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TEMPORAL AND SPATIAL DYNAMICS OF BACTERIA IN RIVERS AFFECTED BY POLLUTION

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Abstract This paper presents a convection-reaction-diffusion system; our objective is to apply this model in quantifying and analyzing the spatial arrangement of bacteria resistant to antibiotics within a river. By means of our research, we successfully determine a sufficient condition for the presence of a non-constant positive solution for the associated elliptic system through Leray-Schauder's degree theory. To further support our findings, we have included numerical simulations that align with our theoretical analysis.

Keywords Reaction-diffusion equations, nonlinear PDE of elliptic type, antibiotic-resistant bacteria.

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1. Introduction

Rivers, like all other natural systems on Earth, contain a diverse array of bacteria. While some of these bacteria are resistant to antibiotics, others are not. A recent study has shown that human activities are the leading cause of the heightened concentration of antibiotic-resistant bacteria in rivers around the world [21]. Human exposure to antibiotic-resistant bacteria may arise from polluted drinking water and the consumption of infected freshwater fish. [3]. High-risk areas for the presence of these bacteria are often located near wastewater treatment facilities and sewage dumps [9,30]. The World Health Organization reports that diseases caused by antibiotic-resistant bacteria result in 250,000 deaths annually, making the fight against antibiotic resistance a critical public health priority [29].

The scientific literature includes only a limited number of models to understand the behavior of bacteria in rivers, such as those proposed by [10, 14, 18, 19]. Our research presents a new mathematical model that predicts the distribution of bacteria in a river, considering various factors such as the biological interactions between bacteria, the movement of bacteria within the river, the speed of the river flow, and the impact of human activities. Our model consists of a

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non-autonomous convection-reaction-diffusion system:

$$\begin{cases} \dfrac{\partial R_s}{\partial t} = d\dfrac{\partial^2 R_s}{\partial x^2} - b\dfrac{\partial R_s}{\partial x} - \alpha\dfrac{R_s(R_I + L_I)}{N} + G(L)R_I + r\left(1 - \dfrac{N}{K}\right)R_s, \\ \dfrac{\partial R_I}{\partial t} = d\dfrac{\partial^2 R_I}{\partial x^2} - b\dfrac{\partial R_I}{\partial x} + \alpha\dfrac{R_s(R_I + L_I)}{N} - G(L)R_I + r\left(1 - \dfrac{N}{K}\right)R_I, \\ \dfrac{\partial L_s}{\partial t} = d\dfrac{\partial^2 L_s}{\partial x^2} - b\dfrac{\partial L_s}{\partial x} + F_s(x,t) - \gamma L_s - \alpha\dfrac{L_s(R_I + L_I)}{N} + G(L)L_I + r\left(1 - \dfrac{N}{K}\right)L_s, \\ \dfrac{\partial L_I}{\partial t} = d\dfrac{\partial^2 L_I}{\partial x^2} - b\dfrac{\partial L_I}{\partial x} + F_I(x,t) - \gamma L_I + \alpha\dfrac{L_s(R_I + L_I)}{N} - G(L)L_I + r\left(1 - \dfrac{N}{K}\right)L_I, \\ N = R_s + R_I + L_s + L_I, \\ L = L_s + L_I. \end{cases} \tag{1.1}$$

For $t > 0$ and $x \in (0, +\infty)$, consider the bacterial population densities at time t represented by R_s, R_I, L_s, and L_I. The river contains two different types of bacteria: River bacteria (R) and land bacteria (L). The bacteria in both the river and land are divided into two categories: Resistant (R_I, L_I) and non-resistant (or susceptible) (R_s, L_s). In the model, the parameters α, γ, r, and K are all positive constants, with the following interpretations: α denotes the rate at which the antibiotic resistance gene is transmitted, γ is the death rate of land bacteria, r is the birth-death rate controlled by the river's carrying capacity K. Additionally, d and b are positive constants representing the diffusion coefficient and flow velocity, respectively. Moreover, the functions F_s and F_I describe the rates at which bacteria migrate into the river from the shoreline. These functions are positive and belong to $C\left(\mathbb{R}_+, L^1(0, +\infty) \cap L^\infty(0, +\infty)\right)$ such that:

$$a_s = \sup_{t \in \mathbb{R}_+} \|F_s(., t)\|_{L^\infty}, \quad m_s = \sup_{t \in \mathbb{R}_+} \|F_s(., t)\|_{L^1}, \tag{1.2}$$

$$a_I = \sup_{t \in \mathbb{R}_+} \|F_I(., t)\|_{L^\infty}, \quad m_I = \sup_{t \in \mathbb{R}_+} \|F_I(., t)\|_{L^1}. \tag{1.3}$$

In addition, there are functions f_s and f_I within the space $L^1(0, +\infty)$ that fulfill the following conditions: $f_s \leq a_s$, $f_I \leq a_I$ and also:

$$\lim_{t \to +\infty} \|F_s(., t) - f_s\|_{L^2}, \tag{1.4}$$

$$\lim_{t \to +\infty} \|F_I(., t) - f_I\|_{L^2}. \tag{1.5}$$

The model (1.1) is investigated with the subsequent non-zero initial conditions:

$$R_s(x, 0) > 0, \quad R_I(x, 0) > 0, \quad L_s(x, 0) > 0, \quad L_I(x, 0) > 0, \tag{1.6}$$

and the boundary conditions are given by:

$$\begin{cases} R_s'(0, t) = 0,\ R_I'(0, t) = 0,\ L_s'(0, t) = 0,\ L_I'(0, t) = 0, \\ R_s'(+\infty, t) = 0,\ R_I'(+\infty, t) = 0,\ L_s'(+\infty, t) = 0,\ L_I'(+\infty, t) = 0. \end{cases} \tag{1.7}$$

The width of the river is relatively small compared to its length, leading to the choice of a one-dimensional model with the variable $x \in (0, +\infty)$. Bacteria do not migrate from the river, thus justifying the choice of boundary conditions. The assumption that land bacteria are not suited for the river environment is reflected in the assumption $\gamma > r$.

Our model builds upon the following model proposed in [18]:

$$\begin{cases} \dfrac{\partial R_s}{\partial t} = BR_s - \alpha R_s(R_I + L_I) + \dfrac{\beta R_I}{L+1} + r\left(1 - \dfrac{\omega}{K}\right) R_s, \\ \dfrac{\partial R_I}{\partial t} = BR_I + \alpha R_s(R_I + L_I) - \dfrac{\beta R_I}{L+1} + r\left(1 - \dfrac{\omega}{K}\right) R_I, \\ \dfrac{\partial L_s}{\partial t} = BL_s + F_s - \gamma L_s - \alpha L_s(R_I + L_I) + \dfrac{\beta L_I}{L+1} + r\left(1 - \dfrac{\omega}{K}\right) L_s, \\ \dfrac{\partial L_I}{\partial t} = BL_I + F_I - \gamma L_I + \alpha L_s(R_I + L_I) - \dfrac{\beta L_I}{L+1} + r\left(1 - \dfrac{\omega}{K}\right) L_I. \end{cases} \tag{1.8}$$

Indeed, representing the transmission of resistance often employs mass-action kinetics, as seen in the term αSI, where α signifies the transmission rate, and S and I denote the non-resistant and resistant bacteria, respectively [11,17,20,24,26]. Nonetheless, this method presupposes that as the quantity of non-resistant bacteria rises, there's a proportional escalation in the number of bacteria acquiring resistance over time, even in scenarios with minimal populations of resistant bacteria. This may not accurately reflect reality. Another method, as exemplified in epidemiological models, is the framework proposed by Keeling and Rohani in their work cited in [12]. They represent the interaction between susceptible (S) and infected (I) populations using $\frac{SI}{S+I}$. This method has been employed in various studies to simulate the transmission of an antibiotic resistance gene among bacteria, including references [5,6,28].

