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THE INTEGRABILITY OF THE GENERALIZED BURGERS-HUXLEY EQUATION*

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Abstract In this paper, we investigate the integrability of the generalized Burgers-Huxley equation. Firstly, we transform the equation into a planar dynamical system using the traveling wave transformation. Furthermore, we determine the type of each singular point and study their integrability. By computing the saddle values and singular point quantities, we finally derive the integrability conditions of the system.

Keywords Integrability, generalized Burgers-Huxley equation, singular point quantity, saddle value.

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1. Introduction

Nonlinear partial differential equations (NPDEs) constitute a vital class of mathematical models across diverse disciplines, including physics, chemistry, life sciences and environmental sciences, etc. (See [7, 29]). The solutions and associated properties of NPDEs are essential elements within the realm of nonlinear science.

The Burgers-Huxley equation exerts substantial influence on practical applications, particularly in elucidating the interaction among convection effects, reaction mechanisms and diffusion transport. The generalized Burgers-Huxley equation is as follows:

$$u_t + \alpha u^n u_x - u_{xx} = \beta u\left(1 - u^n\right)\left(u^n - \gamma\right), \tag{1.1}$$

where $\alpha,\ \beta,\ n \geq 0,\ \gamma$ are real numbers. The equation encompasses a variety of well-established evolutionary equations. When $n = 1,\ \alpha = 0$, equation (1.1) is reduced to the Burgers equation. When $\alpha = 0,\ n = 1,\ \gamma = -1$, equation (1.1) is reduced to the Newwell-Whitehead equation. When $n = 1,\ \beta = 0$, equation (1.1) is reduced to Fitzhugh-Nagumo equation. Mathematically, the generalized Burgers-Huxley equation exhibits rich dynamical behaviors, including traveling wave solutions, shock waves and pattern formation.

Appadu and Tijani [1] obtained the numerical solution of a 1-D generalised Burgers-Huxley equation under specified initial and boundary conditions, considered in three different regimes. Batiha et al. [2] introduced He's variational iteration method to overcome the difficulty arising in calculating Adomian polynomials. Darvishi et al. [6] presented a numerical solution of the generalized Burgers-Huxley equation using the spectral collocation method. Ismail et al. [14]

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obtained the approximate solutions for the Burger's-Huxley and Burger's-Fisher equations by using the Adomian decomposition method. Khattak [15] presented a numerical solution of the generalized Burger's-Huxley equation by means of Kansa's approach. Korkut [16] proposed a highly accurate and simple algorithm based on the Taylor wavelet methods for obtaining the approximate solution of the generalized Burgers-Huxley equation. Wang [26] established the fractal variational formulation for a generalized Burgers-Huxley equation with fractal derivative by using the semi-inverse method. Liang and Wang [19] proposed and explored a new fractional Vakhnenko-Parkes equation with the local fractional derivative for the fractal relaxation medium. Li et al. [18] dealt with the dynamics of a new established fractional-order delayed zooplankton-phytoplankton system. Zhang et al. [31] investigated a class of generalized Burgers-Huxley equation and its singularly perturbed form. Moreover, we can also study its properties by the integrability. See also [28] and references therein for more results.

The integrability problem is one of the main problems in the qualitative theory of planar polynomial differential systems. Despite the fact that integrability represents a stringent constraint and that, in general, differential systems are not integrable, the presence of a first integral enables a detailed understanding of the phase portrait of planar differential systems. The existence of a local analytic first integral is closely associated with the center-focus problem. It is well-established in the literature (See [22,24]) that a singular point in planar smooth dynamical systems qualifies as a center if and only if there exists a local first integral in a neighborhood of that singular point.

The problem of finding the necessary and sufficient conditions for an equilibrium to be a center is referred to the center-focus problem, which is intimately associated with Hilbert's 16th problem. It is known that Dulac et al. obtained the center conditions for quadratic systems. Chen et al. [4] studied dynamics of a class of separable polynomial rigid systems and classify global phase portraits under certain conditions. Cima et al. [5] introduced the notion of strongly persistent centers, together with the condition of the annulation of some generalized moments, for Abel differential equations with trigonometric coefficients as a natural candidate to characterize the centers of composition type for these equations García et al. [11] presented some criteria about the existence and nonexistence of both Puiseux inverse integrating factors and Puiseux first integrals for planar analytic vector fields having a monodromic singularity. Giné [12] studied the centers of the polynomial differential systems with a linear center perturbed by homogeneous polynomials for the degrees $s = 2, 3, 4, 5$. Giné [13] studied the center problem for polynomial Liénard systems of degree n with damping of degree n. Li et al. [17] studied the center-focus problem for a generalized cubic Kukles system with a nilpotent singular point,which consists of a cubic system with an extra 4th-order term. Mironenko and Mironenko [23] traced the relationships between the notion of generalized integral and the notions of reflecting function and Poincaré map (period map) for periodic differential systems. Yang and Tang [30] presented three new results on Abel equations. Zhou and Romanovski [33] gave coefficient conditions for existence of a center in a family of planar quartic polynomial differential systems. Zeng et al. [32] applied Abelian integral method to show the existence of a unique periodic traveling wave solution for a class of nonlinear reaction-diffusion equations. Additionally, several significant results were obtained in [3,8,10,21,25].

