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MULTIPLE SOLUTIONS FOR FRACTIONAL THREE-POINT BOUNDARY VALUE PROBLEMS: A VARIATIONAL APPROACH*

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Abstract We are concerned with the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^C D_t^{\alpha} u(t)) = \lambda f(u(t)), \quad t \in (0,T), \\ u(0) = 0, \quad u(T) = \beta u(\eta), \end{cases}$$

where $\lambda>0$ is a parameter, $\beta \in \mathbb{R}$, $\eta \in (0,T)$. By proving the appropriate fractional derivative space, we establish the variational structure of the proposed problem and derive the existence result for multiple weak solutions using a variant of the mountain pass theorem. To demonstrate the applicability of the main results, we present an example. This paper explores the application of the variational approach in solving fractional three-point BVPs. Our findings are novel and distinct from the conclusions of existing literature, holding significant theoretical value.

Keywords Fractional differential equation, three-point boundary conditions, multiple weak solutions, critical point theorem.

MSC(2010) 34A08, 34B15, 34B10.

1. Introduction

This paper is concerned with the following fractional three-point boundary value problem (BVP):

$$\begin{cases} {}_tD_T^{\alpha}({}_0^C D_t^{\alpha} u(t)) = \lambda f(u(t)), \quad t \in (0,T), \\ u(0) = 0, \quad u(T) = \beta u(\eta), \end{cases} \tag{1.1}$$

where ${}_tD_T^{\alpha}, {}_0^C D_t^{\alpha}$ are the right Riemann-Liouville fractional derivative and the left Caputo fractional derivative, respectively, of order $\alpha \in (1/2, 1]$, λ is a positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\beta \in \mathbb{R}$, $\eta \in (0,T)$.

Fractional differential equations are more appropriate than their integer order counterparts for modeling many real world problems. They describe many phenomena in several fields of engineering and scientific disciplines such as, physical mechanisms [5], medicine [7], image processing [27], control theory [17], electrical engineering [9], rheology [26], etc. For more applications and references we refer to [6, 16].

Qualitative analysis on BVPs for nonlinear ordinary differential equations has always been the study focus among scholars, mainly due to the extensive practical applications of BVPs [1,3].

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Over the last two decades, there has been a growing interest in studying nonlinear fractional differential equations with various boundary conditions, including local boundary conditions (two point boundary condition) and nonlocal boundary conditions (multi-point boundary conditions, integral boundary conditions). For some recent development on the topic, we refer the reader to the papers [2, 11, 13, 29, 31, 32] and the references therein. Especially, variational approach play an important part in studying fractional BVPs. For example, in 2011, Jiao and Zhou [14] proved the following fractional derivative space:

Proposition 1.1 ([14])**.** *Let $\alpha \in (0,1]$ and $p \in (1,\infty)$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space, where*

$$E_0^{\alpha,p} = \{u : [0,T] \to \mathbb{R} | u, {}_0^C D_t^\alpha u \in L^p([0,T],\mathbb{R}),\ u(0) = u(T) = 0\},$$

is defined by the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$||u||_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^C D_t^\alpha u(t)|^p dt\Big)^{1/p},\ u \in E_0^{\alpha,p}.$$

In [15], the authors established the variational structure for the following fractional Dirichlet BVP in space $E_0^{\alpha,p}$,

$$\begin{cases} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where ${}_tD_T^\alpha, {}_0^C D_t^\alpha$ are the right Riemann-Liouville fractional derivative and the left Caputo fractional derivative, respectively, of order $\alpha \in (1/2, 1]$. The existence results of the solutions for BVP (1.2) is obtained by using the mountain pass theorem.

In 2017, Tian and Nieto [24] proved the following fractional derivative space:

Proposition 1.2 ([24])**.** *Let $\alpha \in (0,1]$ and $p \in (1,\infty)$. The fractional derivative space $E^{\alpha,p}$ is a reflexive and separable Banach space, where*

$$E^{\alpha,p} = \{u : [0,T] \to \mathbb{R}^N : u \text{ is absolutely continuous and } {}_0^C D_t^\alpha u \in L^p([0,T],\mathbb{R}^N)\}$$

is defined by the closure of $C^\infty([0,T],\mathbb{R}^N)$ with the norm

$$||u||_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^C D_t^\alpha u(t)|^p dt\Big)^{1/p}.$$

In [24], the authors established the variational structure for the following fractional differential equation supplement with Sturm-Liouville boundary conditions in space $E^{\alpha,p}$,

$$\begin{cases} -\frac{d}{dt}(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))) = \lambda f(u(t)), \quad \text{a.e. } t \in [0,T], \\ au(0) - b(\frac{1}{2}{}_0D_t^{-\beta}(u'(0)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(0))) = 0, \\ cu(T) + d(\frac{1}{2}{}_0D_t^{-\beta}(u'(T)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(T))) = 0, \end{cases} \tag{1.3}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $\beta \in [0,1)$, respectively, $a, c > 0$, $b, d \geq 0$, $f : \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function and λ is a positive parameter. The existence of an unbounded sequence of solutions for BVP (1.3) is obtained by using a multiple critical-point theorem.