The authors in [18] consider the relationship between contamination and the number of land bacteria, proposing that as the number of polluting bacteria, L, increases, the loss rate of the resistance gene, $G(L)$, should decrease. This assumption is based on the idea that if a sewage plant contains antibiotic-resistant bacteria, the presence of the relevant antibiotic would likely prevent resistance loss. The authors express this association through the function $G(L) = \beta/(L+1)$. Nevertheless, there exist various functions akin to the mentioned one. In our model, we integrate the term $G(L)$ to represent the bacteria's loss rate. Furthermore, we posit that G is a $\mathcal{C}^1$-function characterized by the following attributes:

(H1) $G(L) > 0$ for every $L > 0$,

(H2) $G'(L) \leq 0$ for every $L > 0$,

(H3) $\lim_{L \to +\infty} G(L) = 0$.

The aim of this study is to examine the system (1.1). Our primary focus is to determine the existence of steady states that are not constant, which will be achieved through the application of Leray-Schauder's degree theory.

To study the problem of (1.1), we introduce the following scaling transformations:

$$r_s = \frac{R_s}{K}, \quad r_i = \frac{R_I}{K}, \quad l_s = \frac{L_s}{K}, \quad l_i = \frac{L_I}{K}.$$

We obtain the following system:

$$\begin{cases} \dfrac{\partial r_s}{\partial t} = d\dfrac{\partial^2 r_s}{\partial x^2} - b\dfrac{\partial r_s}{\partial x} - \alpha\dfrac{r_s(r_i+l_i)}{n} + g(l)r_i + r\,(1-n)\,r_s, \\ \dfrac{\partial r_i}{\partial t} = d\dfrac{\partial^2 r_i}{\partial x^2} - b\dfrac{\partial r_i}{\partial x} + \alpha\dfrac{r_s(r_i+l_i)}{n} - g(l)r_i + r\,(1-n)\,r_i, \\ \dfrac{\partial l_s}{\partial t} = d\dfrac{\partial^2 l_s}{\partial x^2} - b\dfrac{\partial l_s}{\partial x} + \dfrac{F_s(x,t)}{K} - \gamma l_s - \alpha\dfrac{l_s(r_i+l_i)}{n} + g(l)l_i + r\,(1-n)\,l_s, \\ \dfrac{\partial l_i}{\partial t} = d\dfrac{\partial^2 l_i}{\partial x^2} - b\dfrac{\partial l_i}{\partial x} + \dfrac{F_I(x,t)}{K} - \gamma l_i + \alpha\dfrac{l_s(r_i+l_i)}{n} - g(l)l_i + r\,(1-n)\,l_i, \\ n = r_s + r_i + l_s + l_i, \quad l = l_s + l_i, \quad g(l) = G(Kl_s + Kl_i), \\ r_s'(0,t) = r_i'(0,t) = l_s'(0,t) = l_i'(0,t) = 0, \\ r_s'(+\infty,t) = r_i'(+\infty,t) = l_s'(+\infty,t) = l_i'(+\infty,t) = 0. \end{cases} \tag{1.9}$$

In [18], the transfer of resistance was modeled using the mass action law. In this study, the transfer of antibiotic resistance is approached through the terms $\alpha\frac{r_s(r_i+l_i)}{n}$ and $\alpha\frac{l_s(r_i+l_i)}{n}$, making the mathematics is more complex, but leading to results that are more biologically significant. In [18], non-constant steady states for $G(L) = \beta/(L+1)$ are established under the conditions:

$$\alpha < \frac{\beta}{r(K/2+1)+\beta} < 1, \qquad 2\beta < \alpha^2 K(K/2+1).$$

However, these conditions have no significant biological meaning. In this study, we prove that non-constant steady states exist for all functions G that satisfy (H1)-(H3), given the simple condition $\alpha > G(K)$, through the selection of a suitable homotopy.

The rest of this paper unfolds as follows: Part two will be devoted to demonstrating the global existence of solutions. In part three, we will establish the presence of non-constant steady states. Furthermore, we will provide numerical simulations to verify our theoretical findings. Lastly, we will draw conclusions from our study.

2. Existence of solutions on a global scale

This section is dedicated to establishing the global existence and uniqueness of classical solutions. We aim to demonstrate that for any fixed time $T > 0$, the norms $\|r_s(.,t)\|_{L^1}$, $\|r_i(.,t)\|_{L^1}$, $\|l_s(.,t)\|_{L^1}$ and $\|l_i(.,t)\|_{L^1}$ are bounded for $0 \le t < T$.

Let $F = (F_1, F_2, F_3, F_4)$ denote the right-hand side of (1.9), and $X = \left(L^1(0,+\infty)\right)^4$. According to Amman's local existence theory [1], for any initial condition in X, there exists a $T_{\max} > 0$ such that (1.9) possesses a unique classical solution:

$$(r_s(.,t), r_i(.,t), l_s(.,t), l_i(.,t))$$

in $C([0,T_{\max}), X)$. Furthermore, if $T_{\max} < +\infty$, then:

$$\lim_{t\to T_{\max}} \left(\|r_s(.,t)\|_{L^1} + \|r_i(.,t)\|_{L^1} + \|l_s(.,t)\|_{L^1} + \|l_i(.,t)\|_{L^1}\right) = +\infty.$$

In other words, if for any fixed time $T > 0$, $\|r_s(.,t)\|_{L^1}$, $\|r_i(.,t)\|_{L^1}$, $\|l_s(.,t)\|_{L^1}$ and $\|l_i(.,t)\|_{L^1}$ are bounded for $0 \leq t < T$, then the solution is global [23,25,27].

In the first step, we give a positively invariant set.

Theorem 2.1. *Let Λ be the set defined by:*

$$\Lambda = \left\{ r_s \geq 0,\ r_i \geq 0,\ l_s \geq 0,\ l_i \geq 0\ :\ r_s + r_i \leq 1, \quad l_s + l_i \leq \frac{a_s + a_I}{K(\gamma - r)} \right\}.$$

Hence, Λ remains a region of positive invariance under the flow generated by (1.9).

Proof. Firstly, for all $r_s \geq 0$, $r_i \geq 0$, $l_s \geq 0$, $l_i \geq 0$, the functions F_1, F_2, F_3 and F_4 verify:

$$\begin{cases} F_1(0, r_i, l_s, l_i, t) \geq 0, \\ F_2(r_s, 0, l_s, l_i, t) \geq 0, \\ F_3(r_s, r_i, 0, l_i, t) \geq 0, \\ F_4(r_s, r_i, l_i, 0, t) \geq 0. \end{cases}$$

From Theorem 14.11 of [25], we deduce that $[0, +\infty)^4$ is positively invariant for the system (1.9).

