Journal of Applied Analysis and Computation
Volume 16, Number 3, June 2026, 1552–1572
Website:http://www.jaac-online.com
DOI:10.11948/20250207

BIFURCATION OF RADIAL POSITIVE SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE PRESCRIBED MEAN CURVATURE EQUATION IN THE FRIEDMANN-LEMAîTRE-ROBERTSON-WALKER SPACETIME*

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Abstract In this paper, we investigate the bifurcation of radial positive solutions of the nonlinear Dirichlet problem associated with the prescribed mean curvature equation in the Friedmann-Lemaître-Robertson-Walker spacetime

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\Big)=\lambda\Big(\dfrac{Nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}}-Nf(\varphi^{-1}(v))H(|x|,\varphi^{-1}(v))\Big), \text{ in } \mathcal{B},\\ v=0, \quad \text{on } \partial\mathcal{B}, \end{cases}$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$, λ is a positive parameter, the function f belongs to $C^\infty(I)$ and satisfies $f>0$, I is an open interval in $\mathbb{R}$, φ is the function defined by $\varphi(s)=\int_0^s \frac{dt}{f(t)}$, φ^{-1} represents the inverse function of φ, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, and the function $H:[0,1]\times I\to\mathbb{R}$ is continuous, which is referred to as the mean curvature function. Our findings demonstrate the existence of at least one, two or three radial positive solutions to the aforementioned problem. The proofs are mainly based on the directions of the bifurcation.

Keywords Bifurcation, radial positive solutions, singular ϕ-Laplacian, mean curvature equation, Friedmann-Lemaître-Robertson-Walker spacetime.

MSC(2010) 34B08, 34B18, 34C23.

1. Introduction

The study of problems related to a spatially homogeneous and isotropic universe often involves the following Dirichlet problem for the prescribed mean curvature equation in the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime:

$$\begin{cases} \operatorname{div}\Big(\dfrac{\nabla u}{f(u)\sqrt{f^2(u)-|\nabla u|^2}}\Big)+\dfrac{f'(u)}{\sqrt{f^2(u)-|\nabla u|^2}}\Big(N+\dfrac{|\nabla u|^2}{f^2(u)}\Big)=NH(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $f\in C^\infty(I)$ and satisfies $f>0$, I is an open interval in $\mathbb{R}$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, and $H:[0,1]\times I\to\mathbb{R}$ is continuous, which is referred

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[*] The authors were supported by the Lanzhou Youth Science and Technology Talents Innovation Project (2024-QN-61) and the Young Doctor Project of the Department of Education of Gansu Province (2026-QB-019).

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to as the mean curvature function. In cosmology, the function $f(t)$ is called the *scale factor* or *warping function* and it is interpreted as the radius of the universe at time t, and the sign of its derivative indicates if universe is expanding or contracting at given time [2,5,22,41,45]. Observe that for the particular case $f(t) \equiv 1$, we recover the Dirichlet problem for the prescribed mean curvature equation in the Minkowski space:

$$\begin{cases} \operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = NH(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This kind of problem plays a very important role in fields such as the prescribed mean curvature problem in differential geometry, general relativity, cosmology and the Born-Infeld model in electrodynamics, see [1,2,5,7,11,13,15,22,24,25,32,39,41,42,44–47,51].

Around 2010, scholars such as Brubaker, Pelesko [10], Cano-Casanova, López-Gómez, Takimoto [14], Hung, Cheng, Wang, Chuang [29], as well as Pan and Xing [43] commenced research on the global bifurcation structure of solutions of the Dirichlet problem for the prescribed mean curvature equation in the Euclidean space (changing the "-" in the denominator on the left side of the equation in (1.2) to a "+"). Their work aimed to address problems such as the modeling of micro-electromechanical system devices. Since then, as the theory of the mean curvature equation has become increasingly mature and its application value has become more evident, several scholars and research teams, including Dai and Wang [18–20,23], Huang and Wang [27,28], Lee, Sim and Yang [31,48], the academic group led by professor Ma [35–38], and Zhang and Feng [50], among others, have employed various theories and methods. These include the theory of linear compact operators, the time-mapping analysis method, bifurcation theory, and the Sturm comparison theorem. They have carried out a detailed and comprehensive exploration of the global bifurcation structure of the positive (nodal) solutions to the Dirichlet problem of the prescribed mean curvature equation in the Euclidean and the Minkowski spaces within bounded domains.

For example, by using the global bifurcation theory, Dai [18] and Ma, Gao, Lu [35] have obtained the global structure of positive solutions of (1.2) when $\Omega = B(R) := \{x \in \mathbb{R}^N : |x| < R\}$, and Dai and Wang [23] have also obtained the global structure of radial nodal solutions for the same problem. Dai [19,20] has studied the global structure of positive solutions of (1.2) when Ω is a general domain in $\mathbb{R}^N$, Ma and Xu [37] have also considered the same problem but their nonlinear term H is dependent on the ∇u. Huang, Wang [27,28] and Zhang, Feng [50] have acquired the global structure of positive solution curves of (1.2) when $\Omega := (-L, L)$ and H is autonomous by employing the time-mapping analysis method. Yang, Lee and Sim [48] have investigated the existence and multiplicity of nodal radial solutions of (1.2) on the exterior domain of a ball, i.e., $\Omega := \{x \in \mathbb{R}^N : |x| > R\}$, $R > 0$ by using the global bifurcation theory, the modified Gronwall-Bellman inequality and the modified Picone identity. Lee, Sim and Yang [31] have also obtained the S-shaped and Σ-shaped bifurcation curves for the one dimensional nonautonomous problem (1.2). Ma, He, Su [36] and Ma, Xu [38] have obtained the S-shaped connected components of positive solutions and nodal solutions of (1.2) when $\Omega := (0, L)$.

Based on the research on the global bifurcation structure of solutions of (1.2), it becomes possible to study the global bifurcation structure of solutions of (1.1). But compared with the research on the prescribed mean curvature equation in Minkowski space, the research on the prescribed mean curvature equation in FLRW spacetime is relatively scarce. Only in recent years, by using the Schauder fixed point theorem, Bereanu, de la Fuente, Romero and Torres [2,25] have considered the existence of radial solutions of the nonlinear Dirichlet problem for the prescribed mean curvature equation in the Friedmann-Lemaître-Robertson-Walker spacetime

$$\begin{cases} \operatorname{div}\Big(\dfrac{\nabla u}{f(u)\sqrt{f^2(u)-|\nabla u|^2}}\Big)+\dfrac{f'(u)}{\sqrt{f^2(u)-|\nabla u|^2}}\Big(N+\dfrac{|\nabla u|^2}{f^2(u)}\Big)=NH(|x|,u), & \text{in } \mathcal{B}, \\ u=0, \quad \text{on } \partial\mathcal{B}, \end{cases} \tag{1.3}$$

where $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$. By using the Rabinowitz's global bifurcation theory, Dai, Romero and Torres [22], Luo and Dai [32] have obtained the global bifurcation of radial positive solutions and radial nodal solutions of (1.3). Xu and Ma [47] have concerned the non-spurious solutions of (1.3) by using the lower and upper solutions. Mawhin and Torres [41, 45] have provided some sufficient conditions for the existence of radial solutions of a Neumann problem by the Leray-Schauder degree theory. Ma, Zhao and Su [39, 51] have concerned the global structure of radial positive solution and the existence of infinitely many radially symmetric sign-changing solutions of a quasilinear indefinite Neumann problem via the bifurcation techniques. Bereanu and Torres [5] have studied the existence of solutions of a Neumann problem with a one-dimensional mean curvature operator in FLRW spacetimes by using the variational approach.

Motivated by the papers [18, 22, 27, 28, 35, 36, 38, 50], in this paper, we intend to use the bifurcation theory to study the direction of bifurcation of radial positive solutions for (1.3) when the nonlinear term H satisfies the asymptotically linear, sublinear, and superlinear growth conditions at $u = 0$, respectively, and as an application, obtain the existence of at least one, two, and three radial positive solutions.