In recent years, based on the fractional derivative spaces $E_0^{\alpha,p}$ and $E^{\alpha,p}$ proved in references [14] and [24], some scholars have carried out a series of studies on the fractional Dirichlet BVPs and the fractional Sturm-Liouville BVPs by applying critical point theory, and established many profound results. For some recent works on the topic, we refer the reader to a series of papers [8, 10, 19, 22, 25, 30] and the references cited therein. More recently, scholars have also proved the corresponding fractional derivative spaces in order to establish the variational structure of the generalized fractional BVPs. For example, Sousa et al. [23] proved a fractional derivative space ψ-fractional spaces $\mathbb{H}_p^{\alpha,\beta;\psi}$ and investigated a class of fractional two-point BVPs involving ψ-Hilfer fractional derivative with the help of mountain pass theorem. Ledesma and Nyamoradi [18] presented a fractional derivative space ${}^kE_0^{\alpha,\nu;\psi}$ and used the linking theorem for the study of a class of impulsive fractional Dirichlet BVPs with (k,ψ)-Hilfer fractional derivative. However, all of these works have used variational methods to discuss the fractional two-point BVPs. Therefore, the following natural questions arise:

- Can the variational method be applied to study fractional nonlocal BVPs?
- Would it be possible to select an appropriate fractional derivative space to establish the variational structure of BVP (1.1)?
- Is it possible to apply the critical point theorem to give the existence result of the solution to BVP (1.1)?

Motivated by [14,15,20,21,24], this paper aims to answer the above questions one by one. The main contributions of this study are highlighted as follows. (1) A new reflexive and separable fractional derivative space $\mathbb{E}_\beta^{\alpha,p}$ was proved. (2) The existence of solutions for the fractional multi-point BVP was discussed for the first time using variational methods. (3) A variational structure for the fractional BVP (1.1) was established in $\mathbb{E}_\beta^{\alpha,p}$, and two weak solutions were obtained for (1.1) utilizing a variant of the mountain pass theorem contained in [4] (see Theorem 2.1 below).

2. Preliminaries

In this section, we briefly recollect some basic definitions of the fractional calculus, and some lemmas, and present a critical point theorem which will be applied in the next section.

Definition 2.1 ([16])**.** Let $\nu > 0$, $\mathfrak{u} \in C[0,T]$. Then the left and right Riemann-Liouville fractional integrals ${}_0D_t^{-\nu}\mathfrak{u}(t)$ and ${}_tD_T^{-\nu}\mathfrak{u}(t)$ are respectively defined by

$$ {}_0D_t^{-\nu}\mathfrak{u}(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\theta)^{\nu-1}\mathfrak{u}(\theta)d\theta, \quad t \in [0,T], $$
$$ {}_tD_T^{-\nu}\mathfrak{u}(t) = \frac{1}{\Gamma(\nu)}\int_t^T (\theta-t)^{\nu-1}\mathfrak{u}(\theta)d\theta, \quad t \in [0,T]. $$

Definition 2.2 ([16])**.** Let $\nu \in (0,1)$, $\mathfrak{u} \in C[0,T]$. Then the left and right Riemann-Liouville fractional derivatives ${}_0D_t^{\nu}\mathfrak{u}(t)$ and ${}_tD_T^{\nu}\mathfrak{u}(t)$ are respectively defined by

$$ {}_0D_t^{\nu}\mathfrak{u}(t) = \frac{d}{dt}{}_0D_t^{\nu-1}\mathfrak{u}(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t (t-\theta)^{-\nu}\mathfrak{u}(\theta)d\theta, \quad t \in [0,T], $$

$$ {}_tD_T^{\nu}\mathfrak{u}(t) = -\frac{d}{dt}{}_tD_T^{\nu-1}\mathfrak{u}(t) = \frac{-1}{\Gamma(1-\nu)}\frac{d}{dt}\int_t^T (\theta-t)^{-\nu}\mathfrak{u}(\theta)d\theta, \quad t\in[0,T]. $$

