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SOLVING NONLINEAR TIME-FRACTIONAL KAWAHARA AND MODIFIED KAWAHARA EQUATIONS USING THE LAPLACE RESIDUAL POWER SERIES METHOD

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Abstract In this study, we efficiently obtain numerical solutions for the nonlinear time-fractional Kawahara and modified Kawahara equations using the Laplace residual power series method (L-RPSM), which combines the Laplace transform with the residual power series approach. We demonstrate the procedure, validity, and applicability of the proposed method by solving both equations with arbitrary initial conditions. To validate the theoretical results, we apply three distinct test problems. The results are presented in tabular form and visually represented in both two- and three-dimensional graphs, showing that the solutions converge to the exact results. Additionally, the results highlight the impact of the time-Caputo fractional operator on the obtained solutions and demonstrate that the proposed method offers more accurate approximations compared to other methods, such as the Adomian decomposition method (ADM), the variational iteration method (VIM), the residual power series method (RPSM), and the homotopy perturbation method (HPM).

Keywords Caputo derivative, Laplace residual power series method, nonlinear time-fractional Kawahara equation, nonlinear time-fractional modified Kawahara equation.

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1. Introduction

Fractional calculus is a branch of mathematics that focuses on the study of integrals and derivatives of arbitrary orders of functions. Over the past three decades, the theory of fractional

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calculus has advanced rapidly, largely due to the significant contributions of researchers such as Giuseppe Francesco Brioschi and the Riemann-Liouville formulation. These developments have become a foundational basis for various applied fields, including fractional geometry, fractional differential equations (FDE), and fractional dynamics. As a result, fractional calculus has enabled the effective modeling and analysis of dynamic phenomena across a wide range of scientific and technical disciplines, including physics [8], engineering [13], biology [1], economics [11], and artificial intelligence [23].

In comparison to other fractional derivatives, such as those by Riemann-Liouville [24], Hilfer [18], Caputo-Fabrizio [5], Atangana-Baleanu [26], and Grünwald-Letnikov [6], the Caputo fractional derivative (C-FD) [24, 25], introduced by Michele Caputo in 1967, is unique in that it interprets the initial and boundary conditions in the same way as integer-order derivatives, meaning that they are equivalent. This property makes the Caputo fractional derivative particularly suitable for practical applications.

The Jordanian mathematician Omar Abu Arqub introduced the Residual Power Series Method (RPSM) [7] in 2013 as an novel numerical technique for solving certain types of differential and integral equations, both fractional and non-fractional. This method involves expressing the solution as a power series. Eriqet and Al Ajou further advanced this approach by developing the Laplace Residual Power Series Method (L-RPSM) [15], which combines the Laplace transform with the RPS technique. The L-RPSM provides solutions in the Laplace domain and applies the RPS method within this domain, using the residual error limit in the Laplace domain and a power term to calculate the coefficients for the series solution [2, 22, 28, 31]. Numerous methods have been developed to solve both fractional and non-fractional differential equations [14, 27], each with its own set of advantages and limitations. Some methods offer high accuracy but involve complex, time-consuming calculations that can lead to difficulties or errors. Other methods are simpler and faster but may lack precision. In contrast, the L-RPSM strikes a good balance between accuracy, speed, and simplicity, making it a reliable and efficient method for obtaining both exact and approximate solutions to these equations.

In this paper, the Laplace Residual Power Series Method (L-RPSM) is presented as an effective technique for solving the time-fractional Kawahara and modified Kawahara equations, defined as follows:

$$ {}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)+\delta\,\mathfrak{u}(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)+\sigma\,\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)+\mu\,\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)=0, \quad \zeta\in\mathbb{R},\ \varrho\in[0,T],\ \alpha\in(0,1], $$

and

$$ {}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)+\delta\,\mathfrak{u}^{2}(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)+\sigma\,\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)+\mu\,\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)=0, \quad \zeta\in\mathbb{R},\ \varrho\in[0,T],\ \alpha\in(0,1]. $$

Here, ${}^{C}\mathcal{D}_{\varrho}^{\alpha}$ denotes the Caputo fractional derivative of order α with respect to time ϱ. The operators $\mathcal{D}_{\zeta}^{(n)}$ represent the n-th order partial derivatives with respect to the spatial variable ζ, i.e., $\mathcal{D}_{\zeta}^{(n)}=\frac{\partial^n}{\partial\zeta^n}$ for $n\in\mathbb{N}$. The parameters δ, σ, and μ are arbitrary real constants.

The proposed equations [4, 10, 19, 30] are advanced versions of their classical counterparts, incorporating fractional derivatives with respect to time to study complex dynamic phenomena. By including fractional temporal derivatives, these equations enhance the modeling of systems with memory effects and non-local interactions, offering a more accurate representation of complex behaviors that conventional models may fail to capture. Their applicability spans various fields, such as fluid dynamics, plasma physics, and material science, where they improve the ability to model intricate dynamics and interactions.

While various semi-analytical techniques have been employed to solve fractional differential equations, they often suffer from limited accuracy and higher absolute errors, particularly when applied to complex models such as the time-fractional Kawahara and modified Kawahara equations, which involve strong nonlinearity and high-order spatial derivatives. To address this limitation, the present study utilizes the L-RPSM, which effectively combines the Laplace transform with the residual power series approach. This integration enables more accurate determination of the series coefficients, significantly reducing absolute errors compared to existing methods. The proposed method is thus particularly well-suited for obtaining precise solutions to nonlinear time-fractional partial differential equations (PDEs) that exhibit complex dynamical behavior.

The organization of the paper is as follows: Section 2 introduces the fundamental concepts of the C-FD, fractional power series, and Laplace transformation. Section 3 provides a detailed application of the L-RPSM to the time-fractional Kawahara and modified Kawahara equations. Section 4 presents three numerical examples and includes a comparative analysis of the proposed method to highlight its effectiveness. Finally, Section 5 summarizes the key findings of the study.

2. Main concepts

To fully understand the content of this paper, it is essential to introduce some key fundamental concepts. This section presents the definitions, theorems, lemmas, and properties related to the C-FD, fractional power series, and Laplace transformation.

Definition 2.1. Let $n \in \mathbb{N}$ be the smallest integer such that $n-1 < \alpha < n$. The Caputo fractional derivative of order α with respect to time ϱ for a sufficiently smooth function $\mathfrak{u}(\zeta,\varrho)$ is defined as follows [24, 25]:

$$^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\varrho}(\varrho - s)^{n-\alpha-1}\mathcal{D}_s^{(n)}\mathfrak{u}(\zeta,s)\,ds, \quad \text{for } n-1<\alpha<n,$$

where $\Gamma(\cdot)$ is the Gamma function, and $\mathcal{D}_s^{(n)} = \frac{\partial^n}{\partial s^n}$ denotes the classical integer-order derivative with respect to s. In the particular case where $\alpha = n \in \mathbb{N}$, the Caputo derivative reduces to the classical derivative:

$$\mathcal{D}_{\varrho}^{(n)}\mathfrak{u}(\zeta,\varrho) = \frac{\partial^n \mathfrak{u}(\zeta,\varrho)}{\partial \varrho^n}.$$

Definition 2.2. Let $\alpha > 0$, $\varrho > 0$, and let $\mathfrak{u}(\zeta,\varrho)$ be a sufficiently smooth function. The Riemann–Liouville fractional integral of order α with respect to the time variable ϱ is defined as follows [24, 25]:

$$\mathcal{I}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho) = \frac{1}{\Gamma(\alpha)}\int_0^{\varrho}(\varrho - s)^{\alpha-1}\mathfrak{u}(\zeta,s)\,ds, \quad \alpha > 0. \tag{2.1}$$

When $\alpha = 0$, the operator reduces to the original function:

$$\mathcal{I}_{\varrho}^{0}\mathfrak{u}(\zeta,\varrho) = \mathfrak{u}(\zeta,\varrho).$$

Definition 2.3. The power series about $\varrho = \varrho_0$ is expressed as follows [16]:

$$\sum_{n=0}^{\infty} a_n(\zeta)\,(\varrho-\varrho_0)^n = a_0(\zeta) + a_1(\zeta)(\varrho-\varrho_0) + a_2(\zeta)(\varrho-\varrho_0)^2 + a_3(\zeta)(\varrho-\varrho_0)^3 + \dots, \tag{2.2}$$

where $\zeta \in J$, $\varrho \in (\varrho_0;\infty)$ and $a_n(\zeta)$ are the coefficients of the series.

Theorem 2.1. *Suppose the function* $\mathfrak{u}$ *has a power series representation at* $\varrho = 0$ [16], *given by*

$$\mathfrak{u}(\zeta, \varrho) = \sum_{n=0}^{\infty} a_n(\zeta)\varrho^n. \tag{2.3}$$

If $d_\varrho^n \mathfrak{u}$ *are continious functions on* $J \times (0; \mathcal{R})$, *then the coefficients* $a_n(\zeta)$ *in Eq.* (2.3) *are determined by:*

$$a_n(\zeta) = \frac{d_\varrho^n \mathfrak{u}(\zeta, 0)}{n!}, \tag{2.4}$$

where $\mathcal{R} = \min_{c \in J} \mathcal{R}_c$, $\mathcal{R}_c$ *is the radius of convergence of the power series represented in Eq.* (2.3).

According to Eqs. (2.3) and (2.4), we obtain:

$$\mathfrak{u}(\zeta, \varrho) = \sum_{n=0}^{\infty} \frac{d_\varrho^n \mathfrak{u}(\zeta, 0)}{n!} \varrho^n, \quad \zeta \in J, \quad 0 \leq \varrho < \mathcal{R}. \tag{2.5}$$

Definition 2.4. Assume that $\mathfrak{u}$ is a piecewise continuous (PC) function on $J \times [0; \infty)$ and of exponential order λ. The Laplace transform (LT) of $\mathfrak{u}(\zeta, \varrho)$ is defined as follows [29]:

$$\mathcal{U}(\zeta, \mathfrak{s}) = \mathcal{L}\left[\mathfrak{u}(\zeta, \varrho)\right] = \int_0^{+\infty} e^{-\mathfrak{s}\varrho} \mathfrak{u}(\zeta, \varrho) d\varrho, \quad \mathfrak{s} > \lambda, \tag{2.6}$$

and the inverse LT of $\mathcal{U}(\zeta, \mathfrak{s})$ is defined as:

$$\mathfrak{u}(\zeta, \varrho) = \mathcal{L}^{-1}\left[\mathcal{U}(\zeta, \mathfrak{s})\right] = \int_{l-i\infty}^{l+i\infty} e^{\mathfrak{s}\varrho} \mathcal{U}(\zeta, \mathfrak{s}) d\mathfrak{s}, \quad l = Re(\mathfrak{s}) > l_0, \tag{2.7}$$

where l_0 is a point in the right half-plane where the integral converges.

