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A NEW MULTIDIMENSIONAL HILBERT-TYPE INEQUALITY WITH TWO INTERNAL VARIABLES INVOLVING ONE PARTIAL SUM

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Abstract This study explores a new multidimensional Hilbert-type inequality involving one partial sum, by utilizing transfer formula and Hermite-Hadamard's inequality. The kernel $\frac{1}{\left(u(m)+\|v(k)\|_\alpha\right)^\lambda}(\lambda > 0)$ in the new inequality has two general internal variables compared with previous work, and the best value is achieved with certain parameters. Finally, the equivalent forms, the operator expressions and some particular cases are presented.

Keywords Multidimensional Hilbert-type inequality, weight function, best value, parameter, partial sum.

MSC(2010) 26D15.

1. Introduction

For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0,\ 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following Hardy-Hilbert's inequality stated in [3] (Theorem 315):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where, $\pi/\sin(\frac{\pi}{p})$ is the best value.

In 2006, Krnić et al. [10] provided an expansion of (1.1) by using the Euler-Maclaurin's summation formula as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{1.2}$$

where, $\lambda_i \in (0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda\in(0,4]$. The best value $B(\lambda_1,\lambda_2)$ is expressed as the following beta function:

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}dt \ \ (u,v>0). \tag{1.3}$$

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The half-discrete Hilbert-type inequality with a nonhomogeneous kernel was first described by Hardy et al. in 1934 (cf. [3], Theorem 351), the outcome can be located: If the function $K(t)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1$, $0<\phi(\lambda)=\int_0^\infty K(x)x^{\lambda-1}dx<\infty$, $f(x)\geq 0$, $0<\int_0^\infty f^p(x)<\infty$, then

$$\sum_{m=1}^{\infty} n^{p-2}\left(\int_0^\infty K(nx)f(x)dx\right)^p<\phi^p\left(\frac{1}{q}\right)\int_0^\infty f^p(x)dx. \tag{1.4}$$

Based on inequality (1.4), You et al. obtained some special functions, such as hyperbolic functions (cf. [38]) and the cotangent function (cf. [33]). In 2016, Hong and Wen [8] discussed the equivalent statements of the general form of (1.1) and explored their best values. Subsequently, the expanded theoretical research on multiple Hilbert-type integral inequalities was obtained by [7]. The quasi-homogeneous kernel involved in half-discrete Hilbert-type inequality was also discussed in [5]. Based on Hong's theory, further studies about half-discrete inequalities with their equivalent statements were also conducted by [4, 6, 13, 34, 37].

Recently, some applications of Hilbert-type inequalities were obtained by [1, 2, 9, 36]. Additionally, Hong et al. [9] introduced a half-discrete multidimensional Hilbert-type inequality with a homogeneous kernel using weight functions, the transfer formula, and Hermite–Hadamard's inequality. Some further and interested results were obtained by [14–23, 25–32].

Utilizing the extension transfer formula and the approach from and [9], this article introduces a new multidimensional Hilbert-type inequality involving one partial sum. Its kernel $\frac{1}{\left(u(m)+\|v(k)\|_\alpha\right)^\lambda}(\lambda>0)$ has two general internal variables than the kernel of previous work [9]. The equivalent statements outline the comparable expressions for the best value associated with certain parameters. The equivalent forms, the operator expressions and some particular cases are also presented.

2. Some lemmas

In what follows, let $p>1, q>1, \frac{1}{p}+\frac{1}{q}=1, \lambda>0$, $\lambda_1\in(0,\lambda)$, $\lambda_2\in(0,\lambda)\cap(0,n]$ $(n\in\mathbf{N}=\{1,2,...\})$, $\alpha\in(0,1]$, $\xi,\eta\in[0,\frac{1}{2}]$, $u(x)>0, u'(x)>0, u''(x)\leq 0$,

$$(-1)^i\frac{d^i}{dx^i}[(u(x))^{\lambda_1}u'(x)]\geq 0\ (x\in(\xi,\infty); i=1,2),$$

$u(\xi^+)=0, u(\infty)=\infty$, and

$$v(y)=(v_1(y_1),\cdots,v_n(y_n)), y\in A_\eta:=\{y=\{y_1,\cdots y_n\}; y_i\in(\eta,\infty)\},$$

such that $v_i(y_i)>0$, $v_i'(y_i)>0$, $v_i''(y_i)\leq 0, v_i'''(y_i)\geq 0, v_i(\eta^+)=0$, $v_i(\infty)=\infty$ $(i=1,\cdots,n)$. $\widetilde{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widetilde{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. We also suppose that for $i\in\mathbf{N}$, $a_i\geq 0, A_m:=\sum_{i=1}^m a_i=o(e^{tu(m)})$ $(t>0; m\to\infty)$, $a_m\geq 0$ $(m\in\mathbf{N})$, $b_k=(b_{k_1},\cdots,b_{k_n})\geq 0$, $(k=(k_1,\cdots,k_n)\in\mathbf{N}^n)$, satisfying

$$0<\sum_{m=1}^\infty\frac{u'(m)A_m^p}{(u(m))^{p\widetilde{\lambda}_1+1}}<\infty, 0<\sum_k\frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}b_k^q}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}<\infty.$$

For $M>0, \psi(u)$ $(u>0)$ is a nonnegative measurable function, the following transfer formula holds (cf. [28]):

$$\int\cdots\int_{\{y\in\mathbf{R}_+^n;0<\sum_{i=1}^n\left(\frac{y_i}{M}\right)^\alpha\leq 1\}}\psi\left(\sum_{i=1}^n\left(\frac{y_i}{M}\right)^\alpha\right)dy_1\cdots dy_n=\frac{M^n\Gamma^n(\frac{1}{\alpha})}{\alpha^n\Gamma(\frac{n}{\alpha})}\int_0^1\psi(u)u^{\frac{n}{\alpha}-1}du. \tag{2.1}$$

Particularly, (i) for $\|y\|_\alpha = M\left[\sum_{k=1}^n \left(\frac{y_i}{M}\right)^\alpha\right]^{\frac{1}{\alpha}}, \psi(u) = \phi\left(Mu^{\frac{1}{\alpha}}\right)$, by (2.1), we derive

$$\begin{aligned}\int_{R_+^n} \phi(\|y\|_\alpha)dy &= \lim_{M\longrightarrow\infty} \int\cdots\int_{\{y\in\mathbf{R}_+^n;0<\sum_{i=1}^n\left(\frac{y_i}{M}\right)^\alpha\le 1\}} \phi\left(M\left[\sum_{k=1}^n \left(\frac{y_i}{M}\right)^\alpha\right]^{\frac{1}{\alpha}}\right) dy_1\cdots dy_n \\ &= \lim_{M\longrightarrow\infty} \frac{M^n\Gamma^n(\frac{1}{\alpha})}{\alpha^n\Gamma(\frac{n}{\alpha})}\int_0^1 \phi\left(Mu^{\frac{1}{\alpha}}\right)u^{\frac{n}{\alpha}-1}du \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \phi(v)\, v^{n-1}dv\ , \left(v = Mu^{\frac{1}{\alpha}}\right).\end{aligned} \tag{2.2}$$

(ii) For $\psi(u) = \phi\left(Mu^{\frac{1}{\alpha}}\right) = 0, 0 < u < \frac{b^\alpha}{M^\alpha}(b > 0)$, by (2.1), we derive

$$\begin{aligned}\int_{\{y\in\mathbf{R}_+^n,\|y\|_\alpha\ge b\}} \phi(\|y\|_\alpha)dy &= \lim_{M\longrightarrow\infty} \frac{M^n\Gamma^n(\frac{1}{\alpha})}{\alpha^n\Gamma(\frac{n}{\alpha})}\int_{\frac{b^\alpha}{M^\alpha}}^1 \phi\left(Mu^{\frac{1}{\alpha}}\right)u^{\frac{n}{\alpha}-1}du \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_b^\infty \phi(v)\, v^{n-1}dv.\end{aligned} \tag{2.3}$$