Furthermore, by summing the first two equations of (1.9), $r_s + r_i$ satisfies:

$$\frac{\partial (r_s + r_i)}{\partial t} = d\frac{\partial^2 (r_s + r_i)}{\partial x^2} - b\frac{\partial (r_s + r_i)}{\partial x} + r\,(1 - n)\,(r_s + r_i). \tag{2.1}$$

As, $r_s \geq 0$, $r_i \geq 0$, $l_s \geq 0$ and $l_i \geq 0$, then:

$$\begin{aligned} d\frac{\partial^2 (r_s + r_i)}{\partial x^2} - b\frac{\partial (r_s + r_i)}{\partial x} + r\,(1 - n)\,(r_s + r_i) \leq\ & d\frac{\partial^2 (r_s + r_i)}{\partial x^2} - b\frac{\partial (r_s + r_i)}{\partial x} \\ & + r\,(1 - r_s - r_i)\,(r_s + r_i). \end{aligned}$$

The strong maximum principle of parabolic systems (see [25]) implies that $(r_s + r_i)(x,t) \leq y(x,t)$, where y is the solution of:

$$\begin{cases} \dfrac{\partial y}{\partial t} = d\dfrac{\partial^2 y}{\partial x^2} - b\dfrac{\partial y}{\partial x} + r\,(1 - y)\,y, \\ y(x,0) = r_s(x,0) + r_i(x,0), \\ y'(+\infty, t) = y'(0,t) = 0. \end{cases}$$

Since, $y(x,0) \leq 1$, then $y(x,t) \leq 1$, so $r_s + r_i \leq 1$.

Alternatively, combining the third and fourth equations of system (1.9), we derive the following:

$$\begin{aligned} \frac{d(l_s + l_i)}{\partial t} &= d\frac{\partial^2 (l_s + l_i)}{\partial x^2} - b\frac{\partial (l_s + l_i)}{\partial x} + \frac{F_s(x,t)}{K} + \frac{F_I(x,t)}{K} \\ &\quad - \gamma(l_s + l_i) + r\,(1 - n)\,(l_s + l_i) \\ &\leq d\frac{\partial^2 (l_s + l_i)}{\partial x^2} - b\frac{\partial (l_s + l_i)}{\partial x} + \frac{a_s + a_I}{K} - (\gamma - r)(l_s + l_i). \end{aligned}$$

Using a similar approach as described earlier, we can readily show that:

$$l_s + l_i \leq \frac{a_s + a_I}{K(\gamma - r)}.$$

This concludes the proof. □

Theorem 2.2. *For any positive initial data* $(r_s^0, r_i^0, l_s^0, l_s^0) \in X \cap \Lambda$, *there exists only one positive classical solution to system (1.9) over the interval* $[0, +\infty)$, *residing in* $C([0, +\infty), X)$.

Proof. Integrating equation (2.1) with respect to x over the interval $(0, +\infty)$ yields:

$$\begin{aligned}\int_0^{+\infty} \frac{\partial (r_s + r_i)}{\partial t} dx &= d \int_0^{+\infty} \frac{\partial^2 (r_s + r_i)}{\partial x^2} dx - b \int_0^{+\infty} \frac{\partial (r_s + r_i)}{\partial x} dx \\ &\quad + r \int_0^{+\infty} (1 - n)(r_s + r_i) dx \\ &\leq d \int_0^{+\infty} \frac{\partial^2 (r_s + r_i)}{\partial x^2} dx - b \int_0^{+\infty} \frac{\partial (r_s + r_i)}{\partial x} dx + r \int_0^{+\infty} (r_s + r_i) dx \\ &= -b(r_s + r_i)(+\infty, t) + b(r_s + r_i)(0, t) + r \int_0^{+\infty} (r_s + r_i) dx \\ &\leq b + r \int_0^{+\infty} (r_s + r_i) dx.\end{aligned}$$

By setting:

$$h(t) = \int_0^{+\infty} (r_s + r_i) dx.$$

We obtain

$$h'(t) \leq b + r h(t).$$

Hence,

$$h(t) \leq \left(h(0) + \frac{b}{r} \right) e^{rt} - \frac{b}{r},$$

namely:

$$\|r_s + r_i\|_{L^1} \leq \left(\|r_s^0 + r_i^0\|_{L^1} + \frac{b}{r} \right) e^{rt} - \frac{b}{r}. \tag{2.2}$$

Similarly, from (1.2), (1.3) and (2.2), we can easily check that:

$$\|l_s + l_i\|_{L^1} \leq \left(\|l_s^0 + l_i^0\|_{L^1} - \frac{m_s + m_I}{K} - b \frac{a_s + a_I}{K(\gamma - r)} \right) e^{-(\gamma - r)t} + \frac{m_s + m_I}{K} + b \frac{a_s + a_I}{K(\gamma - r)}. \tag{2.3}$$

From (2.2) and (2.3), we deduce that for any fixed time $T > 0$, $\|r_s(.,t)\|_{L^1}$, $\|r_i(.,t)\|_{L^1}$, $\|l_s(.,t)\|_{L^1}$ and $\|l_i(.,t)\|_{L^1}$ remain bounded for $0 \leq t < T$, implying the existence of a global classical solution within X. □

In this work, we are mostly interested in large time behavior of our non-autonomous spatio-temporal system. We will determinate the omega-limit set defined by:

$$\Omega = \left\{ y \in X, \ \exists\, t_n \longrightarrow +\infty : \ (r_s(x, t_n), r_i(x, t_n), l_s(x, t_n), l_i(x, t_n)) \longrightarrow y \right\}.$$

Recall our hypothesis concerning the functions F_s and F_I:

$$\lim_{t \to +\infty} \|F_s(., t) - f_s\|_{L^2} = 0,$$

$$\lim_{t\to+\infty}\|F_I(.,t)-f_I\|_{L^2}=0.$$

On the basis of these data and by using the method presented in [8,13], we can demonstrate that the omega-limit set of our parabolic system is simplified to the solutions of the subsequent elliptic system:

$$\begin{cases} d\dfrac{\partial^2 r_s}{\partial x^2}-b\dfrac{\partial r_s}{\partial x}-\alpha\dfrac{r_s(r_i+l_i)}{n}+g(l)r_i+r\,(1-n)\,r_s=0,\\ d\dfrac{\partial^2 r_i}{\partial x^2}-b\dfrac{\partial r_i}{\partial x}+\alpha\dfrac{r_s(r_i+l_i)}{n}-g(l)r_i+r\,(1-n)\,r_i=0,\\ d\dfrac{\partial^2 l_s}{\partial x^2}-b\dfrac{\partial l_i}{\partial x}+f_s(x)-\gamma l_s-\alpha\dfrac{l_s(r_i+l_i)}{n}+g(l)l_i+r\,(1-n)\,l_s=0,\\ d\dfrac{\partial^2 l_i}{\partial x^2}-b\dfrac{\partial l_i}{\partial x}+f_I(x)-\gamma l_i+\alpha\dfrac{l_s(r_i+l_i)}{n}-g(l)l_i+r\,(1-n)\,l_i=0,\\ r_s'(0)=r_i'(0)=l_s'(0)=l_i'(0)=0,\quad r_s'(+\infty)=r_i'(+\infty)=l_s'(+\infty)=l_i'(+\infty)=0,\\ n=r_s+r_i+l_s+l_i,\quad l=l_s+l_i. \end{cases} \tag{2.4}$$

We will not include the proof here as it follows the same methodology as outlined in [18]. For those interested in the details, we recommend reading the work by the author in [18].

3. Existence of non-constant positive steady states

In this section, our objective is to demonstrate the existence of a non-constant positive solution for the elliptic system (2.4). To achieve this, we will employ the Leray–Schauder degree theory. The proof will proceed in two stages: Initially, we will establish the existence of a solution within the bounded interval $(0,a)$, followed by extending this result to $(0,+\infty)$.

In addition, we will provide several integral estimates in the ensuing theorems, which will be necessary for the continuation of our analysis.

