac{h_1^2 h_2^2}{144} \|\delta_x^2 \delta_y^2 u^n\| \|\delta_x^2 v^n\| .
\end{aligned} \tag{4.3}
$$

Using the inverse estimate, we arrive at

$$
\begin{aligned}
T_1 &\leq \|\delta_x u^n\| \|\delta_x v^n\| + \frac{h_1^2}{12} \frac{4}{h_1^2} \|\delta_x u^n\| \|\delta_x v^n\| + \frac{h_2^2}{12} \frac{4}{h_2^2} \|\delta_x u^n\| \|\delta_x v^n\| + \frac{h_1^2 h_2^2}{144} \frac{16}{h_1^2 h_2^2} \|\delta_x u^n\| \|\delta_x v^n\| \\
&= \frac{16}{9} \|\delta_x u^n\| \|\delta_x v^n\| .
\end{aligned}
$$

Similar as T_1, term T_2 has the estimate

$$
\begin{aligned}
T_2 &= |\langle u^n + \frac{h_1^2}{12} \delta_x^2 u^n + \frac{h_2^2}{12} \delta_y^2 u^n + \frac{h_1^2 h_2^2}{144} \delta_x^2 \delta_y^2 u^n, \delta_x^2 (\frac{h_2^2}{12} \delta_y^2 v^n) \rangle_h| \\
&\leq \frac{16}{9} \cdot \frac{h_2^2}{12} \|\delta_x u^n\| \|\delta_y^2 \delta_x v^n\| \\
&\leq \frac{16}{27} \|\delta_x u^n\| \|\delta_x v^n\| .
\end{aligned}
$$

Inserting the estimates for terms T_1 and T_2 in (4.2), we have

$$
|\langle \mathcal{H} u^n, \mathcal{H}_y \delta_x^2 u^n \rangle_h| \leq (\frac{16}{9} + \frac{16}{27}) \|\delta_x u^n\| \|\delta_x v^n\| = \frac{64}{27} \|\delta_x u^n\| \|\delta_x v^n\| .
$$

In addition, we note that

$$
\begin{aligned}
|\langle \delta_x^2\delta_y^2 u^n, H_y\delta_x^2 v^n\rangle_h| &= |\langle \delta_x^2\delta_y^2 u^n, \delta_x^2 v^n + \frac{h_2^2}{12}\delta_x^2\delta_y^2 v^n\rangle_h| \\
&\le \|\delta_x^2\delta_y u^n\|\|\delta_x^2\delta_y v^n\| + \frac{h_2^2}{12}\|\delta_x^2\delta_y^2 u^n\|\|\delta_x^2\delta_y^2 v^n\| \\
&\le \frac{4}{3}\|\delta_x^2\delta_y u^n\|\|\delta_x^2\delta_y v^n\|.
\end{aligned}
$$

Combining the above estimates with Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
|\langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)u^n, \Lambda_h v^n\rangle_h| &\le \frac{64}{27}(\|\delta_x u^n\|\|\delta_x v^n\| + \|\delta_y u^n\|\|\delta_y v^n\|) \\
&\quad + \frac{4}{3}r_n^2(\|\delta_x^2\delta_y u^n\|\|\delta_x^2\delta_y v^n\| + \|\delta_y^2\delta_x u^n\|\|\delta_y^2\delta_x v^n\|) \\
&\le \frac{64}{27}[\|\delta_x u^n\|\|\delta_x v^n\| + \|\delta_y u^n\|\|\delta_y v^n\| \\
&\quad + r_n^2(\|\delta_x^2\delta_y u^n\|\|\delta_x^2\delta_y v^n\| + \|\delta_y^2\delta_x u^n\|\|\delta_y^2\delta_x v^n\|)] \\
&\le \frac{64}{27}\|u^n\|_A\|v^n\|_A.
\end{aligned}
$$

The proof is completed. □

Lemma 4.3. *For any grid function $u^n \in V_h$, we have*

$$
-\langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)u^n, \Lambda_h u^n\rangle_h \ge \frac{8}{27}\|u^n\|_A^2.
$$