Definition 2.3 ([16]). Let $\nu \in (0,1)$, $\mathfrak{u} \in AC[0,T]$. Then the left and right Caputo fractional derivatives ${}_0^CD_t^{\nu}\mathfrak{u}(t)$ and ${}_t^CD_T^{\nu}\mathfrak{u}(t)$ are respectively defined by

$$ {}_0^CD_t^{\nu}\mathfrak{u}(t) = {}_0D_t^{\nu-1}\mathfrak{u}'(t) = \frac{1}{\Gamma(1-\nu)}\int_0^t (t-\theta)^{-\nu}\mathfrak{u}'(\theta)d\theta, \quad t\in[0,T], $$

$$ {}_t^CD_T^{\nu}\mathfrak{u}(t) = -{}_tD_T^{\nu-1}\mathfrak{u}'(t) = \frac{-1}{\Gamma(1-\nu)}\frac{d}{dt}\int_t^T (\theta-t)^{-\nu}\mathfrak{u}'(\theta)d\theta, \quad t\in[0,T]. $$

Lemma 2.1 ([12]). *A closed subspace of a reflexive Banach space is also reflexive.*

Theorem 2.1 ([4]). *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous, coercive, convex, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semi-continuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let $\varphi_\lambda = \Phi - \lambda\Psi$, $\lambda > 0$, and assume that 0 is a local minimum of φ_λ and*

$$ \inf_X \Phi = \Phi(0) = \Psi(0) = 0. $$

Moreover, suppose that there exists a constant $\rho \in \mathbb{R}$ and $x_1 \in X$, with $0 < \Phi(x_1) < \rho$, such that

(i) $\frac{2}{\rho}\sup_{x\in\Phi^{-1}((-\infty,\rho))}\Psi(x) < \frac{\Psi(x_1)}{\Phi(x_1)}$, *(ii)* $\min_{t\in[0,1]}\Psi(tx_1) \geq 0$.

Then, for each

$$ \lambda \in \Big(\frac{\Phi(x_1)}{\Psi(x_1)}, \frac{\rho/2}{\sup_{x\in\Phi^{-1}((-\infty,\rho))}\Psi(x)}\Big), $$

the functional φ_λ admits two critical points $u_1, u_2 \neq 0$ such that

$$ \Phi(u_1) < \rho, \quad \Phi(u_2) < \rho. $$

3. Fractional derivative space

We now define a new fractional derivative space $\mathbb{E}_\beta^{\alpha,p} \subset E^{\alpha,p}$ as follows:

Definition 3.1. Let $\alpha \in (0,1]$. The fractional derivative space

$$ \mathbb{E}_\beta^{\alpha,p} = \{u \in E^{\alpha,p}([0,T]) : u(0) = 0,\ u(T) = \beta u(\eta)\}, $$

is defined by the closure of $C^\infty([0,T],\mathbb{R}^N)$ with the norm $||u||_{\alpha,p}$. If $p = 2$, we write $\mathbb{E}_\beta^{\alpha,p} = \mathbb{E}_\beta^{\alpha}$.

Remark 3.1. For any $u \in \mathbb{E}_\beta^{\alpha,p}$, then $u \in L^p([0,T],\mathbb{R}^N)$ and ${}_0^CD_t^{\alpha}u(t) \in L^p([0,T],\mathbb{R}^N)$.

Lemma 3.1. *Let $\alpha \in (0,1]$ and $p \in (1,\infty)$. For all $u \in \mathbb{E}_\beta^{\alpha,p}$, we have*

$$ ||u||_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}||{}_0^CD_t^{\alpha}u||_{L^p}. \tag{3.1} $$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$ ||u||_\infty \leq \frac{T^{\alpha-(1/p)}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}||{}_0^CD_t^{\alpha}u||_{L^p}. \tag{3.2} $$

Proof. The proof follows the idea in the proof of Proposition 3.2 in [14], and so is omitted. □

Remark 3.2. According to Lemma 3.1, we can consider $\mathbb{E}_{\beta}^{\alpha,p}$ with respect to the norm

$$||u|| = ||{}_0^C D_t^{\alpha} u||_{L^p} = \left(\int_0^T |{}_0^C D_t^{\alpha} u(t)|^p dt \right)^{1/p}.$$