Wang et al. [27] converted the generalized Burgers-Huxley equation into a planar ordinary differential system and examined its integrability and Hopf bifurcation through the computation of singular point quantities at an equilibrium. On the basis of Wang's research, we will explore the integrability of the planar dynamical system at several other equilibria.

This paper is arranged in the following manner. In Section 2, we transform equation (1.1) into a planar dynamical system, then we discuss the properties of the equilibriua and their classifications. In Section 3, on account of the case where $(1,0)$ is the center has already been studied in [27], we only discuss the integrability of the planar dynamical system by calculating the saddle values at $(1,0)$. In Section 4, we calculate the focal values of the equilibrium $(\gamma,0)$ and derive the necessary and sufficient conditions for it to be a center. Moreover, we study the integrability at $(\gamma,0)$.

2. Properties of the equilibria

We will consider the planar dynamical system of equation (1.1). To begin with, by applying a coordinate transformation $\phi = u^n$, we get the following equation:

$$\phi_t + \alpha\phi\phi_x + \frac{(1-\frac{1}{n})\phi_x^2}{\phi} - \phi_{xx} = \beta n\phi(1-\phi)(\phi-\gamma), \tag{2.1}$$

supposing that equation (2.1) has a traveling wave solution of the form

$$\phi(x,t) = \phi(\xi),\ \xi = x - ct,$$

where c denotes the velocity of wave propagation. Replacing the variables in equation (2.1) with the specified traveling wave solution, we get

$$-c\phi\phi' + \alpha\phi^2\phi' + \left(1-\frac{1}{n}\right)\phi'^2 - \phi\phi'' = \beta n\phi^2\left(1-\phi\right)\left(\phi-\gamma\right), \tag{2.2}$$

where $\phi' = \frac{d\phi}{d\xi}$ and $\phi'' = \frac{d^2\phi}{d\xi^2}$. By letting $\phi' = y$ and $d\xi = \phi d\zeta$, equation (2.2) is transformed into the following planar dynamic system

$$\begin{cases} \dfrac{d\phi}{d\zeta} = \phi y = P\left(\phi,y\right), \\ \dfrac{dy}{d\zeta} = \dfrac{n-1}{n}y^2 + \phi\left(\alpha\phi - c\right)y + n\beta\phi^2\left(\phi-1\right)\left(\phi-\gamma\right) = Q\left(\phi,y\right). \end{cases} \tag{2.3}$$

Clearly, $(1,0)$ and $(\gamma,0)$ are two equilibria of system (2.3). we denote the equilibrium $(1,0)$, $(\gamma,0)$ by A, B, respectively. We aim to discuss the integrability of system (2.3) at equilibria A, B, respectively.

For system (2.3), the coefficient matrix of the linearized system at the point (ϕ_0,y_0) is presented as follows:

$$J_{(\phi_0,y_0)} = \begin{pmatrix} \dfrac{\partial P(\phi,y)}{\partial\phi} & \dfrac{\partial P(\phi,y)}{\partial y} \\ \dfrac{\partial Q(\phi,y)}{\partial\phi} & \dfrac{\partial Q(\phi,y)}{\partial y} \end{pmatrix}_{(\phi_0,y_0)}, \tag{2.4}$$

where

$$\frac{\partial P(\phi,y)}{\partial\phi} = y,$$
$$\frac{\partial P(\phi,y)}{\partial y} = \phi,$$

$$\begin{aligned}\frac{\partial Q(\phi,y)}{\partial \phi} &= y\left(-c+2\alpha\phi\right)-n\beta\phi\left(-2\gamma+3\phi+3\gamma\phi-4\phi^2\right),\\ \frac{\partial Q(\phi,y)}{\partial y} &= \frac{2(n-1)y}{n}+\phi\left(-c+\alpha\phi\right).\end{aligned}$$

In the following, we will examine the properties of the equilibrium A, B individually.

2.1. The type of equilibrium A

For system (2.3), the coefficient matrix of the linearized system at point A is as follows:

$$J_A=\begin{pmatrix} 0 & 1 \\ n\beta\left(1-\gamma\right) & -c+\alpha \end{pmatrix}. \tag{2.5}$$

The characteristic equation of matrix J_A is as follows:

$$\lambda^2+(c-\alpha)\lambda+n\beta\left(\gamma-1\right)=0, \tag{2.6}$$

so the trace of the matrix J_A is $\alpha-c$ and the determinant of the matrix J_A is $n\beta\left(\gamma-1\right)$.