Lemma 2.1. *Let* $\mathfrak{u}$ *and* $\mathfrak{v}$ *be PC functions on* $J \times [0; +\infty)$ *with exponential orders* λ_1 *and* λ_2, *respectively, where* $\lambda_1 < \lambda_2$. *Suppose that* $\mathcal{U}(\zeta, \mathfrak{s}) = \mathcal{L}\left[\mathfrak{u}(\zeta, \varrho)\right]$ *and* $\mathcal{V}(\zeta, \mathfrak{s}) = \mathcal{L}\left[\mathfrak{v}(\zeta, \varrho)\right]$, *then* [12]:

- $\mathcal{L}\left[a\mathfrak{u}(\zeta, \varrho) + b\mathfrak{v}(\zeta, \varrho)\right] = a\mathcal{U}(\zeta, \mathfrak{s}) + b\mathcal{V}(\zeta, \mathfrak{s}), \quad \zeta \in J, \quad \mathfrak{s} > \lambda_1, \quad a, b \in \mathbb{R}.$
- $\mathcal{L}\left[e^{a\varrho}\mathfrak{u}(\zeta, \varrho)\right] = \mathcal{U}(\zeta, \mathfrak{s} - a), \quad \zeta \in J, \quad \mathfrak{s} > a + \lambda_1.$
- $\lim_{\mathfrak{s} \longrightarrow +\infty} \mathcal{V}(\zeta, \mathfrak{s}) = \mathfrak{v}(\zeta, 0), \quad \zeta \in J.$

Lemma 2.2. *Suppose that* $\mathfrak{u}$ *is a PC function of exponential order* λ, *and* $\mathcal{U}(\zeta, \mathfrak{s}) = \mathcal{L}\left[\mathfrak{u}(\zeta, \varrho)\right]$. *Then the following properties hold* [29]:

- $\mathcal{L}\left[I_\varrho^\alpha \mathfrak{u}(\zeta, \varrho)\right] = \dfrac{\mathcal{U}(\zeta, \mathfrak{s})}{\mathfrak{s}^\alpha}, \quad \alpha > 0.$
- $\mathcal{L}\left[\mathcal{D}_\varrho^\alpha \mathfrak{u}(\zeta, \varrho)\right] = \mathfrak{s}^\alpha \mathcal{U}(\zeta, \mathfrak{s}) - \sum_{k=0}^{m-1} \mathfrak{s}^{\alpha-k-1} \mathfrak{u}^k(\zeta, 0), \quad m - 1 < \alpha \leq m.$
- $\mathcal{L}\left[\mathcal{D}_\varrho^{n\alpha} \mathfrak{u}(\zeta, \varrho)\right] = \mathfrak{s}^{n\alpha} \mathcal{U}(\zeta, \mathfrak{s}) - \sum_{k=0}^{n-1} \mathfrak{s}^{(n-k)\alpha-1} \mathcal{D}_\varrho^{k\alpha} \mathfrak{u}(\zeta, 0), \quad 0 < \alpha \leq 1,$ *where* $\mathcal{D}_\varrho^{n\alpha} = \mathcal{D}_\varrho^\alpha . \mathcal{D}_\varrho^\alpha ... \mathcal{D}_\varrho^\alpha$ *(applied n-times).*

Theorem 2.2. *Let* $\mathfrak{u}$ *be a PC function on* $J \times [0; \infty)$ *and of exponential order* λ. *Assume that the fractional expansion of* $\mathcal{U}(\zeta, \mathfrak{s}) = \mathcal{L}\left[\mathfrak{u}(\zeta, \varrho)\right]$ *is given by* [29]:

$$\mathcal{U}(\zeta, \mathfrak{s}) = \sum_{n=0}^{+\infty} \frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}, \quad 0 < \alpha \leq 1, \quad \zeta \in J, \quad \mathfrak{s} > \lambda, \tag{2.8}$$

then the coefficients $a_n(\zeta)$ *are given by* $\mathcal{D}_\varrho^{n\alpha} \mathfrak{u}(\zeta, 0)$.

Remark 2.1. The inverse LT of the expansion in Eq. (2.8) takes the following form [29]:

$$\mathfrak{u}(\zeta,\varrho)=\sum_{n=0}^{+\infty}\frac{\mathcal{D}_{\varrho}^{n\alpha}\mathfrak{u}(\zeta,0)}{\Gamma(n\alpha+1)}\varrho^{n\alpha},\quad 0<\alpha\leq 1,\quad \varrho\geq 0. \tag{2.9}$$

Theorem 2.3. *Let $\mathfrak{u}$ be a PC function on $J\times[0;+\infty)$ and of exponential order λ. Suppose that $\mathcal{U}(\zeta,\mathfrak{s})=\mathcal{L}\left[\mathfrak{u}(\zeta,\varrho)\right]$ can be expressed by the expansion in Eq. (2.8). If*

$$\left|\mathfrak{s}\mathcal{L}\left[d_{\varrho}^{(n+1)}\mathfrak{u}(\zeta,\varrho)\right]\right|\leq M(\zeta)$$

on $J\times(\lambda;\eta]$, then the remainder $\mathcal{R}_n$ of the MFPS in Eq. (2.8) satisfies the following inequality [7]:

$$|\mathcal{R}_n(\zeta,\mathfrak{s})|\leqslant\frac{M(\zeta)}{\mathfrak{s}^{(n+1)\alpha+1}},\quad \zeta\in J,\quad \lambda<\mathfrak{s}\leq\eta. \tag{2.10}$$

3. The L-RPS methodology

The main concept behind the L-RPS approach is to first apply the LT to the equation being studied, then solve it using a fractional power series expansion. After determining the coefficients of the series using the same method as the standard RPS technique, we apply the inverse Laplace transform to obtain the solution to the original equation. In this section, we will apply the steps of the L-RPSM to generate approximate solutions for both the Kawahara and modified Kawahara equations.

3.1. The L-RPS solution to the time-fractional Kawahara equation

The time-fractional Kawahara equation is expressed as follows:

$${}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)+\delta\,\mathfrak{u}(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)+\sigma\,\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)+\mu\,\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)=0,\quad \zeta\in\mathbb{R},\ \varrho\in[0,T],\ \alpha\in(0,1]. \tag{3.1}$$

With the initial condition:

$$\mathfrak{u}(\zeta,0)=\varphi(\zeta). \tag{3.2}$$

To obtain the solution for the initial value problem (3.1)-(3.2) using the L-RPS method, we follow these steps:

Step 1. Apply the Laplace transform (LT) to both sides of Eq. (3.1):

$$\mathcal{L}\left[{}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)\right]+\delta\mathcal{L}\left[\mathfrak{u}(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)\right]+\sigma\mathcal{L}\left[\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)\right]+\mu\mathcal{L}\left[\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)\right]=0. \tag{3.3}$$

Step 2. Using the fact that

$$\mathcal{L}({}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho))=\mathfrak{s}^{\alpha}\mathcal{L}(\mathfrak{u}(\zeta,\varrho))-\mathfrak{s}^{\alpha-1}\mathfrak{u}(\zeta,0)$$

and applying the initial condition (3.2), we can rewrite Eq. (3.3) as:

$$\mathcal{U}(\zeta,\mathfrak{s})=\frac{\varphi(\zeta)}{\mathfrak{s}}-\frac{\delta}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}(\zeta,\mathfrak{s})\right]\right]-\frac{\sigma}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}(\zeta,\mathfrak{s})-\frac{\mu}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}(\zeta,\mathfrak{s}). \tag{3.4}$$

Where

$$\mathcal{U}(\zeta,\mathfrak{s})=\mathcal{L}(\mathfrak{u}(\zeta,\varrho)).$$

Step 3. We can express the transformed function $\mathcal{U}(\zeta, \mathfrak{s})$ as the following expansion to obtain the solution of Eq. (3.4):

$$\mathcal{U}(\zeta, \mathfrak{s}) = \sum_{n=0}^{\infty} \frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}. \tag{3.5}$$

Thus, according to the initial condition (3.2), the expansion (3.5) can be written as:

$$\mathcal{U}(\zeta, \mathfrak{s}) = \frac{\varphi(\zeta)}{\mathfrak{s}} + \sum_{n=1}^{\infty} \frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}. \tag{3.6}$$

Therefore, the k-th truncated series solutions of Eq. (3.4) is:

$$\mathcal{U}_k(\zeta, \mathfrak{s}) = \frac{\varphi(\zeta)}{\mathfrak{s}} + \sum_{n=1}^{k} \frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}. \tag{3.7}$$

Step 4. Define the Laplace residual functions (L-RF), $\mathcal{L}$Res, for Eq. (3.4) as:

$$\begin{aligned}\mathcal{L}\text{Res}(\zeta, \mathfrak{s}) =& \mathcal{U}(\zeta, \mathfrak{s}) - \frac{\varphi(\zeta)}{\mathfrak{s}} + \frac{\delta}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta, \mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}(\zeta, \mathfrak{s})\right]\right] + \frac{\sigma}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}(\zeta, \mathfrak{s}) \\ &+ \frac{\mu}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}(\zeta, \mathfrak{s}).\end{aligned} \tag{3.8}$$

The k-th LRF, $\mathcal{L}\text{Res}_k$ of Eq. (3.4) is:

$$\begin{aligned}\mathcal{L}\text{Res}_k(\zeta, \mathfrak{s}) =& \mathcal{U}_k(\zeta, s) - \frac{\varphi(\zeta)}{\mathfrak{s}} + \frac{\delta}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}_k(\zeta, \mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}_k(\zeta, \mathfrak{s})\right]\right] + \frac{\sigma}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}_k(\zeta, \mathfrak{s}) \\ &+ \frac{\mu}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}_k(\zeta, s).\end{aligned} \tag{3.9}$$

Remark 3.1. Using the standard RPSM, both the L-RF and the k-th LRF satisfy the following properties:

- $\mathcal{L}\text{Res}(\zeta, \mathfrak{s}) = 0$ and $\lim_{k \to +\infty} \mathcal{L}\text{Res}_k(\zeta, \mathfrak{s}) = \mathcal{L}\text{Res}(\zeta, \mathfrak{s})$, for $\mathfrak{s} > 0$.
- $\lim_{\mathfrak{s} \to +\infty} \mathfrak{s}\mathcal{L}\text{Res}(\zeta, \mathfrak{s}) = \lim_{\mathfrak{s} \to +\infty} \mathfrak{s}\mathcal{L}\text{Res}_k(\zeta, \mathfrak{s}) = 0$.
- $\lim_{\mathfrak{s} \to +\infty} \mathfrak{s}^{k\alpha+1}\mathcal{L}\text{Res}(\zeta, \mathfrak{s}) = \lim_{\mathfrak{s} \to +\infty} \mathfrak{s}^{k\alpha+1}\mathcal{L}\text{Res}_k(\zeta, \mathfrak{s}) = 0, \quad 0 < \alpha \leq 1$ and $k = 0, 1, 2, \ldots$.

For further discussion of these properties, see reference [2].

Step 5. We calculate the coefficients $a_1(\zeta), a_2(\zeta), a_3(\zeta), \ldots$ of the MFPS (3.7). To determine the first unknown coefficient, $a_1(\zeta)$, we substitute the first truncated series solution

$$\mathcal{U}_1(\zeta, \mathfrak{s}) = \frac{\varphi(\zeta)}{\mathfrak{s}} + \frac{a_1(\zeta)}{\mathfrak{s}^{\alpha+1}}, \tag{3.10}$$

into the first LRF. We obtain:

$$\begin{aligned}\mathcal{L}\text{Res}_1(\zeta, \mathfrak{s}) =& \frac{a_1(\zeta)}{\mathfrak{s}^{1+\alpha}} + \frac{\delta}{\mathfrak{s}^{\alpha}}\left(\frac{\varphi(\zeta)\varphi'(\zeta)}{\mathfrak{s}} + \frac{a_1(\zeta)\varphi'(\zeta)}{\mathfrak{s}^{1+\alpha}} + \frac{\varphi(\zeta)a_1'(\zeta)}{\mathfrak{s}^{1+\alpha}} + \frac{\Gamma(1+2\alpha)a_1(\zeta)a_1'(\zeta)}{\mathfrak{s}^{1+2\alpha}\Gamma(1+\alpha)^2}\right) \\ &+ \frac{\sigma}{\mathfrak{s}^{\alpha}}\left(\frac{\varphi^{(3)}(\zeta)}{\mathfrak{s}} + \frac{a_1^{(3)}(\zeta)}{\mathfrak{s}^{1+\alpha}}\right) - \frac{\mu}{\mathfrak{s}^{\alpha}}\left(\frac{\varphi^{(5)}(\zeta)}{\mathfrak{s}} + \frac{a_1^{(5)}(\zeta)}{\mathfrak{s}^{1+\alpha}}\right).\end{aligned}$$

By using the condition

$$\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{s}^{\alpha+1}\mathcal{L}\mathrm{Res}_1(\zeta,\mathfrak{s})=0,$$

we obtain:

$$a_1(\zeta)=-\delta\varphi(\zeta)\varphi'(\zeta)-\sigma\varphi^{(3)}(\zeta)+\mu\varphi^{(5)}(\zeta). \tag{3.11}$$

Thus, the first approximate solution to Eq. (3.4) is:

$$\mathcal{U}_1(\zeta,\mathfrak{s})=\frac{\varphi(\zeta)}{\mathfrak{s}}+\frac{\left(-\delta\varphi(\zeta)\varphi'(\zeta)-\sigma\varphi^{(3)}(\zeta)+\mu\varphi^{(5)}(\zeta)\right)}{\mathfrak{s}^{\alpha+1}}. \tag{3.12}$$