The rest of the paper is organized as follows. In Section 2 we give the main result of this paper. In Section 3 we state some preliminary results, including the properties of the superior limit of a sequence of connected components, the equivalent representation of the studied problem and some properties of its solutions. In Section 4 we study the direction of connected component of radial positive solutions. Section 5 is devoted to proving the main result. Finally in Section 6, we shall use the result in Section 2 and the properties in Section 3 to give some further results and remarks.

For other results concerning the problem associated to prescribed mean curvature equation in Minkowski space, we refer the reader to [3, 4, 6, 8, 9, 12, 16, 17, 30, 40, 49] and references therein.

2. The main result

We adopt the method developed in [22], define the function $\varphi : I \to \mathbb{R}$ as $\varphi(s) = \int_0^s \frac{dt}{f(t)}$, and this φ is an increasing diffeomorphism from I onto $J := \varphi(I)$, with $\varphi(0) = 0$. By making the substitution $v = \varphi(u)$, we can transform the Dirichlet problem (1.3) into the problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\Big)=\dfrac{Nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}}-Nf(\varphi^{-1}(v))H(|x|,\varphi^{-1}(v)), & \text{in } \mathcal{B}, \\ v=0, \quad \text{on } \partial\mathcal{B}. \end{cases}$$

When examining the bifurcation of the solutions of (1.3), essentially, we are addressing the following parameter-dependent problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\Big)=\lambda\Big(\dfrac{Nf'(\varphi^{-1}(v))}{\sqrt{1-|\nabla v|^2}}-Nf(\varphi^{-1}(v))H(|x|,\varphi^{-1}(v))\Big), & \text{in } \mathcal{B}, \\ v=0, \quad \text{on } \partial\mathcal{B}. \end{cases} \tag{2.1}$$

By converting to radial coordinates, we can transform the above problem into the following mixed boundary value problem

$$\begin{cases} -\left(r^{N-1}\phi(v')\right)' = \lambda N r^{N-1}\Big[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(r,\varphi^{-1}(v))\Big], \quad r\in(0,1), \\ v'(0)=v(1)=0, \end{cases} \tag{2.2}$$

here, $\phi(s)=\frac{s}{\sqrt{1-s^2}}$, $\phi:(-1,1)\to\mathbb{R}$ is an increasing homeomorphism with $\phi(0)=0$, we say the ϕ is *singular*, λ is a positive parameter, the function f belongs to $C^\infty(I)$ and satisfies $f>0$, I is an open interval in $\mathbb{R}$, the function $\varphi(s)=\int_0^s \frac{dt}{f(t)}$, φ^{-1} represents the inverse function of φ, and the function $H:[0,1]\times I\to\mathbb{R}$ is continuous.

Let $X=C[0,1]$ with the norm $||v||_\infty := \max\limits_{r\in[0,1]}|v(r)|$. Let $E=\{v\in C^1[0,1]: v'(0)=v(1)=0\}$ with the norm $||v|| := ||v'||_\infty$. We say that a pair $(\lambda,v)\in(0,\infty)\times C^1[0,1]$ is a solution of (2.2) if $||v'||_\infty<1$, $r^{N-1}\phi(v')\in C^1[0,1]$, and (2.2) is satisfied. For (2.2), because the graph of v is spacelike (i.e. $||v'||_\infty<1$), we can deduce that $||v||_\infty<1$. This shows that the non-negative values of v fall within the interval $[0,1]$. Therefore, when discussing (2.2), we always assume that $\varphi^{-1}([0,1])\subset I$. This assumption is equivalent to the condition

$$I_f := \left[0, \int_0^1 f(\varphi^{-1}(s))ds\right] \subset I.$$

Let λ_k denote the k-th eigenvalue of the eigenvalue problem presented as follows

$$\begin{cases} (r^{N-1}v')' + \lambda r^{N-1}v = 0, \quad r\in(0,1), \\ v'(0)=v(1)=0. \end{cases} \tag{2.3}$$

And let ϕ_k be the eigenfunction corresponding to λ_k. It is a well-established fact that

$$0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots, \quad \lim_{k\to\infty}\lambda_k=\infty,$$

and that ϕ_k has exactly $k-1$ zeros in (0,1), see [21, 26].

Throughout this paper, we make the following assumptions:
(A1) $I_f\subset I$;
(F1) $f'(t)\geq 0$, $\frac{f'(t)}{f(t)}\geq H(r,t)$ for any $r\in[0,1], t\in I_f$ and $\frac{f'(t)}{f(t)}>H(r,t)$ for $r\in[0,1], t\in I_f\backslash\{0\}$;
(F2) there exist $f_0\in(0,\infty)$, $H_0\in(-\infty,+\infty)$ with $f_0+H_0\in(0,\infty)$, $\delta\in(0,\frac{1}{32})$, $\theta\in C([0,\infty),[0,\infty))$ with $\theta(t)>0$ for $t\in(0,\varphi^{-1}(\delta)]$, $\xi\in C([0,1]\times[0,\infty),[0,\infty))$ with $\xi(r,t)>0$ for $(r,t)\in[0,1]\times(0,\varphi^{-1}(\delta)]$ such that

$$Nf'(t) = f_0\varphi(t) - \theta(t) \quad \text{for} \quad t\in[0,\varphi^{-1}(\delta)],$$

and

$$Nf(t)H(r,t) = -H_0\varphi(t) - \xi(r,t) \quad \text{for} \quad r\in[0,1], t\in[0,\varphi^{-1}(\delta)],$$

where $\lim\limits_{t\to0^+}\frac{\theta(t)}{\varphi(t)}=0$ and $\lim\limits_{t\to0^+}\frac{\xi(r,t)}{\varphi(t)}=0$ uniformly for $r\in[0,1]$;
(F3) there exists $s_0: s_0\in(\frac{1}{32},\frac{1}{12})$, such that

$$\min_{s\in[\varphi^{-1}(s_0),\varphi^{-1}(4s_0)]} \frac{Nf'(s)-Nf(s)H(r,s)}{\varphi(s)} \geq \frac{27(f_0+H_0)}{5\sqrt{5}\lambda_1}\cdot\eta_1 \quad \text{uniformly for } r\in[0,1],$$

where η_1 is the first positive eigenvalue of the problem

$$\begin{cases}(r^{N-1}v'(r))' + \eta r^{N-1}v(r) = 0, \quad r \in \Big(0, \dfrac{3}{4}\Big),\\ v'(0) = v\Big(\dfrac{3}{4}\Big) = 0.\end{cases}$$

The main result of this paper is as follows.

Theorem 2.1. *Suppose that assumptions (A1) and (F1)-(F3) are all satisfied. Then, there exist* $\lambda_* \in (0, \frac{\lambda_1}{f_0+H_0})$ *and* $\lambda^* > \frac{\lambda_1}{f_0+H_0}$ *such that*
(i) (2.1) has at least one radial positive solution if $\lambda = \lambda_*$*;*
(ii) (2.1) has at least two radial positive solutions if $\lambda_* < \lambda \le \frac{\lambda_1}{f_0+H_0}$*;*
(iii) (2.1) has at least three radial positive solutions if $\frac{\lambda_1}{f_0+H_0} < \lambda < \lambda^*$*;*
(iv) (2.1) has at least two radial positive solutions if $\lambda = \lambda^*$*;*
(v) (2.1) has at least one radial positive solution if $\lambda > \lambda^*$*;*
(vi) $\lim_{\lambda\to\infty} \|v\|_\infty = 1$ *and* $\lim_{\lambda\to\infty} \|v\| = 1$.

Remark 2.1. Let (λ, v) be a solution of (2.2). Given that $|v'| < 1$, we deduce that $||v||_\infty < 1$. This finding implies that the bifurcation diagrams are predominantly determined by the behavior of the expression $Nf'(s) - Nf(s)H(r,s)$ in the neighborhood of $s = 0$.