1. If $\left(c-\alpha\right)^2-4n\beta\left(\gamma-1\right)>0$, equation (2.6) has two different real eigenvalues, i.e.,

$$\lambda^A_{1,2}=\frac{1}{2}\left(-c+\alpha\pm\sqrt{\left(c-\alpha\right)^2-4n\beta\left(\gamma-1\right)}\right). \tag{2.7}$$

For this case, further analysis can be derived:
(1a) If $n\beta\left(\gamma-1\right)>0$, equation (2.6) has different real eigenvalues with the same sign.
(i) When $\alpha-c>0$, equation (2.6) has two positive real eigenvalues, so A is an unstable node.
(ii) When $\alpha-c<0$, equation (2.6) has two negative real eigenvalues, so A is a stable node.
(1b) If $n\beta\left(\gamma-1\right)<0$, equation (2.6) has two different real eigenvalues with opposite signs, so A is a saddle.

2. If $\left(c-\alpha\right)^2-4n\beta\left(\gamma-1\right)=0$, equation (2.6) has two iterative real eigenvalues, i.e., $\lambda^A_{1,2}=\frac{1}{2}\left(-c+\alpha\right)$, and A is a degenerate node point. On this situation, further analysis can be obtained:
(2a) If $\alpha-c\geq 0$, then A is an unstable degenerate node.
(2b) If $\alpha-c<0$, then A is a stable degenerate node.

3. If $\left(c-\alpha\right)^2-4n\beta\left(\gamma-1\right)<0$, equation (2.6) has two conjugate complex eigenvalues, i.e.,

$$\lambda^A_{1,2}=\frac{1}{2}\left(-c+\alpha\pm i\sqrt{-\left(c-\alpha\right)^2+4n\beta\left(\gamma-1\right)}\right). \tag{2.8}$$

(3a) If $\alpha-c<0$, then A is a stable focus.
(3b) If $\alpha-c>0$, then A is an unstable focus.
(3c) If $c=\alpha$, the equilibrium A of the linearized system of system (2.3) is a center. Thus, the equilibrium A of system (2.3) might turn into either a center or a focus.

2.2. The type of equilibrium B

For system (2.3), the coefficient matrix of the linearized system at point B is as follows:

$$J_B = \begin{bmatrix} 0 & \gamma \\ n\beta\left(-1+\gamma\right)\gamma^2 & \gamma\left(-c+\alpha\gamma\right) \end{bmatrix}. \tag{2.9}$$

The characteristic equation of matrix J_B is as follows:

$$\lambda^2 + \left(c\gamma - \alpha\gamma^2\right)\lambda + n\beta\gamma^3\left(1-\gamma\right) = 0, \tag{2.10}$$

so the trace of the matrix J_B is $\alpha\gamma^2 - c\gamma$ and the determinant of the matrix J_B is $n\beta\gamma^3\left(1-\gamma\right)$.

1. If $\left(c\gamma - \alpha\gamma^2\right)^2 - 4n\beta\gamma^3\left(1-\gamma\right) > 0$, equation (2.10) has two different real eigenvalues, i.e.,

$$\lambda_{1,2}^B = \frac{1}{2}\left(\alpha\gamma^2 - c\gamma \pm \sqrt{\left(c\gamma-\alpha\gamma^2\right)^2 - 4n\beta\gamma^3\left(1-\gamma\right)}\right). \tag{2.11}$$

   For this case, further analysis can be derived:
   ($1a$) If $n\beta\gamma^3\left(1-\gamma\right) > 0$, equation (2.10) has different real eigenvalues with the same sign.
   (i) When $\alpha\gamma^2 - c\gamma > 0$, equation (2.10) has two positive real eigenvalues, so B is an unstable node.
   (ii) When $\alpha\gamma^2 - c\gamma < 0$, equation (2.10) has two negative real eigenvalues, so B is a stable node.
   ($1b$) If $n\beta\gamma^3\left(1-\gamma\right) < 0$, equation (2.10) has two different real eigenvalues with opposite signs, so B is a saddle.

2. If $\left(c\gamma - \alpha\gamma^2\right)^2 - 4n\beta\gamma^3\left(1-\gamma\right) = 0$, equation (2.10) has two iterative real eigenvalues, i.e., $\lambda_{1,2}^B = \frac{1}{2}\left(\alpha\gamma^2 - c\gamma\right)$, and B is a degenerate node. In this situation, further analysis can obtain
   ($2a$) If $\alpha\gamma^2 - c\gamma \geq 0$, then B is an unstable degenerate node.
   ($2b$) If $\alpha\gamma^2 - c\gamma < 0$, then B is a stable degenerate node.

3. If $\left(c\gamma - \alpha\gamma^2\right)^2 - 4n\beta\gamma^3\left(1-\gamma\right) < 0$, equation (2.10) has two conjugate complex eigenvalues, i.e.,

$$\lambda_{1,2}^B = \frac{1}{2}\left(\alpha\gamma^2 - c\gamma \pm i\sqrt{-\left(c\gamma-\alpha\gamma^2\right)^2 + 4n\beta\gamma^3\left(1-\gamma\right)}\right). \tag{2.12}$$

   ($3a$) If $\alpha\gamma^2 - c\gamma < 0$, then B is a stable focus.
   ($3b$) If $\alpha\gamma^2 - c\gamma > 0$, then B is an unstable focus.
   ($3c$) If $\alpha\gamma^2 - c\gamma = 0$, the equilibrium B of the linearized system of system (2.3) is a center. Thus, the equilibrium B of system (2.3) might turn into either a center or a focus, the analysis of the equilibrium B is a center-focus problem.