Lemma 3.1. *Every positive solution to the elliptic system (2.4) within the finite interval* $(0,a)$ *exhibits boundedness.*

Proof. First, by adding the first two equations of (2.4), we get:

$$-d(r_s+r_i)''+b(r_s+r_i)'=r(1-n)(r_s+r_i)\le r(1-r_s-r_i)(r_s+r_i).$$

Therefore, according to the strong maximum principle, it follows that $r_s+r_i\le 1$. Similarly, we can see that:

$$\begin{aligned}-d(l_s+l_i)''+b(l_s+l_i)'&=f_s(x)+f_I(x)-\gamma(l_s+l_i)+r(1-n)(l_s+l_i)\\ &\le\frac{\|f_s\|_{L^\infty}}{K}+\frac{\|f_I\|_{L^\infty}}{K}-(\gamma-r)(l_s+l_i).\end{aligned}$$

Thus,

$$w+z\le\frac{\|f_s\|_{L^\infty}+\|f_I\|_{L^\infty}}{K(\gamma-r)}.$$

□

The preceding Lemma yields the subsequent result.

Lemma 3.2. *If* $(r_s, r_i, l_s, l_i) \in \left(L^2(0,a)\right)^4$ *represents a positive solution to the elliptic system (2.4), then:*

$$(r_s, r_i, l_s, l_i) \in \left(H^2(0,a)\right)^4.$$

Proof. From the first equation of (2.4), we have:

$$\int_0^a (dr_s'' - br_s')^2 dx = \int_0^a \left(-\alpha \frac{r_s(r_i + l_i)}{n} + r_i g(l) + r(1-n)r_s\right)^2 dx,$$

which implies:

$$\begin{aligned}\int_0^a (dr_s'')^2 dx + \int_0^a (br_s')^2 dx = &\int_0^a \alpha^2 \frac{r_s^2(r_i+l_i)^2}{n^2} dx + \int_0^a (r_i)^2 (g(l))^2 dx \\ &+ r^2 \int_0^a (1-n)^2 r_s^2 dx - 2\alpha \int_0^a \frac{r_s r_i (l_s + l_i)}{n} g(l) dx \\ &- 2\alpha r \int_0^a \frac{r_s^2 (l_s + l_i)}{n} (1-n)\, dx + 2r \int_0^a r_s r_i g(l)(1-n) dx.\end{aligned}$$

In the proof of Lemma 3.1, we have established that: $r_s + r_i \le 1$ and

$$l_s + l_i \le \frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)},$$

hence,

$$\begin{aligned}\int_0^a (dr_s'')^2 dx + \int_0^a (br_s')^2 dx \le &\ \alpha^2 \int_0^a r_s^2 dx + (g(0))^2 \int_0^a r_i^2 dx + 2rg(0)\|r_s\|_{L^2}\|r_i\|_{L^2} \\ &+ 2\alpha r \frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)} \int_0^a r_s^2 dx + r^2 \int_0^a r_s^2 dx \\ &+ r^2 \left(\frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)}\right)^2 \int_0^a r_s^2 dx.\end{aligned}$$

We deduce that:

$$\|r_s'\|_{L^2}^2 + \|r_i''\|_{L^2}^2 \le c\left(\|r_s\|_{L^2}^2 + \|r_i\|_{L^2}^2\right) + 2rg(0)\|r_s\|_{L^2}\|r_i\|_{L^2} < +\infty,$$

where:

$$c = \alpha^2 + (g(0))^2 + r^2 + 2\alpha r \frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)} + r^2 \left(\frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)}\right)^2.$$

Therefore, $u \in H^2(0,a)$. In the same manner, it leads r_i, l_s, $l_i \in H^2(0,a)$. □

The remainder of this section is focused on establishing the existence of a non-constant positive solution to the system (2.4) by employing the Leray-Schauder theorem.

Theorem 3.1 (Leray-Schauder [7,15]). *Let Z be a real Banach space, Ω a bounded, open subset of Z, and $\pi : [s_1, s_2] \times \bar{\Omega} \to Z$ defined as $\pi(\tau, u) = u - T(\tau, u)$, where T is a compact map. We also assume that:*

$$\pi(\tau, u) =\neq 0 \quad (\tau, u) \in [s_1, s_2] \times \partial\Omega.$$

If

$$\deg(\pi, \Omega, 0) \neq 0,$$

then $\pi(\tau, u) = 0$ possesses a solution in Ω for all $s_1 \le \tau \le s_2$.

Our strategy is to write (2.4) as a parametric system to apply the Leray-Schauder theorem. Let's examine the system (S_τ) parametrized by $0 \le \tau \le 1$, formulated as follows:

$$\begin{cases} -dr_s'' + \tau b r_s' = -\alpha \dfrac{r_s(r_i+l_i)}{n} + g(l)r_i + r\left(1-r_s-r_i\right)r_s - \tau r(l_s+l_i)r_s, \\ -dr_i'' + \tau b r_i' = \alpha \dfrac{r_s(r_i+l_i)}{n} - g(l)r_i + r\left(1-r_s-r_i\right)r_i - \tau r(l_s+l_i)r_i, \\ -dl_s'' + \tau b l_s' = \tau f_s(x) - \tau\gamma l_s - \alpha \dfrac{l_s(r_i+l_i)}{n} + g(l)l_i + r\left(1-l_s-l_i\right)l_s - \tau r(l_s+l_i)l_s, \\ -dl_i' + \tau b l_i' = \tau f_I(x) - \tau\gamma l_i + \alpha \dfrac{l_s(r_i+l_i)}{n} - g(l)l_i + r\left(1-l_s-l_i\right)l_i - \tau r(r_s+r_i)l_i, \\ r_s'(0) = r_i'(0) = l_s'(0) = l_i'(0) = 0, \quad r_s'(a) = r_i'(a) = l_s'(a) = l_i'(a) = 0. \end{cases} \tag{3.1}$$

Such that if we take $\tau = 1$ we obtain the system (2.4). Let's start by investigating the system (S_0). In fact, $(r_s^*, r_i^*, l_s^*, l_i^*)$ defined by:

$$r_s^* = l_s^* = \frac{g(1)}{\alpha}, \quad r_i^* = l_i^* = 1 - \frac{g(1)}{\alpha} \tag{3.2}$$

is a solution of (S_0).

Lemma 3.3. *If $\alpha > G(K)$, then $(r_s^*, r_i^*, l_s^*, l_i^*)$ stands as the only strictly positive solution to (S_0).*

Proof. Consider a positive solution (r_s, r_i, l_s, l_i) of (S_0), then $r_s + r_i$ adheres to the equation:

$$-d(r_s+r_i)'' = r\left(1-r_s-r_i\right)(r_s+r_i), \tag{3.3}$$

alongside the boundary conditions:

$$(r_s+r_i)'(0) = (r_s+r_i)'(a) = 0.$$

We multiply the equation (3.3) by $(1 - r_s - r_i)$ and we integrate over $(0, a)$, we obtain:

$$r\int_0^a (1-r_s-r_i)^2(r_s+r_i)dx = -\int_0^a ((r_s+r_i)')^2 dx \le 0,$$

which implies that $r_s + r_i = 1$. Furthermore, $l_s + l_i$ verifies:

$$\begin{cases} -d(l_s+l_i)'' = r\left(1-l_s-l_i\right)(l_s+l_i), \\ (l_s+l_i)'(0) = (l_s+l_i)'(a) = 0. \end{cases} \tag{3.4}$$

Hence, $l_s + l_i = 1$. Moreover, by summing the first and the third equations of (S_0), we get:

$$\begin{cases} -(r_s+l_s)'' = \dfrac{\alpha(r_i+l_i)}{2}\left(\dfrac{2g(1)}{\alpha} - r_s - l_s\right), \\ (r_s+l_s)'(0) = (r_s+l_s)'(a) = 0. \end{cases} \tag{3.5}$$