By employing Theorem 2.1, we can also analyze the radial positive solutions of the nonlinear Dirichlet problem for the prescribed mean curvature equation in the Minkowski spacetime:

$$\begin{cases}\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda N H(|x|, u), \quad \text{in } \mathcal{B},\\ u = 0, \quad \text{on } \partial\mathcal{B}.\end{cases} \tag{2.4}$$

Here, $\mathcal{B}$ is the unit ball in $\mathbb{R}^N$, and λ is a positive parameter. The following assumptions are made:
(F'1) $H(r,t) \le 0$ for any $r \in [0,1], t \in [0,1]$ and $H(r,t) < 0$ for $r \in [0,1], t \in (0,1]$;
(F'2) there exist $H_0 \in (0, +\infty)$, $\delta \in (0, \frac{1}{32})$, $\xi \in C([0,1] \times [0,\infty), [0,\infty))$ with $\xi(r,t) > 0$ for $(r,t) \in [0,1] \times (0, \delta]$ such that

$$NH(r,t) = -H_0 t - \xi(r,t) \quad \text{for} \quad r \in [0,1], t \in [0,\delta],$$

where $\lim_{t\to 0^+} \frac{\xi(r,t)}{t} = 0$ uniformly for $r \in [0,1]$;
(F'3) there exists $s_0 : s_0 \in (\frac{1}{32}, \frac{1}{12})$, such that

$$\min_{s\in[s_0,4s_0]} \frac{-NH(r,s)}{s} \ge \frac{27H_0}{5\sqrt{5}\lambda_1} \cdot \eta_1 \quad \text{uniformly for } r \in [0,1].$$

And we obtain the following consequence.

Corollary 2.1. *Suppose that assumptions (F'1)-(F'3) are satisfied. Then, there exist* $\lambda_{**} \in (0, \frac{\lambda_1}{H_0})$ *and* $\lambda^{**} > \frac{\lambda_1}{H_0}$ *such that*
(i) (2.4) has at least one positive radial solution if $\lambda = \lambda_{**}$*;*
(ii) (2.4) has at least two positive radial solutions if $\lambda_{**} < \lambda \le \frac{\lambda_1}{H_0}$*;*
(iii) (2.4) has at least three positive radial solutions if $\frac{\lambda_1}{H_0} < \lambda < \lambda^{**}$*;*

(iv) (2.4) has at least two positive radial solutions if $\lambda = \lambda^{**}$*;*

(v) (2.4) has at least one positive radial solution if $\lambda > \lambda^{**}$*;*

(vi) $\lim\limits_{\lambda\to\infty} \|u\|_\infty = 1$ *and* $\lim\limits_{\lambda\to\infty} \|u\| = 1$.

Remark 2.2. It should be emphasized that we have studied comparable results regarding problem (2.4) with $N = 1$ in our earlier work. For in-depth information, we refer the reader to [38].

3. Preliminary results

3.1. Superior limit and connected component

Definition 3.1 ([33]). Let S be a Banach space and $\{C_n \mid n = 1, 2, \cdots\}$ be a family of subsets of S. Then the the *superior limit* $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in S \mid \text{there exist } \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x\}.$$

Lemma 3.1 (Lemma 2.4, [33], Lemma 2.2, [34]). *Assume that*

(i) there exist $z_n \in C_n$, $n = 1, 2, \cdots$, *and* $z^* \in S$, *such that* $z_n \to z^*$*;*

(ii) $\lim\limits_{n\to\infty} r_n = \infty$, *where* $r_n = \sup\{||x|| : \ x \in C_n\}$*;*

(iii) for every $R > 0$, $\big(\bigcup\limits_{n=1}^{\infty} C_n\big) \cap B_R$ *is a relative compact set of* S*, where*

$$B_R = \{x \in S \mid ||x|| \le R\}.$$

Then there exists an unbounded connected component $\mathcal{C}$ *in* $\mathcal{D}$ *with* $z^* \in \mathcal{C}$.

3.2. Equivalent representation and some properties of solutions

For simplicity, we define the function

$$e(r, s, t) = \frac{N f'(\varphi^{-1}(s))}{\sqrt{1 - t^2}} - N f(\varphi^{-1}(s)) H(r, \varphi^{-1}(s)).$$

Following the approach of [16], we now introduce a function $\widetilde{f} : [0, 1] \times \mathbb{R} \times (-1, 1) \to \mathbb{R}$. For $r \times t \in [0, 1] \times (-1, 1)$, it is defined as follows:

$$\widetilde{f}(r, s, t) = \begin{cases} e(r, s, t), & \text{if } 0 \le s \le 1, \\ (2 - s)e(r, 1, t), & \text{if } 1 < s < 2, \\ 0, & \text{if } s \ge 2, \\ -\widetilde{f}(r, -s, t), & \text{if } s < 0. \end{cases} \tag{3.1}$$

It is important to note that, in the context of positive solutions, problem (2.2) is equivalent to the same problem with e replaced by $\widetilde{f}$. Subsequently, we will replace e with $\widetilde{f}$. For the sake of simplicity, we will still denote the modified function as e. Next, we define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(y) = \begin{cases} (1 - y^2)^{\frac{3}{2}}, & \text{if } |y| \le 1, \\ 0, & \text{if } |y| > 1. \end{cases} \tag{3.2}$$

Based on the above definitions, we obtain the following conclusion.

Lemma 3.2. *A function $v \in C^1[0,1]$ is a positive solution of (2.2) if and only if it is a positive solution of the following problem*

$$\begin{cases} -(r^{N-1}v')' = r^{N-1}\left[\lambda e(r,v,v')h(v') - \dfrac{N-1}{r}v'^3\right], \quad r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{3.3}$$

Proof. Clearly, a positive solution $v \in C^1[0,1]$ of (2.2) is a positive solution of (3.3). Conversely, assume that $v \in C^1[0,1]$ is a positive solution of (3.3). We aim to prove that $\|v'\|_\infty < 1$. Suppose, for the sake of contradiction, that $\|v'\|_\infty \geq 1$. Then, there exists an interval $[b,c] \subseteq [0,1]$ such that either $v'(b) = 0$, $0 < |v'(r)| < 1$ for $r \in (b,c)$ and $|v'(c)| = 1$, or $|v'(b)| = 1$, $0 < |v'(r)| < 1$ for $r \in (b,c)$ and $v'(c) = 0$. Assume the former case holds. The function v satisfies the equation

$$-(r^{N-1}\phi(v')) = \lambda r^{N-1}e(r,v,v')$$

on $[b,c)$. For each $r \in (b,c)$, integrating over the interval $[b,r]$, we get

$$\left|\phi(v'(r))\right| = \left|\frac{1}{r^{N-1}}\int_b^r \lambda t^{N-1}e(t,v,v')dt\right| \leq M$$

for some constant $M > 0$. Thus,

$$|v'(r)| \leq \phi^{-1}(M)$$

for all $r \in [b,c)$. Since $\phi^{-1}(M) < 1$, taking the limit as $r \to c^-$, we have $|v'(c)| < 1$, which is a contradiction. Therefore, $\|v'\|_\infty < 1$ and v is a positive solution of (2.2). □

Lemma 3.3. *Suppose that the assumptions (A1) and (F1) are satisfied. Let v be a nontrivial solution of (2.2). Then $v > 0$ on $[0,1)$ and v is strictly decreasing.*

Proof. First, we know that

$$\phi(v') = -\frac{\lambda N}{r^{N-1}}\int_0^r \tau^{N-1}\Big[\frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(\tau,\varphi^{-1}(v))\Big]d\tau. \tag{3.4}$$

From $f'(t) \geq 0$, $\frac{f'(t)}{f(t)} \geq H(r,t)$ for all $r \in [0,1]$ and $t \in I_f$, we can conclude that $v' \leq 0$. This implies that v is decreasing. Since $v(1) = 0$, we have $v \geq 0$ on $[0,1]$. Moreover, since v is non-zero and $v(0) > 0$, from (3.4), we can further conclude that $v' < 0$ on $(0,1]$. Thus, v is strictly decreasing and $v > 0$ on $[0,1)$. □

Lemma 3.4. *Suppose that the assumptions (A1) and (F1) are satisfied. Let v be a positive solution of (2.2). Then*

$$\frac{1}{4}\|v\|_\infty \leq v(r) \leq \|v\|_\infty, \quad r \in \left[0, \frac{3}{4}\right]. \tag{3.5}$$

Proof. Since $(r^{N-1}\phi(v'))' = -\lambda N r^{N-1}\Big[\frac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(r,\varphi^{-1}(v))\Big]$, by conditions (A1) and (F1), we deduce that $(r^{N-1}\phi(v'))' \leq 0$. Combining this with the monotonicity of r^{N-1} and ϕ, it follows that v' is decreasing on $[0,1]$. Since $v'(0) = v(1) = 0$ and $v(r) > 0$ on $(0,1)$, we know that $v(0) > 0$ and $v'(1) < 0$. Hence, v is concave on $[0,1]$. From this and Lemma 3.3, we have

$$v(r) \geq (1-r)||v||_\infty$$

for all $r \in [0,1]$. Therefore,

$$\frac{1}{4}\|v\|_\infty \le v(r) \le \|v\|_\infty, \quad r \in \left[0, \frac{3}{4}\right].$$

□

Then, we present a property of concave functions.