Consequently, for system (2.3), the equilibria $A,\ B$ might be saddle, center or focus respectively, we will investigate the integrability of system (2.3) at the equilibria $A,\ B$ in the following sections.

3. The integrability of equilibrium A

Here, we let $c = \alpha,\ \gamma \neq 1$. When $\gamma < 1$, A is a saddle; when $\gamma > 1$, A is a center or focus, which was studied in [27]. Thus, we're going to study the integrability of system (2.3) when A is a saddle. In this case, we apply the transformation $\phi = x + 1,\ y = y$, getting the system

$$\begin{cases} \dfrac{dx}{d\zeta} = (x+1)y, \\ \dfrac{dy}{d\zeta} = \dfrac{n-1}{n}y^2 + \alpha x\left(x+1\right)y + n\beta x\left(x-\gamma+1\right)\left(x+1\right)^2. \end{cases} \tag{3.1}$$

Thus, the equilibrium A of system (2.3) becomes the origin of system (3.1). For convenience, we set $\gamma = 1 - \frac{\rho^2}{n\beta}$, with ρ is a nonzero real number.

Under the transformation

$$x = \frac{\eta - \xi}{\rho},\ y = \xi + \eta,\ d\zeta = -\frac{1}{\rho}dT, \tag{3.2}$$

system (3.1) becomes

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}T} = \xi + \displaystyle\sum_{i+j=2}^{4} a_{ij}^{A}\xi^i\eta^j, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}T} = -\eta + \displaystyle\sum_{i+j=2}^{4} b_{ij}^{A}\xi^i\eta^j, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} & a_{20}^A = -\frac{n^2\beta - n\alpha\rho - \rho^2 + 4n\rho^2}{2n\rho^3},\quad a_{11}^A = \frac{n^2\beta + \rho^2 + n\rho^2}{n\rho^3}, \\ & a_{02}^A = -\frac{n^2\beta + n\alpha\rho - \rho^2 + 2n\rho^2}{2n\rho^3},\quad a_{30}^A = \frac{2n\beta - \alpha\rho + \rho^2}{2\rho^4}, \\ & a_{21}^A = -\frac{6n\beta - \alpha\rho + 3\rho^2}{2\rho^4},\quad a_{12}^A = \frac{6n\beta + \alpha\rho + 3\rho^2}{2\rho^4}, \\ & a_{03}^A = -\frac{2n\beta + \alpha\rho + \rho^2}{2\rho^4},\quad a_{40}^A = -\frac{n\beta}{2\rho^5},\quad a_{31}^A = \frac{2n\beta}{\rho^5}, \\ & a_{22}^A = -\frac{3n\beta}{\rho^5},\quad a_{13}^A = a_{31}^A,\quad a_{04}^A = a_{40}^A, \\ & b_{ij}^A = a_{ij}^A\ (ij = 20,\ 11,\ 02,\ 30,\ 21,\ 12,\ 03,\ 40,\ 31,\ 22,\ 13,\ 04). \end{aligned} \tag{3.4}$$

We calculate the saddle values at the origin of system (3.3), getting

$$\begin{aligned} \nu_1^A &= \frac{-\alpha\left(n^2\beta + \rho^2\right)}{n\rho^5}, \\ \nu_2^A &= \frac{\alpha\left(n^2\beta + \rho^2\right)\left(5n^4\beta^2 + 6n^2\alpha^2\rho^2 - 14n^2\beta\rho^2 + 11n^3\beta\rho^2 - 43\rho^4 + 91n\rho^4 - 49n^2\rho^4\right)}{3n^3\rho^{11}}. \end{aligned} \tag{3.5}$$

Notice that $n^2\beta + \rho^2 > 0$, it is easy to find that $\nu_1^A = \nu_2^A = 0$ if and only if $\alpha = 0$.

Theorem 3.1. *All saddle values at the origin of system* (3.3) *are zero if and only if* $\alpha = 0$. *Moreover, the system* (3.1) *is integrable near the origin when* $\alpha = 0$.

Proof. It is obvious that $\alpha = 0$ is a necessary condition for the integrability of the origin of system (3.1), our goal is to demonstrate that the condition is also sufficient.