Multiplying (3.5) by $\left(\frac{2g(1)}{\alpha} - r_s - l_s\right)$ and integrating over $(0, a)$, it follows:

$$\int_0^a (r_i+l_i)\left(\frac{2g(1)}{\alpha} - r_s - l_s\right)^2 dx = -\int_0^a ((r_s+l_s)')^2 dx,$$

it leads to $r_s + l_s = \frac{2g(1)}{\alpha}$. Consequently, the first equation of (S_0) becomes:

$$-dr_s'' = (g(1) - \alpha r_s), \quad r_s'(0) = r_s'(a) = 0.$$

Which gives $r_s = \frac{g(1)}{\alpha}$, hence $r_i = 1 - r_s = 1 - \frac{g(1)}{\alpha}$. Similarly, we will obtain l_s and l_i. □

Define $\left(I - d\frac{d^2}{dx^2}\right)^{-1}$ as the inverse operator of $\left(I - d\frac{d^2}{dx^2}\right)$, with homogeneous Neumann boundary conditions, where I represents the identity operator. Consequently, the system (S_τ) is equivalent to:

$$\begin{cases} \left(I - d\dfrac{d^2}{dx^2}\right)^{-1}\left[r_s - \alpha\dfrac{r_s(r_i + l_i)}{n} + g(l)r_i + r\left(1 - r_s - r_i\right)r_s + \tau\eta_1(r_s, r_i, l_s, l_i)\right] = r_s, \\ \left(I - d\dfrac{d^2}{dx^2}\right)^{-1}\left[r_i + \alpha\dfrac{r_s(r_i + l_i)}{n} - g(l)r_i + r\left(1 - r_s - r_i\right)r_i + \tau\eta_2(r_s, r_i, l_s, l_i)\right] = r_i, \\ \left(I - d\dfrac{d^2}{dx^2}\right)^{-1}\left[l_s - \dfrac{\alpha l_s(r_i + l_i)}{n} + g(1)l_i + r\left(1 - l_s - l_i\right)l_i + \tau\eta_3(r_s, r_i, l_s, l_i)\right] = l_s, \\ \left(I - d\dfrac{d^2}{dx^2}\right)^{-1}\left[l_i + \dfrac{\alpha l_s(r_i + l_i)}{n} - g(l)l_i + r\left(1 - l_s - l_i\right)l_i + \tau\eta_4(r_s, r_i, l_s, l_i)\right] = l_i, \\ r_s'(0) = r_i'(0) = l_s'(0) = l_i'(0) = 0, \quad r_s'(a) = r_i'(a) = l_s'(a) = l_i'(a) = 0, \end{cases}$$

where,

$$\begin{aligned} \eta_1(r_s, r_i, l_s, l_i) &= -br_s' - r(l_s + l_i)r_s, \\ \eta_2(r_s, r_i, l_s, l_i) &= -br_i' - r(l_s + l_i)r_i, \\ \eta_3(r_s, r_i, l_s, l_i) &= -bl_s' + f_s(x) - \gamma l_s - r(r_s + r_i)l_s, \\ \eta_4(r_s, r_i, l_s, l_i) &= -bl_i' + f_I(x) - \gamma l_i - r(r_s + r_i)l_i. \end{aligned}$$

Now, we define the mapping T_τ from $\left(W^{1,4}(0,a)\right)^4$ to itself, by:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = T_\tau \begin{pmatrix} r_s \\ r_i \\ l_s \\ l_i \end{pmatrix},$$

where $0 \leq \tau \leq 1$ and,

$$\begin{aligned} \varphi_1 &= \left(I - d\frac{d^2}{dx^2}\right)^{-1}\left[r_s - \alpha\frac{r_s(r_i + l_i)}{n} + g(l)r_i + r\left(1 - r_s - r_i\right)r_s + \tau\eta_1(r_s, r_i, l_s, l_i)\right], \\ \varphi_2 &= \left(I - d\frac{d^2}{dx^2}\right)^{-1}\left[r_i + \alpha\frac{r_s(r_i + l_i)}{n} - g(l)r_i + r\left(1 - r_s - r_i\right)r_i + \tau\eta_2(r_s, r_i, l_s, l_i)\right], \\ \varphi_3 &= \left(I - d\frac{d^2}{dx^2}\right)^{-1}\left[l_s - \frac{\alpha l_s(r_i + l_i)}{n} + g(1)l_i + r\left(1 - l_s - l_i\right)l_i + \tau\eta_3(r_s, r_i, l_s, l_i)\right], \\ \varphi_4 &= \left(I - d\frac{d^2}{dx^2}\right)^{-1}\left[l_i + \frac{\alpha l_s(r_i + l_i)}{n} - g(l)l_i + r\left(1 - l_s - l_i\right)l_i + \tau\eta_4(r_s, r_i, l_s, l_i)\right]. \end{aligned}$$

As the embeddings $W^{2,2}(0,a) \hookrightarrow W^{1,4}(0,a) \hookrightarrow L^\infty(0,a)$ are compact, the mapping T_τ becomes compact. It can be observed that (r_s, r_i, l_s, l_i) serves as a solution to (S_τ) if and only if it acts as a fixed point of T_τ.

Let us now determinate the topological degree identified in Theorem 3.1. Given that the degree remains invariant under homotopy, then:

$$deg(T_\tau, \Omega, 0) = deg(T_0, \Omega, 0). \tag{3.6}$$

Obviously, The linearization of T_0 around the fixed point $(r_s^*, r_i^*, l_s^*, l_i^*)$ is expressed as:

$$DT_\tau(r_s^*, r_i^*, l_s^*, l_i^*) = \left(I - d\frac{d^2}{dx^2}\right)^{-1} \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix},$$

where:

$$\begin{aligned}
A_{1,1} &= 1 - \frac{\alpha - g(1)}{2\alpha}(2\alpha - g(1)) - \frac{rg(1)}{\alpha}, \quad A_{1,2} = \frac{g(1)}{2\alpha}(2\alpha - 2r - g(1)),\\
A_{1,3} &= \frac{\alpha - g(1)}{2\alpha}(2Kg'(1) + g(1)), \quad A_{1,4} = -\frac{g(1)}{2} + J_{1,3},\\
A_{2,1} &= \frac{\alpha - g(1)}{2\alpha}(2\alpha - 2r - g(1)), \quad A_{2,2} = 1 - r - \frac{g(1)}{2\alpha}(2\alpha + 2r - g(1)),\\
A_{2,3} &= -J_{1,3}, \quad A_{2,4} = -A_{1,4}, \quad A_{3,1} = \frac{\alpha - g(1)}{2\alpha}g(1),\\
A_{3,2} &= \frac{(g(1))^2}{2\alpha}, \quad A_{3,3} = 1 - \frac{rg(1)}{\alpha} + \frac{\alpha - g(1)}{2\alpha}(g(1) - 2\alpha + 2Kg'(1)),\\
A_{3,4} &= -\frac{g(1)}{2\alpha}(\alpha + 2r) + A_{1,3}, \quad A_{4,1} = -\frac{\alpha - g(1)}{2\alpha}g(1), \quad A_{4,2} = -\frac{(g(1))^2}{2\alpha},\\
A_{4,3} &= \frac{\alpha - g(1)}{2\alpha}(2\alpha - g(1) - 2Kg'(1) - 2r), \quad A_{4,4} = A_{4,3} + \frac{3g(1)}{2} + 1 - \alpha.
\end{aligned}$$