Next, to determine the second unknown coefficient, $a_2(\zeta)$, we substitute the second truncated series solution

$$\mathcal{U}_2(\zeta,\mathfrak{s})=\frac{\varphi(\zeta)}{\mathfrak{s}}+\frac{\left(-\delta\varphi(\zeta)\varphi'(\zeta)-\sigma\varphi^{(3)}(\zeta)+\mu\varphi^{(5)}(\zeta)\right)}{\mathfrak{s}^{\alpha+1}}+\frac{a_2(\zeta)}{\mathfrak{s}^{2\alpha+1}}, \tag{3.13}$$

into the second LRF. Using

$$\lim_{\mathfrak{s}\longrightarrow\infty} \mathfrak{s}^{2\alpha+1}\mathcal{L}\mathrm{Res}_2(\zeta,\mathfrak{s})=0,$$

we obtain:

$$\begin{aligned}a_2(\zeta)=&\,2\delta^2\varphi(\zeta)\left(\varphi'(\zeta)\right)^2+\delta^2\varphi(\zeta)^2\varphi''(\zeta)+3\delta\sigma\left(\varphi''(\zeta)\right)^2+5\delta\sigma\varphi'(\zeta)\varphi^{(3)}(\zeta)-10\delta\mu\left(\varphi^{(3)}(\zeta)\right)^2\\&+2\delta\sigma\varphi(\zeta)\varphi^{(4)}(\zeta)-15\delta\mu\varphi''(\zeta)\varphi^{(4)}(\zeta)7\delta\mu\varphi'(\zeta)\varphi^{(5)}(\zeta)+\sigma^2\varphi^{(6)}(\zeta)\\&-2\delta\mu\varphi(\zeta)\varphi^{(6)}(\zeta)-2\sigma\mu\varphi^{(8)}(\zeta)+\mu^2\varphi^{(10)}(\zeta).\end{aligned}$$

Therfore, the second L-RPS approximate solution of Eq. (3.4) is:

$$\begin{aligned}\mathcal{U}_2(\zeta,\mathfrak{s})=&\,\frac{\varphi(\zeta)}{\mathfrak{s}}+\frac{1}{\mathfrak{s}^{\alpha+1}}\left(-\varphi(\zeta)\varphi'(\zeta)-\sigma\varphi^{(3)}(\zeta)+\mu\varphi^{(5)}(\zeta)\right)+\frac{1}{\mathfrak{s}^{2\alpha+1}}\Big(2\delta^2\varphi(\zeta)\left(\varphi'(\zeta)\right)^2\\&+\delta^2\varphi(\zeta)^2\varphi''(\zeta)+3\delta\sigma\left(\varphi''(\zeta)\right)^2+5\delta\sigma\varphi'(\zeta)\varphi^{(3)}(\zeta)-10\delta\mu\left(\varphi^{(3)}(\zeta)\right)^2+2\delta\sigma\varphi(\zeta)\varphi^{(4)}(\zeta)\\&-15\delta\mu\varphi''(\zeta)\varphi^{(4)}(\zeta)-7\delta\mu\varphi'(\zeta)\varphi^{(5)}(\zeta)+\sigma^2\varphi^{(6)}(\zeta)-2\delta\mu\varphi(\zeta)\varphi^{(6)}(\zeta)-2\sigma\mu\varphi^{(8)}(\zeta)\\&+\mu^2\varphi^{(10)}(\zeta)\Big).\end{aligned} \tag{3.14}$$

Thus, by applying the previous steps to define the coefficient $a_k(\zeta)$, we obtain:

$$\begin{aligned}&a_3(\zeta)\\=&-33\delta^2\sigma\varphi'(\zeta)\varphi''(\zeta)^2-\delta^3\varphi(\zeta)^3\varphi^{(3)}(\zeta)-25\delta^2\sigma\varphi'(\zeta)^2\varphi^{(3)}(\zeta)-\frac{\delta^2\sigma\Gamma(1+2\alpha)\varphi'(\zeta)^2\varphi^{(3)}(\zeta)}{\Gamma(1+\alpha)^2}\\&+210\delta^2\mu\varphi''(\zeta)^2\varphi^{(3)}(\zeta)+150\delta^2\mu\varphi'(\zeta)\varphi^{(3)}(\zeta)^2+225\delta^2\mu\varphi'(\zeta)\varphi''(\zeta)\varphi^{(4)}(\zeta)-40\delta\sigma^2\varphi^{(3)}(\zeta)\\&\times\varphi^{(4)}(\zeta)-\frac{\delta\sigma^2\Gamma(1+2\alpha)\varphi^{(3)}(\zeta)\varphi^{(4)}(\zeta)}{\Gamma(1+\alpha)^2}+49\delta^2\mu\varphi'(\zeta)^2\varphi^{(5)}(\zeta)+\frac{\delta^2\mu\Gamma(1+2\alpha)\varphi'(\zeta)^2\varphi^{(5)}(\zeta)}{\Gamma(1+\alpha)^2}\\&-27\delta\sigma^2\varphi''(\zeta)\varphi^{(5)}(\zeta)+274\delta\sigma\mu\varphi^{(4)}(\zeta)\varphi^{(5)}(\zeta)+\frac{\delta\sigma\mu\Gamma(1+2\alpha)\varphi^{(4)}(\zeta)\varphi^{(5)}(\zeta)}{\Gamma(1+\alpha)^2}-12\delta\sigma^2\varphi'(\zeta)\\&\times\varphi^{(6)}(\zeta)+193\delta\sigma\mu\varphi^{(3)}(\zeta)\varphi^{(6)}(\zeta)+\frac{\delta\sigma\mu\Gamma(1+2\alpha)\varphi^{(3)}(\zeta)\varphi^{(6)}(\zeta)}{\Gamma(1+\alpha)^2}-469\delta\mu^2\varphi^{(5)}(\zeta)\varphi^{(6)}(\zeta)\end{aligned}$$

$$
\begin{aligned}
&-\frac{\delta\mu^2\Gamma(1+2\alpha)\varphi^{(5)}(\zeta)\varphi^{(6)}(\zeta)}{\Gamma(1+\alpha)^2}+93\delta\sigma\mu\varphi''(\zeta)\varphi^{(7)}(\zeta)-345\delta\mu^2\varphi^{(4)}(\zeta)\varphi^{(7)}(\zeta)-\frac{\delta^2\varphi(\zeta)^2}{\Gamma(1+\alpha)^2}\\
&\times(7\delta\Gamma(1+\alpha)^2\varphi'(\zeta)\varphi''(\zeta)+\delta\Gamma(1+2\alpha)\varphi'(\zeta)\varphi''(\zeta)+3\sigma\Gamma(1+\alpha)^2\varphi^{(5)}(\zeta)-3\mu\Gamma(1+\alpha)^2\\
&\times\varphi^{(7)}(\zeta))+30\delta\sigma\mu\varphi'(\zeta)\varphi^{(8)}(\zeta)-185\delta\mu^2\varphi^{(3)}(\zeta)\varphi^{(8)}(\zeta)-\sigma^3\varphi^{(9)}(\zeta)-70\delta\mu^2\varphi''(\zeta)\varphi^{(9)}(\zeta)\\
&-18\delta\mu^2\varphi'(\zeta)\varphi^{(10)}(\zeta)+3\sigma^2\mu\varphi^{(11)}(\zeta)-\frac{1}{\Gamma(1+\alpha)^2}\delta\varphi(\zeta)\Big(4\delta^2\Gamma(1+\alpha)^2\varphi'(\zeta)^3+\delta^2\Gamma(1+2\alpha)\\
&\times\varphi'(\zeta)^3+31\delta\sigma\Gamma(1+\alpha)^2\varphi''(\zeta)\varphi^{(3)}(\zeta)+\delta\sigma\Gamma(1+2\alpha)\varphi''(\zeta)\varphi^{(3)}(\zeta)+19\delta\sigma\Gamma(1+\alpha)^2\varphi'(\zeta)\varphi^{(4)}(\zeta)\\
&+\delta\sigma\Gamma(1+2\alpha)\varphi'(\zeta)\varphi^{(4)}(\zeta)-105\delta\mu\Gamma(1+\alpha)^2\varphi^{(3)}(\zeta)\varphi^{(4)}(\zeta)-64\delta\mu\Gamma(1+\alpha)^2\varphi''(\zeta)\varphi^{(5)}(\zeta)\\
&-\delta\mu\Gamma(1+2\alpha)\varphi''(\zeta)\varphi^{(5)}(\zeta)-25\delta\mu\Gamma(1+\alpha)^2\varphi'(\zeta)\varphi^{(6)}(\zeta)-\delta\mu\Gamma(1+2\alpha)\varphi'(\zeta)\varphi^{(6)}(\zeta)\\
&+3\sigma^2\Gamma(1+\alpha)^2\varphi^{(7)}(\zeta)-6\sigma\mu\Gamma(1+\alpha)^2\varphi^{(9)}(\zeta)+3\mu^2\Gamma(1+\alpha)^2\varphi^{(11)}(\zeta)\Big)-3\sigma\mu^2\varphi^{(13)}(\zeta)\\
&+\mu^3\varphi^{(15)}(\zeta)\\
&\vdots
\end{aligned}
$$

Consequently, the solution to Eq. (3.4) as an infinite series is:

$$
\begin{aligned}
\mathcal{U}(\zeta,\mathfrak{s})=&\frac{\varphi(\zeta)}{\mathfrak{s}}+\frac{1}{\mathfrak{s}^{\alpha+1}}\left(-\varphi(\zeta)\varphi'(\zeta)-\sigma\varphi^{(3)}(\zeta)+\mu\varphi^{(5)}(\zeta)\right)+\frac{1}{\mathfrak{s}^{2\alpha+1}}\Big(2\delta^2\varphi(\zeta)\left(\varphi'(\zeta)\right)^2\\
&+\delta^2\varphi(\zeta)^2\varphi''(\zeta)+3\delta\sigma\left(\varphi''(\zeta)\right)^2+5\delta\sigma\varphi'(\zeta)\varphi^{(3)}(\zeta)-10\delta\mu\left(\varphi^{(3)}(\zeta)\right)^2+2\delta\sigma\varphi(\zeta)\varphi^{(4)}(\zeta)\\
&-15\delta\mu\varphi''(\zeta)\varphi^{(4)}(\zeta)-7\delta\mu\varphi'(\zeta)\varphi^{(5)}(\zeta)+\sigma^2\varphi^{(6)}(\zeta)-2\delta\mu\varphi(\zeta)\varphi^{(6)}(\zeta)-2\sigma\mu\varphi^{(8)}(\zeta)\\
&+\mu^2\varphi^{(10)}(\zeta)\Big)+\frac{1}{\mathfrak{s}^{1+3\alpha}}\Big(-33\delta^2\sigma\varphi'(\zeta)\varphi''(\zeta)^2-\delta^3\varphi(\zeta)^3\varphi^{(3)}(\zeta)-25\delta^2\sigma\varphi'(\zeta)^2\varphi^{(3)}(\zeta)\\
&-\frac{\delta^2\sigma\Gamma(1+2\alpha)\varphi'(\zeta)^2\varphi^{(3)}(\zeta)}{\Gamma(1+\alpha)^2}+210\delta^2\mu\varphi''(\zeta)^2\varphi^{(3)}(\zeta)+150\delta^2\mu\varphi'(\zeta)\varphi^{(3)}(\zeta)^2\\
&+225\delta^2\mu\varphi'(\zeta)\varphi''(\zeta)\varphi^{(4)}(\zeta)-40\delta\sigma^2\varphi^{(3)}(\zeta)\varphi^{(4)}(\zeta)-\frac{\delta\sigma^2\Gamma(1+2\alpha)\varphi^{(3)}(\zeta)\varphi^{(4)}(\zeta)}{\Gamma(1+\alpha)^2}\\
&+49\delta^2\mu\varphi'(\zeta)^2\varphi^{(5)}(\zeta)+\frac{\delta^2\mu\Gamma(1+2\alpha)\varphi'(\zeta)^2\varphi^{(5)}(\zeta)}{\Gamma(1+\alpha)^2}+\cdots-3\sigma\mu^2\varphi^{(13)}(\zeta)+\mu^3\varphi^{(15)}(\zeta)\Big)\\
&+\ldots. \qquad (3.15)
\end{aligned}
$$