When $\alpha = 0$, the system (3.1) can be reduced to the form

$$\begin{cases} \dfrac{dx}{d\zeta} = (1+x)\, y, \\ \dfrac{dy}{d\zeta} = \frac{n-1}{n} y^2 + x\,(1+x)^2 \left(nx\beta + \rho^2\right), \end{cases} \tag{3.6}$$

system (3.6) has the integrating factor $(x+1)^{\frac{2-3n}{n}}$ and the corresponding first integral

$$H(x,y) = \frac{1}{2}(1+x)^{\frac{2}{n}} \left[-\frac{y^2}{(1+x)^2} + \frac{n^2 (1+x)^2 \beta}{1+n} + n\left(n\beta - \rho^2\right) - \frac{2n\,(1+x)\left(2n\beta - \rho^2\right)}{2+n} \right]. \tag{3.7}$$

Thus, system (3.1) is integrable at the origin if and only if $\alpha = 0$, $\gamma = 1 - \frac{\rho^2}{n\beta}$, $c = \alpha$. □

4. The integrability and center-focus problem of system (2.3) at equilibrium B

4.1. Center-focus problem of the equilibrium B

For system (2.3), if the equilibrium B is a center of the linearized system of system (2.3), then point B could potentially be either a center or a focus of system (2.3). This classification issue is referred to the center-focus problem. Next, we investigate the center-focus problem of system (2.3) through the computation and analysis of focal values, constructing integrating factors or first integral.

If equilibrium B is a center of the linearized system of system (2.3), we let $c = \alpha\gamma$, $\beta = -\frac{\rho^2}{n(-1+\gamma)\gamma^3}$, where ρ is a nonzero real number. In this case, J_B has a pair of pure imaginary eigenvalues $\lambda_{1,2}^B = \pm\rho i$, so the study of point B is a center-focus problem, which will be studied in detail.

Performing the transformation $\phi = x + \gamma,\ y = y$, system (2.3) is transformed to

$$\begin{cases} \dfrac{dx}{d\zeta} = y(x+\gamma), \\ \dfrac{dy}{d\zeta} = \frac{(-1+n)y^2}{n} + xy\alpha\,(x+\gamma) - \dfrac{x\,(-1+x+\gamma)\,(x+\gamma)^2 \rho^2}{(-1+\gamma)\,\gamma^3}. \end{cases} \tag{4.1}$$

The origin of system (4.1) corresponds to the equilibrium B of system (2.3).

Under the transformation

$$x = \frac{y_1 \gamma}{\rho},\ y = x_1,\ d\zeta = \frac{1}{\rho} d\tau, \tag{4.2}$$

system (4.1) becomes

$$\begin{cases} \dfrac{dx}{d\zeta} = -y + \text{h.o.t.}, \\ \dfrac{dy}{d\zeta} = x + \text{h.o.t.}, \end{cases} \tag{4.3}$$

where we write $x,\ y,\ \zeta$ instead of $x_1,\ y_1,\ \tau$ respectively.

Under the complex transformation

$$z = x + iy,\ w = x - iy,\ T = i\zeta,\ i = \sqrt{-1}, \tag{4.4}$$

system (4.3) becomes its concomitant complex system, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}T} = z + \displaystyle\sum_{i+j=2}^{4} a_{ij}^{B_1} z^i w^j = Z(z, w), \\ \dfrac{\mathrm{d}w}{\mathrm{d}T} = -w - \displaystyle\sum_{i+j=2}^{4} b_{ij}^{B_1} w^i z^j = -W(z, w), \end{cases} \tag{4.5}$$

where $z,\ w,\ T \in \mathbb{C}$, and

$$\begin{aligned}
&a_{20}^{B_1} = -\frac{n\alpha\gamma^2(\gamma-1) + i(\rho - 4n\rho - \gamma\rho + 5n\gamma\rho)}{4n(-1+\gamma)\rho^2},\ a_{11}^{B_1} = \frac{i(-1-n+\gamma+2n\gamma)}{2n(-1+\gamma)\rho}, \\
&a_{02}^{B_1} = \frac{n\alpha\gamma^2(\gamma-1) - i(\rho - 2n\rho - \gamma\rho + 3n\gamma\rho)}{4n(-1+\gamma)\rho^2},\ a_{30}^{B_1} = \frac{\rho - 3\gamma\rho - i\alpha\gamma^2(1-\gamma)}{8(-1+\gamma)\rho^3}, \\
&a_{21}^{B_1} = \frac{-3\rho + 9\gamma\rho + i\alpha\gamma^2(1-\gamma)}{8(-1+\gamma)\rho^3},\ a_{12}^{B_1} = \frac{3\rho - 9\gamma\rho + i\alpha\gamma^2(1-\gamma)}{8(-1+\gamma)\rho^3}, \\
&a_{03}^{B_1} = \frac{-\rho + 3\gamma\rho - i\alpha\gamma^2(1-\gamma)}{8(-1+\gamma)\rho^3},\ a_{40}^{B_1} = \frac{i\gamma}{16(-1+\gamma)\rho^3},\ a_{31}^{B_1} = -\frac{i\gamma}{4(-1+\gamma)\rho^3}, \\
&a_{22}^{B_1} = \frac{3i\gamma}{8(-1+\gamma)\rho^3},\ a_{13}^{B_1} = a_{31}^{B_1},\ a_{04}^{B_1} = a_{40}^{B_1}, \\
&b_{ij}^{B_1} = \overline{a_{ij}^{B_1}}\ (ij = 20,\ 11,\ 02,\ 30,\ 21,\ 12,\ 03,\ 40,\ 31,\ 22,\ 13,\ 04).
\end{aligned}$$

Next, we calculate the singular point quantities at the origin of system (4.5) to determine the focal values of the equilibrium B of system (2.3).