Lemma 3.5. *Let* $\nu \in (0,1)$ *and* $\beta_0 \in (0, \frac{1-\nu}{8})$ *be given. Define the interval* $I_{\nu,\beta_0} := \left[0, 1-\frac{4\beta_0}{1-\nu}\right]$. *Then*

$$|v'(s)| \le 1-\nu, \qquad \forall\ v \in \mathcal{A}, \quad \forall\ s \in I_{\nu,\beta_0},$$

where

$$\mathcal{A} := \{v \in E \,|\, v \text{ is concave on } [0,1],\ v'(1) > -1,\ ||v||_\infty \le 4\beta_0\}.$$

Proof. Suppose on the contrary that there exists $s_0 \in I_{\nu,\beta_0}$ such that $|v'(s_0)| > 1-\nu$. Given that $v \in C^1[0,1]$ and v is concave on the interval $[0,1]$, the v' is decreasing. As a result, we have $v'(s_0) < \nu - 1$. Applying the Lagrange Mean-Value Theorem to v on the interval $[s_0, 1]$, we can conclude that there exists $\xi \in (s_0, 1)$ such that

$$v'(\xi) = \frac{v(1)-v(s_0)}{1-s_0} = \frac{-v(s_0)}{1-s_0}.$$

Consequently, we obtain the following inequalities:

$$v(s_0) = v'(\xi)(s_0-1) > v'(s_0)(s_0-1) > (\nu-1)\Big(-\frac{4\beta_0}{1-\nu}\Big) = 4\beta_0 \ge ||v||_\infty.$$

This leads to a contradiction. □

If we set $\nu = \frac{1}{3}$ and $\beta_0 = \frac{1}{24} \in (0, \frac{1}{12})$, we obtain the following result.

Corollary 3.1. *For any concave function* $v \in E$ *such that*

$$v'(1) > -1, \quad ||v||_\infty \le \frac{1}{6},$$

we have

$$|v'(r)| \le \frac{2}{3}, \qquad r \in \left[0, \frac{3}{4}\right].$$

4. The direction of bifurcation

Let $h \in X$ be given. It is well-known that the solution v of the problem

$$\begin{cases} (r^{N-1}v')' + r^{N-1}g(r) = 0, & r \in (0,1), \\ v'(0) = v(1) = 0 \end{cases} \tag{4.1}$$

can be represented as

$$v(r) = \int_0^1 G(r,s)s^{N-1}g(s)ds := \mathcal{L}(g)(r),$$

where, when $N=2$, the Green's function of (4.1) is given by

$$G(t,s)=\begin{cases}\ln\dfrac{1}{t}, & 0\le s\le t\le 1,\\ \ln\dfrac{1}{s}, & 0\le t\le s\le 1,\end{cases}$$

and when $N\ge 3$, the Green's function of (4.1) is given by

$$G(t,s)=\begin{cases}\dfrac{1}{2-N}\big[1-t^{2-N}\big], & 0\le s\le t\le 1,\\ \dfrac{1}{2-N}\big[1-s^{2-N}\big], & 0\le t\le s\le 1.\end{cases}$$

It is straightforward to verify that $\mathcal{L}:X\to E$ is compact and continuous. Moreover, equation (2.3) is equivalent to

$$v=\lambda\mathcal{L}(v),$$

so the eigenvalues of (2.3) are the characteristic values of $\mathcal{L}$.

If condition (F2) holds, then

$$\lim_{\varphi^{-1}(v)\to 0^+}\frac{Nf'(\varphi^{-1}(v))}{v}=f_0,\qquad \lim_{\varphi^{-1}(v)\to 0^+}\frac{Nf(\varphi^{-1}(v))H(r,\varphi^{-1}(v))}{v}=-H_0,$$

and when $\varphi^{-1}(v)\to 0^+$, we have

$$e(r,v,v')=\left(f_0-\frac{\theta(\varphi^{-1}(v))}{v}+H_0+\frac{\xi(r,\varphi^{-1}(v))}{v}\right)v,$$

where $\theta(\varphi^{-1}(v))$ and $\xi(r,\varphi^{-1}(v))$ are continuous functions, and

$$\lim_{\varphi^{-1}(v)\to 0^+}\frac{\theta(\varphi^{-1}(v))}{v}=0,\quad \lim_{\varphi^{-1}(v)\to 0^+}\frac{\xi(r,\varphi^{-1}(v))}{v}=0. \tag{4.2}$$

Let $k(y)=h(y)-1$ for $y\in\mathbb{R}$. We know that

$$\lim_{y\to 0}\frac{k(y)}{y}=0. \tag{4.3}$$

Define the operator $\mathcal{H}:\mathbb{R}\times E\to E$ by

$$\begin{aligned}\mathcal{H}(\lambda,v)=&\mathcal{L}\left(\lambda\left(\left(f_0-\frac{\theta(\varphi^{-1}(v))}{v}+H_0+\frac{\xi(r,\varphi^{-1}(v))}{v}\right)k(v')-\frac{\theta(\varphi^{-1}(v))}{v}+\frac{\xi(r,\varphi^{-1}(v))}{v}\right)v\right.\\ &\left.-\frac{N-1}{r}v'^3\right).\end{aligned}$$

Obviously, $\mathcal{H}$ is completely continuous. From (4.2) and (4.3), we can conclude that

$$\lim_{\|v\|\to 0}\frac{\|\mathcal{H}(\lambda,v)\|}{\|v\|}=0$$

uniformly with respect to λ varying in bounded intervals. For any λ, $(\lambda, v) \in \mathbb{R}\times E$, with $v > 0$, is a solution of the equation

$$v = \lambda\,(f_0 + H_0)\,\mathcal{L}\,(v) + \mathcal{H}(\lambda, v), \tag{4.4}$$

if and only if v is a positive solution of (3.3).

Denote by $\mathcal{S}$ the closure in $\mathbb{R}\times E$ of the set of all nontrivial solutions (λ, v) of (4.4) with $\lambda > 0$. Let $P = \{v \in E : v(r) \geq 0, r \in [0, 1]\}$. Then P is a positive cone of E and int $P \neq \emptyset$. Since $\|v'\|_\infty < 1$ for $(\lambda, v) \in \mathcal{S}$, it implies that $\|v\|_\infty < 1$ for all $(\lambda, v) \in \mathcal{S}$.

Therefore, the following result can be derived from [22].