Lemma 3.2. *$\mathbb{E}_{\beta}^{\alpha,p}$ is a Banach space.*

Proof. Obviously, $\mathbb{E}_{\beta}^{\alpha,p}$ is a normed linear space. Let $\{u_n\}$ be any Cauchy sequence in space $\mathbb{E}_{\beta}^{\alpha,p}$, then $\{u_n\}$ is also a Cauchy sequence in $E^{\alpha,p}$. Utilizing the completeness of $E^{\alpha,p}$, we known that $\{u_n\}$ has a convergent subsequence, which we may still denote as $\{u_n\}$. Then there exists a $u \in E^{\alpha,p}$ such that $\lim_{n\to\infty} u_n = u$ holds in $E^{\alpha,p}$. Noting that $||\cdot||_{\alpha,p}$ and $||\cdot||$ are equivalent norms, we can deduce that

$$\lim_{n\to\infty} u_n(0) = u(0) = 0, \quad \lim_{n\to\infty} (u_n(T) - \beta u_n(\eta)) = u(T) - \beta u(\eta) = 0.$$

Therefore, $\mathbb{E}_{\beta}^{\alpha,p}$ is a Banach space. □

Lemma 3.3. *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The space $\mathbb{E}_{\beta}^{\alpha,p}$ is a reflexive and separable Banach space.*

Proof. In view of $\mathbb{E}_{\beta}^{\alpha,p}$ is the closed subspace of reflexive Banach space $E^{\alpha,p}$, it follows from Lemma 2.1 that $\mathbb{E}_{\beta}^{\alpha,p}$ is a reflexive Banach space. We now show that $\mathbb{E}_{\beta}^{\alpha,p}$ is separable. In fact, since $E^{\alpha,p}$ is separable, there exists a countable subset $\Omega \subset E^{\alpha,p}$ such that for any $x \in E^{\alpha,p}$, there exists $\{y_n\} \subset \Omega$ satisfying $y_n \to x$, as $n \to \infty$. For any $y \in \Omega$, we define $\varphi : \Omega \to \mathbb{E}_{\beta}^{\alpha,p}$ by

$$(\varphi y)(t) = \begin{cases} y(t) - \dfrac{\eta - t}{\eta} y(0), & \text{if } 0 \leq t \leq \eta, \\ y(t) + [\beta y(\eta) - y(T)] \dfrac{t - \eta}{T - \eta}, & \text{if } \eta \leq t \leq T. \end{cases}$$

Let $z = \varphi(y)$, then

$$z(0) = (\varphi y)(0) = 0, \quad z(\eta) = (\varphi y)(\eta) = y(\eta),$$

and

$$z(T) = (\varphi y)(T) = \beta y(\eta) = \beta z(\eta),$$

this implies $z \in \mathbb{E}_{\beta}^{\alpha,p}$. Let $\Omega_1 = \{z = \varphi(y) : y \in \Omega\}$, then $\Omega_1 \subset \mathbb{E}_{\beta}^{\alpha,p}$ is a countable subset. We claim that Ω_1 is also the dense subset of $\mathbb{E}_{\beta}^{\alpha,p}$, that is, for any $x \in \mathbb{E}_{\beta}^{\alpha,p}$, we need to find a sequence $\{z_n\} \subset \Omega_1$ such that $||z_n - x|| \to 0$ as $n \to \infty$. Indeed, since $x \in E^{\alpha,p}$ there exists a sequence $\{y_n\} \subset \Omega$ such that

$$||y_n - x|| \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Let $z_n = \varphi(y_n)$, then we have

$$\begin{aligned} & ||z_n - y_n|| \\ = & \left(\int_0^T |{}_0^C D_t^{\alpha} (z_n - y_n)|^p dt \right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)}\left\{\int_0^\eta \left|\frac{y_n(0)}{\eta}t^{1-\alpha}\right|^p dt + \int_\eta^T \left|\frac{y_n(0)}{\eta^\alpha} + \frac{\beta y_n(\eta)-y_n(T)}{T-\eta}(t-\eta)^{1-\alpha}\right|^p dt\right\}^{1/p} \\
&\leq \frac{1}{\Gamma(2-\alpha)}\left\{\frac{|y_n(0)|^p\eta^{1-p\alpha}}{p-p\alpha+1} + 2^{p-1}\int_\eta^T \left[\frac{|y_n(0)|^p}{\eta^{\alpha p}} + \frac{|\beta y_n(\eta)-y_n(T)|^p}{(T-\eta)^p}(t-\eta)^{p-p\alpha}\right] dt\right\}^{1/p} \\
&= \frac{1}{\Gamma(2-\alpha)}\left\{\frac{|y_n(0)|^p\eta^{1-p\alpha}}{p-p\alpha+1} + 2^{p-1}\left[\frac{|y_n(0)|^p}{\eta^{\alpha p}}(T-\eta) + \frac{|\beta y_n(\eta)-y_n(T)|^p}{p-p\alpha+1}(T-\eta)^{1-p\alpha}\right]\right\}^{1/p} \qquad (3.4) \\
&\to \frac{1}{\Gamma(2-\alpha)}\left\{\frac{|x(0)|^p\eta^{1-p\alpha}}{p-p\alpha+1} + 2^{p-1}\left[\frac{|x(0)|^p}{\eta^{\alpha p}}(T-\eta) + \frac{|\beta x(\eta)-x(T)|^p}{p-p\alpha+1}(T-\eta)^{1-p\alpha}\right]\right\}^{1/p} \\
&= 0, \ n\to\infty.
\end{aligned}
$$