(iii) For $\psi(u) = \phi\left(Mu^{\frac{1}{\alpha}}\right) = 0, u > \frac{b^\alpha}{M^\alpha}(b > 0)$, by (2.1), we derive

$$\begin{aligned}\int_{\{y\in\mathbf{R}_+^n,\|y\|_\alpha\le b\}} \phi(\|y\|_\alpha)dy &= \lim_{M\longrightarrow\infty} \frac{M^n\Gamma^n(\frac{1}{\alpha})}{\alpha^n\Gamma(\frac{n}{\alpha})}\int_0^{\frac{b^\alpha}{M^\alpha}} \phi\left(Mu^{\frac{1}{\alpha}}\right)u^{\frac{n}{\alpha}-1}du \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^b \phi(v)\, v^{n-1}dv.\end{aligned} \tag{2.4}$$

Lemma 2.1. *Suppose that $\lambda > 0, \alpha \in (0, 1]$. We construct the following function*

$$h(y) := \frac{\|v(y)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(y_i)}{(u(m) + \|v(y)\|_\alpha)^\lambda},\ \ (y \in A_\eta, y_i \in (\eta, \infty)).$$

Then, $\frac{\partial}{\partial y_j}h(y) < 0,\ \frac{\partial^2}{\partial y_j^2}h(y) > 0$ $(j = 1, \cdots, n)$.

Proof. Since $\lambda > 0, \alpha \in (0, 1], \eta \in [0, \frac{1}{2}], y \in A_\eta$, we obtain

$$g(y) := \frac{1}{(u(m) + \|v(y)\|_\alpha)^\lambda} = \frac{1}{[u(m) + (\sum_{i=1}^n v_i^\alpha(y_i))^{\frac{1}{\alpha}}]^\lambda},$$

$$f_1(y) := \|v(y)\|_\alpha^{\lambda_2-n} = \left(\sum_{i=1}^n v_i^\alpha(y_i)\right)^{\frac{\lambda_2-n}{\alpha}},\ \ f_2(y) := \prod_{i=1}^n v_i'(y_i).$$

For $j = 1, \cdots, n, v_j'(y_j) > 0, v_j''(y_j) \le 0, v_j'''(y_j) \ge 0, v_j(\eta^+) = 0$, we derive

$$\frac{\partial}{\partial y_j}g(y) = \frac{-\lambda(\sum_{i=1}^n v_i^\alpha(y_i))^{\frac{1}{\alpha}-1}v_j^{\alpha-1}(y_j)\, v_j'(y_j)}{[u(m) + (\sum_{i=1}^n v_i^\alpha(y_i))^{1/\alpha}]^{\lambda+1}} < 0,$$

$$\frac{\partial}{\partial y_j}f_1(y) = (\lambda_2 - n)\left(\sum_{i=1}^{n} v_i^{\alpha}(y_i)\right)^{\frac{\lambda_2-n}{\alpha}-1} v_j^{\alpha-1}(y_j)\, v_j'(y_j) \le 0,$$

$$\frac{\partial}{\partial y_j}f_2(y) = v_j''(y_j)\prod_{i=1(i\neq j)}^{n} v_i'(y_i) \le 0,\ \ \frac{\partial^2}{\partial y_j^2}f_1(y) \ge 0,\ \ \frac{\partial^2}{\partial y_j^2}f_2(y) \ge 0.$$

We still can find that $\frac{\partial^2}{\partial y_j^2}g(y) > 0$, and then in the same way,

$$\frac{\partial}{\partial y_j}h(y) = f_1(y)f_2(y)\frac{\partial}{\partial y_j}g(y) + g(y)\frac{\partial}{\partial y_j}(f_1(y)f_2(y)) < 0,$$

$$\frac{\partial^2}{\partial y_j^2}h(y) = \frac{\partial}{\partial y_j}[f_1(y)f_2(y)\frac{\partial}{\partial y_j}g(y)] + \frac{\partial}{\partial y_j}[g(y)\frac{\partial}{\partial y_j}(f_1(y)f_2(y))]$$
$$> 0\ \ (y_j \in (\eta, \infty),\ j = 1, \cdots, n).$$

The lemma has been shown. □

Lemma 2.2. *For $c > 0, b = \min_{1\le i\le n}\{v_i(1)\}, e = \max_{1\le i\le n}\{v_i(1)\}$, there exists a constant $a \in \mathbf{R}_+$, such that the following inequalities hold:*

$$0 < \frac{e^{-c}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}(\frac{1}{c} - A(c))$$
$$< \sum_k \|v(k)\|_\alpha^{-c-n}\prod_{i=1}^{n} v_i'(k_i)$$
$$< \frac{1}{c}\left(ca + \frac{\Gamma^n(\frac{1}{\alpha})}{b^c\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right), \tag{2.5}$$

where $A(c) := e^c\int_0^\infty \frac{(w+e)^{n-1}-w^{n-1}}{(w+e)^{c+n}}dw = \frac{1}{1+c}O(1), a := \sum_{i=1}^n M_i \in \mathbf{R}_+$,

$$M_i = \sum_{k_1=1}^{\infty}\cdots\sum_{k_{i-1}=1}^{\infty}\sum_{k_{i+1}=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left[v_1^{\alpha}(k_1) + \cdots v_{i-1}^{\alpha}(k_{i-1}) + v_i^{\alpha}(1)\right.$$
$$\left. + v_{i+1}^{\alpha}(k_{1i+1}) + \cdots + v_n^{\alpha}(k_n)\right]^{\frac{1}{\alpha}(-c-n)} v_i'(1)\prod_{j=1(j\neq i)}^{n} v_j'(k_j)\ \ (i = 1, \cdots, n).$$

Proof. Since $c > 0, \alpha \in (0, 1], v_i'(y_i) > 0,\ v_i''(y_i) \le 0, v_i'''(y_i) \ge 0$, and $j = 1, \cdots, n$, we derive

$$\frac{\partial}{\partial y_j}\left(\|v(y)\|_\alpha^{-c-n}\prod_{i=1}^{n} v_i'(y_i)\right) < 0,$$
$$\frac{\partial^2}{\partial y_j^2}\left(\|v(y)\|_\alpha^{-c-n}\prod_{i=1}^{n} v_i'(y_i)\right) > 0.$$

Setting $k' = (k_1', \cdots, k_n'), \left(k_i' = \{2, 3, \cdots\}, i = 1, \cdots, n\right)$, by (2.3), we have

$$\sum_k \|v(k)\|_\alpha^{-c-n}\prod_{i=1}^{n} v_i'(k_i) \le a + \sum_{k'}\left\|v(k')\right\|_\alpha^{-c-n}\prod_{i=1}^{n} v_i'(k_i')$$

$$
\begin{aligned}
&< a + \int_{\left\{y\in\mathbf{R}_+^n; y_i\ge 1\right\}} \frac{\prod_{i=1}^n v_i'(y_i)}{\|v(y)\|_\alpha^{c+n}} dy \\
&= a + \int_{\left\{u\in\mathbf{R}_+^n; u_i\ge v_i(1)\right\}} \|u\|_\alpha^{-c-n}\, du \\
&\le a + \int_{\left\{u\in\mathbf{R}_+^n; u_i\ge b\right\}} \|u\|_\alpha^{-c-n}\, du \\
&\le a + \int_{\left\{u\in\mathbf{R}_+^n; \|u\|_\alpha\ge b\right\}} \|u\|_\alpha^{-c-n}\, du \\
&= a + \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \int_b^\infty x^{-c-n} x^{n-1} dx \\
&= \frac{1}{c}\left(ca + \frac{\Gamma^n(\frac{1}{\alpha})}{b^c\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right).
\end{aligned}
$$