Lemma 4.1. *Suppose that the assumptions (A1), (F1) and (F2) are satisfied. Then there exists an unbounded connected component* $\mathcal{C}$ *in* $\mathcal{S}$ *that bifurcates from* $\left(\frac{\lambda_1}{f_0+H_0}, 0\right)$ *such that* $\mathcal{C} \subseteq \left(([0,+\infty)\times \textit{int } P) \cup \left\{\left(\frac{\lambda_1}{f_0+H_0}, 0\right)\right\}\right)$. *In addition,* $\mathcal{C}$ *connects* $\left(\frac{\lambda_1}{f_0+H_0}, 0\right)$ *to infinity in the* λ *direction.*

Lemma 4.2. *Suppose that the assumptions (A1), (F1) and (F2) are satisfied. Let* $\{(\lambda_n, v_n)\}$ *be a sequence of positive solutions of (4.4) such that* $\|v_n\| \to 0$ *and* $\lambda_n \to \frac{\lambda_1}{f_0+H_0}$. *Let* ϕ_1 *be the first eigenfunction of (2.3) with* $\|\phi_1\| = 1$. *Then there is a subsequence of* $\{v_n\}$, *still denoted by* $\{v_n\}$, *such that* $\frac{v_n}{\|v_n\|}$ *converges uniformly to* ϕ_1 *on on the interval* $[0, 1]$.

Proof. Define $u_n = \frac{v_n}{\|v_n\|}$. Then $\|u_n\| = 1$, and thus $\|u_n\|_\infty$ is bounded. By the Arzela-Ascoli theorem, there exists a subsequence of $\{u_n\}$ that converges uniformly to some $u \in X$. We continue to denote this subsequence by $\{u_n\}$. For each (λ_n, v_n), we have

$$\begin{aligned} v_n(r) = \int_0^1 G(r,s)s^{N-1}\Bigg(\lambda_n\left(\frac{Nf'(\varphi^{-1}(v_n))}{\sqrt{1-v_n'^2}} - Nf(\varphi^{-1}(v_n))H(s,\varphi^{-1}(v_n))\right)h(v_n'(s)) \\ -\frac{N-1}{s}(v_n'(s))^3\Bigg)\,ds. \end{aligned} \tag{4.5}$$

Dividing both sides of (4.5) by $\|v_n\|$, we get

$$\begin{aligned} u_n(r) = \int_0^1 G(r,s)s^{N-1}\Bigg(\lambda_n\left(\frac{Nf'(\varphi^{-1}(v_n))}{v_n\sqrt{1-v_n'^2}} - \frac{Nf(\varphi^{-1}(v_n))H(s,\varphi^{-1}(v_n))}{v_n}\right)h(v_n'(s))u_n \\ -\frac{N-1}{s}\frac{(v_n'(s))^3}{\|v_n\|}\Bigg)\,ds. \end{aligned}$$

Since $\|v_n\| \to 0$, we have $\|v_n\|_\infty \to 0$. Using assumptions (F2) and (3.2), we know that

$$\frac{Nf'(\varphi^{-1}(v_n))}{v_n\sqrt{1-v_n'^2}} - \frac{Nf(\varphi^{-1}(v_n))H(s,\varphi^{-1}(v_n))}{v_n} \to f_0 + H_0$$

and

$$h(v_n'(s)) \to 1$$

as $n \to \infty$, uniformly for $s \in [0, 1]$. By Lebesgue's dominated convergence theorem, we obtain

$$u(r) = \frac{\lambda_1}{f_0 + H_0}\int_0^1 G(r,s)s^{N-1}(f_0 + H_0)u(s)ds.$$

Therefore, u is a non-trivial solution of (2.3) corresponding to $\lambda = \lambda_1$, and $u \equiv \phi_1$. □

Lemma 4.3. *Suppose that the assumptions (A1), (F1) and (F2) are satisfied. Let* $\mathcal{C}$ *be defined as in Lemma 4.1. Then, there exists* $\sigma > 0$ *such that for any* $(\lambda, v) \in \mathcal{C}$ *with* $\left|\lambda - \frac{\lambda_1}{f_0+H_0}\right| + \|v\| \le \sigma$*, we have* $\lambda > \frac{\lambda_1}{f_0+H_0}$.

Proof. By (F1) and (F2), there exists a sufficiently small $\delta > 0$ such that

$$0 \le Nf'(t) - Nf(t)H(r,t) \le f_0\varphi(t) + H_0\varphi(t) \quad \text{for } (r,t) \in [0,1] \times [0, \varphi^{-1}(\delta)].$$

From the definition of h, we obtain

$$0 \le \frac{Nf'(t) - Nf(t)H(r,t)}{\varphi(t)} h < f_0 + H_0 \quad \text{for } (r,t) \in [0,1] \times (0, \varphi^{-1}(\delta)). \tag{4.6}$$

Set $\sigma = \delta$. Then, for any $(\lambda, v) \in \mathcal{C}$ satisfying $\left|\lambda - \frac{\lambda_1}{f_0+H_0}\right| + \|v\| \le \sigma$, v is a positive solution of the following problem:

$$\begin{cases} -(r^{N-1}v')' = r^{N-1}\left(\lambda\left(\dfrac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(r,\varphi^{-1}(v))\right)h(v'(r)) \right. \\ \qquad\qquad\qquad \left. -\dfrac{N-1}{r}v'^3\right), \quad r \in (0,1), \\ v'(0) = v(1) = 0, \end{cases} \tag{4.7}$$

and

$$0 \le v \le \delta.$$

Suppose there exists a sequence $\{(\lambda_n, v_n)\}$ such that $(\lambda_n, v_n) \in \mathcal{C}$, $|\lambda_n - \frac{\lambda_1}{f_0+H_0}| + ||v_n|| \le \sigma$, $\lambda_n \to \lambda$, and $||v_n|| \to 0$. Define $u_n = \frac{v_n}{||v_n||}$. By Lemma 4.2, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that u_n converges uniformly to u on $[0,1]$, where $u > 0$ and satisfies $||u|| = 1$. Dividing both sides of the equation of (4.7) with $(\lambda, v) = (\lambda_n, v_n)$ by $||v_n||$, we obtain

$$\begin{cases} -(r^{N-1}u_n')' = r^{N-1}\left(\lambda_n\left(\dfrac{Nf'(\varphi^{-1}(v_n))}{v_n\sqrt{1-v_n'^2}} - \dfrac{Nf(\varphi^{-1}(v_n))H(r,\varphi^{-1}(v_n))}{v_n}\right)h(v_n'(r))u_n \right. \\ \qquad\qquad\qquad \left. -\dfrac{N-1}{r}u_n'(v_n')^2\right), \quad r \in (0,1), \\ u_n'(0) = u_n(1) = 0. \end{cases}$$

Since $\|v_n\| \to 0$ implies $\|v_n\|_\infty \to 0$, we have

$$-(r^{N-1}u')' < \lambda r^{N-1}(f_0 + H_0)u,$$

along with

$$u'(0) = u(1) = 0.$$

Moreover, we know that

$$\begin{cases} -(r^{N-1}\phi_1'(r))' = \lambda_1 r^{N-1}\phi_1(r), \quad r \in (0,1), \\ \phi_1'(0) = \phi_1(1) = 0. \end{cases}$$

Therefore,

$$\begin{aligned}\lambda_1 \int_0^1 t^{N-1}\phi_1(t)u(t)dt &= \int_0^1 t^{N-1}\phi_1'(t)u'(t)dt\\ &< \lambda(f_0+H_0)\int_0^1 t^{N-1}\phi_1(t)u(t)dt.\end{aligned}$$

This implies $\lambda > \frac{\lambda_1}{f_0+H_0}$. □

Lemma 4.4. *Suppose that the assumptions (A1), (F1) and (F2) are satisfied. Let $\mathcal{C}$ be defined as in Lemma 4.1. Then, there exists $\lambda_\diamondsuit > 0$ such that $Proj_{\mathbb{R}}\mathcal{C} = [\lambda_\diamondsuit, \infty) \subset (0,\infty)$.*

Proof. Suppose, for the sake of contradiction, that $\lambda_\diamondsuit = 0$. Then, there exists a sequence $\{(\mu_n, v_n)\} \subset \mathcal{C}$ with $v_n > 0$ such that

$$\lim_{n\to\infty}(\mu_n, v_n) = (0, v^*) \ \text{ in } \mathbb{R}\times X$$

for some $v^* \geq 0$. Consider the equation

$$\begin{cases} -(r^{N-1}\phi(v_n'))' = \mu_n r^{N-1}\left(\dfrac{Nf'(\varphi^{-1}(v_n))}{\sqrt{1-v_n'^2}} - Nf(\varphi^{-1}(v_n))H(r,\varphi^{-1}(v_n))\right), \quad r\in(0,1),\\ v_n'(0) = v_n(1) = 0.\end{cases}$$

After passing to a subsequence (and relabeling if necessary), we can show that $v_n \to 0$.