To determine the topological degree, we need to identify the negative or zero eigenvalues $-\mu$ of the operator $I - DT_\tau(r_s^*, r_i^*, l_s^*, l_i^*)$ for $\tau = 0$. Essentially, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ represents an eigenfunction corresponding to the eigenvalue $-\mu$ if and only if $[I - DT_0(r_s^*, r_i^*, l_s^*, l_i^*)]\,\psi = -\mu\psi$, or equivalently, $[DT_0(r_s^*, r_i^*, l_s^*, l_i^*)]\,\psi = (1+\mu)\psi$. This equivalence leads to the following system:

$$\begin{cases}
-d(1+\mu)\psi_1'' = (A_{1,1} - \mu - 1)\,\psi_1 + A_{1,2}\psi_2 + A_{1,3}\psi_3 + A_{1,4}\psi_4,\\
-d(1+\mu)\psi_2'' = A_{2,1}\psi_1 + (A_{2,2} - \mu - 1)\psi_2 + A_{2,3}\psi_3 + A_{2,4}\psi_4,\\
-d(1+\mu)\psi_3'' = A_{3,1}\psi_1 + A_{3,2}\psi_2 + (A_{3,3} - \mu - 1)\psi_3 + A_{3,4}\psi_4,\\
-d(1+\mu)\psi_4'' = A_{4,1}\psi_1 + A_{4,2}\psi_2 + A_{4,3}\psi_3 + (A_{4,4} - \mu - 1)\psi_4,
\end{cases} \tag{3.7}$$

possesses infinitely many real eigenvalues: $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \longrightarrow +\infty$.

Using these values as a foundation, we will determine the eigenvalues of $I - DT_0(r_s^*, r_i^*, l_s^*, l_i^*)$.

Lemma 3.4. *If $\alpha > G(K)$, then there exists a positive integer m such that the negative eigenvalues $-\mu_k$ of $I - DT_0(r_s^*, r_i^*, l_s^*, l_i^*)$ are characterized by:*

$$-\mu_k^{(1)} = -\frac{r - d\lambda_k}{1 + d\lambda_k} < 0,$$

for every $k \in \{1, 2, \ldots, m\}$.

Proof. By referring to [4, 16, 31], the system (3.7) is the same as:

$$\left[\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} - (\mu+1)(d\lambda_k+1)\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

The determinant of the above matrix was obtained through lengthy calculations, which is presented as follows:

$$\begin{aligned} N(\mu) = {} & (1 - r - (\mu+1)(d\lambda_k+1))^2 \left(1 - (\mu+1)(d\lambda_k+1) - \frac{(\alpha - g(1))^2}{\alpha}\right) \\ & \times (1 - (\alpha - g(1)) - (\mu+1)(d\lambda_k+1)). \end{aligned}$$

Hence, we have the following eigenvalue of multiplicity two:

$$\mu_k^{(1)} = \frac{r - d\lambda_k}{1 + d\lambda_k} \tag{3.9}$$

and the negative eigenvalues:

$$\begin{aligned} \mu_k^{(2)} &= -\frac{(\alpha - g(1))^2 + d\alpha\lambda_k}{\alpha(1 + d\lambda_k)} < 0, \\ \mu_k^{(3)} &= -\frac{(\alpha - g(1)) + d\lambda_k}{1 + d\lambda_k} < 0. \end{aligned}$$

Considering the formula for $\mu_k^{(1)}$ in (3.9), it is evident that there exists a positive integer m such that $\mu_k^{(1)} > 0$ for all $k \in 1, 2, \ldots m$. This concludes the proof. □

Now, we can summarize the theorem guaranteeing the existence of a non-constant positive solution to the elliptic system (2.4) over the interval $(0, \infty)$.

Theorem 3.2. *If $\alpha > G(K)$, then there exists a non-constant positive solution of the elliptic system (2.4) in $(0, \infty)$.*

Proof. The proof proceeds in two steps. Initially, we establish existence within the bounded interval $(0, a)$, and subsequently, we extend this result to the interval $(0, +\infty)$.

Step 1. Let us consider Ω an open set in $(W^{1,4}(0, a))^4$ defined by:

$$\Omega = \left\{ (r_s, r_i, l_s, l_i) \in (W^{1,4}(0,a))^4 : \begin{array}{c} C_1 < r_s < M_1 \\ C_2 < r_i < M_2 \\ C_3 < l_s < M_3 \\ C_4 < l_i < M_4 \end{array} \right\},$$

such that C_i, M_i, for $i = 1, \dots, 4$, are positive constants independent of a being selected such that $u - T_\tau u \neq 0$ in $\partial\Omega$. From Lemma 3.1, any positive solution (r_s, r_i, l_s, l_i) of the system (2.4) satisfies:

$$r_s + r_i \leq 1 \quad \text{and} \quad l_s + l_i \leq \frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)}.$$

Thus, we simply choose $M_1,\ M_2 > 1$ and:

$$M_3,\ M_4 > \frac{\|f_s\|_{L^\infty} + \|f_I\|_{L^\infty}}{K(\gamma - r)}.$$

Furthermore, if we take $C_1 < r_s^*$, $C_2 < r_i^*$, $C_3 < l_s^*$ and $C_4 < l_i^*$, then $(r_s^*, r_i^*, l_s^*, l_i^*)$ is the only constant solution of $u - T_\tau u = 0$ in $\Omega \cup \partial\Omega$. Moreover, employing the standard Leray–Schauder degree theory [22], we obtain:

$$\deg(T_0, \Omega, 0) = \sum_{\theta \in (I - T_0)(0)} (-1)^{\psi_j(\theta)}$$

where $\psi_j(\theta)$ is the sum of the algebraic multiplicities of negative eigenvalues of $I - DT_0(r_s^*, r_i^*, l_s^*, l_i^*)$. According to Lemma 3.4:

$$\deg(T_0, \Omega, 0) = (-1)^{2(m+1)} = 1 \neq 0.$$

Which prove the existence in $(0, a)$.

Step 2. As seen in step 1, for every $n \in \mathbb{N}^*$, there exists a solution of (2.4) in $\big(W^{1,4}(0, n)\big)^4$, let us note this solution by $\tilde{U}_n = (\tilde{r}_{s,n}, \tilde{r}_{i,n}, \tilde{l}_{s,n}, \tilde{l}_{i,n})$ and we define a sequence $(U_n)_{n\geq 1}$ by:

$$U_n = \begin{cases} \tilde{U}_n, & \text{for } x \in (0, n), \\ 0, & \text{for } x \in [n, +\infty). \end{cases}$$

The right-hand side of the elliptic system (2.4) is represented by (G_1, G_2, G_3, G_4). For any $n \in \mathbb{N}^*$, U_n satisfies:

$$\begin{cases} d\displaystyle\int_0^{+\infty} r'_{s,n}\phi'_1 dx - b\int_0^{+\infty} r_{s,n}\phi'_1 dx = \int_0^{+\infty} G_1(x, r_{s,n}, r_{i,n}, l_{s,n}, l_{i,n})\phi_1\, dx, \\ d\displaystyle\int_0^{+\infty} r'_{i,n}\phi'_2 dx - b\int_0^{+\infty} r_{i,n}\phi'_2 dx = \int_0^{+\infty} G_2(x, r_{s,n}, r_{i,n}, l_{s,n}, l_{i,n})\phi_2\, dx, \\ d\displaystyle\int_0^{+\infty} l'_{s,n}\phi'_3 dx - b\int_0^{+\infty} l_{s,n}\phi'_3 dx = \int_0^{+\infty} G_3(x, r_{s,n}, r_{i,n}, l_{s,n}, l_{i,n})\phi_3\, dx, \\ d\displaystyle\int_0^{+\infty} l'_{i,n}\phi'_4 dx - b\int_0^{+\infty} l_{i,n}\phi'_4 dx = \int_0^{+\infty} G_4(x, r_{s,n}, r_{i,n}), l_{s,n}, l_{i,n})\phi_4\, dx. \end{cases} \tag{3.10}$$

According to Lemma 3.1 and Lemma 3.2, $(U_n)_{n\geq 1}$ is a bounded sequence in $\big(H^2(0, +\infty)\big)^4$. $\big(H^2(0, +\infty)\big)^4$ is a separable Hilbert space, hence $(U_n)_{n\geq 1}$ admits a weakly convergent subsequence (U_{n_k}) in $\big(H^2(0, +\infty)\big)^4$ i.e. $U_{n_k} \rightharpoonup U$ (see [2]).