Supposed that $f(x) := (x+d)^{\frac{1}{\alpha}} - x^{\frac{1}{\alpha}} - d^{\frac{1}{\alpha}} (x, d \ge 0)$, we have

$$
f'(x) := \frac{1}{\alpha}[(x+d)^{\frac{1}{\alpha}-1} - x^{\frac{1}{\alpha}-1}] \ge 0 \ (\alpha \in (0,1]).
$$

For $f(0) = 0$, we find $(x+d)^{\frac{1}{\alpha}} \ge x^{\frac{1}{\alpha}} + d^{\frac{1}{\alpha}}$. Then,

$$
\begin{aligned}
0 &< M_n \\
&< \int_{\left\{y_i\ge\frac{1}{2}, i=1,\cdots,n-1\right\}} \left[\sum_{j=1}^{n-1} v_j^\alpha(y_j) + v_n^\alpha(1)\right]^{-\frac{1}{\alpha}(c+n)} v_n'(1)\prod_{i=1}^{n-1} v_i'(y_i) dy \\
&\overset{u=v(y)}{=} v_n'(1)\int_{\left\{u_i\ge v_i(\frac{1}{2}), i=1,\cdots,n-1\right\}} \left(\sum_{j=1}^{n-1} u_j^\alpha + v_n^\alpha(1)\right)^{-\frac{1}{\alpha}(c+n)} du \\
&\le v_n'(1)\int_{\{u_i\ge b, i=1,\cdots,n-1\}} \left(\sum_{j=1}^{n-1} u_j^\alpha + v_n^\alpha(1)\right)^{-\frac{1}{\alpha}(c+n)} du \\
&\le v_n'(1)\int_{\left\{\|u\|_\alpha\ge b\right\}} \left[\left(\sum_{j=1}^{n-1} u_j^\alpha\right)^{\frac{1}{\alpha}} + v_n(1)\right]^{-c-n} du \\
&= v_n'(1)\frac{\Gamma^{n-1}(\frac{1}{\alpha})}{\alpha^{n-2}\Gamma(\frac{n-1}{\alpha})}\int_b^\infty (x+v_n(1))^{-c-n} x^{n-2} dx \\
&\le v_n'(1)\frac{\Gamma^{n-1}(\frac{1}{\alpha})}{\alpha^{n-2}\Gamma(\frac{n-1}{\alpha})}\int_b^\infty x^{-c-2} dx \\
&< \infty.
\end{aligned}
$$

Above all, M_i $(i = 1, 2, \cdots, n)$ is a positive constant, then, a is a positive constant.

In the same way, since the series is decreasing, we derive

$$
\sum_k \|v(k)\|_\alpha^{-c-n} \prod_{i=1}^n v_i'(k_i) > \int_{\left\{y\in\mathbf{R}_+^n; y_i\ge 1\right\}} \|v(y)\|_\alpha^{-c-n} \prod_{i=1}^n v_i'(y_i) dy
$$

$$
\begin{aligned}
&= \int_{\{u\in\mathbf{R}_+^n;u_i\geq v_i(1)\}} \|u\|_\alpha^{-c-n}\,du \\
&\geq \int_{\{u\in\mathbf{R}_+^n;u_i\geq e\}} \|u\|_\alpha^{-c-n}\,du \\
&= \int_{\{w\in\mathbf{R}_+^n;w_i\geq 0\}} \|w+e\|_\alpha^{-c-n}\,dw\ (w=u-e).
\end{aligned}
$$

Setting $\phi(w) := w^{-c-n}$, by (2.2), we have

$$
\begin{aligned}
&\int_{\{w\in\mathbf{R}_+^n;w_i\geq 0\}} \|w+e\|_\alpha^{-c-n}\,dw \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \phi(w+e)w^{n-1}dw \\
&= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left[\int_0^\infty \frac{(w+e)^{n-1}dw}{(w+e)^{c+n}} - \int_0^\infty \frac{(w+e)^{n-1}-w^{n-1}}{(w+e)^{c+n}}dw\right] \\
&= \frac{e^{-c}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(\frac{1}{c}-A(c)\right) \\
&> 0,
\end{aligned}
$$

where, we indicate that

$$A(c) = e^c\int_0^\infty \frac{(w+e)^{n-1}-w^{n-1}}{(w+e)^{c+n}}dw.$$

Fot $n=1$, we find $A(c)=0$; for $n\in\mathbf{N}\backslash\{1\}$, by the mid-value theorem, we have

$$
\begin{aligned}
0 &< A(c) \\
&= (n-1)e^{1+c}\int_0^\infty \frac{(w+\theta_w e)^{n-2}dw}{(w+e)^{c+n}}\ (\theta_w\in(0,1)) \\
&\leq (n-1)e^{1+c}\int_0^\infty \frac{(w+e)^{n-2}}{(w+e)^{c+n}}dw \\
&= \frac{n-1}{1+c} \\
&< \infty,
\end{aligned}
$$

namely, $A(c) = \frac{1}{1+c}O(1)$. Hence, we have

$$
\begin{aligned}
\sum_k \|v(k)\|_\alpha^{-c-n}\prod_{i=1}^n v_i'(k_i) &> \int_{\{w\in\mathbf{R}_+^n;w_i\geq 0\}} \|w+e\|_\alpha^{-c-n}\,dw \\
&= \frac{e^{-c}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}(\frac{1}{c}-A(c)) \\
&> 0.
\end{aligned}
$$

Inequalities (2.4) are proved.

The proof of this lemma is complete. □

Lemma 2.3. *We define the weight functions as follows:*

$$\widetilde{\omega}_\lambda(\lambda_2, m) := u^{\lambda-\lambda_2}(m)\sum_k \frac{\|v(k)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(k_i)}{(u(m)+\|v(k)\|_\alpha)^\lambda} \quad (m \in \mathbf{N}), \tag{2.6}$$

$$\omega_\lambda(\lambda_1, k) := \|v(k)\|_\alpha^{\lambda-\lambda_1} \sum_{m-=1}^\infty \frac{u^{\lambda_1-1}(m)u'(m)}{(u(m)+\|v(k)\|_\alpha)^\lambda} \quad (k \in \mathbf{N}^n). \tag{2.7}$$

(i) For $\lambda_2 \le n, 0 < \lambda_2 < \lambda$, the following inequality holds:

$$\widetilde{\omega}_\lambda(\lambda_2, m) < \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda-\lambda_2), m \in \mathbf{N}. \tag{2.8}$$

(ii) For $\lambda_1 < 1, 0 < \lambda_1 < \lambda$, the following expression holds:

$$\begin{aligned} &0 < B(\lambda_1, \lambda-\lambda_1)\left(1 - O(\frac{1}{\|v(k)\|_\alpha^{\lambda_1}})\right) \\ &< \omega_\lambda(\lambda_1, k) \\ &< B(\lambda_1, \lambda-\lambda_1), k \in \mathbf{N}^n. \end{aligned} \tag{2.9}$$