Step 6. By applying the inverse LT to both sides of Eq. (3.15), we express the L-RPS solution of the IVP (3.1)-(3.2) as follows:

$$
\begin{aligned}
&\mathfrak{u}(\zeta,\varrho)\\
&=\varphi(\zeta)+\frac{\varrho^{\alpha}}{\Gamma(1+\alpha)}\left(-\varphi(\zeta)\varphi'(\zeta)-\sigma\varphi^{(3)}(\zeta)+\mu\varphi^{(5)}(\zeta)\right)+\frac{\varrho^{2\alpha}}{\Gamma(1+2\alpha)}\Big(2\delta^2\varphi(\zeta)\left(\varphi'(\zeta)\right)^2\\
&+\delta^2\varphi(\zeta)^2\varphi''(\zeta)+3\delta\sigma\left(\varphi''(\zeta)\right)^2+5\delta\sigma\varphi'(\zeta)\varphi^{(3)}(\zeta)-10\delta\mu\left(\varphi^{(3)}(\zeta)\right)^2+2\delta\sigma\varphi(\zeta)\varphi^{(4)}(\zeta)\\
&-15\delta\mu\varphi''(\zeta)\varphi^{(4)}(\zeta)-7\delta\mu\varphi'(\zeta)\varphi^{(5)}(\zeta)+\sigma^2\varphi^{(6)}(\zeta)-2\delta\mu\varphi(\zeta)\varphi^{(6)}(\zeta)-2\sigma\mu\varphi^{(8)}(\zeta)\\
&+\mu^2\varphi^{(10)}(\zeta)\Big)+\frac{\varrho^{3\alpha}}{\Gamma(1+3\alpha)}\Big(-33\delta^2\sigma\varphi'(\zeta)\varphi''(\zeta)^2-\delta^3\varphi(\zeta)^3\varphi^{(3)}(\zeta)-25\delta^2\sigma\varphi'(\zeta)^2\varphi^{(3)}(\zeta)
\end{aligned}
$$

$$- \frac{\delta^2\sigma\Gamma(1+2\alpha)\varphi'(\zeta)^2\varphi^{(3)}(\zeta)}{\Gamma(1+\alpha)^2} + 210\delta^2\mu\varphi''(\zeta)^2\varphi^{(3)}(\zeta) + 150\delta^2\mu\varphi'(\zeta)\varphi^{(3)}(\zeta)^2$$
$$+ 225\delta^2\mu\varphi'(\zeta)\varphi''(\zeta)\varphi^{(4)}(\zeta) - 40\delta\sigma^2\varphi^{(3)}(\zeta)\varphi^{(4)}(\zeta) - \frac{\delta\sigma^2\Gamma(1+2\alpha)\varphi^{(3)}(\zeta)\varphi^{(4)}(\zeta)}{\Gamma(1+\alpha)^2}$$
$$+ 49\delta^2\mu\varphi'(\zeta)^2\varphi^{(5)}(\zeta) + \frac{\delta^2\mu\Gamma(1+2\alpha)\varphi'(\zeta)^2\varphi^{(5)}(\zeta)}{\Gamma(1+\alpha)^2} + \cdots - 3\sigma\mu^2\varphi^{(13)}(\zeta) + \mu^3\varphi^{(15)}(\zeta)\Big)$$
$$+ \ldots.$$

3.2. The L-RPS solution of the time-fractional modified Kawahara equation

The time-fractional modified Kawahara equation is given by:

$$ {}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho) + \delta\,\mathfrak{u}^2(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho) + \sigma\,\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho) + \mu\,\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho) = 0, \quad \zeta\in\mathbb{R},\ \varrho\in[0,T],\ \alpha\in(0,1], \tag{3.16}$$

with the initial condition:

$$\mathfrak{u}(\zeta,0) = \psi(\zeta). \tag{3.17}$$

To find the L-RPS solution for the IVP (3.16)-(3.17), we follow the steps from the previous section.

First, we apply the LT to Eq. (3.16):

$$\mathcal{L}\left[{}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)\right] + \delta\mathcal{L}\left[\mathfrak{u}^2(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)\right] + \sigma\mathcal{L}\left[\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)\right] + \mu\mathcal{L}\left[\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)\right] = 0. \tag{3.18}$$

Then, using the initial condition (3.17) and Lemma 2, we rewrite Eq. (3.18) as:

$$\mathcal{U}(\zeta,\mathfrak{s}) = \frac{\psi(\zeta)}{\mathfrak{s}} - \frac{\delta}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta,\mathfrak{s})\right]\right]^2\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}(\zeta,\mathfrak{s})\right]\right] - \frac{\sigma}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}(\zeta,s) - \frac{\mu}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}(\zeta,\mathfrak{s}), \tag{3.19}$$

where, $\mathcal{U}(\zeta,\mathfrak{s}) = \mathcal{L}(\mathfrak{u}(\zeta,\varrho))$.

Next, the k-th truncated series solutions of Eq. (3.19), $\mathcal{U}_k(\zeta,\mathfrak{s})$, is expressed as:

$$\mathcal{U}_k(\zeta,\mathfrak{s}) = \frac{\psi(\zeta)}{\mathfrak{s}} + \sum_{n=1}^{k}\frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}. \tag{3.20}$$

Thus, the L-RF $\mathcal{L}$Res and the $k-$th LRF ($\mathcal{L}\text{Res}_k$) for Eq. (3.19) are given by:

$$\begin{aligned}\mathcal{L}\text{Res}\,(\zeta,\mathfrak{s}) &= \mathcal{U}(\zeta,\mathfrak{s}) - \frac{\psi(\zeta)}{\mathfrak{s}} + \frac{\delta}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}(\zeta,\mathfrak{s})\right]\right] + \frac{\sigma}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}(\zeta,\mathfrak{s})\\ &\quad + \frac{\mu}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}(\zeta,\mathfrak{s}),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathcal{L}\text{Res}_k\,(\zeta,\mathfrak{s}) &= \mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{\psi(\zeta)}{\mathfrak{s}} + \frac{\delta}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}_k(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}_k(\zeta,\mathfrak{s})\right]\right] + \frac{\sigma}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}_k(\zeta,\mathfrak{s})\\ &\quad + \frac{\mu}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}_k(\zeta,\mathfrak{s}).\end{aligned} \tag{3.22}$$

By performing similar calculations as in Step 5, we obtain the solution of Eq. (3.19) as:

$$\mathcal{U}(\zeta,\mathfrak{s})$$
$$= \frac{\psi(\zeta)}{s} + \frac{1}{s^{1+\alpha}}\left(-\delta\psi(\zeta)^2\psi'(\zeta) - \sigma\psi^{(3)}(\zeta) - \mu\psi^{(5)}(\zeta)\right) + \frac{1}{s^{1+2\alpha}}\Big(4\delta^2\psi(\zeta)^3\psi'(\zeta)^2$$

$$+\delta^2\psi(\zeta)^4\psi''(\zeta)+12\delta\sigma\psi'(\zeta)^2\psi''(\zeta)+6\delta\sigma\psi(\zeta)\psi''(\zeta)^2+30\delta\mu\psi''(\zeta)^3$$
$$+10\delta\sigma\psi(\zeta)\psi'(\zeta)\psi^{(3)}(\zeta)+120\delta\mu\psi'(\zeta)\psi''(\zeta)\psi^{(3)}(\zeta)+20\delta\mu\psi(\zeta)\psi^{(3)}(\zeta)^2+2\delta\sigma\psi(\zeta)^2\psi^{(4)}(\zeta)$$
$$+30\delta\mu\psi'(\zeta)^2\psi^{(4)}(\zeta)+30\delta\mu\psi(\zeta)\psi''(\zeta)\psi^{(4)}(\zeta)+14\delta\mu\psi(\zeta)\psi'(\zeta)\psi^{(5)}(\zeta)+\sigma^2\psi^{(6)}(\zeta)$$
$$+2\delta\mu\psi(\zeta)^2\psi^{(6)}(\zeta)+2\mu\sigma\psi^{(8)}(\zeta)+\mu^2\psi^{(10)}(\zeta)\Big)+\ldots. \tag{3.23}$$

Finally, the L-RPS solution of the IVP (3.16)-(3.17) is given by:

$$\begin{aligned}
&u(\zeta,\varrho)\\
&=\psi(\zeta)+\frac{\varrho^{\alpha}}{\Gamma(1+\alpha)}\left(-\delta\psi(\zeta)^2\psi'(\zeta)-\sigma\psi^{(3)}(\zeta)-\mu\psi^{(5)}(\zeta)\right)+\frac{\varrho^{2\alpha}}{\Gamma(1+2\alpha)}\Big(4\delta^2\psi(\zeta)^3\psi'(\zeta)^2\\
&\quad+\delta^2\psi(\zeta)^4\psi''(\zeta)+12\delta\sigma\psi'(\zeta)^2\psi''(\zeta)+6\delta\sigma\psi(\zeta)\psi''(\zeta)^2+30\delta\mu\psi''(\zeta)^3\\
&\quad+10\delta\sigma\psi(\zeta)\psi'(\zeta)\psi^{(3)}(\zeta)+120\delta\mu\psi'(\zeta)\psi''(\zeta)\psi^{(3)}(\zeta)+20\delta\mu\psi(\zeta)\psi^{(3)}(\zeta)^2+2\delta\sigma\psi(\zeta)^2\psi^{(4)}(\zeta)\\
&\quad+30\delta\mu\psi'(\zeta)^2\psi^{(4)}(\zeta)+30\delta\mu\psi(\zeta)\psi''(\zeta)\psi^{(4)}(\zeta)+14\delta\mu\psi(\zeta)\psi'(\zeta)\psi^{(5)}(\zeta)+\sigma^2\psi^{(6)}(\zeta)\\
&\quad+2\delta\mu\psi(\zeta)^2\psi^{(6)}(\zeta)+2\mu\sigma\psi^{(8)}(\zeta)+\mu^2\psi^{(10)}(\zeta)\Big)+\ldots.
\end{aligned} \tag{3.24}$$

4. Discussion and numerical results

In this section, we evaluate the effectiveness of the proposed Laplace Residual Power Series (L-RPS) method in solving the time-fractional Kawahara and modified Kawahara equations. We present three numerical examples, each with known exact solutions, while keeping the parameters δ, σ and μ fixed. Error estimations and comparisons with other methods are provided to demonstrate the accuracy of the proposed approach.