Proof. Note that

$$
-\langle \mathcal{H}u^n, \Lambda_h u^n\rangle_h = -\langle \mathcal{H}u^n, (\mathcal{H}_y\delta_x^2 + H_x\delta_y^2)u^n)\rangle_h.
$$

Using Lemma 4.1 and Lemma 3.2 in [50], we can obtain that

$$
\begin{aligned}
-\langle \mathcal{H}u^n, \mathcal{H}_y\delta_x^2 u^n\rangle_h &= -\langle \mathcal{H}_x(\mathcal{H}_y u^n), \delta_x^2(\mathcal{H}_y u^n)\rangle_h \\
&= \|\mathcal{H}_y u^n\|_{\mathcal{H}_x}^2 \\
&\ge \frac{8}{27}\|\delta_x u^n\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
-\langle \delta_x^2\delta_y^2 u^n, \mathcal{H}_y\delta_x^2 u^n\rangle_h &= -\langle \delta_x^2\delta_y^2 u^n, \delta_x^2 u^n + \frac{h_2^2}{12}\delta_x^2\delta_y^2 u^n\rangle_h \\
&= \langle \delta_x^2\delta_y u^n, \delta_x^2\delta_y u^n\rangle_h - \frac{h_2^2}{12}\|\delta_x^2\delta_y^2 u^n\|^2 \\
&\ge \|\delta_x^2\delta_y u^n\|^2 - \frac{1}{3}\|\delta_x^2\delta_y u^n\|^2 \\
&= \frac{2}{3}\|\delta_x^2\delta_y u^n\|^2.
\end{aligned}
$$

Combining the above estimates, we have

$$
-\langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)u^n, \Lambda_h u^n\rangle_h \ge \frac{8}{27}(\|\delta_x u^n\|^2 + \|\delta_y u^n\|^2) + \frac{2}{3}r_n^2(\|\delta_x^2\delta_y u^n\|^2 + \|\delta_y^2\delta_x u^n\|^2)
$$

$$\geq \frac{8}{27}[\|\delta_x u^n\|^2 + \|\delta_y u^n\|^2 + r_n^2(\|\delta_x^2\delta_y u^n\|^2 + \|\delta_y^2\delta_x u^n\|^2)]$$
$$= \frac{8}{27}\|u^n\|_A^2.$$

The proof is completed. □

Now we present the convergence result of the transformed L1 compact ADI scheme.

Theorem 4.1. *Assume that the problem* (2.1) *has a unique solution satisfies* (2.4) *and the exact solution* $V_{i,j}^n = v(x_i, y_j, s_n)$ *is smooth in the domain* $\Omega \times [0, T^\alpha]$. *Let* $\{v_{i,j}^n \mid (x_i, y_j) \in \Omega_h, 1 \leq n \leq N\}$ *be the solution of the difference scheme* (2.11)-(2.12), (2.19). *Then there exist positive constant* C *and* τ_2 *such that, when* $\tau_s \leq \tau_2$ *the error* $e^n = V^n - v^n$ *satisfies*

$$\|e^n\|_{H^1} \leq C\left(\tau_s^{\min\{2\alpha, 2-\alpha\}} + h_1^4 + h_2^4\right), \quad 1 \leq n \leq N,$$

where C *is independent of* τ_s, h_1 *and* h_2.

Proof. Subtracting (2.17) from (2.16), we get

$$(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)D_\tau^\alpha e_{ij}^n = \varepsilon\Lambda_h e_{i,}^n + R_{4,ij}^n, \tag{4.4}$$

where $R_{4,ij}^n$ is as defined in (2.16), which can be rewritten as

$$R_{4,ij}^n = R_{3,ij}^n - \left(\mathcal{H}Q_{ij}^n + r_n^2\delta_x^2\delta_y^2 Q_{ij}^n\right) + r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V_{ij}^n.$$

Equation (4.4) equals to

$$(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)D_\tau^\alpha e_{ij}^n = \varepsilon\Lambda_h e_{ij}^n + R_{3,ij}^n - \left(\mathcal{H}Q_{ij}^n + r_n^2\delta_x^2\delta_y^2 Q_{ij}^n\right) + r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V_{ij}^n. \tag{4.5}$$