Since

$$||z_n - x|| \leq ||z_n - y_n|| + ||y_n - x||.$$

This together with (3.3) and (3.4), it follows

$$||z_n - x|| \to 0, \quad \text{as } n \to \infty.$$

Hence, $\mathbb{E}_\beta^{\alpha,p}$ is a separable Banach space. □

Lemma 3.4. *Let $\alpha \in (0,1]$, $p \in (1,\infty)$. Assume that $\alpha > 1/p$ and the sequence $\{u_n\}$ converges weakly in $\mathbb{E}_\beta^{\alpha,p}$, i.e., $u_n \rightharpoonup u$. Then $u_n \to u$ in $C([0,T])$, i.e., $||u_n - u||_\infty \to 0$ as $n \to \infty$.*

Proof. The proof of this Lemma is not particularly difficult but will not be reproduced here. For a rigorous proof of this Lemma the reader is referred to Proposition 3.3 [14] or Proposition 5.5 [24]. □

4. Existence result of BVP (1.1)

In this section, we prove problem (1.1) admits at least two positive weak solutions by using Theorem 2.1. To begin with, we denote the fractional derivative space $X = \mathbb{E}_\beta^\alpha$ endowed with the norm

$$||u||_{\mathbb{E}_\beta^\alpha} = \Big(\int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt\Big)^{1/2}. \qquad (4.1)$$

In order to apply Theorem 2.1 to study problem (1.1), we define the functionals $\Phi, \Psi : X \to \mathbb{R}$ as follows:

$$\Phi(u) = \frac{1}{2}||u||^2_{\mathbb{E}_\beta^\alpha}, \quad \Psi(u) = \int_0^T F(u(t))dt, \quad \text{for all } u \in X, \qquad (4.2)$$

where $F(u) = \int_0^u f(s)ds$. Standard arguments show that $\Phi \in C^1(X,\mathbb{R})$ and convex, Ψ has continuous Gâteaux derivatives. For any $v(t) \in X$, their Gâteaux derivatives at the point $u(t) \in X$ are the functional $\Phi'(u), \Psi'(u) \in X^*$, respectively, given by

$$\langle \Phi'(u), v\rangle = \int_0^T ({}_0^C D_t^\alpha u(t))({}_0^C D_t^\alpha v(t))dt, \qquad (4.3)$$

$$\langle \Psi'(u), v\rangle = \int_0^T f(u(t))v(t)dt. \qquad (4.4)$$

We say that a function $u \in X$ is a weak solution of problem (1.1) if

$$\int_0^T ({}_0^C D_t^\alpha u(t))({}_0^C D_t^\alpha v(t))dt - \lambda \int_0^T f(u(t))v(t)dt = 0,$$

holds for all $v \in X$.

Lemma 4.1. *If function $u \in X$ is a critical point of the functional $\Phi - \lambda\Psi$, then u is a weak solution of the problem* (1.1).