Example 4.1. Let us consider the Kawahara equation with the parameters $\delta=1$, $\sigma=1$, and $\mu=-1$, i.e.,

$${}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)+\mathfrak{u}(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)+\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)-\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)=0. \tag{4.1}$$

The exact solution of (4.1) at $\alpha=1$, is given by [20]:

$$\mathfrak{u}(\zeta,\varrho)=\frac{105}{169}\sinh^4\left(\frac{1}{2\sqrt{13}}\left(\zeta-\frac{36}{169}\varrho\right)\right), \tag{4.2}$$

so the initial condition is :

$$\mathfrak{u}(\zeta,0)=\frac{105}{169}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right). \tag{4.3}$$

According to the L-RPSM, we define:

$$\begin{aligned}
\mathcal{U}(\zeta,\mathfrak{s})=&\frac{105}{169\mathfrak{s}}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right)-\frac{1}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}(\zeta,\mathfrak{s})\right]\right]-\frac{1}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}(\zeta,\mathfrak{s})\\
&+\frac{1}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}(\zeta,\mathfrak{s}),
\end{aligned} \tag{4.4}$$

and the k-th truncated series solutions of Eq. (4.4) is:

$$\mathcal{U}_k(\zeta,\mathfrak{s}) = \frac{105}{169\mathfrak{s}}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right) + \sum_{n=1}^{k}\frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}, \tag{4.5}$$

therfore, the k-th LRF of Eq. (4.4) is:

$$\begin{aligned}\mathcal{L}\mathrm{Res}_k(\zeta,\mathfrak{s}) =& \mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{105}{169s}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right) + \frac{1}{\mathfrak{s}^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}_k(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_\zeta^{(1)}\mathcal{U}_k(\zeta,\mathfrak{s})\right]\right]\\ &+ \frac{1}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(3)}\mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{1}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(5)}\mathcal{U}_k(\zeta,\mathfrak{s}).\end{aligned} \tag{4.6}$$

By following the steps of the L-RPSM, we obtain:

$$\begin{aligned}
a_0(\zeta) =& \frac{105}{169}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right),\\
a_1(\zeta) =& \frac{7560\,\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)}{28561\sqrt{13}},\\
a_2(\zeta) =& \frac{136080\left(-3+2\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\right)\mathrm{sech}^6\left(\frac{\zeta}{2\sqrt{13}}\right)}{62748517},\\
a_3(\zeta) =& \frac{204120}{137858491849\Gamma(1+\alpha)^2}\Bigg(765\sqrt{13}\Gamma(1+\alpha)^2\,\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&- 650\sqrt{13}\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\,\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&- 9\sqrt{13}\cosh\left(\frac{2\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\,\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&+ 6\sqrt{13}\cosh\left(\frac{3\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\,\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&- 420\sqrt{13}\Gamma(1+2\alpha)\,\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&+ 280\sqrt{13}\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\Gamma(1+2\alpha)\,\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\Bigg),\\
&\vdots
\end{aligned}$$

It follows that the solution of Eq. (4.4) is:

$$\begin{aligned}\mathcal{U}(\zeta,\mathfrak{s}) =& \frac{105}{169\mathfrak{s}}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right) + \frac{136080\mathrm{sech}^6\left(\frac{\zeta}{2\sqrt{13}}\right)}{62748517\mathfrak{s}^{\alpha+1}}\left(-3+2\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\right)\\
&+ \frac{136080\mathrm{sech}^6\left(\frac{\zeta}{2\sqrt{13}}\right)\left(-3+2\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\right)}{62748517\mathfrak{s}^{2\alpha+1}}\\
&+ \frac{204120}{137858491849\Gamma(1+\alpha)^2\mathfrak{s}^{1+3\alpha}}\Bigg(765\sqrt{13}\Gamma(1+\alpha)^2\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&- 650\sqrt{13}\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\mathrm{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\end{aligned}$$

$$- 9\sqrt{13}\cosh\left(\frac{2\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)$$
$$+ 6\sqrt{13}\cosh\left(\frac{3\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)$$
$$- 420\sqrt{13}\Gamma(1+2\alpha)\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right) + 280\sqrt{13}\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\Gamma(1+2\alpha)$$
$$\times\,\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\Bigg) + \ldots.$$

Finally, the L-RPS solution of Example 4.1 is given by:

$$\begin{aligned}
&\mathfrak{u}(\zeta,\varrho)\\
=&\frac{105}{169s}\sinh^4\left(\frac{1}{2\sqrt{13}}\zeta\right) + \frac{136080\varrho^{\alpha}\text{sech}^6\left(\frac{\zeta}{2\sqrt{13}}\right)}{62748517\Gamma(1+\alpha)}\left(-3+2\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\right)\\
&+\frac{136080\varrho^{2\alpha}\text{sech}^6\left(\frac{\zeta}{2\sqrt{13}}\right)}{62748517\Gamma(1+2\alpha)}\left(-3+2\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\right) + \frac{204120\varrho^{3\alpha}}{137858491849\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\\
&\times\Bigg(765\sqrt{13}\Gamma(1+\alpha)^2\times\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right) - 650\sqrt{13}\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\\
&\times\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right) - 9\sqrt{13}\cosh\left(\frac{2\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&\times\tanh\left(\frac{\zeta}{2\sqrt{13}}\right) + 6\sqrt{13}\cosh\left(\frac{3\zeta}{\sqrt{13}}\right)\Gamma(1+\alpha)^2\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\\
&-420\sqrt{13}\Gamma(1+2\alpha)\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right) + 280\sqrt{13}\cosh\left(\frac{\zeta}{\sqrt{13}}\right)\Gamma(1+2\alpha)\\
&\times\text{sech}^{10}\left(\frac{\zeta}{2\sqrt{13}}\right)\tanh\left(\frac{\zeta}{2\sqrt{13}}\right)\Bigg) + \ldots.
\end{aligned}$$

Example 4.2. Consider the Kawahara equation with the parameters $\delta = 1$, $\sigma = 4$, and $\mu = -\frac{1}{13}$. In this case, the equation is written as follows:

$$^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho) + \mathfrak{u}(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho) + 4\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho) - \frac{1}{13}\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho) = 0, \tag{4.7}$$

with the exact solution [3] at ($\alpha = 1$) given by [3]:

$$\mathfrak{u}(\zeta,\varrho) = \frac{1}{2197}\left(-1093 - 3360\tanh^2\left(\frac{\zeta+\varrho}{13}\right) + 1680\tanh^4\left(\frac{\zeta+\varrho}{13}\right)\right), \tag{4.8}$$

and the initial conditions is:

$$\mathfrak{u}(\zeta,0) = \frac{1}{2197}\left(-1093 - 3360\tanh^2\left(\frac{\zeta}{13}\right) + 1680\tanh^4\left(\frac{\zeta}{13}\right)\right). \tag{4.9}$$

According to the L-RPSM, we obtain:

$$\mathcal{U}(\zeta,\mathfrak{s}) = \frac{1}{2197\mathfrak{s}}\left(-1093 - 3360\tanh^2\left(\frac{\zeta}{13}\right) + 1680\tanh^4\left(\frac{\zeta}{13}\right)\right)$$

Table 1. Comparison of exact and L-RPS solutions for Example 4.1 at $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$.

ϱ	ζ	Exact sol $\alpha = 1$	L-RPS Sol $\alpha = 1$	L-RPS Sol $\alpha = 0.75$	L-RPS Sol $\alpha = 0.5$	L-RPS Sol $\alpha = 0.25$
0.05	2	0.53458855	0.53458855	0.53568954	0.53798070	0.54232157
	4	0.34674710	0.34674710	0.34808873	0.35092749	0.35652622
	6	0.17885304	0.17885304	0.17979212	0.18179522	0.18582112
	8	0.07815209	0.07815209	0.07863702	0.07967646	0.08179023
	10	0.03057957	0.03057957	0.03078812	0.03123648	0.03215533
0.1	2	0.53543692	0.53543692	0.53700903	0.53970224	0.54386908
	4	0.34777780	0.34777780	0.34970859	0.35309365	0.35858293
	6	0.17957344	0.17957344	0.18093004	0.18333508	0.18731956
	8	0.07852376	0.07852376	0.07922589	0.08047914	0.08258338
	10	0.03073933	0.03073933	0.03104169	0.03158369	0.03250214
0.5	2	0.54211410	0.54211410	0.54444574	0.54671601	0.54850403
	4	0.35605052	0.35605052	0.35911455	0.36227467	0.36500106
	6	0.18541057	0.18541057	0.18763294	0.18998227	0.19207125
	8	0.08155237	0.08155237	0.08272482	0.08398347	0.08512297
	10	0.03204554	0.03204554	0.03255646	0.03311093	0.03362139

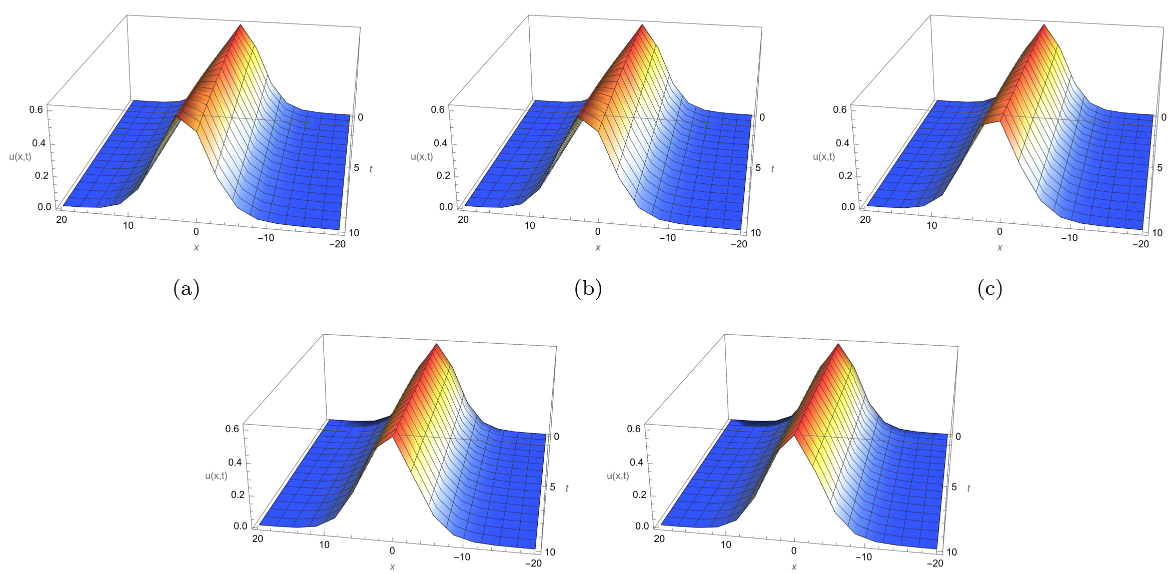


Figure 1. 3D plots of exact and 4th order L-RPS solutions of Example 4.1 for $-20 \leq \zeta \leq 20$ and $0 \leq \varrho \leq 10$. (a) Exact solution $\mathfrak{u}(\zeta, \varrho)$ $(\alpha = 1)$; (b) $\mathfrak{u}_4(\zeta, \varrho)$ $(\alpha = 1)$; (c) $\mathfrak{u}_4(\zeta, \varrho)$ $(\alpha = 0.75)$; (d) $\mathfrak{u}_4(\zeta, \varrho)$ $(\alpha = 0.5)$, and (e) $\mathfrak{u}_4(\zeta, \varrho)$ $(\alpha = 0.25)$.

$$-\frac{1}{\mathfrak{s}^{\alpha}}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_{\zeta}^{(1)}\mathcal{U}(\zeta,\mathfrak{s})\right]\right]-\frac{1}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(3)}\mathcal{U}(\zeta,\mathfrak{s})+\frac{1}{\mathfrak{s}^{\alpha}}\mathcal{D}_{\zeta}^{(5)}\mathcal{U}(\zeta,\mathfrak{s}), \tag{4.10}$$

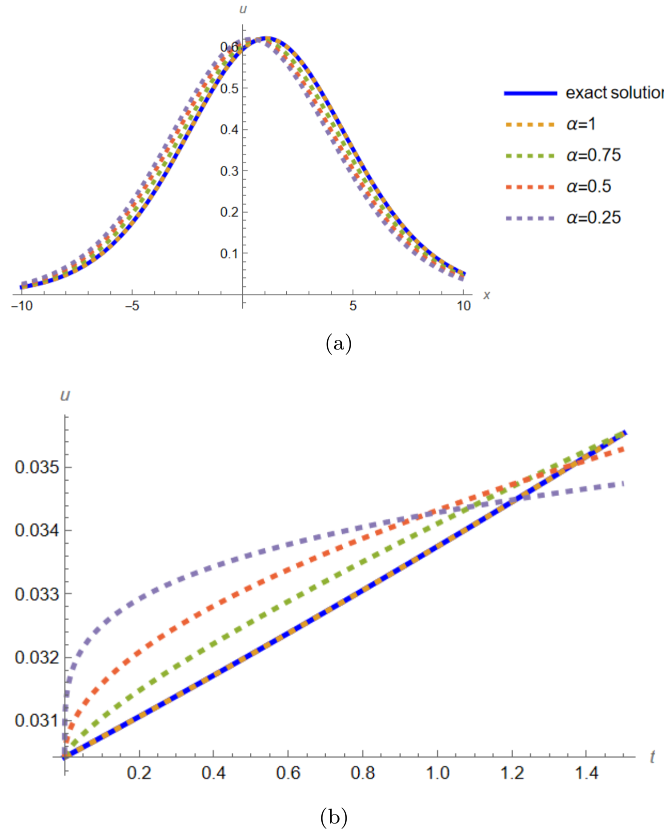


Figure 2. Graphs of the exact and 4th order L-RPS solutions for Example 4.1 at different values of α ($\alpha = 1$, $\alpha = 0.75$, $\alpha = 0.5$, and $\alpha = 0.25$). (a) $\zeta = 10$ and $0 \leq \varrho \leq 1.5$; (b) $\varrho = 5$ and $-10 \leq \zeta \leq 10$.