Taking the inner product of (4.4) with $-\Lambda_h e^n$, we arrive at

$$\begin{aligned}-a_{n,0}\langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)e^n, \Lambda_h e^n\rangle_h \leq& -\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})\langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)e^k, \Lambda_h e^n\rangle_h\\ &-a_{n,n-1}\langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)e^0, \Lambda_h e^n\rangle_h + \langle(\mathcal{H} + r_n^2\delta_x^2\delta_y^2)Q^n, \Lambda_h e^n\rangle_h\\ &-\langle R_3^n, \Lambda_h e^n\rangle_h - \langle r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V^n, \Lambda_h e^n\rangle_h.\end{aligned}$$

Using Lemma 4.2 and Lemma 4.3, we obtain

$$\begin{aligned}\frac{8}{27}a_{n,0}\|e^n\|_A^2 \leq& \frac{64}{27}\left(\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})\|e^k\|_A\|e^n\|_A + a_{n,n-1}\|e^0\|_A\|e^n\|_A\right)\\ &+\frac{64}{27}\|Q^n\|_A\|e^n\|_A + \left(\|\nabla_h R_3^n\| + \|\nabla_h r_n^2\delta_x^2\delta_y^2\partial_s^\alpha V^n\|\right)\|\nabla_h e^n\|.\end{aligned}$$

Since $R_{3,ij}^n$ can be written as

$$R_{3,ij}^n = -\frac{7h_1^4}{720}\frac{\partial^6 V}{\partial x^6}|_{x_i,y_j} - \frac{7h_2^4}{720}\frac{\partial^6 V}{\partial y^6}|_{x_i,y_j} + \mathcal{O}(h_1^6 + h_2^6).$$

Then, we have

$$\delta_x R_{3,ij}^n = \frac{1}{h_1}\left[R_{3,i+\frac{1}{2}j}^n - R_{3,i-\frac{1}{2}j}\right]$$

$$= \frac{\partial R^n_{3,ij}}{\partial x} + \frac{h_1^2}{3}\frac{\partial^3 R^n_{3,ij}}{\partial x^3} + \mathcal{O}(h_1^4)$$
$$= \mathcal{O}(h_1^4 + h_2^4 + h_1^2 h_2^4).$$

Similarly, we can obtain $\delta_y R^n_{3,ij} = \mathcal{O}(h_1^4 + h_2^4 + h_2^2 h_1^4)$. Since h_1 and h_2 are sufficient small, we have

$$\|\nabla_h R_3\| = \sqrt{\|\delta_x R_3\|^2 + \|\delta_y R_3\|^2} \le C_5(h_1^4 + h_2^4). \tag{4.6}$$

By using Cauchy-Schwartz inequality, (4.6) and (3.9), we arrive

$$a_{n,0}\|e^n\|_A^2 \le 8\left[\sum_{k=1}^{n-1}(a_{n,n-k-1} - a_{n,n-k})\left(\frac{1}{8}\|e^k\|_A^2 + 2\|e^n\|_A^2\right) + a_{n,n-1}\left(\frac{1}{8}\|e^0\|_A^2 + 2\|e^n\|_A^2\right)\right]$$
$$+8\|Q^n\|_A\|e^n\|_A + (C_5h_1^4 + C_5h_2^4 + C_4\tau_s^{2\alpha})\|e^n\|_A.$$

Rearranging the above inequality, we obtain

$$D_\tau^\alpha\|e^n\|_A^2 \le 16a_{n,0}\|e^n\|_A^2 + \|e^n\|_A(8\|Q^n\|_A + C_5h_1^4 + C_5h_2^4 + C_4\tau_s^{2\alpha}).$$

From Lemma 3.4, there exists a constant τ_2, when $\tau_s \le \tau_2$, it holds

$$\|e^n\|_A \le 2E_\alpha(2\lambda^* r_n)\left[\|e^0\|_A + C^{**}\left(\tau_s^{\min\{2-\alpha,2\alpha\}} + h_1^4 + h_2^4\right)\right],$$

where $C^{**} = \max\{C_4,\ C_5\}$ and this completes the proof. □

5. Numerical experiments

In this section, we present examples in 2D for the sub-diffusion equation (1.1)-(1.3) to demonstrate numerically the effectiveness and accuracy of both the ADI scheme (2.13)-(2.14) and the compact ADI scheme (2.19). To verify the convergence rates in both time and space of the proposed schemes, we measure the accuracy of the numerical solution using the maximal H^1 error

$$e(\tau, h) = \max_{1\le n\le N}\|V^n - v^n\|_{H^1}.$$

Here the H^1-norm is as defined in (3.1) and we fix $h_1 = h_2 = h$ throughout this section.