Proof. For $u \in X$, it follows from (4.3) and (4.4) that

$$\langle (\Phi - \lambda\Psi)'(u), v \rangle = \int_0^T ({}_0^C D_t^\alpha u(t))({}_0^C D_t^\alpha v(t))dt - \lambda \int_0^T f(u(t))v(t)dt,$$

for all $v \in X$. Furthermore, since u is a critical point of the functional $\Phi - \lambda\Psi$, this implies that

$$\int_0^T ({}_0^C D_t^\alpha u(t))({}_0^C D_t^\alpha v(t))dt - \lambda \int_0^T f(u(t))v(t)dt = 0, \quad \forall v \in X.$$

Thus, the proof is complete. □

Theorem 4.1. *Assume that the following conditions hold:*

(C_1) *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function and $f(u) = 0$ for all $u < 0$, such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 0.$$

(C_2) *There exist positive constants c, d with $C(\alpha)d < \dfrac{c}{\sigma(\alpha)}$, such that*

$$\frac{4\sigma^2(\alpha)F(c)}{c^2} < \frac{F(d)}{C^2(\alpha)d^2}.$$

Then, for any

$$\lambda \in \Lambda = \left(\frac{C^2(\alpha)d^2}{TF(d)}, \frac{c^2}{4\sigma^2(\alpha)TF(c)} \right),$$

the problem (1.1) *admits at least two weak solutions $u_1, u_2 \neq 0$ such that $||u_i||_\infty < c$, $i = 1, 2$, where*

$$C(\alpha) = \frac{2}{\Gamma(2-\alpha)} \Big[\frac{1}{\eta^2} \frac{1}{3-2\alpha} \Big(\frac{\eta}{2}\Big)^{3-2\alpha} + \frac{1}{\eta^2} \Big(\frac{\eta}{2}\Big)^{2-2\alpha} \frac{2T-\eta}{2}$$
$$+ \frac{1}{3-2\alpha} \Big(\frac{\beta-1}{T-\eta}\Big)^2 \Big(\frac{T-\eta}{2}\Big)^{3-2\alpha} + \frac{2}{2-\alpha} \frac{\beta-1}{\eta(T-\eta)} \Big(\frac{\eta}{2}\Big)^{1-\alpha} \Big(\frac{T-\eta}{2}\Big)^{2-\alpha} \Big]^{1/2},$$

$$\sigma(\alpha) = \frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{1/2}\Gamma(\alpha)}.$$

In order to prove Theorem 4.1, we first prove three auxiliary lemmas.

Lemma 4.2. *The functional $\Phi : X \to \mathbb{R}$ is weakly lower semi-continuous and coercive.*

Proof. Let $\{u_n\} \subset X$ satisfies $u_n \to u$ as $n \to \infty$, we have $||u_n||_{\mathbb{E}^\alpha_\beta} \to ||u||_{\mathbb{E}^\alpha_\beta}$ as $n \to \infty$, which follows that $\Phi(u_n) \to \Phi(u)$ as $n \to \infty$. So Φ is lower semi-continuous. Besides Φ is concave, it follows that $\Phi : X \to \mathbb{R}$ is weakly lower semi-continuous. Moreover, by the definition of Φ, it follows readily that

$$\lim_{||u||_{\mathbb{E}^\alpha_\beta} \to \infty} \Phi(u) = +\infty,$$

that is, Φ is coercive. The lemma is proved. □

Lemma 4.3. *The functional $\Phi' : X \to X^*$ admits a continuous inverse on X^*.*

Proof. We first show that Φ' is coercive. In fact, for every $u \in X \backslash \{0\}$, by (4.3), we see that

$$\big\langle \Phi'(u), u \big\rangle = \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt = ||u||^2_{\mathbb{E}^\alpha_\beta},$$

which implies that Φ' is coercive. Moreover, for given $u, v \in X$, we find that

$$\big\langle \Phi'(u) - \Phi'(v), u - v \big\rangle = ||u - v||^2_{\mathbb{E}^\alpha_\beta}.$$

This implies, Φ' is uniformly monotone. In view of ([28], Theorem 26.A (d)), we see that $(\Phi')^{-1}$ exists and is continuous on X^*. The proof is completed. □

Lemma 4.4. *$\Psi : X \to \mathbb{R}$ is weakly upper semi-continuous and $\Psi' : X \to X^*$ is a continuous and compact functional.*

Proof. Easily, we can obtain that Ψ is weakly upper semi-continuous. In order to prove Ψ' is continuous and compact, we first show that Ψ' is strongly continuous on X. In fact, let $\{u_n\} \subset X$ satisfies $u_n \rightharpoonup u$ as $n \to \infty$, then by Lemma 3.4, we know that $\{u_n\}$ converges uniformly to u on $C([0, T])$. Since the functions $f \in C(\mathbb{R}, \mathbb{R})$, it follows that $f(u_n) \to f(u)$, as $n \to \infty$. Then we obtain $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$, that is, Ψ' is strongly continuous on X. Furthermore, by ([28], Proposition 26.2) we can infer that Ψ' is a compact operator. This completes the proof. □

By using Theorem 2.1, we are now turning to the proof of Theorem 4.1.