and the k-th truncated series solutions of Eq. (4.10) is:

$$\mathcal{U}_k(\zeta,\mathfrak{s}) = \frac{1}{2197\mathfrak{s}}\left(-1093 - 3360\tanh^2\left(\frac{\zeta}{13}\right) + 1680\tanh^4\left(\frac{\zeta}{13}\right)\right) + \sum_{n=1}^{k}\frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}, \tag{4.11}$$

hence, the kth LRF of Eq. (4.10) is given by:

$$\begin{aligned}\mathcal{L}\mathrm{Res}_k(\zeta,\mathfrak{s}) =& \mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{1}{2197\mathfrak{s}}\left(-1093 - 3360\tanh^2\left(\frac{\zeta}{13}\right) + 1680\tanh^4\left(\frac{\zeta}{13}\right)\right)\\ &+ \frac{1}{\mathfrak{s}^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}_k(\zeta,\mathfrak{s})\right]\mathcal{L}^{-1}\left[\mathcal{D}_\zeta\mathcal{U}_k(\zeta,\mathfrak{s})\right]\right] + \frac{4}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(3)}\mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{1}{13\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(5)}\mathcal{U}_k(\zeta,\mathfrak{s}).\end{aligned}$$

By following the steps of the L-RPSM, we obtain:

$$\begin{aligned}a_0(\zeta) &= -\frac{1}{2197}\left(-1093 - 3360\tanh^2\left(\frac{\zeta}{13}\right) + 1680\tanh^4\left(\frac{\zeta}{13}\right)\right),\\ a_1(\zeta) &= \frac{2100\,\mathrm{sech}^9\left(\frac{\zeta}{13}\right)}{10604499373}\left(562190\sinh\left(\frac{\zeta}{13}\right) - 406635\sinh\left(\frac{3\zeta}{13}\right) - 65657\sinh\left(\frac{5\zeta}{13}\right)\right),\\ a_2(\zeta) &= \frac{2625}{102371786028181514}\Bigg(-968520561505 + 2149482614426\cosh\left(\frac{2\zeta}{13}\right)\end{aligned}$$

Table 2. Error comparisons of the L-RPSM with other methods at $\alpha = 1$ [9,20,21].

ζ	ϱ	L-RPSM	RPSM	VIM	HPM	ADM
-20	0.1	2.57×10^{-16}	1.58×10^{-11}	3.97×10^{-11}	1.01×10^{-8}	6.04×10^{-9}
	0.2	8.2×10^{-15}	8.28×10^{-11}	3.13×10^{-10}	4.04×10^{-8}	2.67×10^{-8}
	20	6.21×10^{-14}	5.38×10^{-10}	1.05×10^{-9}	9.06×10^{-8}	7.30×10^{-8}
	0.4	2.61×10^{-13}	2.82×10^{-9}	2.49×10^{-9}	1.60×10^{-7}	1.14×10^{-7}
	0.5	7.96×10^{-13}	7.28×10^{-9}	4.85×10^{-9}	2.49×10^{-7}	6.02×10^{-7}
-10	0.1	1.36×10^{-15}	8.29×10^{-10}	2.46×10^{-9}	1.53×10^{-6}	4.34×10^{-8}
	0.2	4.25×10^{-14}	4.83×10^{-9}	1.92×10^{-8}	6.11×10^{-6}	9.53×10^{-8}
	0.3	3.15×10^{-13}	1.63×10^{-8}	6.48×10^{-8}	1.37×10^{-5}	3.03×10^{-7}
	0.4	1.29×10^{-12}	7.04×10^{-8}	1.53×10^{-7}	2.43×10^{-5}	7.24×10^{-7}
	0.5	3.84×10^{-12}	3.47×10^{-7}	2.98×10^{-7}	3.78×10^{-5}	2.19×10^{-6}
0	0.1	5.55×10^{-16}	2.00×10^{-10}	1.74×10^{-10}	5.42×10^{-6}	1.49×10^{-8}
	0.2	5.52×10^{-14}	8.48×10^{-10}	1.77×10^{-9}	2.16×10^{-5}	5.04×10^{-8}
	0.3	6.29×10^{-13}	3.18×10^{-9}	9.03×10^{-9}	4.87×10^{-5}	2.38×10^{-7}
	0.4	3.53×10^{-12}	9.48×10^{-9}	2.82×10^{-8}	8.67×10^{-5}	6.45×10^{-7}
	0.5	1.35×10^{-11}	4.03×10^{-8}	6.90×10^{-8}	1.35×10^{-4}	1.39×10^{-6}
10	0.1	1.42×10^{-15}	6.03×10^{-10}	2.41×10^{-9}	1.54×10^{-6}	5.83×10^{-8}
	0.2	4.68×10^{-14}	2.94×10^{-9}	1.94×10^{-8}	6.17×10^{-6}	1.00×10^{-7}
	0.3	3.64×10^{-13}	9.89×10^{-9}	6.58×10^{-8}	1.39×10^{-5}	4.89×10^{-7}
	0.4	1.57×10^{-12}	4.28×10^{-8}	1.56×10^{-7}	2.48×10^{-5}	9.45×10^{-7}
	0.5	4.9×10^{-12}	1.38×10^{-7}	3.06×10^{-7}	3.89×10^{-5}	5.34×10^{-6}
20	0.1	2.58×10^{-16}	2.28×10^{-11}	3.98×10^{-11}	1.02×10^{-8}	4.93×10^{-9}
	0.2	8.26×10^{-15}	8.68×10^{-11}	3.16×10^{-10}	4.10×10^{-8}	9.56×10^{-9}
	0.3	6.28×10^{-14}	5.03×10^{-10}	1.07×10^{-9}	9.27×10^{-8}	3.48×10^{-8}
	0.4	2.65×10^{-13}	2.00×10^{-9}	2.54×10^{-9}	1.65×10^{-7}	7.039×10^{-8}
	0.5	8.11×10^{-13}	7.28×10^{-9}	4.99×10^{-9}	2.59×10^{-7}	2.94×10^{-7}

$$- 1840463521748 \cosh\left(\frac{4\zeta}{13}\right) + 199599361013 \cosh\left(\frac{6\zeta}{13}\right) + 51862187765 \cosh\left(\frac{8\zeta}{13}\right)$$
$$+ 4310841649 \cosh\left(\frac{10\zeta}{13}\right) \operatorname{sech}^{14}\left(\frac{\zeta}{13}\right)\Bigg),$$
$$\vdots$$

Consequently, the solution of Eq. (4.10) obtained by L-RPSM is given as follows:

$$\mathcal{U}(\zeta, \mathfrak{s})$$
$$= -\frac{1}{2197\,\mathfrak{s}}\left(-1093 - 3360 \tanh^2\left(\frac{\zeta}{13}\right) + 1680 \tanh^4\left(\frac{\zeta}{13}\right)\right) + \frac{2100 \operatorname{sech}^9\left(\frac{\zeta}{13}\right)}{10604499373\,\mathfrak{s}^{1+\alpha}}$$

$$\times\left(562190\sinh\left(\frac{\zeta}{13}\right)-406635\sinh\left(\frac{3\zeta}{13}\right)-65657\sinh\left(\frac{5\zeta}{13}\right)\right)+\frac{2625}{102371786028181514\,\mathfrak{s}^{2+\alpha}}$$
$$\times\Bigg(-968520561505+2149482614426\cosh\left(\frac{2\zeta}{13}\right)-1840463521748\cosh\left(\frac{4\zeta}{13}\right)$$
$$+199599361013\cosh\left(\frac{6\zeta}{13}\right)+51862187765\cosh\left(\frac{8\zeta}{13}\right)+4310841649\cosh\left(\frac{10\zeta}{13}\right)$$
$$\times\operatorname{sech}^{14}\left(\frac{\zeta}{13}\right)\Bigg)+\ldots.$$

Finally, the L-RPS solution for Example 4.2 is given by:

$$u(\zeta,\varrho)=-\frac{1}{2197}\left(-1093-3360\tanh^2\left(\frac{\zeta}{13}\right)+1680\tanh^4\left(\frac{\zeta}{13}\right)\right)+\frac{2100\,\varrho^{\alpha}\operatorname{sech}^9\left(\frac{\zeta}{13}\right)}{10604499373\,\Gamma(1+\alpha)}$$
$$\times\left(562190\sinh\left(\frac{\zeta}{13}\right)-406635\sinh\left(\frac{3\zeta}{13}\right)-65657\sinh\left(\frac{5\zeta}{13}\right)\right)$$
$$+\frac{2625\,\varrho^{\alpha}}{102371786028181514\,\Gamma(1+2\alpha)}\Bigg(-968520561505+2149482614426\cosh\left(\frac{2\zeta}{13}\right)$$
$$-1840463521748\cosh\left(\frac{4\zeta}{13}\right)+199599361013\cosh\left(\frac{6\zeta}{13}\right)+51862187765\cosh\left(\frac{8\zeta}{13}\right)$$
$$+4310841649\cosh\left(\frac{10\zeta}{13}\right)\operatorname{sech}^{14}\left(\frac{\zeta}{13}\right)\Bigg)+\ldots.$$

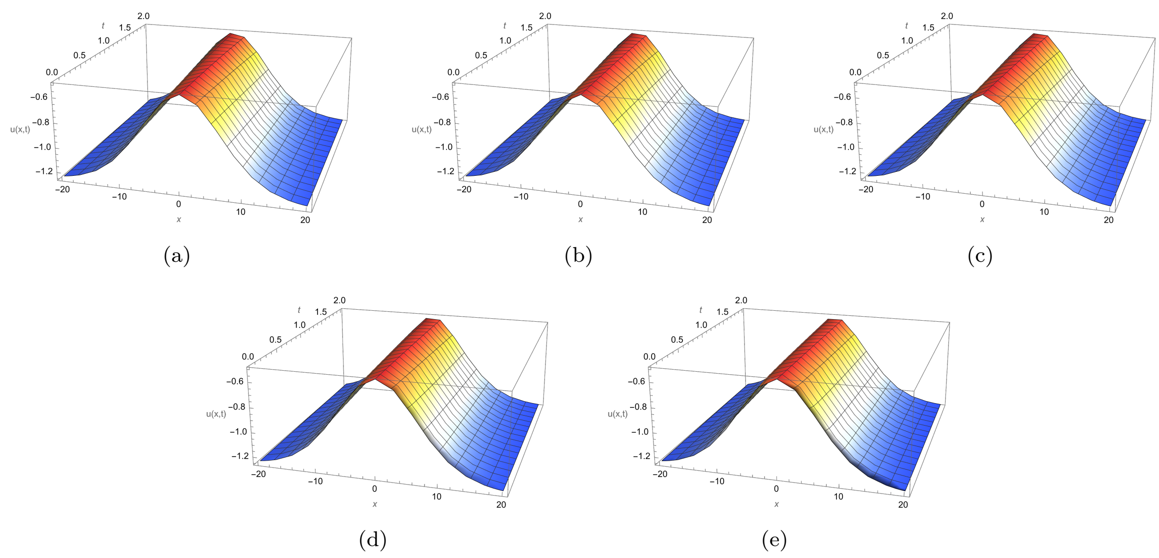


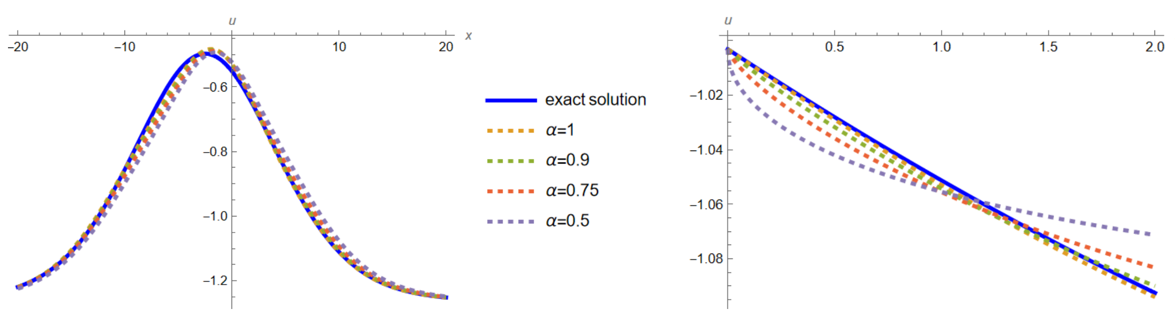
Figure 3. 3D-plots of exact and 3rd-order L-RPS solutions for Example 4.2, with $-20\le\zeta\le 20$ and $0\le\varrho\le 2$: (a) Eexact solution $\mathfrak{u}(\zeta,\varrho)$ $(\alpha=1)$; (b) $\mathfrak{u}_3(\zeta,\varrho)$ $(\alpha=1)$; (c) $\mathfrak{u}_3(\zeta,\varrho)$ $(\alpha=0.9)$; (d) $\mathfrak{u}_3(\zeta,\varrho)$ $(\alpha=0.75)$ and (e) $\mathfrak{u}_3(\zeta,\varrho)$ $(\alpha=0.5)$.