Now, let n tend to $+\infty$ in (3.10), we get:

$$\begin{cases} d\displaystyle\int_0^{+\infty} r_s'\phi_1' dx - b\int_0^{+\infty} r_s\phi_1' dx = \int_0^{+\infty} G_1(x, r_s, r_i, l_s, l_i)\phi_1\, dx, \\ d\displaystyle\int_0^{+\infty} r_i'\phi_2' dx - b\int_0^{+\infty} r_i\phi_2' dx = \int_0^{+\infty} G_2(x, r_s, r_i, l_s, l_i)\phi_2\, dx, \\ d\displaystyle\int_0^{+\infty} l_s'\phi_3' dx - b\int_0^{+\infty} l_s\phi_3' dx = \int_0^{+\infty} G_3(x, r_s, r_i, l_s, l_i)\phi_3\, dx, \\ d\displaystyle\int_0^{+\infty} l_i'\phi_4' dx - b\int_0^{+\infty} l_i\phi_4' dx = \int_0^{+\infty} G_4(x, r_s, r_i), l_s, l_i)\phi_4\, dx. \end{cases}$$

What constitutes the variational form of our elliptic system. This concludes the demonstration. □

4. Numerical investigations

This section employs numerical techniques to analyze system (1.9) and confirm Theorem 3.2 through numerical illustrations. The parameters and the form of G chosen in this simulation are solely for demonstrative purposes. The rates of bacteria inflow into the river from the shoreline are specified as follows:

$$F_s(x,t) = \begin{cases} \dfrac{t}{t+1}e^{-x} & \text{if} \quad t \in [10k, 10k+1], \quad k \in \mathbb{N}, \\ 0 \quad \text{otherwise}, \end{cases} \tag{4.1}$$

$$F_I(x,t) = \begin{cases} \dfrac{1}{3}\dfrac{t}{t+1}e^{-x} & \text{if} \quad t \in [10k, 10k+1], \quad k \in \mathbb{N}, \\ 0 \quad \text{otherwise}. \end{cases} \tag{4.2}$$

It is easy to verify that these functions, with $f_I(x) = e^{-x}$ and $f_s(x) = \frac{1}{3}e^{-x}$ satisfy the conditions (1.4)-(1.5) (see Figure 1). We have used the function $G(L) = e^{-L}$. The other parameters are listed in Table 1.

Table 1. Parameter configurations.

Parameter	Values
α	1
K	50
r	0.05
d	1
b	1
γ	0.5

5. Conclusions

Polluted rivers can serve as a source of antibiotic-resistant bacteria due to the presence of high levels of antibiotics and other chemicals in the water. These chemicals can select for antibiotic-

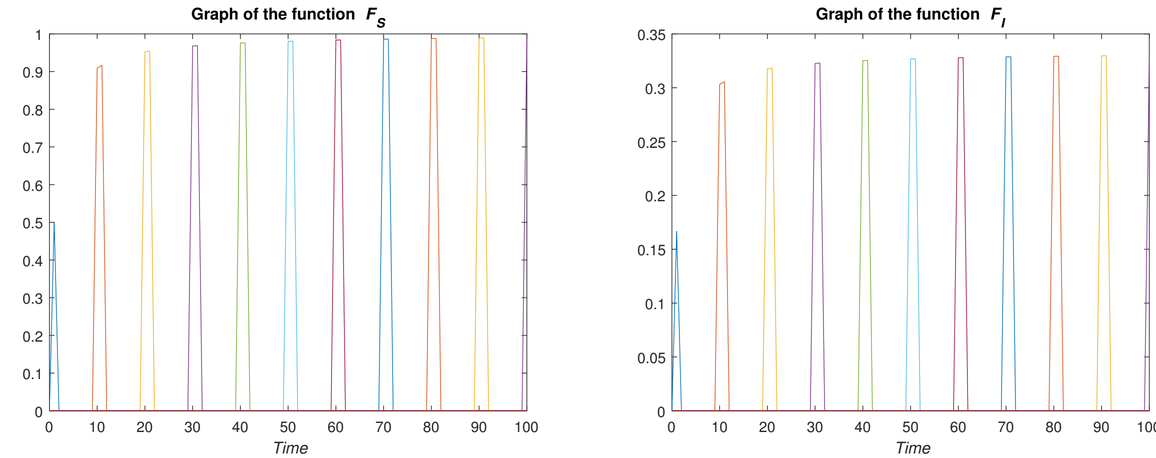

Figure 1. (a) Arrival of antibiotic-resistant (AR) bacteria from land at $x = 0$ (F_I). (b) Arrival of antibiotic-resistant (NAR) bacteria from land at $x = 0$ (F_s).

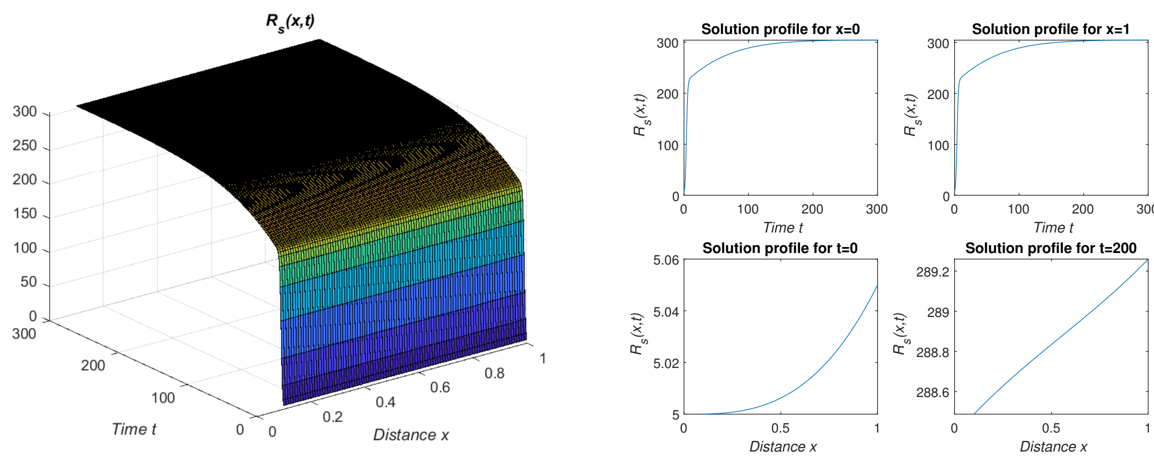

Figure 2. Evolution of R_s in relation to both temporal and spatial variables.

resistant bacteria, promoting their growth and spread. Additionally, polluted rivers can also serve as a means of transportation for these bacteria, allowing them to spread to other locations.