Example 5.1. Consider the sub-diffusion equation (1.1)-(1.3) with $\varepsilon = 1$ on the domain $\Omega = [0,\pi]\times[0,\pi]$ up to the time $T = 1$. The source term and the initial and boundary conditions are determined by the exact analytical solution

$$u(x,y,t) = E_{\alpha,1}(-2t^\alpha)\sin x \sin y.$$

At first, we present experiments to verify the convergence rates in time of the ADI scheme (2.13)-(2.14). We fix $h = h_1 = h_2 = \pi/200$ and compute the numerical solutions of the sub-diffusion equation using the ADI scheme with various time step sizes with $N = 32, 64, 128, 256, 512$. The order of convergence in the time direction is approximated by

$$\text{rate} = \log_2\left(\frac{e(2\tau,h)}{e(\tau,h)}\right).$$

The numerical errors and orders of convergence for Example 5.1 with different values of α are reported in Table 1. We see that the order of convergence in time is approximate $\min\{2\alpha, 2-\alpha\}$, which is in accordance with Theorem 3.3. To verify the spatial second-order accuracy of the ADI scheme (2.10), we set $N = 20000$ and $M = 4, 8, 16, 32$. The numerical errors and orders of convergence in space are presented in Table 2 where the second-order accuracy is obvious.

In order to show the efficiency of the ADI scheme, we compare this scheme with the method without using ADI scheme, i.e. removing the small term $r_n^2\delta_x^2\delta_y^2 D_\tau^\alpha v_{ij}^n$, and fixing $N = 2000$. Moreover, we compare our method with the uniform L1 method, i.e., the time differential operator is approximated by

$$D_\tau^\alpha u(x,y,t_n) = \sum_{k=1}^{n} a_{n,n-k}\big(u(x,y,t_k) - u(x,y,t_{k-1})\big),$$

where

$$a_{n,n-k} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}ds, \quad \tau = T/N, t_k = k\tau, k = 0,1,\dots,N.$$

The errors and CPU time using different α with $M = 8, 16, 32, 64$ are shown in Table 5 and 6. It can be observed that in terms of computational accuracy, our method exhibits greater stability and produces smaller errors than the uniform L1 method. In addition, the TL1 scheme combined with the ADI technique (ADI method) requires significantly less computation time(s) than the methods without ADI (non-ADI method).

Table 1. Temporal convergence rates in the H^1-norm of the ADI scheme for Example 5.1.

	α=0.2		α=0.5		α=0.8	
N	error	order	error	order	error	order
32	2.0545e-01	*	2.4815e-02	*	8.5477e-03	*
64	1.5800e-01	0.3789	1.2344e-02	1.0074	3.5292e-03	1.2183
128	1.2152e-01	0.3788	6.1505e-03	1.0051	1.4710e-03	1.2153
256	9.3304e-02	0.3811	3.0709e-03	1.0020	6.2227e-04	1.2067
512	7.1490e-02	0.3842	1.5377e-03	0.9979	2.6960e-04	1.1904

Table 2. Spatial convergence rates in the H^1-norm of the ADI scheme for Example 5.1.

	α=0.5		α=0.8	
M	error	order	error	order
4	2.9984e-02	*	3.2512e-02	*
8	7.5115e-03	1.9970	8.0714e-03	2.0101
16	1.9066e-03	1.9781	2.0162e-03	2.0012
32	5.0620e-04	1.9132	5.0587e-04	1.9948

Secondly, we verify the convergence rates in both time and space directions of the compact ADI scheme (2.19). We fix $h = h_1 = h_2 = \pi/200$ with $N = 64, 128, 256, 512$ and report the numerical errors and orders of convergence in time for Example 5.1 in Table 3 where the $\min\{2\alpha, 2-\alpha\}$-order accuracy is evident. To verify the spatial fourth-order accuracy, we set $N = M^{(4/\min\{2-\alpha,2\alpha\})}$, where $M = M_1 = M_2$. The order of convergence in space is approximated by

$$\text{rate}_h = \log_{\frac{M^{(2)}}{M^{(1)}}} \left(\frac{e(\tau^{(1)}, h^{(1)})}{e(\tau^{(2)}, h^{(2)})} \right).$$

The numerical errors are presented in Table 4 where the fourth-order accuracy is obvious.