According to the method for computing focal values in the literature [9,20], we conclude that the focal values of the origin of system (4.3) are zero if and only if the singular point quantities at the origin of system (4.5) are zero.

The first three singular point quantities at the origin of system (4.5) are as follows:

$$\begin{aligned}
\mu_1 &= \frac{i\alpha\gamma^2(-1+\gamma+n\gamma)}{4n(-1+\gamma)\rho^3}, \\
\mu_2 &= -\frac{i\alpha\gamma^2(-1+\gamma+n\gamma)h_1}{48n^3(-1+\gamma)^3\rho^7}, \\
\mu_3 &= \frac{i\alpha\gamma^2(-1+\gamma+n\gamma)h_2}{4608n^5(-1+\gamma)^5\rho^{11}},
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
h_1 =& 6n^2\alpha^2\gamma^4 - 12n^2\alpha^2\gamma^5 + 6n^2\alpha^2\gamma^6 + 43\rho^2 - 91n\rho^2 + 49n^2\rho^2 - 86\gamma\rho^2 + 168n\gamma\rho^2 - 87n^2\gamma\rho^2 \\
&+ 43\gamma^2\rho^2 - 77n\gamma^2\rho^2 + 33n^2\gamma^2\rho^2, \\
h_2 =& 304n^4\alpha^4\gamma^8 - 1216n^4\alpha^4\gamma^9 + 1824n^4\alpha^4\gamma^{10} - 1216n^4\alpha^4\gamma^{11} + 304n^4\alpha^4\gamma^{12} + 5243n^2\alpha^2\gamma^4\rho^2
\end{aligned}$$

$$
\begin{aligned}
&- 12251n^3\alpha^2\gamma^4\rho^2 + 6889n^4\alpha^2\gamma^4\rho^2 - 20972n^2\alpha^2\gamma^5\rho^2 + 48152n^3\alpha^2\gamma^5\rho^2 - 26355n^4\alpha^2\gamma^5\rho^2 \\
&+ 31458n^2\alpha^2\gamma^6\rho^2 - 70950n^3\alpha^2\gamma^6\rho^2 + 36468n^4\alpha^2\gamma^6\rho^2 - 20972n^2\alpha^2\gamma^7\rho^2 + 46448n^3\alpha^2\gamma^7\rho^2 \\
&- 21427n^4\alpha^2\gamma^7\rho^2 + 5243n^2\alpha^2\gamma^8\rho^2 - 11399n^3\alpha^2\gamma^8\rho^2 + 4425n^4\alpha^2\gamma^8\rho^2 + 22068\rho^4 - 92993n\rho^4 \\
&+ 142334n^2\rho^4 - 93800n^3\rho^4 + 22413n^4\rho^4 - 88272\gamma\rho^4 + 356110n\gamma\rho^4 - 520359n^2\gamma\rho^4 \\
&+ 326048n^3\gamma\rho^4 - 73710n^4\gamma\rho^4 + 132408\gamma^2\rho^4 - 510372n\gamma^2\rho^4 + 706535n^2\gamma^2\rho^4 - 415531n^3\gamma^2\rho^4 \\
&+ 87450n^4\gamma^2\rho^4 - 88272\gamma^3\rho^4 + 324386n\gamma^3\rho^4 - 421329n^2\gamma^3\rho^4 + 229828n^3\gamma^3\rho^4 - 44667n^4\gamma^3\rho^4 \\
&+ 22068\gamma^4\rho^4 - 77131n\gamma^4\rho^4 + 92819n^2\gamma^4\rho^4 - 46545n^3\gamma^4\rho^4 + 8784n^4\gamma^4\rho^4.
\end{aligned}
$$

We find the first three singular point quantities at the origin for system (4.5) are all zero if and only if one of the following two conditions is satisfied:

$$
\begin{aligned}
&K_1 : \ \alpha = 0, \\
&K_2 : \ \gamma = \frac{1}{1+n}.
\end{aligned} \tag{4.7}
$$

Theorem 4.1. *All singular point quantities at the origin of system* (4.5) *are zero if and only if condition K_1 or K_2 is satisfied. Moreover, the system* (4.1) *is integrable near the origin when condition K_1 or K_2 is satisfied.*

Proof. The necessity of Theorem 4.1 is obvious, so we need to prove the sufficiency.