Proof. (i) For $\lambda_2 \le n, 0 < \lambda_2 < \lambda$, by Lemma 2.1 and Hermite-Hadamard's inequality (cf. [11]), setting $w = v(y)$, $dw = \prod_{i=1}^n v_i'(y_i)dy$, we have

$$\begin{aligned} \widetilde{\omega}_\lambda(\lambda_2, m) &< u^{\lambda-\lambda_2}(m)\int_{A_{1/2}} \frac{\|v(y)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(y_i)}{(u(m)+\|v(y)\|_\alpha)^\lambda}dy \\ &\le u^{\lambda-\lambda_2}(m)\int_{A_\eta} \frac{\|v(y)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(y_i)}{(u(m)+\|v(y)\|_\alpha)^\lambda}dy \\ &= u^{\lambda-\lambda_2}(m)\int_{w\in\mathbf{R}_+^n} \frac{\|w\|_\alpha^{\lambda_2-n}}{(u(m)+\|w\|_\alpha)^\lambda}dw. \end{aligned}$$

Setting $\phi(s) := \frac{s^{\lambda_2-n}}{(x+s)^\lambda}$, by (2.2), it follows that

$$\begin{aligned} \widetilde{\omega}_\lambda(\lambda_2, m) &< u^{\lambda-\lambda_2}(m)\int_{u\in\mathbf{R}_+^n} \phi(\|w\|_\alpha)dw \\ &= u^{\lambda-\lambda_2}(m)\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \frac{s^{\lambda_2-1}}{(u(m)+s)^\lambda}ds \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\int_0^\infty \frac{t^{\lambda_2-1}}{(1+t)^\lambda}dt \ (t = s/u(m)) \\ &= \frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2, \lambda-\lambda_2), \end{aligned}$$

and then (2.8) is proved.

(ii) For $(-1)^i \frac{d^i}{dx^i}[(u(x))^{\lambda_1}u'(x)] \ge 0$ $(i = 1, 2)$, we still can find that

$$\frac{u^{\lambda_1-1}(x)u'(x)}{(u(x)+\|v(k)\|_\alpha)^\lambda} = \frac{u^{\lambda_1}(x)u'(x)}{(u(x)+\|v(k)\|_\alpha)^\lambda}\frac{1}{u(x)}$$

$(u(x) > 0, u'(x) > 0, u''(x) \le 0)$ is strictly decreasing and strictly convex in (ξ, ∞), by Hermite-Hadamard's inequality and (2.7), setting $s = \frac{u(x)}{\|v(k)\|_\alpha}$, we have

$$\begin{aligned}
\omega_\lambda(\lambda_1, k) &< \|v(k)\|_\alpha^{\lambda-\lambda_1} \int_{\frac{1}{2}}^{\infty} \frac{u^{\lambda_1-1}(x)u'(x)}{(u(x)+\|v(k)\|_\alpha)^\lambda} dx \\
&\le \|v(k)\|_\alpha^{\lambda-\lambda_1} \int_{\xi}^{\infty} \frac{u^{\lambda_1-1}(x)u'(x)}{(u(x)+\|v(k)\|_\alpha)^\lambda} dx \\
&= \int_0^\infty \frac{s^{\lambda_1-1}}{(s+1)^\lambda} ds \\
&= B(\lambda_1, \lambda-\lambda_1), \\
\omega_\lambda(\lambda_1, k) &> \|v(k)\|_\alpha^{\lambda-\lambda_1} \int_{1}^{\infty} \frac{u^{\lambda_1-1}(x)u'(x)}{(u(x)+\|v(k)\|_\alpha)^\lambda} dx \\
&= B(\lambda_1, \lambda-\lambda_1)\left(1 - O(\frac{1}{\|v(k)\|_\alpha^{\lambda_1}})\right) \\
&> 0,
\end{aligned}$$

where,

$$\begin{aligned}
0 &< O(\frac{1}{\|v(k)\|_\alpha^{\lambda_1}}) \\
&:= \frac{\|v(k)\|_\alpha^{\lambda-\lambda_1}}{B(\lambda_1, \lambda-\lambda_1)} \int_\xi^1 \frac{u^{\lambda_1-1}(x)u'(x)dx}{(u(x)+\|v(k)\|_\alpha)^\lambda} \\
&\le \frac{\|v(k)\|_\alpha^{\lambda-\lambda_1}}{B(\lambda_1, \lambda-\lambda_1)} \int_\xi^1 \frac{u^{\lambda_1-1}(x)du(x)}{\|v(k)\|_\alpha^\lambda} \\
&= \frac{[B(\lambda_1, \lambda-\lambda_1)]^{-1} u^{\lambda_1}(1)}{\lambda_1 \|v(k)\|_\alpha^{\lambda_1}},
\end{aligned}$$

and then (2.9) are proved.

This proves the lemma. □

Lemma 2.4. *For $t > 0$, we have the following inequality:*

$$\sum_{m=1}^{\infty} e^{-tu(m)} a_m \le t \sum_{m=1}^{\infty} e^{-tu(m)} u'(m) A_m. \tag{2.10}$$

Proof. For $x \in (\xi, \infty)$, setting $g(x) = e^{-tu(x)}$, we obtain

$$g'(x) = -tu'(x)e^{-tu(x)}.$$

Since $u''(x) \le 0$, then $-g'(x)$ is decreasing in (ξ, ∞). In view of $e^{-tu(m)}A_m = o(1)$ $(m \to \infty)$, by Abel summation formula and the differentiation mid-value theorem, we have $0 < \theta_m < 1$ and

$$\sum_{m=1}^{\infty} e^{-tu(m)} a_m = \lim_{m\to\infty} e^{-tu(m)} A_m + \sum_{m=1}^{\infty} (e^{-tu(m)} - e^{-tu(m+1)}) A_m$$

$$
\begin{aligned}
&= -\sum_{m=1}^{\infty}(e^{-tu(m+1)} - e^{-tu(m)})A_m \\
&= -\sum_{m=1}^{\infty}(g(m+1) - g(m))A_m \\
&= \sum_{m=1}^{\infty}(-g'(m+\theta_m))A_m \\
&\leq \sum_{m=1}^{\infty}(-g'(m))A_m \\
&= t\sum_{m=1}^{\infty} e^{-tu(m)}u'(m)A_m.
\end{aligned}
$$

Hence, we have (2.10).

The lemma is proved. □

Lemma 2.5. *The following inequality holds:*

$$
\begin{aligned}
I_\lambda &:= \sum_k \sum_{m=1}^{\infty} \frac{u'(m)A_m b_k}{(u(m)+\|v(k)\|_\alpha)^\lambda} \\
&\leq \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
&\quad \times \left[\sum_{m=1}^{\infty}\frac{(u(m))^{p(1-\widetilde{\lambda}_1)-1}A_m^p}{(u'(m))^{-1}}\right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}\, b_k^q}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\right]^{\frac{1}{q}}. \qquad (2.11)
\end{aligned}
$$

Proof. By employing the Hölder's inequality (cf. [11]), we obtain

$$
\begin{aligned}
I_\lambda &= \sum_k \sum_{m=1}^{\infty} \frac{1}{(u(m)+\|v(k)\|_\alpha)^\lambda}\left[\frac{\|v(k)\|_\alpha^{(\lambda_2-n)/p}(\prod_{i=1}^n v_i'(k_i))^{1/p}}{(u(m))^{(\lambda_1-1)/q}(u'(m))^{1/q}}u'(m)A_m\right] \\
&\quad \times \left[\frac{(u(m))^{(\lambda_1-1)/q}(u'(m))^{1/q}}{\|v(k)\|_\alpha^{(\lambda_2-n)/p}(\prod_{i=1}^n v_i'(k_i))^{1/p}}b_k\right] \\
&\leq \left\{\sum_{m=1}^{\infty}\left[\sum_k \frac{(u(m))^{(\lambda_1-1)(1-p)}}{(u(m)+\|v(k)\|_\alpha)^\lambda}\frac{\|v(k)\|_\alpha^{\lambda_2-n}\prod_{i=1}^n v_i'(k_i)}{(u'(m))^{p-1}}\right](u'(m)A_m)^p\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_k \left[\sum_{m=1}^{\infty}\frac{\|v(k)\|_\alpha^{(\lambda_2-n)(1-q)}}{(u(m)+\|v(k)\|_\alpha)^\lambda}\frac{(u(m))^{\lambda_1-1}u'(m)}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\right]b_k^q\right\}^{\frac{1}{q}} \\
&= \left[\sum_{m=1}^{\infty}\widetilde{\omega}_\lambda(\lambda_2,m)\frac{(u(m))^{p(1-\widetilde{\lambda}_1)-1}}{(u'(m))^{-1}}A_m^p\right]^{\frac{1}{p}}\left[\sum_k \omega_\lambda(\lambda_1,k)\frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}b_k^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

By (2.8) and (2.8), (2.11) follows.