Since $v_n'(0)=0$, by integrating the first order differential equation for $\phi(v_n')$, we have

$$\phi(v_n'(r)) = -\frac{\mu_n}{r^{N-1}}\int_0^r s^{N-1}\left(\frac{Nf'(\varphi^{-1}(v_n))}{\sqrt{1-v_n'^2}} - Nf(\varphi^{-1}(v_n))H(s,\varphi^{-1}(v_n))\right)ds, \quad r\in(0,1).$$

Combined with assumption (F1), we obtain

$$\lim_{n\to\infty}\|v_n'\|_\infty = 0. \tag{4.8}$$

Next, consider the equation

$$\begin{cases} -(r^{N-1}v_n')' = r^{N-1}\left(\mu_n\left(\dfrac{Nf'(\varphi^{-1}(v_n))}{\sqrt{1-v_n'^2}} - Nf(\varphi^{-1}(v_n))H(r,\varphi^{-1}(v_n))\right)h(v_n'(r))\right.\\ \qquad\qquad\qquad \left. -\dfrac{N-1}{r}v_n'^3\right), \quad r\in(0,1),\\ v_n'(0)=v_n(1)=0.\end{cases}$$

For each n, define $u_n = \frac{v_n}{\|v_n\|}$. Then we get

$$\begin{cases} -(r^{N-1}u_n')' = r^{N-1}\left(\mu_n\left(\dfrac{Nf'(\varphi^{-1}(v_n))}{v_n\sqrt{1-v_n'^2}} - \dfrac{Nf(\varphi^{-1}(v_n))H(r,\varphi^{-1}(v_n))}{v_n}\right)h(v_n'(r))u_n\right.\\ \qquad\qquad\qquad \left. -\dfrac{N-1}{r}u_n'(v_n')^2\right), \quad r\in(0,1),\\ u_n'(0)=u_n(1)=0.\end{cases} \tag{4.9}$$

Similar to the proof of Lemma 4.2, by using (4.8) and (4.9), we find that $\mu_n \to \frac{\lambda_1}{f_0+H_0}$. This contradicts the fact that $\mu_n \to 0$. Thus, $\lambda_\diamondsuit > 0$. □

Lemma 4.5. *Suppose that the assumptions (A1), (F1) and (F3) are satisfied. Let* $(\lambda, v) \in \mathcal{C}$ *with* $\|v\|_\infty = 4s_0$. *Then* $\lambda < \frac{\lambda_1}{f_0+H_0}$.

Proof. Let $(\lambda, v) \in \mathcal{C}$. By Lemma 3.4, we have

$$s_0 \le v(r) \le 4s_0, \quad r \in \left[0, \frac{3}{4}\right]. \tag{4.10}$$

Fix $s_0 = \frac{1}{24}$. From Corollary 3.1, for any $(\lambda, v) \in \mathcal{C}$ with $\|v\|_\infty = 4s_0$, we obtain

$$0 \le |v'(r)| \le \frac{2}{3}, \quad r \in \left[0, \frac{3}{4}\right].$$

Suppose, for the sake of contradiction, that $\lambda \ge \frac{\lambda_1}{f_0+H_0}$. Then for $r \in \left[0, \frac{3}{4}\right]$, using (4.10) and (F3), we have

$$\begin{aligned}
&\lambda\left(\frac{Nf'(\varphi^{-1}(v))}{v\sqrt{1-v'^2}} - \frac{Nf(\varphi^{-1}(v))H(r,\varphi^{-1}(v))}{v}\right)h(v') \\
&\ge \lambda\left(\frac{Nf'(\varphi^{-1}(v))}{v} - \frac{Nf(\varphi^{-1}(v))H(r,\varphi^{-1}(v))}{v}\right)h(v') \\
&\ge \frac{\lambda_1}{f_0+H_0} \cdot \frac{27(f_0+H_0)}{5\sqrt{5}\lambda_1} \cdot \eta_1 \cdot \frac{5\sqrt{5}}{27} \\
&= \eta_1,
\end{aligned}$$

where η_1 is the first positive eigenvalue of the problem

$$\begin{cases} (r^{N-1}u'(r))' + \eta r^{N-1}u(r) = 0, \quad r \in \left(0, \dfrac{3}{4}\right), \\ u'(0) = u(\dfrac{3}{4}) = 0. \end{cases}$$

Let u be the corresponding eigenfunction of η_1. Then

$$u(r) > 0, \quad r \in \left[0, \frac{3}{4}\right).$$

Since v is a solution of

$$-(r^{N-1}v')' = r^{N-1}\left[\lambda\left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(r,\varphi^{-1}(v))\right)h(v') - \frac{N-1}{r}v'^3\right]$$

on $\left[0, \frac{3}{4}\right]$, by the result in [21], we know that v has at least one zero on $\left[0, \frac{3}{4}\right]$. This clearly contradicts the fact that from the previous conditions, we expect v to be non-zero on this interval. □

Lemma 4.6. *Suppose that the assumptions (A1) and (F1) are satisfied. Let* $\mathcal{C}$ *be defined as in Lemma 4.1. Then* $\lim\limits_{(\lambda,v)\in\mathcal{C},\lambda\to\infty} \|v\| = 1$ *and* $\lim\limits_{(\lambda,v)\in\mathcal{C},\lambda\to\infty} \|v\|_\infty = 1$.

Proof. We will prove this lemma in four steps.

Step 1. We claim that there exists a constant $B > 0$ such that for every $(\lambda, v) \in \mathcal{C}$, if $\lambda \ge B$, then

$$\|v\|_\infty \ge \rho_*$$

for some $\rho_* > 0$. Suppose, for the sake of contradiction, that there exists a sequence $\{(\mu_n, v_n)\} \subset \mathcal{C}$ such that

$$(\mu_n, v_n) \to (\infty, 0) \ \text{ in } (0, +\infty) \times X.$$

Similar to the proof of Lemma 4.4, we can show that v_n' as $n \to \infty$. By considering (4.9) and passing to a subsequence (and relabeling if necessary), we find that $u_n \to u_*$ in X for some $u_* \in X$, and

$$\begin{cases} -\left(r^{N-1}u_*'(r)\right)' = r^{N-1}\mu_n(f_0 + H_0)u_*(r), & r \in (0,1), \\ u_*'(0) = u_*(1) = 0. \end{cases}$$

This implies that $\mu_n \to \frac{\lambda_1}{f_0+H_0}$, which contradicts the fact that $\mu_n \to \infty$. Thus, the claim is valid.

Step 2. Fix $\varepsilon \in \left(0, \frac{1}{4}\right)$. Then there exists $\beta = \beta(\varepsilon) > 0$ such that for $(\lambda, v) \in \mathcal{C}$ with $\lambda > B$, we have

$$\min_{r \in [0, 1-\varepsilon]} v(r) \geq \beta \rho_*. \tag{4.11}$$

Step 3. From (4.11) and (F1), there exists a constant $M_1 > 0$ such that

$$\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - Nf(\varphi^{-1}(v))H(r, \varphi^{-1}(v)) \geq M_1 > 0 \qquad \text{for } r \in [0, 1-\varepsilon].$$

Therefore, we have

$$\lim_{\lambda \to \infty} \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} \left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - Nf(\varphi^{-1}(v))H(s, \varphi^{-1}(v)) \right) ds = +\infty$$

uniformly for $r \in [\varepsilon_1, 1-\varepsilon]$, where $\varepsilon_1 \in \left(0, \frac{1-\varepsilon}{4}\right)$ is arbitrarily fixed. Combining this with the relation

$$v'(r) = -\phi^{-1} \left(\frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} \lambda \left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - Nf(\varphi^{-1}(v))H(s, \varphi^{-1}(v)) \right) ds \right),$$

we conclude that

$$v' \to -1 \ \text{ in } C[\varepsilon_1, 1-\varepsilon] \ \text{ as } \lambda \to +\infty. \tag{4.12}$$