The spatiotemporal dispersion of antibiotic-resistant bacteria in contaminated rivers is subject to significant variation, contingent upon numerous factors, encompassing pollution levels, coexisting bacterial species, and environmental parameters like temperature and water flow.

Within our paper, a mathematical model has been formulated for the propagation of bacteria within a river, incorporating the influence of pollution along the shoreline resulting from hospital wastewater or agricultural discharges. The model, has been referred to as a non-autonomous partial differential system which may provides a comprehensive description of the spatio-temporal dynamics of bacteria in the river environment.

Firstly, we have demonstrated the well-posedness of the problem by establishing the existence of global solutions in $C\left(\mathbb{R}_+, \left(L^1(0,+\infty)\right)^4\right)$ using maximum principles. From a biological perspective, the integrals $\int_0^{+\infty} R_s(x,t)dx$, $\int_0^{+\infty} R_I(x,t)dx$, $\int_0^{+\infty} L_s(x,t)dx$, and $\int_0^{+\infty} L_I(x,t)dx$ correspond to the bacterial species densities in the river at time t.

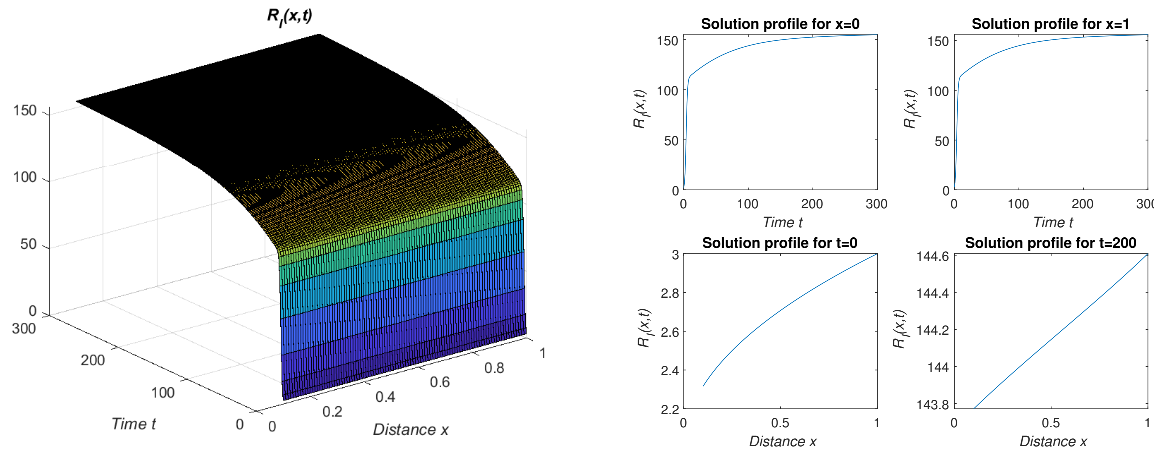

Figure 3. Evolution of R_I in relation to both temporal and spatial variables.

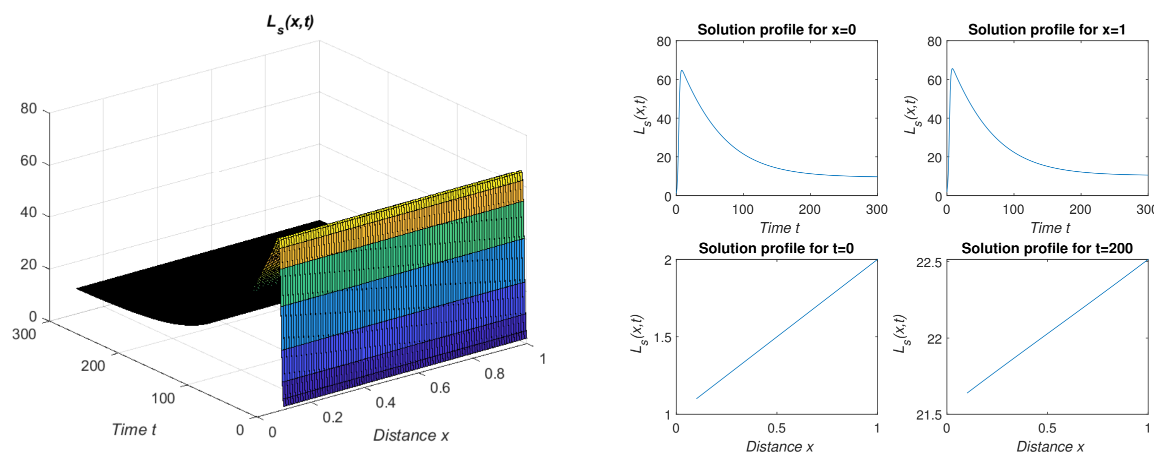

Figure 4. Evolution of L_s in relation to both temporal and spatial variables.

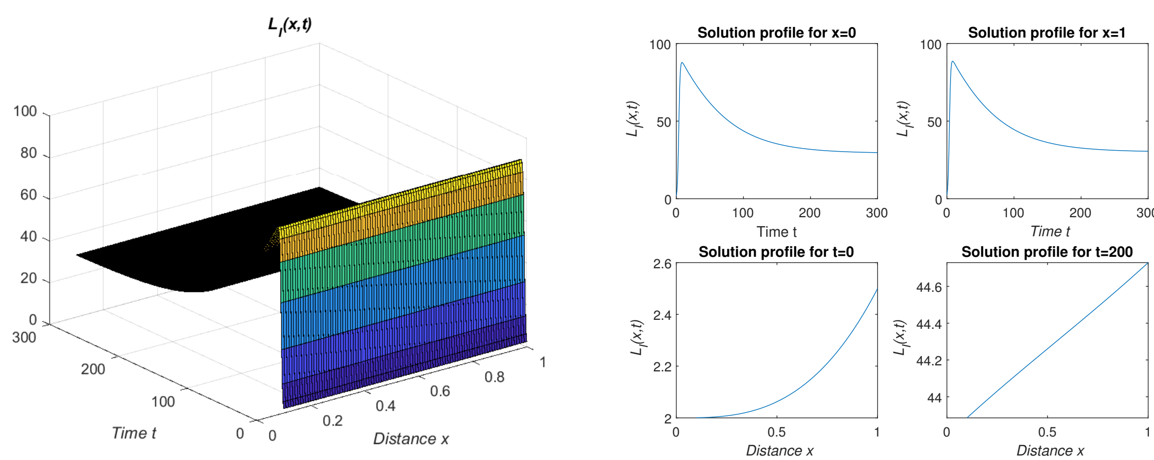

Figure 5. Evolution of L_I in relation to both temporal and spatial variables.

In the second part, our focus was on the qualitative analysis of solutions, as the system's flow is Characterized by the solutions of the corresponding elliptic system, we confirmed the presence of a non-constant steady state across all functions G that satisfied (H1)-(H3), provided that α was greater than $G(K)$. To do so, we demonstrated the smoothness and regularity of solutions to the elliptic system and we used Leray-Schauder's degree theory. From a biological perspective, this suggests that when the transmission rate of the antibiotic-resistant gene exceeds its loss rate, both resistant and non-resistant bacteria can coexist over time. The environmental conditions where bacteria reside can dictate whether the acquisition or loss rate of the antibiotic-resistant gene predominates. Since maintaining the antibiotic resistance gene demands energy, bacteria may shed this gene more frequently to conserve energy in the absence of antibiotic exposure. Consequently, regions immediately downstream of hospitals, where antibiotic residues are more likely to enter the river, may harbor persistent resistant bacteria compared to other river sections.

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