The proof of this lemma is complete. □

Remark 2.1. Replacing λ (λ_1) by $\lambda+1$ (λ_1+1), in view of the assumption, $\frac{u^{\lambda_1}(x)u'(x)}{(u(x)+\|v(k)\|_\alpha)^\lambda}$ is strictly decreasing and strictly convex in (ξ,∞), then inequality $\omega_{\lambda+1}(\lambda_1+1,k) < B(\lambda_1+1,\lambda-\lambda_1)$ is valid and by the assumption, the following inequality follows:

$$\begin{aligned} I_{\lambda+1} &= \sum_k \sum_{m=1}^{\infty} \frac{u'(m)A_m b_k}{(u(m)+\|v(k)\|_\alpha)^{\lambda+1}} \\ &< \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2,\lambda+1-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+1,\lambda-\lambda_1) \\ &\quad \times \left[\sum_{m=1}^{\infty} \frac{u'(m)A_m^p}{(u(m))^{p\widetilde{\lambda}_1+1}}\right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}\, b_k^q}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\right]^{\frac{1}{q}}. \end{aligned} \tag{2.12}$$

3. Main results

Theorem 3.1. *We have a new multidimensional Hilbert-type inequality with two general internal variables involving one partial sum as follows:*

$$\begin{aligned} I &= \sum_k \sum_{m=1}^{\infty} \frac{a_m b_k}{(u(m)+\|v(k)\|_\alpha)^{\lambda}} \\ &< \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2,\lambda+1-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+1,\lambda-\lambda_1) \\ &\quad \times \left[\sum_{m=1}^{\infty} \frac{u'(m)A_m^p}{(u(m))^{p\widetilde{\lambda}_1+1}}\right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} b_k^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, we obtain

$$\begin{aligned} I &< \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1+1,\lambda_2) \\ &\quad \times \left[\sum_{m=1}^{\infty} \frac{u'(m)A_m^p}{(u(m))^{p\widetilde{\lambda}_1+1}}\right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\lambda_2)-n}\, b_k^q}{(\prod_{i=1}^n v_i'(k_i))^{q-1}}\right]^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

where the value

$$\lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1+1,\lambda_2)$$

is the best.

Proof. For $\lambda>0$, we have

$$\frac{1}{(u(m)+\|v(k)\|_\alpha)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-(u(m)+\|v(k)\|_\alpha)t} dt.$$

By employing (2.10) and Lebesgue term by term integration theorem (cf. [12]), we obtain

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda)}\sum_k\sum_{m=1}^{\infty} a_m b_k \left[\int_0^\infty t^{\lambda-1}e^{-\left(u(m)+\|v(k)\|_\alpha\right)t}dt\right] \\
&= \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\left(\sum_{m=1}^{\infty} e^{-u(m)t}a_m\right)\sum_k e^{-\|v(k)\|_\alpha t}b_k dt \\
&\le \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\left(t\sum_{m=1}^{\infty} e^{-u(m)t}u'(m)A_m\right)\sum_k e^{-\|v(k)\|_\alpha t}b_k dt \\
&= \frac{1}{\Gamma(\lambda)}\sum_k\sum_{m=1}^{\infty} u'(m)A_m b_k \left[\int_0^\infty t^{\lambda}e^{-\left(u(m)+\|v(k)\|_\alpha\right)t}dt\right] \\
&= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}\sum_k\sum_{m=1}^{\infty}\frac{u'(m)A_m b_k}{\left(u(m)+\|v(k)\|_\alpha\right)^{\lambda+1}} \\
&= \lambda I_{\lambda+1}.
\end{aligned}$$

Then by (2.11), we obtain (3.1). Particularly, for $\lambda_1+\lambda_2=\lambda$, we obtain (3.2).

For any $0<\varepsilon<p\lambda_1$, we set

$$\begin{aligned}
\widehat{a}_m &:= (u(m))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(m)\ (m\in\mathbf{N}),\\
\widehat{b}_k &:= \|v(k)\|_\alpha^{\lambda_2-\frac{\varepsilon}{q}-n}\prod_{i=1}^{n}v_i'(k_i)\ \ (k\in\mathbf{N}^n).
\end{aligned}$$

Since the function

$$(u(t))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(t) = [(u(t))^{\lambda_1}u'(t)](u(t))^{-\frac{\varepsilon}{p}-1},$$

is strictly decreasing and strictly convex, by Hermite-Hadamard's inequality, we find

$$\begin{aligned}
\widehat{A}_m &:= \sum_{i=1}^{m}\widehat{a}_i \\
&= \sum_{i=1}^{m}(u(t))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(i) \\
&< \int_{\frac{1}{2}}^{m-\frac{1}{2}}(u(t))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(t)dt \\
&\le \int_{\xi}^{m}(u(t))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(t)dt \\
&= \frac{(u(m))^{\lambda_1-\frac{\varepsilon}{p}}}{\lambda_1-\frac{\varepsilon}{p}}\ (m\in\mathbf{N}).
\end{aligned}$$

We observe $\widehat{A}_m = o(e^{tu(m)})$ $(t>0; m\to\infty)$.

If there exists a positive constant

$$M \le \lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1+1,\lambda_2),$$

such that (3.2) is valid when we replace

$$\lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1+1,\lambda_2)$$

by M, then in particular, we obtain

$$\begin{aligned}\widehat{I} &:= \sum_k \sum_{m=1}^{\infty} \frac{\widehat{a}_m \widehat{b}_k}{(u(m)+\|v(k)\|_\alpha)^\lambda} \\ &< M\left[\sum_{m=1}^{\infty} \frac{u'(m)\widehat{A}_m^p}{(u(m))^{p\lambda_1+1}}\right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\lambda_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} \widehat{b}_k^q\right]^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

By (2.4), we obtain

$$\begin{aligned}\widehat{J} &:= \left[\sum_{m=1}^{\infty} \frac{u'(m)\widehat{A}_m^p}{(u(m))^{p\lambda_1+1}}\right]^{\frac{1}{p}} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\lambda_2)-n}}{(\prod_{i=1}^n v_i'(k_i))^{q-1}} \widehat{b}_k^q\right]^{\frac{1}{q}} \\ &\le \frac{1}{\lambda_1-\frac{\varepsilon}{p}}\left[(u(1))^{-\varepsilon-1}u'(1)+\sum_{m=2}^{\infty}(u(m))^{-\varepsilon-1}u'(m)\right]^{\frac{1}{p}} \left(\sum_k \frac{\prod_{i=1}^n v_i'(k_i)}{\|v(k)\|_\alpha^{\varepsilon+n}}\right)^{\frac{1}{q}} \\ &< \frac{1}{\lambda_1-\frac{\varepsilon}{p}}\left[(u(1))^{-\varepsilon-1}u'(1)+\int_1^{\infty}(u(t))^{-\varepsilon-1}du(t)\right]^{\frac{1}{p}} \left(\sum_k \frac{\prod_{i=1}^n v_i'(k_i)}{\|v(k)\|_\alpha^{\varepsilon+n}}\right)^{\frac{1}{q}} \\ &\le \frac{1}{\varepsilon}\frac{\left[\varepsilon(u(1))^{-\varepsilon-1}u'(1)+(u(1))^{-\varepsilon}\right]^{\frac{1}{p}}}{\lambda_1-\frac{\varepsilon}{p}}\left[\varepsilon a+\frac{\Gamma^n(\frac{1}{\alpha})}{b^\varepsilon\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right]^{\frac{1}{q}}. \end{aligned}$$