Proof. (**Theorem 4.1.**) From Lemma 4.2 and Lemma 4.3 that Φ is a weakly lower semi-continuous, coercive, convex and continuously Gâteaux differentiable functional, and its Gâteaux derivative admits a continuous inverse on X^*. By Lemma 4.4, we also obtain that Ψ is a weakly upper semi-continuous and continuously Gâteaux differentiable functional, and its Gâteaux derivative is compact. Let $\varphi_\lambda = \Phi - \lambda\Psi$. By using (C_1) and a standard computation, we can obtain 0 is a local minimum for φ_λ. Setting $\rho = \frac{c^2}{2\sigma^2(\alpha)}$, it then follows from (3.2) that

$$\frac{2}{\rho} \sup_{u \in \Phi^{-1}((-\infty, \rho))} \Psi(x) \le \frac{2}{\rho} \int_0^T \max_{|\zeta| < c} F(\zeta) dt = \frac{2TF(c)}{\rho} = \frac{4\sigma^2(\alpha)}{c^2} TF(c). \tag{4.5}$$

We consider the function $\omega : [0, T] \to \mathbb{R}$ given by

$$\omega(t) = \begin{cases} \dfrac{2d}{\eta} t, & t \in [0, \eta/2], \\ d, & t \in [\eta/2, (T + \eta)/2], \\ d\Big[\dfrac{(2 - \beta)T - \beta\eta}{T - \eta} + \dfrac{2(\beta - 1)t}{T - \eta}\Big], & t \in [(T + \eta)/2, T], \end{cases}$$

with

$$\omega'(t) = \begin{cases} \dfrac{2d}{\eta}, & t \in (0, \eta/2), \\ 0, & t \in (\eta/2, (T+\eta)/2), \\ \dfrac{2(\beta-1)}{T-\eta} d, & t \in ((T+\eta)/2, T). \end{cases}$$

Recalling the definition of Caputo fractional derivative, we then see that

$$\begin{aligned} {}_0^C D_t^\alpha \omega(t) &= \frac{1}{\Gamma(1-\alpha)} \Big(\int_0^t (t-s)^{-\alpha} \omega'(s) ds \Big) \\ &= \frac{2d}{\Gamma(2-\alpha)} \begin{cases} \dfrac{1}{\eta} t^{1-\alpha}, & t \in [0, \eta/2], \\ \dfrac{1}{\eta} \Big(\dfrac{\eta}{2}\Big)^{1-\alpha}, & t \in (\eta/2, (T+\eta)/2], \\ \dfrac{1}{\eta} \Big(\dfrac{\eta}{2}\Big)^{1-\alpha} + \dfrac{(\beta-1)}{T-\eta} \Big(t - \dfrac{T+\eta}{2}\Big)^{1-\alpha}, & t \in ((T+\eta)/2, T]. \end{cases} \end{aligned}$$

Obviously, one has that $\omega(t) \in X$,

$$\begin{aligned} \Phi(\omega) &= \frac{1}{2} \Big(\int_0^T |{}_0^C D_t^\alpha \omega(t)|^2 dt \Big) = \frac{1}{2} C^2(\alpha) d^2, \\ \Psi(\omega) &= \int_0^T F(\omega(t)) dt \geq \frac{T}{2} F(d), \end{aligned}$$

which yields

$$\frac{\Psi(\omega)}{\Phi(\omega)} \geq \frac{T F(d)}{C^2(\alpha) d^2}. \tag{4.6}$$

Moreover, from $C(\alpha) d < \frac{c}{\sigma(\alpha)}$ one get $\Phi(\omega) < \rho$. Substituting (4.6) into (4.5) and then making use of condition (C_2), we obtain that

$$\frac{2}{\rho} \sup_{u \in \Phi^{-1}((-\infty, \rho))} \Psi(x) < \frac{\Psi(\omega)}{\Phi(\omega)}.$$

Finally, since $f(u) = 0$ for all $u < 0$, for which $F(\zeta) \geq 0$ for all $\zeta \in \mathbb{R}$. Therefore, $\Psi(u) \geq 0$ for all $u \in X$. Especially, $\min_{t \in [0,1]} \Psi(t\omega) \geq 0$. Thus, all assumptions of Theorem 2.1 are satisfied, and hence φ_λ has two critical points, that is, problem (1.1) has two distinct weak solutions, which are positive. The proof is completed. □

Remark 4.1. In contrast to reference [24], the fractional derivative space $\mathbb{E}_\beta^{\alpha,p}$ presented in this paper incorporates boundary conditions for BVPs. This approach has the advantage of allowing the energy functional of the BVPs to be established without considering the interference from boundary conditions, thereby simplifying the computation process for analyzing the existence of critical points.