Example 4.3. Consider the following modified Kawahara equation for the parameters $\delta=1$, $\sigma=1$, and $\mu=-100$:

$${}^{C}\mathcal{D}_{\varrho}^{\alpha}\mathfrak{u}(\zeta,\varrho)+\mathfrak{u}^2(\zeta,\varrho)\mathcal{D}_{\zeta}^{(1)}\mathfrak{u}(\zeta,\varrho)+\mathcal{D}_{\zeta}^{(3)}\mathfrak{u}(\zeta,\varrho)-100\mathcal{D}_{\zeta}^{(5)}\mathfrak{u}(\zeta,\varrho)=0, \tag{4.12}$$

Table 3. Comparison of the exact and L-RPS solutions for Example 4.2 at $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 0.9$, and $\alpha = 1$.

ϱ	ζ	Exact sol $\alpha = 1$	L-RPS Sol $\alpha = 1$	Abs Err $\alpha = 1$	L-RPS Sol $\alpha = 0.9$	L-RPS Sol $\alpha = 0.75$	L-RPS Sol $\alpha = 0.5$
0.01	2	-0.533059	-0.533007	5.21×10^{-5}	-0.533196	-0.533724	-0.536112
	4	-0.628142	-0.628081	6.03×10^{-5}	-0.628421	-0.629364	-0.63356
	6	-0.756239	-0.756209	3×10^{-5}	-0.756626	-0.757781	-0.762867
	8	-0.88333	-0.883333	3.84×10^{-6}	-0.888745	-0.889882	-0.894844
	10	-1.00362	-1.00364	1.93×10^{-5}	-1.00399	-1.00494	-1.00907
0.05	2	-0.534448	-0.534183	2.64×10^{-4}	-0.534786	-0.536157	-0.54065
	4	-0.630482	-0.630183	2.99×10^{-4}	-0.631251	-0.633658	-0.641826
	6	-0.75893	-0.758785	1.45×10^{-4}	-0.760087	-0.763002	-0.772919
	8	-0.890849	-0.890872	2.2×10^{-5}	-0.892148	-0.894989	-0.903599
	10	-1.00567	-1.00577	9.67×10^{-5}	-1.00684	-1.0092	-1.01625
0.1	2	-0.536218	-0.53568	5.38×10^{-4}	-0.536629	-0.538618	-0.544318
	4	-0.633425	-0.632835	5.9×10^{-4}	-0.634497	-0.637924	-0.6473
	6	-0.762294	-0.762017	2.77×10^{-4}	-0.764025	-0.768119	-0.778945
	8	-0.893989	-0.89404	5.08×10^{-5}	-0.895939	-0.899366	-0.910077
	10	-1.00823	-1.00842	1.94×10^{-4}	-1.01004	-1.01329	-1.02148



(a) (b)

Figure 4. Graphs of the exact and 3rd-order L-RPS solutions for Example 4.2 at different values of α ($\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.75$, and $\alpha = 0.5$). (a) $\zeta = 10$ and $0 \le \varrho \le 2$; (b) $\varrho = 2.5$, and $-20 \le \zeta \le 20$.

with the exact solution at ($\alpha = 1$) [20]:

$$\mathfrak{u}(\zeta, \varrho) = \frac{3}{10\sqrt{10}} \operatorname{sech}^2 \left(\gamma \left(\zeta - \frac{126}{625} \varrho \right) \right), \tag{4.13}$$

and the initial condition is:

$$\mathfrak{u}(\zeta, 0) = \frac{3}{10\sqrt{10}} \operatorname{sech}^2 (\gamma \zeta), \tag{4.14}$$

where $\gamma = \frac{1}{20\sqrt{5}}$.

According to the L-RPSM, we get:

$$\mathcal{U}(\zeta,\mathfrak{s}) = \frac{3}{10\sqrt{10}\mathfrak{s}}\operatorname{sech}^2(\gamma\zeta) - \frac{1}{\mathfrak{s}^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}(\zeta,\mathfrak{s})\right]^2\mathcal{L}^{-1}\left[\mathcal{D}_\zeta^{(1)}\mathcal{U}(\zeta,\mathfrak{s})\right]\right] - \frac{1}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(3)}\mathcal{U}(\zeta,\mathfrak{s}) + \frac{100}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(5)}\mathcal{U}(\zeta,\mathfrak{s}), \tag{4.15}$$

and the k-th turncated series solution of Eq. (4.15) is:

$$\mathcal{U}_k(\zeta,\mathfrak{s}) = \frac{3}{10\sqrt{10}}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right) + \sum_{n=1}^{k}\frac{a_n(\zeta)}{\mathfrak{s}^{n\alpha+1}}, \tag{4.16}$$

therfore, the k-th LRF of Eq. (4.15) is:

$$\mathcal{L}\mathrm{Res}_k(\zeta,\mathfrak{s}) = \mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{3}{10\sqrt{10}}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right) + \frac{1}{\mathfrak{s}^\alpha}\mathcal{L}\left[\mathcal{L}^{-1}\left[\mathcal{U}_k(\zeta,\mathfrak{s})\right]^2\mathcal{L}^{-1}\left[\mathcal{D}_\zeta^{(1)}\mathcal{U}_k(\zeta,\mathfrak{s})\right]\right] + \frac{1}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(3)}\mathcal{U}_k(\zeta,\mathfrak{s}) - \frac{100}{\mathfrak{s}^\alpha}\mathcal{D}_\zeta^{(5)}\mathcal{U}_k(\zeta,\mathfrak{s}).$$

By following the steps of L-RPSM, we obtain:

$$\begin{aligned}
a_0(\zeta) =& \frac{3}{10\sqrt{10}}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right),\\
a_1(\zeta) =& \frac{3\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)}{312500\sqrt{2}},\\
a_2(\zeta) =& \frac{3\operatorname{sech}^4\left(\frac{\zeta}{20\sqrt{5}}\right)\left(-2+\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\right)}{3906250000\sqrt{10}},\\
a_3(\zeta) =& \frac{3}{390625\times 10^{10}}\left(874\sqrt{2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 573\sqrt{2}\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\right.\\
&\times\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right) tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 6\sqrt{2}\cosh\left(\frac{\zeta}{5\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\\
&+\sqrt{2}\cosh\left(\frac{3\zeta}{10\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 450\sqrt{2}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\\
&\left.\times\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) + 270\sqrt{2}\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\right),\\
&\vdots
\end{aligned}$$

Consequently, the solution of Eq. (4.15) obtained by L-RPSM is given as follows:

$$\begin{aligned}
&\mathcal{U}(\zeta,\mathfrak{s})\\
=&\frac{3}{10\sqrt{10}\,s}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right) + \frac{3}{312500\sqrt{2}s^{1+\alpha}}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\\
&+\frac{3}{3906250000\sqrt{10}\,s^{1+2\alpha}}\operatorname{sech}^4\left(\frac{\zeta}{20\sqrt{5}}\right)\left(-2+\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\right) + \frac{3}{390625\times 10^{10}s^{1+3\alpha}}\\
&\times\left(874\sqrt{2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 573\sqrt{2}\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\right.
\end{aligned}$$

$$- 6\sqrt{2}\cosh\left(\frac{\zeta}{5\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) + \sqrt{2}\cosh\left(\frac{3\zeta}{10\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)$$
$$\times \tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 450\sqrt{2}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) + 270\sqrt{2}\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)$$
$$\times \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\Bigg) + \ldots.$$

Finally, the L-RPS solution for Example 4.3 is given by:

$$\begin{aligned}
&\mathfrak{u}(\zeta,\varrho)\\
&= \frac{3}{10\sqrt{10}}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right) + \frac{3\varrho^{\alpha}}{312500\sqrt{2}\Gamma(1+\alpha)}\operatorname{sech}^2\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\\
&\quad + \frac{3\varrho^{2\alpha}}{3906250000\sqrt{10}\,\Gamma(1+2\alpha)}\operatorname{sech}^4\left(\frac{\zeta}{20\sqrt{5}}\right)\left(-2+\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\right)\\
&\quad + \frac{3\varrho^{3\alpha}}{390625\times 10^{10}\Gamma(1+3\alpha)}\Bigg(874\sqrt{2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 573\sqrt{2}\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\\
&\quad \times \operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 6\sqrt{2}\cosh\left(\frac{\zeta}{5\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\\
&\quad + \sqrt{2}\cosh\left(\frac{3\zeta}{10\sqrt{5}}\right)\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right) - 450\sqrt{2}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\\
&\quad \times \tanh\left(\frac{\zeta}{20\sqrt{5}}\right) + 270\sqrt{2}\cosh\left(\frac{\zeta}{10\sqrt{5}}\right)\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\operatorname{sech}^8\left(\frac{\zeta}{20\sqrt{5}}\right)\tanh\left(\frac{\zeta}{20\sqrt{5}}\right)\Bigg)\\
&\quad + \ldots.
\end{aligned}$$

Tables 1, 3, and 4 present comparisons between the exact solutions and L-RPS solutions for Examples 4.1, 4.2, and 4.3, respectively, at different values of α. The results confirm that the solutions obtained using the L-RPSM converge to the exact solutions and also illustrate the impact of different values of α on the solutions.

Table 2, corresponding to Example 4.1, and Table 5, related to the modified Kawahara equation ($\delta = 1$, $\sigma = 0.001$, and $\mu = -1$), provide an error comparison analysis of the L-RPSM in relation to other methods at $\alpha = 1$. These tables demonstrate the effectiveness of the L-RPSM compared to other methods for solving the Kawahara and Modified Kawahara equations.

Figure 1 displays 3D plots of the exact solution and 4th-order L-RPS solutions for Example 4.1 at defferent values of α, ζ, and ϱ, while Figure 3 and 5 present 3D plots of the exact solutions and 3rd-order L-RPS solutions for Examples 4.2 and 4.3, respectively, at different values of α, ζ, and ϱ. In contrast, Figure 2 illustrates 2D graphs comparing the exact solution with 4th-order L-RPS solutions for Example 4.1 at various values of α, ζ, and ϱ. Meanwhile, Figures 4 and 6 show 2D graphs of the exact solutions along with 3rd-order L-RPS solutions for Example 4.2 and Example 4.3, respectively, at different values of α, ζ, and ϱ. These visualizations reaffirm the earlier results regarding the convergence of the proposed method and further emphasize the influence of the fractional derivative α on the obtained solutions.

5. Conclusion

In summary, this paper demonstrates the successful application of the L-RPS approach to approximate solutions for the nonlinear time-fractional Kawahara and modified Kawahara equa-

Table 4. Comparison of the exact and L-RPS solutions for Example 4.3 at $\alpha = 0.25,\ \alpha = 0.5,\ \alpha = 0.75$, and $\alpha = 1$.