Table 3. Temporal convergence rates in the H^1-norm of the compact ADI scheme for Example 5.1.

	α=0.2		α=0.5		α=0.8	
N	error	order	error	order	error	order
32	2.0545e-01	*	2.4804e-02	*	8.5349e-03	*
64	1.5800e-01	0.3789	1.2333e-02	1.0081	3.5161e-03	1.2203
128	1.2151e-01	0.3788	6.1386e-03	1.0065	1.4578e-03	1.2197
256	9.3295e-02	0.3812	3.0589e-03	1.0049	6.0909e-04	1.2173
512	7.1480e-02	0.3843	1.5256e-03	1.0036	2.5640e-04	1.2146

Table 4. Spatial convergence rates in the H^1-norm of the compact ADI scheme for Example 5.1.

	α=0.5		α=0.8	
M	error	order	error	order
8	2.4978e-04	*	1.6565e-04	*
10	1.0237e-04	3.9972	6.6858e-05	4.0661
12	4.9385e-05	3.9983	3.1940e-05	4.0516
14	2.6661e-05	3.9989	1.7133e-05	4.0406
16	1.5630e-05	3.9992	9.9976e-06	4.0339
18	9.7584e-06	3.9994	6.2204e-06	4.0287
20	6.4028e-06	3.9995	4.0706e-06	4.0246

Table 5. Different numerical schemes with different $\alpha = 0.5$ and $N = 2000$.

	L1 method		non-ADI method		ADI method	
M	error	CPU time(s)	error	CPU time(s)	error	CPU time(s)
8	2.7060e+00	1.54	9.6514e-03	5.68	9 .8496e-03	5.10
16	2.7147e+00	2.51	2.4223e-03	6.09	2.6344e-03	7.13
32	2.7169e+00	19.08	6.0994e-04	22.02	8.7167e-04	7.45
64	2.7174e+00	396.98	1.6150e-04	410.14	5.1569e-04	14.22

Table 6. Different numerical schemes with different $\alpha = 0.8$ and $N = 2000$.

	L1 method		non-ADI method		ADI method	
M	error	CPU time(s)	error	CPU time(s)	error	CPU time(s)
8	2.7062e+00	1.94	1.1323e-02	6.63	1.1326e-02	6.76
16	2.7149e+00	3.34	2.9090e-03	7.96	2.9121e-03	6.97
32	2.7171e+00	18.75	8.0029e-04	22.29	8.0338e-04	9.59
64	2.7176e+00	399.79	2.8037e-04	397.16	2.8329e-04	16.01

Example 5.2. In this example, we consider the sub-diffusion equation (1.1)-(1.3) with $\varepsilon = 1$. The spatial domain is $\Omega = [0, \pi] \times [0, \pi]$ and the time interval is $[0, T] = [0, 1]$. The corresponding forcing term is given as

$$f(x, y, t) = \left[\Gamma(1 + \alpha) + \frac{6}{\Gamma(4 - \alpha)} t^{3-\alpha} + 2(t^\alpha + t^3)\right] \sin x \sin y,$$

and the initial and boundary conditions are determined by the exact solution $u(x, y, t) = (t^\alpha + t^3) \sin x \sin y$.