When condition K_1 holds, system (4.1) becomes

$$
\begin{cases}
\dfrac{dx}{d\zeta} = (x+\gamma)\, y, \\
\dfrac{dy}{d\zeta} = \dfrac{(n-1)\, y^2}{n} - \dfrac{x\,(-1+x+\gamma)\,(x+\gamma)^2\,\rho^2}{(-1+\gamma)\,\gamma^3},
\end{cases} \tag{4.8}
$$

which has the integrating factor $(x+\gamma)^{\frac{2-3n}{n}}$ and the corresponding first integral

$$
H_1(x,y) = \frac{1}{2}\,(x+\gamma)^{\frac{2}{n}} \left\{ -\frac{y^2}{(x+\gamma)^2} - \frac{n\left[(2+n)\,x^2 + (1+n-\gamma)\,(n\gamma - 2x)\right]\rho^2}{(1+n)\,(2+n)\,(\gamma-1)\,\gamma^3} \right\}. \tag{4.9}
$$

Therefore, if condition K_1 is satisfied, the origin of system (4.1) (or the equilibrium B of system (2.3)) is a center.

When condition K_2 holds, the system (4.1) becomes

$$
\begin{cases}
\dfrac{dx}{d\zeta} = y\left(x + \dfrac{1}{n+1}\right), \\
\dfrac{dy}{d\zeta} = \dfrac{(n-1)y^2}{n} + xy\alpha\left(x + \dfrac{1}{1+n}\right) + \dfrac{x(1+n)(n(x-1)+x)(1+x+nx)^2\rho^2}{n}.
\end{cases} \tag{4.10}
$$

Thus, we need to find a sufficient number of invariant algebraic curves in order to determine the first integrals. Here, we identify the following three invariant algebraic curves:

$$
\begin{aligned}
&f_1 = \frac{1}{1+n} + x, \\
&f_{2,3} = 1 + \frac{(n+1)x\left[n-1-(n+1)x\right]}{n} - \frac{y\left[n\alpha \pm \sqrt{n^2\alpha^2 + 4\rho^2(n+1)^5}\right]}{2n(1+n)^2\rho^2}.
\end{aligned} \tag{4.11}
$$

On account of $\frac{\partial H_2(x,y)}{\partial \zeta}=0$, we get a corresponding first integral

$$\begin{aligned} H_2(x,y)=&\frac{(1+x+nx)}{1+n}\left\{1+\frac{(n+1)x\left[n-1-(n+1)x\right]}{n}\right.\\ &\left.+\frac{y\left[-n\alpha+\sqrt{n^2\alpha^2+4(1+n)^5\rho^2}\right]}{2n(1+n)^2\rho^2}\right\}^{k_1}\\ &\times\left\{1+\frac{(n+1)x\left[n-1-(n+1)x\right]}{n}-\frac{y\left[n\alpha+\sqrt{n^2\alpha^2+4(1+n)^5\rho^2}\right]}{2n(1+n)^2\rho^2}\right\}^{k_2}, \end{aligned} \tag{4.12}$$

where $k_{1,2}=\frac{n\left[-\sqrt{n^2\alpha^2+4(1+n)^5\rho^2}\mp n\alpha\right]}{2(-1+n)\sqrt{n^2\alpha^2+4(1+n)^5\rho^2}}$.

This demonstrates that the origin is a center for the system (4.1), and the system (4.1) is integrable at the origin if and only if condition K_2 is satisfied. ☐

4.2. The integrability of system (2.3) when the equilibrium B is a saddle

When $c=\alpha\gamma$, $\beta=\frac{\rho^2}{n(\gamma-1)\gamma^3}$, where ρ is nonzero real number, the eigenvalues of J_B are $\lambda_{1,2}^B=\pm\rho$, so the equilibrium B of the system (2.3) is a saddle. We perform the transformation

$$\phi=x+\gamma,\ y=y, \tag{4.13}$$

then system (2.3) can be written as

$$\begin{cases} \dfrac{dx}{d\zeta}=y(x+\gamma),\\ \dfrac{dy}{d\zeta}=\dfrac{(-1+n)\,y^2}{n}+xy\alpha\,(x+\gamma)+\dfrac{x\,(-1+x+\gamma)\,(x+\gamma)^2\,\rho^2}{(-1+\gamma)\,\gamma^3}. \end{cases} \tag{4.14}$$

The origin of system (4.14) corresponds to the equilibrium B of system (2.3).

Under the transformation

$$x=\frac{(y_2-x_2)\,\gamma}{\rho},\ y=x_2+y_2,\ d\zeta=-\frac{1}{\rho}d\varphi, \tag{4.15}$$

system (4.14) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\zeta}=x+\displaystyle\sum_{i+j=2}^{4}a_{ij}^{B_2}x^iy^j,\\ \dfrac{\mathrm{d}y}{\mathrm{d}\zeta}=-y+\displaystyle\sum_{i+j=2}^{4}b_{ij}^{B_2}x^iy^j, \end{cases} \tag{4.16}$$

where we write $x,\ y,\ \zeta$ instead of $x_2,\ y_2,\ \varphi$ respectively, and

$$a_{20}^{B_2}=\frac{-n\alpha\gamma^2+n\alpha\gamma^3-\rho+4n\rho+\gamma\rho-5n\gamma\rho}{2n(-1+\gamma)\rho^2},\ a_{11}^{B_2}=\frac{-1-n+\gamma+2n\gamma}{n(-1+\gamma)\rho},$$