By the arbitrariness of ε and ε_1, we obtain

$$\lim_{(\lambda, v) \in \mathcal{C}, \lambda \to \infty} \|v\| = 1.$$

Step 4. Since

$$-v'(r) \geq 0, \quad r \in (0, 1],$$

and using (4.12), for $(\lambda, v) \in \mathcal{C}$, we have

$$\lim_{\lambda \to \infty} \|v\|_\infty = \lim_{\lambda \to \infty} v(0) = \lim_{\lambda \to \infty} \int_0^1 -v'(s)ds \geq \lim_{\lambda \to \infty} \int_{\varepsilon_1}^{1-\varepsilon} -v'(s)ds = 1 - \varepsilon - \varepsilon_1.$$

By the arbitrariness of ε and ε_1, we get

$$\lim_{\lambda \to \infty} \|v\|_\infty \geq 1.$$

On the other hand,

$$v(0) = \int_0^1 -v'(s)ds \leq 1.$$

Hence,

$$\lim_{\lambda\to\infty} \|v\|_\infty = 1.$$

□

5. Proof of the main result

Proof of Theorem 2.1. By Lemma 4.1 and Lemma 4.3, there exists an unbounded connected component $\mathcal{C}$ in the set of radial positive solutions of (2.1). This component bifurcates from the point corresponding to $\left(\frac{\lambda_1}{f_0+H_0}, 0\right)$ and extends right-handedly in the $(\lambda, \|v\|_\infty)$-plane.

Lemma 4.6 implies that $\lim\limits_{(\lambda,v)\in\mathcal{C},\lambda\to\infty} \|v\| = 1$ and $\lim\limits_{(\lambda,v)\in\mathcal{C},\lambda\to\infty} \|v\|_\infty = 1$. Thus, there must be a point $(\lambda_0, v_0) \in \mathcal{C}$ for which $\|v_0\|_\infty = 4s_0$. Invoking Lemma 4.5, we conclude that $\lambda < \frac{\lambda_1}{f_0+H_0}$.

Lemma 4.4 states that for all $(\lambda, v) \in \mathcal{C}$, there exists a $\lambda_\diamondsuit$ such that $\lambda \geq \lambda_\diamondsuit$. By combining Lemma 4.3, 4.5, and 4.6, we can deduce that the connected component $\mathcal{C}$ passes through points $\left(\frac{\lambda_1}{f_0+H_0}, v_1\right)$ and $\left(\frac{\lambda_1}{f_0+H_0}, v_2\right)$ with $\|v_1\|_\infty < 4s_0 < \|v_2\|_\infty$. Additionally, there exist $\underline{\lambda}$ and $\overline{\lambda}$ satisfying $0 < \underline{\lambda} < \frac{\lambda_1}{f_0+H_0} < \overline{\lambda}$, and the following two conditions:

(i) for $\lambda \in \left(\frac{\lambda_1}{f_0+H_0}, \overline{\lambda}\right]$, there exist u and v such that $(\lambda, v), (\lambda, u) \in \mathcal{C}$, and $\|u\|_\infty < \|v\|_\infty < 4s_0$;

(ii) for $\lambda \in \left[\underline{\lambda}, \frac{\lambda_1}{f_0+H_0}\right)$, there exist u and v such that $(\lambda, v), (\lambda, u) \in \mathcal{C}$, and $\|u\|_\infty < 4s_0 < \|v\|_\infty$.

Define $\lambda^* = \sup\{\overline{\lambda} : \overline{\lambda}$ satisfies (i)$\}$ and $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda}$ satisfies (ii)$\}$. Employing standard arguments in the theory of bifurcation and existence of solutions, we can show that (2.2) admits a positive solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$. This completes the proof. □

6. Further results and remarks

In this section, we discuss the existence of radial positive solutions of (2.1) under two cases: $f_0 + H_0 = \infty$ and $f_0 + H_0 = 0$. Our analysis is based on Theorem 2.1 and makes use of the properties of the superior limit of a sequence of connected components as presented by Ma and An in [33, 34]. Additionally, we analyze the assumptions made in this paper.

Theorem 6.1. *Suppose that (A1), (F1),*
(F4) $f_0 + H_0 = \infty$,
(F5) there exists $m_0 \in \left(\frac{1}{12}, \frac{1}{2}\right)$ *such that for* $\frac{1}{12} < s \leq m_0$, *we have*

$$Nf'(\varphi^{-1}(s)) - Nf(\varphi^{-1}(s))H(r, \varphi^{-1}(s)) \geq \frac{m_0 M_2}{\Lambda\sqrt{(1-(\frac{1}{2}+m_0)^2)^3}} \quad \textit{uniformly for } r \in [0,1],$$

where Λ *is the positive parameter corresponding to* $\|s\|_\infty = \frac{1}{12}$, *when* $N = 2$, $M_2 = \frac{16}{1+2\ln 2}$, *when* $N \geq 3$, $M_2 = \frac{(2-N)(8N\cdot 2^N)}{8-N\cdot 2^N}$. *Then there exists* $0 < \lambda_{**} < \Lambda$ *such that*

(i) (2.1) has at least one radial positive solution if $0 < \lambda < \lambda_{**}$*;*
(ii) (2.1) has at least two radial positive solutions if $\lambda = \lambda_{**}$*;*
(iii) (2.1) has at least three radial positive solutions if $\lambda_{**} < \lambda < \Lambda$*;*
(iv) (2.1) has at least two radial positive solutions if $\lambda = \Lambda$*;*
(v) (2.1) has at least one radial positive solution if $\lambda > \Lambda$*;*
(vi) $\lim_{\lambda\to\infty} \|v\|_\infty = 1$ *and* $\lim_{\lambda\to\infty} \|v\| = 1$.

Proof. For the sake of brevity, we still define

$$e(r,s,t) = \frac{Nf'(\varphi^{-1}(s))}{\sqrt{1-t^2}} - Nf(\varphi^{-1}(s))H(r,\varphi^{-1}(s)).$$

For each $n \in \mathbb{N}$, we define a function $f^{[n]} : [0,1] \times \mathbb{R} \times (-1,1) \to \mathbb{R}$ as follows:

$$f^{[n]}(r,s,t) = \begin{cases} e(r,s,t), & s \in \left(\frac{1}{n},\infty\right), \\ ne(r,\frac{1}{n},t)s, & s \in \left[0,\frac{1}{n}\right]. \end{cases}$$

Then, for each $n \in \mathbb{N}$, $f^{[n]}$ is a continuous function and

$$\limsup_{n\to\infty} \left[f^{[n]}(r,s,t) - e(r,s,t)\right] = 0 \text{ uniformly for } s \in [0,\infty),$$

and

$$(f^{[n]})_0 = \lim_{s\to 0} \frac{f^{[n]}(r,s,t)}{s} = ne(r,\frac{1}{n},t).$$

Using the same method as in the proof of Lemma 4.1, we can show that for each $n \in \mathbb{N}$, the closure of the set of positive solutions of the auxiliary problem

$$\begin{cases} -(r^{N-1}v')' = r^{N-1}\left[\lambda f^{[n]}(r,v,v')h(v') - \frac{N-1}{r}v'^3\right], & r \in (0,1), \\ v'(0) = v(1) = 0 \end{cases}$$

has a connected component $\mathcal{C}^{[n]}$ that joins $\left(\frac{\lambda_1}{ne\left(r,\frac{1}{n},t\right)},0\right)$ to infinity in the λ direction.

From (F4), we have

$$\lim_{n\to\infty} (f^{[n]})_0 = \lim_{n\to\infty} \frac{e\left(r,\frac{1}{n},t\right)}{\frac{1}{n}} = \infty,$$

and

$$\lim_{n\to\infty} \frac{\lambda_1}{ne\left(r,\frac{1}{n},t\right)} = 0.$$

Using Lemma 3.1, the set $\limsup_{n\to\infty} \mathcal{C}^{[n]}$ contains an unbounded connected component $\mathcal{C}$ such that

$$(0,0) \in \mathcal{C} \subset \limsup_{n\to\infty} \mathcal{C}^{[n]},$$

which joins (0,0) to infinity.