Similar to Example 5.1, in Table 7 and Table 9, we test the convergence with respect to the time step size by fixing $M_1 = M_2 = 250$, $N = 64, 128, 256, 512$ for the proposed ADI scheme and $M_1 = M_2 = 150$, $N = 500, 600, 700, 800, 900$ for the compact ADI scheme, respectively. We observe that the temporal convergence rates are approximate $\min\{2\alpha, 2 - \alpha\}$ order which is in accordance with Theorem 3.3 and Theorem 4.1. In Table 8 and Table 10, we show the performance of the ADI scheme and the compact ADI scheme with respect to accuracy in space, respectively. We set $N = 20000$, and $M = 4, 8, 16, 32$ for the calculation of the ADI scheme. Meanwhile, we set $M = 2 \times k, k = 4, 5, \ldots, 10$, $N = M^{(4/\min\{2-\alpha, 2\alpha\})}$ for the calculation of the compact ADI scheme. Again, we observe that the ADI scheme achieves second-order accuracy in space and the compact ADI scheme achieves fourth-order accuracy in space.

We compare our method with the TL1 scheme without using technique (TL1 method), the uniform L1 scheme (L1 method) and the grided mesh L1 scheme (GL1 method). The time differential operator approximated by the non-uniform L1 scheme can be written as

$$D_\tau^\alpha u(x, y, t_n) = \sum_{k=1}^{n} a_{n,n-k}\big(u(x, y, t_k) - u(x, y, t_{k-1})\big),$$

where

$$a_{n,n-k} = \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \frac{(t - s)^{-\alpha}}{\Gamma(1 - \alpha)} ds, \quad \tau_k = t_k - t_{k-1}, t_k = T(k/N)^\delta, k = 1, \ldots, n,$$

where we choose $\delta = 2$. We define $M = 64$ and $N = 100, 200, 400, 800$. The errors and convergence orders of different numerical methods have been presented in Table 11 for $\alpha = 0.5$. The L1 method does not show good accuracy and efficiency in the calculation. Our method can achieve computational accuracy comparable to that of the GL1 method. Additionally, we set $M = 64$, $N = 200, 400, 800, 1600$ to compare the computational time costs of the above methods and present the results in Figure 1. It can be observed that the computational speed of our method is markedly superior to that of other methods.

Table 7. Temporal convergence rates in the H^1-norm of the ADI scheme for Example 5.2.

	α=0.2		α=0.5		α=0.8	
N	error	order	error	order	error	order
64	5.2642e-01	*	4.3479e-02	*	1.3042e-02	*
128	3.8283e-01	0.4595	2.1994e-02	0.9832	6.0589e-03	1.1061
256	2.8617e-01	0.4198	1.1150e-02	0.9801	2.7784e-03	1.1248
512	2.1647e-01	0.4027	5.6350e-03	0.9845	1.2700e-03	1.1294

Table 8. Spatial convergence rates in the H^1-norm of the ADI, scheme for Example 5.2.

	α=0.5		α=0.8	
M	error	order	error	order
4	1.6431e-01	*	1.4378e-01	*
8	4.1071e-02	2.0002	3.6309e-02	1.9854
16	1.0041e-02	2.0322	9.1271e-03	1.9921
32	2.2694e-03	2.1454	2.3120e-03	1.9810

Table 9. Temporal convergence rates in the H^1-norm of the compact ADI scheme for Example 5.2.

	α=0.2		α=0.5		α=0.8	
N	error	order	error	order	error	order
500	2.3021e-01	*	5.9571e-03	*	1.2828e-03	*
600	2.1410e-01	0.3978	4.9809e-03	0.9817	1.0384e-03	1.1596
700	2.0141e-01	0.3965	4.2808e-03	0.9826	8.6805e-04	1.1623
800	1.9105e-01	0.3956	3.7539e-03	0.9835	7.4305e-04	1.1645
900	1.8236e-01	0.3950	3.3431e-03	0.9842	6.4768e-04	1.1663

Table 10. Spatial convergence rates in the H^1-norm of the compact ADI scheme for Example 5.2.

	α=0.5		α=0.8	
M	error	order	error	order
8	4.5696e-04	*	8.4402e-04	*
10	1.9007e-04	3.9312	3.5130e-04	3.9281
12	9.2408e-05	3.9554	1.7100e-04	3.9491
14	5.0121e-05	3.9687	9.2873e-05	3.9599
16	2.9471e-05	3.9768	5.4674e-05	3.9680
18	1.8438e-05	3.9821	3.4240e-05	3.9732
20	1.2115e-05	3.9857	2.2519e-05	3.9771

Table 11. Different numerical schemes with different $\alpha = 0.5$ and $M = 64$.