$$a_{02}^{B_2}=-\frac{-n\alpha\gamma^2+n\alpha\gamma^3+\rho-2n\rho-\gamma\rho+3n\gamma\rho}{2n(-1+\gamma)\rho^2},\ a_{30}^{B_2}=-\frac{-\alpha\gamma^2+\alpha\gamma^3+\rho-3\gamma\rho}{2(-1+\gamma)\rho^3},$$

$$
\begin{aligned}
&a_{21}^{B_2} = \frac{-\alpha\gamma^2+\alpha\gamma^3+3\rho-9\gamma\rho}{2(-1+\gamma)\rho^3},\ a_{12}^{B_2} = \frac{-\alpha\gamma^2+\alpha\gamma^3-3\rho+9\gamma\rho}{2(-1+\gamma)\rho^3},\\
&a_{03}^{B_2} = -\frac{-\alpha\gamma^2+\alpha\gamma^3-\rho+3\gamma\rho}{2(-1+\gamma)\rho^3},\ a_{40}^{B_2} = -\frac{\gamma}{2(-1+\gamma)\rho^3},\\
&a_{31}^{B_2} = \frac{2\gamma}{(-1+\gamma)\rho^3},\ a_{22}^{B_2} = -\frac{3\gamma}{(-1+\gamma)\rho^3},\ a_{13}^{B_2} = a_{31}^{B_2},\ a_{04}^{B_2} = a_{40}^{B_2},\\
&b_{20}^{B_2} = \frac{-n\alpha\gamma^2+n\alpha\gamma^3-\rho+2n\rho+\gamma\rho-3n\gamma\rho}{2n(-1+\gamma)\rho^2},\\
&b_{02}^{B_2} = -\frac{-n\alpha\gamma^2+n\alpha\gamma^3+\rho-4n\rho-\gamma\rho+5n\gamma\rho}{2n(-1+\gamma)\rho^2},\\
&b_{ij}^{B_2} = a_{ij}^{B_2}\ (ij = 11,\ 30,\ 21,\ 12,\ 03,\ 40,\ 31,\ 22,\ 13,\ 04).
\end{aligned}
\tag{4.17}
$$

We write the first two saddle values at the origin of the system (4.16)

$$
\begin{aligned}
\nu_1^B &= -\frac{\alpha\gamma^2\left(-1+\gamma+n\gamma\right)}{n\left(-1+\gamma\right)\rho^3},\\
\nu_2^B &= \frac{\alpha\gamma^2(-1+\gamma+n\gamma)m_1}{3n^3(-1+\gamma)^3\rho^7},
\end{aligned}
\tag{4.18}
$$

where

$$
\begin{aligned}
m_1 = {} & (-43+91n-49n^2+86\gamma-168n\gamma+87n^2\gamma-43\gamma^2+77n\gamma^2-33n^2\gamma^2)\rho^2\\
& +6n^2\alpha^2(-1+\gamma)^2\gamma^4.
\end{aligned}
\tag{4.19}
$$

By calculation, the first two saddle values at the origin of system (4.16) are all zero if and only if one of the following two conditions is satisfied:

$$
M_1:\ \alpha = 0, \quad M_2:\ \gamma = \frac{1}{1+n}. \tag{4.20}
$$

Notice that $n,\ \beta \geq 0,\ \gamma$ are real numbers, we find that the condition M_2 and $\beta = \frac{\rho^2}{n(\gamma-1)\gamma^3}$ are contradictory, so we don't discuss this case.

Theorem 4.2. *All saddle values at the origin of system* (4.16) *are zero if and only if condition* M_1 *is satisfied. Moreover, the system* (4.14) *is integrable near the origin when condition* M_1 *is satisfied.*

Proof. The necessity of Theorem 4.2 is obvious, we need prove the sufficiency of the condition M_1.

When condition M_1 holds, system (4.14) becomes

$$
\begin{cases}
\dfrac{dx}{d\zeta} = \left(x+\gamma\right)y,\\[2mm]
\dfrac{dy}{d\zeta} = \dfrac{\left(n-1\right)y^2}{n} + \dfrac{x\left(-1+x+\gamma\right)\left(x+\gamma\right)^2\rho^2}{\left(-1+\gamma\right)\gamma^3},
\end{cases}
\tag{4.21}
$$

which has the integrating factor $(x+\gamma)^{\frac{2-3n}{n}}$ and the corresponding first integral

$$
H_3\left(x,y\right) = \frac{1}{2}\left(x+\gamma\right)^{\frac{2}{n}}\left\{-\frac{y^2}{\left(x+\gamma\right)^2} + \frac{n\left[\left(2+n\right)x^2+\left(1+n-\gamma\right)\left(n\gamma-2x\right)\right]\rho^2}{\left(1+n\right)\left(2+n\right)\left(\gamma-1\right)\gamma^3}\right\}. \tag{4.22}
$$

Thus, system (4.14) is integrable at the origin if and only if condition M_1 is satisfied.

This completes the proof. $\square$

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