5. Example

Example 5.1. Consider the following fractional three point BVP

$$\begin{cases} {}_t D_1^{3/4} ({}_0^C D_t^{3/4} u(t)) = \lambda f(u(t)), \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = (1/2) u(1/2). \end{cases} \tag{5.1}$$

Corresponding to problem (1.1), here

$$\alpha = 3/4, \quad \beta = \eta = 1/2, \quad T = 1,$$

$$f(u) = \begin{cases} u^2, & \text{if } 0 \le u < 1/2, \\ -(1/2)(u-1), & \text{if } 1/2 \le u < 1, \\ 0, & \text{if } 1 \le u < 6, \text{ and } u < 0, \\ g(u), & \text{if } u \ge 6, \end{cases}$$

where $g : [6, +\infty) \to \mathbb{R}^+$ is a completely arbitrary continuous function with $g(6) = 0$. Obviously, the condition (C_1) of Theorem 4.1 holds. Moreover, we compute that

$$\begin{aligned} C(\alpha) &= \frac{2}{\Gamma(2-\alpha)} \Big[\frac{1}{\eta^2} \frac{1}{3-2\alpha} \Big(\frac{\eta}{2}\Big)^{3-2\alpha} + \frac{1}{\eta^2} \Big(\frac{\eta}{2}\Big)^{2-2\alpha} \frac{2T-\eta}{2} \\ &\quad + \frac{1}{3-2\alpha} \Big(\frac{\beta-1}{T-\eta}\Big)^2 \Big(\frac{T-\eta}{2}\Big)^{3-2\alpha} + \frac{2}{2-\alpha} \frac{\beta-1}{\eta(T-\eta)} \Big(\frac{\eta}{2}\Big)^{1-\alpha} \Big(\frac{T-\eta}{2}\Big)^{2-\alpha} \Big]^{1/2} \\ &= \frac{2}{\Gamma(5/4)} \Big(\frac{1}{3} + \frac{3}{2} + \frac{1}{12} - \frac{2}{5} \Big)^{1/2} \\ &\approx 2.7174, \end{aligned}$$

$$\sigma(\alpha) = \frac{T^{\alpha-\frac{1}{2}}}{(2\alpha-1)^{1/2}\Gamma(\alpha)} = \frac{1}{\sqrt{1/2}\,\Gamma(3/4)} \approx 1.154068,$$

$$F(\zeta) = \begin{cases} \zeta^3/3, & \text{if } \zeta < 1/2, \\ -(1/4)\zeta^2 + (1/2)\zeta + (-7/48), & \text{if } 1/2 \le \zeta < 1, \\ 5/48, & \text{if } 1 \le \zeta < 6. \end{cases}$$

Choose, $d = 1/2$ and $c = 6$, it is easy to verify that f is nonnegative on $\mathbb{R}$ and

$$C(\alpha)d \approx 1.3587 < \frac{c}{\sigma(\alpha)} \approx 5.1990, \quad \frac{4\sigma^2(\alpha)F(c)}{c^2} \approx 0.01542 < \frac{F(d)}{C^2(\alpha)d^2} \approx 0.02257.$$

Therefore, the hypothesis (C_2) of Theorem 4.1 is satisfied. Thus, all the conditions of Theorem 4.1 hold and hence for each $\lambda \in (44.3056, 64.8711)$ the problem (5.1) admits two untrivial weak solutions u_i with $||u_i||_\infty < 6, \ i = 1, 2$.

6. Conclusion

In this paper, we prove a new fractional derivative space, establish the variational structure for a class of fractional three-point BVPs, and successfully apply the critical point theorem to obtain the existence of solutions for fractional three-point BVPs. The main contribution of this paper is to discuss the existence of solutions for a class of fractional nonlocal BVPs by using variational methods. The conclusions for the existence of solutions to fractional BVP (1.1) obtained are different from those obtained using degree theory, fixed point theorem, etc. Our work broadens the scope of the application of critical point theory to fractional BVPs and is of great significance. Based on this study, there are numerous work can be down in the future, such as using variational methods to discuss the existence and multiplicity of solutions for p-Laplacian (Kirchhoff-type) fractional three-point BVPs and impulsive fractional three-point BVPs, and so on.

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