By (2.9), setting $\widehat{\lambda}_1 := \lambda_1-\frac{\varepsilon}{p}\in(0,\lambda)$ in (2.5), since $u^{\widehat{\lambda}_1-1}(t)u'(t)$ is still decreasing and convex in (ξ,∞), we obtain

$$\begin{aligned}\widehat{I} &= \sum_k \sum_{m=1}^{\infty} \frac{(u(m))^{\lambda_1-\frac{\varepsilon}{p}-1}u'(m)}{(u(m)+\|v(k)\|_\alpha)^\lambda}\|v(k)\|_\alpha^{\lambda_2-\frac{\varepsilon}{q}-n}\prod_{i=1}^n v_i'(k_i) \\ &= \sum_k\left[\|v(k)\|_\alpha^{\lambda-\widehat{\lambda}_1}\sum_{m=1}^{\infty}\frac{u^{\widehat{\lambda}_1-1}(m)u'(m)}{(u(m)+\|v(k)\|_\alpha)^\lambda}\right]\|v(k)\|_\alpha^{-\varepsilon-n}\prod_{i=1}^n v_i'(k_i) \\ &= \sum_k \omega_\lambda(\widehat{\lambda}_1,k)\|v(k)\|_\alpha^{-\varepsilon-n}\prod_{i=1}^n v_i'(k_i) \\ &> B(\widehat{\lambda}_1,\lambda-\widehat{\lambda}_1)\sum_k\left(1-O(\frac{1}{\|v(k)\|_\alpha^{\widehat{\lambda}_1}})\right)\frac{\prod_{i=1}^n v_i'(k_i)}{\|v(k)\|_\alpha^{\varepsilon+n}} \\ &= B(\widehat{\lambda}_1,\lambda-\widehat{\lambda}_1)\sum_k\|v(k)\|_\alpha^{-\varepsilon-n}\prod_{i=1}^n v_i'(k_i)-B(\widehat{\lambda}_1,\lambda-\widehat{\lambda}_1)\sum_k\frac{\prod_{i=1}^n v_i'(k_i)}{O(|v(k)||_\alpha^{(\lambda_1+\frac{\varepsilon}{q})+n})} \\ &> B(\widehat{\lambda}_1,\lambda-\widehat{\lambda}_1)\frac{e^{-\varepsilon}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(\frac{1}{\varepsilon}-\frac{1}{\varepsilon+1}O(1)\right)-O_1(1). \end{aligned}$$

Hence, we derive

$$\begin{aligned}
&B(\widehat{\lambda}_1,\lambda-\widehat{\lambda}_1)\frac{e^{-\varepsilon}\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\left(1-\frac{\varepsilon}{\varepsilon+1}O(1)\right)-\varepsilon O_1(1)\\
&<\varepsilon\widehat{I}\\
&<M\frac{\left[\varepsilon(u(1))^{-\varepsilon-1}u'(1)+(u(1))^{-\varepsilon}\right]^{\frac{1}{p}}}{\lambda_1-\frac{\varepsilon}{p}}\left(\varepsilon a+\frac{\Gamma^n(\frac{1}{\alpha})}{b^{\varepsilon}\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon \longrightarrow 0^+$ and using the fact that the beta function is continuous, we derive

$$\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_1,\lambda_2)\leq\frac{M}{\lambda_1}\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{q}},$$

that is,

$$\begin{aligned}
M&\geq\lambda_1\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1,\lambda_2)\\
&=\lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1+1,\lambda_2).
\end{aligned}$$

Therefore, the value

$$M=\lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1+1,\lambda_2)$$

is the best in (3.2).

The proof of this theorem is complete. □

Remark 3.1. For $\widetilde{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\lambda_1+\frac{\lambda-\lambda_1-\lambda_2}{p}$, $\widetilde{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda_2+\frac{\lambda-\lambda_1-\lambda_2}{q}$, $0<\lambda_1,\lambda_2<\lambda$, we have $\widetilde{\lambda}_1+\widetilde{\lambda}_2=\lambda$, $0<\widetilde{\lambda}_1,\widetilde{\lambda}_2<\lambda$, and $B(\widetilde{\lambda}_1+1,\widetilde{\lambda}_2)\in\mathbf{R}_+$. For $\lambda-\lambda_1-\lambda_2\leq 0$, $u''(x)\leq 0$, we obtain $\widetilde{\lambda}_2\leq\lambda_2\leq n$ and

$$(u(x))^{\widetilde{\lambda}_1}u'(x)=[(u(x))^{\lambda_1}u'(x)](u(x))^{\frac{\lambda-\lambda_1-\lambda_2}{p}}\ \ (x\in(\xi,\infty))$$

is decreasing and convex. Therefore, for $\lambda-\lambda_1-\lambda_2\leq 0$, (3.2) is rewritten as:

$$\begin{aligned}
I&<\lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\widetilde{\lambda}_1+1,\widetilde{\lambda}_2)\\
&\quad\times\left[\sum_{m=1}^{\infty}\frac{u'(m)A_m^p}{(u(m))^{p\widetilde{\lambda}_1+1}}\right]^{\frac{1}{p}}\left[\sum_{k}\frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n}b_k^q}{\left[\prod_{i=1}^n v_i'(k_i)\right]^{q-1}}\right]^{\frac{1}{q}}.
\end{aligned}\tag{3.4}$$

Theorem 3.2. *For $\lambda - \lambda_1 - \lambda_2 \le 0$, if the value*

$$\lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda + 1 - \lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1)$$

is the best in (3.1), then $\lambda_1 + \lambda_2 = \lambda$.

Proof. By employing the Hölder's inequality (cf. [11]), we derive

$$\begin{aligned} B(\widetilde{\lambda}_1 + 1, \widetilde{\lambda}_2) &= \int_0^\infty \frac{u^{\widetilde{\lambda}_1} du}{(1+u)^{\lambda+1}} \\ &= \int_0^\infty \frac{u^{\frac{\lambda-\lambda_2}{p}} u^{\frac{\lambda_1}{q}}}{(1+u)^{\lambda+1}} du \\ &\le \left[\int_0^\infty \frac{u^{\lambda-\lambda_2} du}{(1+u)^{\lambda+1}} \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1} du}{(1+u)^{\lambda+1}} \right]^{\frac{1}{q}} \\ &= \left[\int_0^\infty \frac{v^{\lambda_2-1} dv}{(v+1)^{\lambda+1}} \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1} du}{(1+u)^{\lambda+1}} \right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\lambda_2, \lambda + 1 - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1). \end{aligned} \tag{3.5}$$

We compare with the constant factors in (3.1) and (3.4), and by the assumption, it follows that

$$\lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda + 1 - \lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1) \le \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \right)^{\frac{1}{p}} B(\widetilde{\lambda}_1 + 1, \widetilde{\lambda}_2),$$

that is,

$$B(\widetilde{\lambda}_1 + 1, \widetilde{\lambda}_2) \ge B^{\frac{1}{p}}(\lambda_2, \lambda + 1 - \lambda_2) B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1).$$

Hence, (3.5) keeps the form of equality.