Subsequently, by using an argument similar to that in Theorem 2.1 and the following Lemma 6.1, we can obtain the conclusion of Theorem 6.1. □

Lemma 6.1. *Suppose that the assumptions (A1), (F1) and (F5) are satisfied. Let* $(\lambda, v) \in \mathcal{C}$ *with* $\|v\|_\infty = m_0$. *Then* $\lambda < \Lambda$.

Proof. According to Lemma 3.5, if $\|v\|_\infty = m_0$, then $4\beta_0 = m_0$. This implies that $\beta_0 = \frac{m_0}{4}$ and $\frac{1-\nu}{8} > \frac{m_0}{4}$, and thus we know that $2m_0 < 1-\nu < 1$, that is $0 < \nu < 1 - 2m_0$. Therefore, consider the case where $\nu = \frac{1}{2} - m_0$, which corresponds to $1 - \nu = \frac{1}{2} + m_0$.

Based on the properties of Green's function, when $N \geq 3$, we have

$$
\begin{aligned}
m_0 &= \|v\|_\infty \\
&= \max_{r\in[0,1]} \int_0^1 G(r,s)s^{N-1}\left[\lambda\left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(s,\varphi^{-1}(v))\right)h(v') - \frac{N-1}{s}v'^3\right]ds \\
&> \max_{r\in[0,1]} \int_0^1 G(r,s)s^{N-1}\left[\lambda\left(Nf'(\varphi^{-1}(v)) - Nf(\varphi^{-1}(v))H(s,\varphi^{-1}(v))\right)h(v')\right]ds \\
&> \lambda \cdot \frac{m_0 M_2}{\Lambda\sqrt{(1-(\frac{1}{2}+m_0)^2)^3}} \cdot \sqrt{(1-(\frac{1}{2}+m_0)^2)^3}\int_0^{\frac{1}{2}} G(r,s)s^{N-1}ds \\
&= \frac{\lambda}{\Lambda}m_0.
\end{aligned}
$$

Consequently, $\lambda < \Lambda$. Similarly, for $N = 2$, the proof is carried out in a similar way. $\square$

Next, we consider the case where $f_0 + H_0 = 0$. Under this condition, we have

$$
\lim_{n\to\infty}(f^{[n]})_0 = \lim_{n\to\infty}\frac{e\left(r,\frac{1}{n},t\right)}{\frac{1}{n}} = 0,
$$

and

$$
\lim_{n\to\infty}\frac{\lambda_1}{ne\left(r,\frac{1}{n},t\right)} = \infty.
$$

Consequently, from Lemma 3.1, there exists an unbounded component $\mathcal{C}$ in $\mathcal{S}$, which connects $(+\infty, 1)$ to $(+\infty, 0)$. Using [22], we get $\mathcal{C} \cap ([0,+\infty)\times\{0\}) = \emptyset$; thus, this implies $\mathrm{Proj}_{\mathbb{R}}\mathcal{C} = [\Lambda_1, \infty) \subset (0,\infty)$ for some $\Lambda_1 > 0$. Let $\|v\|_\infty = m_1$, corresponding to the positive parameter Λ_1. We present the following result.

Theorem 6.2. *Suppose that (A1), (F1),*
(F6) $f_0 + H_0 = 0$,
(F7) there exists $m_3 \in (m_2, m_2+\varepsilon)$ *such that for* $m_2 < s \leq m_3$, *we have*

$$
Nf'(\varphi^{-1}(s)) - Nf(\varphi^{-1}(s))H(r,\varphi^{-1}(s)) \geq \frac{m_3 M_3}{\Lambda_2\sqrt{(1-(m_3)^2)^3}} \quad \textit{uniformly for } r \in [0,1],
$$

where $\Lambda_2 > \Lambda_1$ *is the positive parameter corresponding to* $\|s\|_\infty = m_2$, $m_2 \in (m_1, 1)$, $\varepsilon > 0$ *and satisfies* $m_2 + \varepsilon < 1$, $M_3 = \left(\int_0^\xi G(r,s)s^{N-1}ds\right)^{-1}$, $\xi \in (0,1)$, $v'(\xi) = -m_3$. *Then there exists* $\Lambda_1 < \lambda_{**} < \Lambda_2$ *such that*
(i) (2.1) has no radial positive solution if $0 < \lambda < \Lambda_1$;
(ii) (2.1) has at least one radial positive solutions if $\lambda = \Lambda_1$;
(iii) (2.1) has at least two radial positive solutions if $\Lambda_1 < \lambda < \lambda_{**}$;
(iv) (2.1) has at least three radial positive solutions if $\lambda = \lambda_{**}$;
(v) (2.1) has at least four radial positive solutions if $\lambda_{**} < \lambda < \Lambda_2$;

(vi) (2.1) has at least three radial positive solutions if $\lambda = \Lambda_2$*;*
(vii) (2.1) has at least two radial positive solutions if $\lambda > \Lambda_2$*;*
(viii) $\lim\limits_{\lambda\to\infty} \|v\|_\infty = 1$ *and* $\lim\limits_{\lambda\to\infty} \|v\| = 1$.

Similarly, by using an argument similar to that of Theorem 2.1 and the following Lemma 6.2, we can obtain the conclusion of Theorem 6.2.

Lemma 6.2. *Suppose that the assumptions (A1), (F1) and (F7) are satisfied. Let* $(\lambda, v) \in \mathcal{C}$ *with* $\|v\|_\infty = m_3$*. Then* $\lambda < \Lambda_2$*.*

Proof. According to the assumptions, we have

$$\begin{aligned}
m_3 &= \|v\|_\infty \\
&= \max_{r\in[0,1]} \int_0^1 G(r,s)s^{N-1}\left[\lambda\left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(s,\varphi^{-1}(v))\right)h(v') - \frac{N-1}{s}v'^3\right]ds \\
&> \max_{r\in[0,1]} \int_0^1 G(r,s)s^{N-1}\left[\lambda\left[Nf'(\varphi^{-1}(v)) - Nf(\varphi^{-1}(v))H(s,\varphi^{-1}(v))\right]h(v')\right]ds \\
&\geq \lambda \cdot \frac{m_3 M_3}{\Lambda_2\sqrt{(1-(m_3)^2)^3}} \cdot \sqrt{(1-(m_3)^2)^3}\int_0^\xi G(r,s)s^{N-1}ds \\
&= \frac{\lambda}{\Lambda_2}m_3.
\end{aligned}$$

Consequently, $\lambda < \Lambda_2$. □

Remark 6.1. By appropriately adjusting the parameters ν and β_0 in Lemma 3.5, we can derive some new conditions. For example, when $\nu = \frac{1}{4}$ and $\beta_0 = \frac{3}{64}$, condition (F3) can be replaced with the following:
(F3") there exists $s_0 : s_0 \in (\frac{1}{32}, \frac{1}{12})$ such that

$$\min_{s\in[\varphi^{-1}(s_0),\varphi^{-1}(4s_0)]} \frac{Nf'(s) - Nf(s)H(r,s)}{\varphi(s)} \geq \frac{64(f_0 + H_0)}{7\sqrt{7}\lambda_1} \cdot \eta_1 \quad \text{uniformly for } r \in [0,1],$$

here η_1 is the first positive eigenvalue of the problem

$$\begin{cases} (r^{N-1}v'(r))' + \eta r^{N-1}v(r) = 0, & r \in \left(0, \dfrac{3}{4}\right), \\ v'(0) = v\left(\dfrac{3}{4}\right) = 0. \end{cases}$$

Acknowledgements. The authors are grateful to the anonymous referees for their useful comments and suggestions.

Author contributions statement. MX completed the main study, TW drafted the paper, and YL checked the proofs and verified the calculation. All authors read and approved the final manuscript.

Competing interests. All authors of this article claim that they have no any competing interests.

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Received July 2025; Accepted December 2025; Available online January 2026.