	L1 method		GL1 method		TL1 method		ADI-TL1 method	
N	error	order	error	order	error	order	error	order
100	4.96e-02	*	5.74e-03	*	4.04e-03	*	2.69e-02	*
200	3.67e-02	0.43	2.92e-03	0.98	1.90e-03	1.09	1.40e-02	1.37
400	2.68e-02	0.45	1.47e-03	0.99	1.10e-03	0.79	6.94e-03	1.05
800	1.94e-02	0.47	7.66e-04	0.94	8.11e-04	0.44	3.24e-03	1.10
Theory	*	0.5	*	1.0	*	1.0	*	1.0

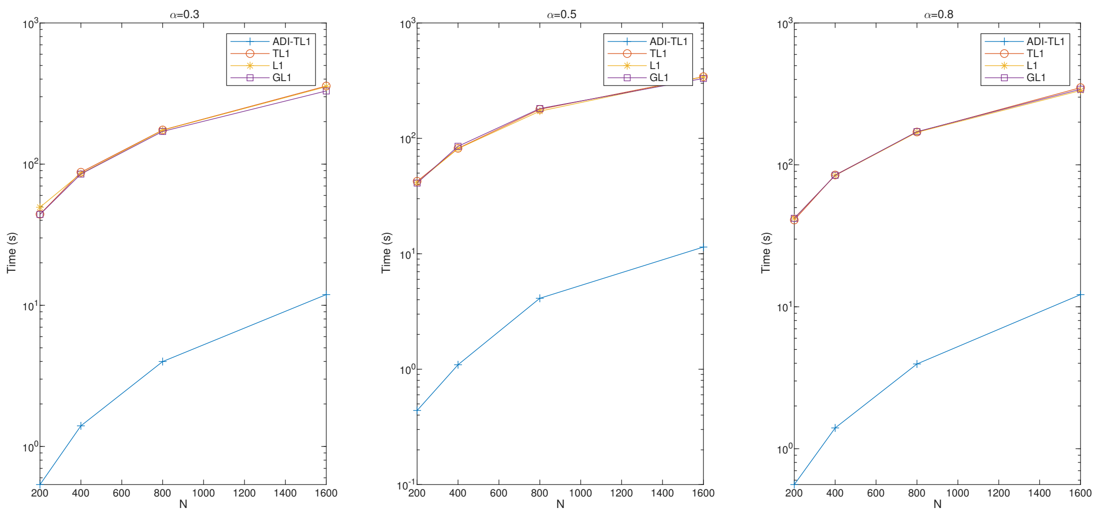

Figure 1. Time contract(s) of the L1 method (L1), grided mesh L1 method (GL1), TL1 method (TL1) and our method (ADI-TL1) with $\alpha = 0.3$ (left), $\alpha = 0.5$ (middle), $\alpha = 0.8$ (right) for Example 5.2.

6. Conclusions

In this paper, we have proposed two fully discrete numerical schemes for solving two-dimensional time fractional sub-diffusion equations. The temporal discretization employs a transformed L1 method with the spatial discretization utilizes the ADI and compact ADI methods. Theoretically, the transformed L1 method applies a variable transformation $t = s^{1/\alpha}$ to rescale the equation. Thereby, our method effectively captures the rapid evolution at the initial time and reduces the cumulative error effect caused by non-local operators in the temporal direction. Moreover, the error estimates of the proposed schemes both in H^1-norm based on an improved discrete Grönwall inequality. From a practical perspective, we introduce the ADI and compact ADI techniques. These methods transform the two-dimensional problem into one-dimensional problems, significantly reducing computational cost and achieving up to fourth-order convergence in space. Comparing to existing methods, our time rescaling strategy significantly improves accuracy in the initial time. Meanwhile, the compact ADI scheme enhances the calculation accuracy, while exceeding the typical second-order accuracy of the former finite difference method. Such technique accelerates the long-time simulations, which can be seen evidently in our numerical experiments. In addition, the numerical experiments show consistency with our theoretical analysis when $0 < \alpha < 1$ and demonstrate the effectiveness of the proposed schemes for small α.

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