We can observe that equality (3.5) is true if and only if there exist constants P and Q, such that they are not both zero and (cf. [11]) $Pu^{\lambda-\lambda_2-1} = Qu^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$. Suppose that $P \ne 0$. Then $u^{\lambda-\lambda_2-\lambda_1} = \frac{Q}{P}$ a.e. in $\mathbf{R}_+$, which implies that $\lambda - \lambda_2 - \lambda_1 = 0$. Hence, we have $\lambda_1 + \lambda_2 = \lambda$.

The proof of this theorem is complete. □

4. Equivalent forms and operator expressions

Theorem 4.1. *We have the following multidimensional Hilbert-type inequality equivalent to (3.1):*

$$\begin{aligned} J &:= \left\{ \sum_k \|v(k)\|_\alpha^{p\widetilde{\lambda}_2 - n} \prod_{i=1}^n v_i'(k_i) \left[\sum_{m=1}^\infty \frac{a_m}{(u(m) + \|v(k)\|_\alpha)^\lambda} \right]^p \right\}^{\frac{1}{p}} \\ &< \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda + 1 - \lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1) \left[\sum_{m=1}^\infty \frac{u'(m)}{(u(m))^{p\widetilde{\lambda}_1 + 1}} A_m^p \right]^{\frac{1}{p}}. \end{aligned} \tag{4.1}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, the following inequality equivalent to (3.2) holds:

$$\left\{\sum_k \|v(k)\|_\alpha^{p\lambda_2-n}\prod_{i=1}^n v_i'(k_i)\left[\sum_{m=1}^\infty \frac{a_m}{(u(m)+\|v(k)\|_\alpha)^\lambda}\right]^p\right\}^{\frac{1}{p}} < \lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1+1,\lambda_2)\left[\sum_{m=1}^\infty \frac{u'(m)}{(u(m))^{p\lambda_1+1}}A_m^p\right]^{\frac{1}{p}}. \tag{4.2}$$

Proof. Assuming (4.1) is valid, by employing Hölder's inequality (cf. [11]), we have

$$\begin{aligned} I &= \sum_k \left[\|v(k)\|_\alpha^{\widetilde{\lambda}_2-\frac{n}{p}}\left(\prod_{i=1}^n v_i'(k_i)\right)^{\frac{1}{p}}\sum_{m=1}^\infty \frac{a_m}{(u(m)+\|v(k)\|_\alpha)^\lambda}\right]\left[\frac{\|v(k)\|_\alpha^{\frac{n}{p}-\widetilde{\lambda}_2} b_k}{\left(\prod_{i=1}^n v_i'(k_i)\right)^{\frac{1}{p}}}\right] \\ &\le J\left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n} b_k^q}{\left(\prod_{i=1}^n v_i'(k_i)\right)^{q-1}}\right]^{\frac{1}{q}}. \end{aligned} \tag{4.3}$$

Then, by (4.1), we obtain (3.1).

Additionally, if (3.1) is valid, then we construct

$$b_k = \|v(k)\|_\alpha^{p\widetilde{\lambda}_2-n}\prod_{i=1}^n v_i'(k_i)\left[\sum_{m=1}^\infty \frac{a_m}{(u(m)+\|v(k)\|_\alpha)^\lambda}\right]^{p-1},\ k\in\mathbf{N}^n.$$

If $J=0$, then (4.1) is true. If $J=\infty$, then (4.1) is not true, implying that $J<\infty$. For $0<J<\infty$, based on (3.1), we obtain

$$\begin{aligned} \sum_k \frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n} b_k^q}{\left(\prod_{i=1}^n v_i'(k_i)\right)^{q-1}} &= J^p \\ &= I \\ &< \lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda+1-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+1,\lambda-\lambda_1) \\ &\quad\times\left[\sum_{m=1}^\infty \frac{u'(m)}{(u(m))^{p\widetilde{\lambda}_1+1}}A_m^p\right]^{\frac{1}{p}} J^{p-1}, \end{aligned}$$

$$\begin{aligned} \left[\sum_k \frac{\|v(k)\|_\alpha^{q(n-\widetilde{\lambda}_2)-n} b_k^q}{\left(\prod_{i=1}^n v_i'(k_i)\right)^{q-1}}\right]^{\frac{1}{p}} &= J \\ &< \lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda+1-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+1,\lambda-\lambda_1) \\ &\quad\times\left[\sum_{m=1}^\infty \frac{u'(m)}{(u(m))^{p\widetilde{\lambda}_1+1}}A_m^p\right]^{\frac{1}{p}}. \end{aligned}$$

Thus, (4.1) is the equivalent form of (3.1).

The proof of this theorem is complete. □

Theorem 4.2. *If $\lambda_1 + \lambda_2 = \lambda$, then the value*

$$\lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2, \lambda+1-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+1, \lambda-\lambda_1)$$

in (4.1) is the best. On contrast, if the same value in (4.1) is the best, then for $\lambda - \lambda_1 - \lambda_2 \leq 0$, we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. Based on Theorem 3.2, for $\lambda_1 + \lambda_2 = \lambda$,

$$\lambda\left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2, \lambda+1-\lambda_2)\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1+1, \lambda-\lambda_1)$$

in (3.1) is the best value. The value in (4.1) remains the best. Alternatively, based on (4.3), it is a contradiction that the value in (3.1) is not the best. In addition, if the value in (4.1) is the best, then using the equivalence between (4.1) and (3.1), and by considering $J^p = I$ as outlined in the proof of Theorem 4.1, then the value in (3.1) is the best. Based on the assumption and Theorem 3.2, it can be concluded that $\lambda_1 + \lambda_2 = \lambda$.

The proof of this theorem is complete. □

Remark 4.1. (i) For $u(m) = (m-\xi)^\gamma$ $(\gamma \in (0,1)), v(k) = k - \eta$, $\xi, \eta \in [0, \frac{1}{2}]$, $m \in \mathbf{N}, k \in \mathbf{N}^n$, then for $\lambda_1 \in (0,\lambda) \cap (0, \frac{1}{\gamma} - 1]$,

$$(u(t))^{\lambda_1}u'(t) = \gamma t^{\gamma(\lambda_1+1)-1} \ (t > \xi)$$

is decreasing and convex. By (3.2) and (4.2), we obtain the following equivalent inequalities:

$$\begin{aligned}
&\sum_k \sum_{m=1}^{\infty} \frac{a_m b_k}{[(m-\xi)^\gamma + \|k-\eta\|_\alpha]^\lambda} \\
&< \lambda\left(\gamma\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} \\
&\times B(\lambda_1+1, \lambda_2)\left[\sum_{m=1}^{\infty}\frac{A_m^p}{(m-\xi)^{p\gamma\lambda_1+1}}\right]^{\frac{1}{p}}\left[\sum_k \|k-\eta\|_\alpha^{q(n-\lambda_2)-n}\, b_k^q\right]^{\frac{1}{q}},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\left\{\sum_k \|k-\eta\|_\alpha^{p\lambda_2-n}\left[\sum_{m=1}^{\infty}\frac{a_m}{[(m-\xi)^\gamma + \|k-\eta\|_\alpha]^\lambda}\right]^p\right\}^{\frac{1}{p}} \\
&< \lambda\left(\gamma\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}} B(\lambda_1+1, \lambda_2)\left[\sum_{m=1}^{\infty}\frac{A_m^p}{(m-\xi)^{p\gamma\lambda_1+1}}\right]^{\frac{1}{p}}.
\end{aligned} \tag{4.5}$$

(ii) For $u(m) = \ln^\gamma(m+1-\xi)$ $(\gamma \in (0,1))$, $v(k) = \|\ln(k+1-\eta)\|_\alpha$, $\xi, \eta \in [0, \frac{1}{2}]$, $m \in \mathbf{N}, k \in \mathbf{N}^n$, then for $\lambda_1 \in (0,\lambda) \cap (0, \frac{1}{\gamma} - 1]$,

$$(u(t))^{\lambda_1}u'(t) = \frac{\gamma}{t}\ln^{\gamma(\lambda_1+1)-1} t \ (t > \xi)$$

is decreasing and convex. by (3.2) and (4.2), we obtain the following equivalent inequalities:

$$\sum_{k}\sum_{m=1}^{\infty}\frac{a_m b_k}{[\ln^{\gamma}(m+1-\xi)+\|\ln(k+1-\eta)\|_{\alpha}]^{\lambda}}$$
$$<\lambda\left(\gamma\frac{\Gamma^{n}(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1+1,\lambda_2)$$
$$\times\left[\sum_{m=1}^{\infty}\frac{[\ln(m+1-\xi)]^{-p\gamma\lambda_1-1}}{m+1-\xi}A_m^p\right]^{\frac{1}{p}}\left[\sum_{k}\frac{\|\ln(k+1-\eta)\|_{\alpha}^{q(n-\lambda_2)-n}}{\left(\prod_{i=1}^{n}(k_i+1-\eta)\right)^{1-q}}b_k^q\right]^{\frac{1}{q}}, \tag{4.6}$$

$$\left\{\sum_{k}\frac{\|\ln(k+1-\eta)\|_{\alpha}^{p\lambda_2-n}}{\prod_{i=1}^{n}(k_i+1-\eta)}\left[\sum_{m=1}^{\infty}\frac{a_m}{[\ln^{\gamma}(m+1-\xi)+\|\ln(k+1-\eta)\|_{\alpha}]^{\lambda}}\right]^{p}\right\}^{\frac{1}{p}}$$
$$<\lambda\left(\frac{\gamma\Gamma^{n}(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}\right)^{\frac{1}{p}}B(\lambda_1+1,\lambda_2)\left[\sum_{m=1}^{\infty}\frac{[\ln(m+1-\xi)]^{-p\gamma\lambda_1-1}A_m^p}{m+1-\xi}\right]^{\frac{1}{p}}. \tag{4.7}$$

The constant in the above inequalities is the best possible.

Setting functions $\phi(m):=u^{-p\widetilde{\lambda}-1}(m),\psi(k):=\frac{\|v(k)\|_{\alpha}^{q(n-\widetilde{\lambda}_2)-n}}{\left(\prod_{i=1}^{n}v_i'(k_i)\right)^{q-1}}$, then

$$\psi^{1-p}(k)=\|v(k)\|_{\alpha}^{p\widetilde{\lambda}_2-n}\prod_{i=1}^{n}v_i'(k_i),\ (m\in\mathbf{N},k\in\mathbf{N}^n).$$

We define the real normed spaces as follows:

$$l_{p,\phi}:=\left\{a=\{a_m\}_{m=1}^{\infty};\|a\|_{p,\phi}:=\left(\sum_{m=1}^{\infty}\phi(m)\left|a_m\right|^p\right)^{\frac{1}{p}}<\infty\right\},$$
$$l_{q,\psi}:=\left\{b=\{b_{k_1,\cdots k_n}\};\|b\|_{q,\psi}:=\left(\sum_{k}\psi(k)\left|b_k\right|^q\right)^{\frac{1}{q}}<\infty\right\},$$
$$l_{p,\psi^{1-p}}:=\left\{c=\{c_{k_1,\cdots k_n}\};\|c\|_{q,\psi}:=\left(\sum_{k}\psi^{1-p}(k)\left|c_k\right|^p\right)^{\frac{1}{p}}<\infty\right\},$$
$$\widehat{l}_{p,\phi}:=\left\{a=\{a_m\}_{m=1}^{\infty};A_m=\sum_{i=1}^{m}a_i(\in l_{p,\phi})=o(e^{tu(m)})(t>0;m\longrightarrow\infty)\right\}.$$

For any $a(>0)\in\widehat{l}_{p,\phi}$, setting $b_k:=\sum_{m=1}^{\infty}\frac{a_m}{\left(u(m)+\|v(k)\|_{\alpha}\right)^{\lambda}},k\in\mathbf{N}^n$, (4.1) is rewritten as follows:

$$\|b\|_{p,\psi^{1-p}}\le\lambda\left(\frac{\Gamma^{n}(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})}B(\lambda_2,\lambda+1-\lambda_2)\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1+1,\lambda-\lambda_1)\left\|A\right\|_{p,\phi}$$
$$<\infty,$$

that is, $b\in l_{p,\psi^{1-p}}$.

Definition 4.1. Define a multidimensional Hilbert–type operator

$$T : \widehat{l}_{p,\phi} \longrightarrow l_{p,\psi^{1-p}}$$

as follows: For any $a \in \widehat{l}_{p,\phi}$, there exists a unique representation $b = Ta \in l_{p,\psi^{1-p}}$, such that for any $k \in \mathbf{N}^n, Tf(k) = b_k$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta, b) := \sum_k b_k \sum_{m=1}^{\infty} \frac{a_m}{(u(m) + \|v(k)\|_\alpha)^\lambda} = I,$$

$$\|T\| := \sup_{a(\neq 0) \in \widehat{l}_{p,\phi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|A\|_{p,\phi}}.$$

By Theorems 3.1, 3.2, 4.1, and 4.2, we have the following theorem.

Theorem 4.3. *If* $a(\geq 0) \in \widehat{l}_{p,\phi}$, $\|b\|_{q,\psi} > 0$, $\|A\|_{p,\varphi} > 0$, *then the following equivalent inequalities hold:*

$$(Ta, b) < \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda + 1 - \lambda_2) \right)^{\frac{1}{p}} \times B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1) \|A\|_{p,\phi} \|b\|_{q,\psi}, \tag{4.8}$$

$$\|Ta\|_{p,\psi^{1-p}} < \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda + 1 - \lambda_2) \right)^{\frac{1}{p}} \times B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1) \|A\|_{p,\phi}. \tag{4.9}$$

Furthermore, if $\lambda_1 + \lambda_2 = \lambda$, *then the value*

$$\lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} B(\lambda_2, \lambda + 1 - \lambda_2) \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1 + 1, \lambda - \lambda_1)$$

in (4.8) and (4.9) is the best. Thus,

$$\|T\| = \lambda \left(\frac{\Gamma^n(\frac{1}{\alpha})}{\alpha^{n-1}\Gamma(\frac{n}{\alpha})} \right)^{\frac{1}{p}} B(\lambda_1 + 1, \lambda_2).$$

In contrast, if the same value in (4.4) or (4.5) is the best, then for $\lambda - \lambda_1 - \lambda_2 \leq 0$, *we have* $\lambda_1 + \lambda_2 = \lambda$.

5. Conclusions

This work uses the transfer formula and weight functions to develop a new multidimensional Hilbert-type inequality involving one partial sum. It has a kernel $\frac{1}{(u(m)+\|v(k)\|_\alpha)^\lambda}(\lambda > 0)$. Theorem 3.1 states the inequality. Theorem 3.2 focuses on the equivalent statements of the best value linked to some parameters. Additionally, Theorem 4.1-4.3 explore the equivalent forms and the operator expressions. This work obtains two general internal variables in the kernel than the kernel in [9].

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