ϱ	ζ	Exact sol $\alpha = 1$	L-RPS Sol $\alpha = 1$	L-RPS Sol $\alpha = 0.75$	L-RPS Sol $\alpha = 0.5$	L-RPS Sol $\alpha = 0.25$
0.01	2	0.09468073	0.09467884	0.09467885	0.09467888	0.09467895
	4	0.09411716	0.09411341	0.09411343	0.09411348	0.09411362
	6	0.09318654	0.09318099	0.09318101	0.09318108	0.09318129
	8	0.09190343	0.09189616	0.09189619	0.09189628	0.09189655
	10	0.09028755	0.09027867	0.09027871	0.09027882	0.09027915
0.05	2	0.09469732	0.09467887	0.09467890	0.09467895	0.09467903
	4	0.09415055	0.09411347	0.09411352	0.09411362	0.09411378
	6	0.09323620	0.09318107	0.09318115	0.09318130	0.09318153
	8	0.09196859	0.09189626	0.09189637	0.09189656	0.09189687
	10	0.09036721	0.09027880	0.09027893	0.09027916	0.09027954
0.1	2	0.09468820	0.09467886	0.09467888	0.09467892	0.09467900
	4	0.09413210	0.09411344	0.09411348	0.0941135	0.09411372
	6	0.09320870	0.09318102	0.09318108	0.09318120	0.09318144
	8	0.09193247	0.09189620	0.09189628	0.09189644	0.09189675
	10	0.09032303	0.09027873	0.09027882	0.09027901	0.09027940

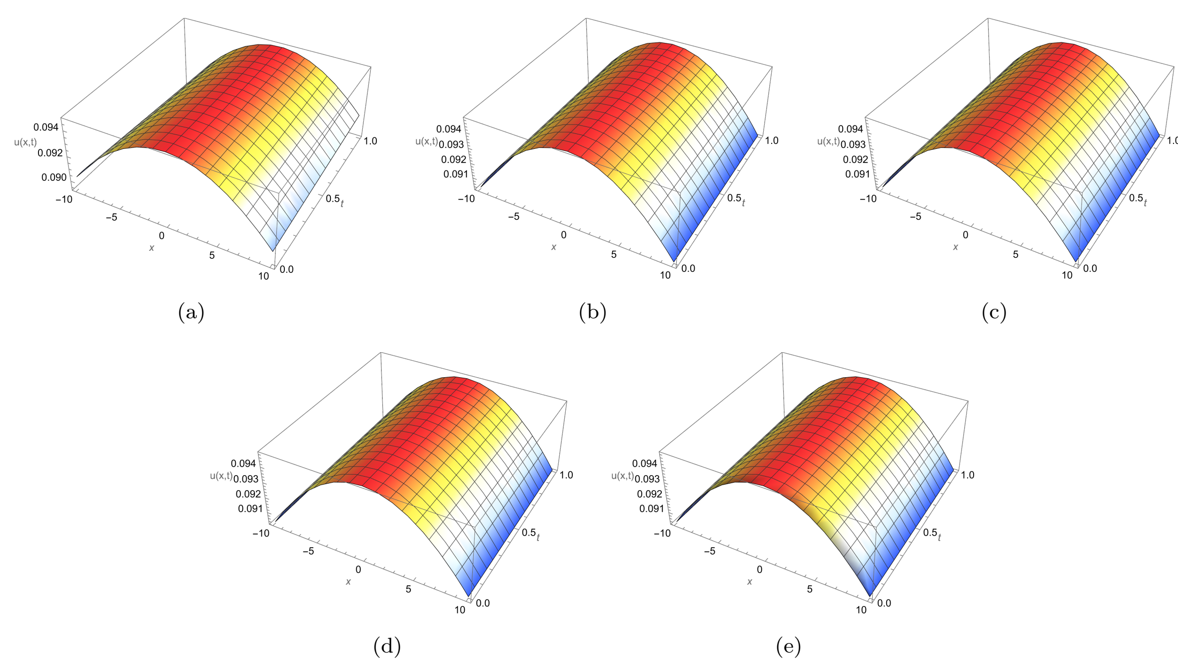


Figure 5. 3D plots of exact and 3rd-order L-RPS solutions for Example 4.3, with $-10 \le \zeta \le 10$ and $0 \le \varrho \le 1$: (a) Exact solution $\mathfrak{u}(\zeta, \varrho)$ $(\alpha = 1)$; (b) $\mathfrak{u}_3(\zeta, \varrho)$ $(\alpha = 1)$; (c) $\mathfrak{u}_3(\zeta, \varrho)$ $(\alpha = 0.75)$; (d) $\mathfrak{u}_3(\zeta, \varrho)$ $(\alpha = 0.5)$ and (e) $\mathfrak{u}_3(\zeta, \varrho)$ $(\alpha = 0.25)$.

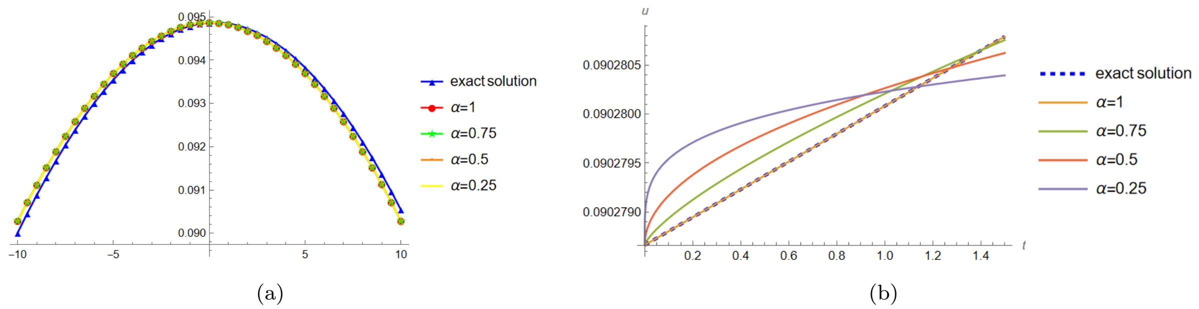


Figure 6. Graphs of the exact and 3rd-order L-RPS solutions for Example 4.3 at different values of α ($\alpha = 1$, $\alpha = 0.75$, $\alpha = 0.5$, and $\alpha = 0.25$): (a) $\zeta = 10$ and $0 \leq \varrho \leq 1.5$; (b) $\varrho = 0.3$ and $-10 \leq \zeta \leq 10$.

Table 5. Comparison of L-RPSM with other methods [9, 17, 20] for $\alpha = 1$, $\delta = 1$, $\sigma = 0.001$, and $\mu = -1$.

ζ	ϱ	L-RPSM	RPSM	VIM	HPM	ADM
-20	0.1	1.85×10^{-7}	8.51×10^{-7}	2.14×10^{-5}	2.14×10^{-5}	4.56×10^{-6}
	0.2	3.71×10^{-7}	2.54×10^{-6}	4.16×10^{-5}	4.16×10^{-5}	7.78×10^{-6}
	0.3	5.58×10^{-7}	4.73×10^{-6}	6.05×10^{-5}	6.05×10^{-5}	1.45×10^{-5}
	0.4	7.46×10^{-7}	5.79×10^{-6}	7.83×10^{-5}	7.83×10^{-5}	4.57×10^{-5}
	0.5	9.35×10^{-7}	4.94×10^{-6}	9.50×10^{-5}	9.50×10^{-5}	8.29×10^{-5}
-10	0.1	9.47×10^{-8}	6.94×10^{-5}	1.79×10^{-3}	1.79×10^{-3}	7.40×10^{-4}
	0.2	1.90×10^{-7}	1.95×10^{-4}	3.49×10^{-3}	3.49×10^{-3}	9.52×10^{-4}
	0.3	2.87×10^{-7}	3.43×10^{-4}	5.09×10^{-3}	5.09×10^{-3}	2.58×10^{-3}
	0.4	3.84×10^{-7}	4.61×10^{-4}	6.59×10^{-3}	6.59×10^{-3}	5.45×10^{-3}
	0.5	4.83×10^{-7}	8.52×10^{-4}	8.00×10^{-3}	8.00×10^{-3}	8.04×10^{-3}
0	0.1	4.74×10^{-10}	2.42×10^{-5}	6.50×10^{-4}	6.50×10^{-4}	1.58×10^{-4}
	0.2	1.90×10^{-9}	5.25×10^{-5}	2.59×10^{-3}	2.59×10^{-3}	3.85×10^{-4}
	0.3	4.27×10^{-9}	8.26×10^{-5}	5.83×10^{-3}	5.83×10^{-3}	6.02×10^{-4}
	0.4	7.59×10^{-9}	3.33×10^{-4}	1.03×10^{-2}	1.03×10^{-2}	9.45×10^{-4}
	0.5	1.19×10^{-8}	6.77×10^{-4}	1.60×10^{-2}	1.60×10^{-2}	3.49×10^{-3}
10	0.1	9.38×10^{-8}	8.96×10^{-6}	1.90×10^{-3}	1.90×10^{-3}	2.04×10^{-4}
	0.2	1.87×10^{-7}	2.80×10^{-5}	3.92×10^{-3}	3.92×10^{-3}	4.24×10^{-4}
	0.3	2.79×10^{-7}	6.44×10^{-5}	6.06×10^{-3}	6.06×10^{-3}	6.34×10^{-4}
	0.4	3.70×10^{-7}	7.06×10^{-5}	8.33×10^{-3}	8.33×10^{-3}	9.00×10^{-4}
	0.5	4.60×10^{-7}	1.72×10^{-4}	1.07×10^{-2}	1.07×10^{-2}	2.56×10^{-3}
20	0.1	1.84×10^{-7}	1.31×10^{-7}	2.28×10^{-5}	2.28×10^{-5}	5.23×10^{-6}
	0.2	3.68×10^{-7}	2.99×10^{-7}	4.70×10^{-5}	4.70×10^{-5}	8.98×10^{-6}
	0.3	5.50×10^{-7}	2.82×10^{-6}	7.28×10^{-5}	7.28×10^{-5}	3.56×10^{-5}
	0.4	7.32×10^{-7}	7.15×10^{-6}	1.00×10^{-4}	1.00×10^{-4}	6.13×10^{-5}
	0.5	9.13×10^{-7}	8.95×10^{-6}	1.29×10^{-4}	1.29×10^{-4}	8.95×10^{-5}

tions using the C-FD. The proposed approach integrates the RPSM with the Laplace transform. A key advantage of the L-RPS method is its ability to generate infinite series solutions with minimal iterations by utilizing the concept of an infinite limit. This feature makes it computationally efficient, as it does not require perturbation, linearization, discretization, or other restrictive conditions. We evaluated the method through three applications to highlight its simplicity, adaptability, and efficacy. The solutions, computed using Mathematica 13, were examined through tables, 2D and 3D graphs, and compared with exact solutions and results from alternative methods. The findings confirm that the proposed approach is accurate, versatile, and efficient, while also emphasizing the impact of the fractional derivative on the obtained solutions. Based on these results, we conclude that the L-RPSM is an effective tool for solving a variety of PDEs and FPDEs.

Future research will aim to generalize this method by combining the Laplace transform with other generalized integral transforms, enhancing its applicability and efficiency for solving a broader range of fractional equations.

Conflict of interest statement. This work does not have any conflicts of interest.

Data availability statement. No datasets were generated or analyzed during the current study.

References

[1] T. Abdeljawad, M. Sher, K. Shah, M. Sarwar, I. Amacha, M. Alqudah and A. Al-Jaser, *Analysis of a class of fractal hybrid fractional differential equation with application to a biological model*, Scientific Reports, 2024, 14(1), 18937.

[2] A. A. Alderremy, R. Shah, N. Iqbal, S. Aly and K. Nonlaopon, *Fractional series solution construction for nonlinear fractional reaction-diffusion Brusselator model utilizing Laplace residual power series*, Symmetry, 2022, 14(9), 1944. DOI: 10.3390/sym14091944.

[3] M. Alquran and I. Jaradat, *A novel scheme for solving Caputo time-fractional nonlinear equations: Theory and application*, Nonlinear Dynamics, 2018, 91, 2389–2395.

[4] M. Alquran and I. Jaradat, *A novel scheme for solving Caputo time-fractional nonlinear equations: Theory and application*, Nonlinear Dynamics, 2018, 91, 2389–2395.

[5] T. M. Atanacković, S. Pilipović and D. Zorica, *Properties of the Caputo-Fabrizio fractional derivative and its distributional settings*, Fractional Calculus and Applied Analysis, 2018, 21(1), 29–44.

[6] F. M. Atici, S. Chang and J. Jonnalagadda, *Grünwald-Letnikov fractional operators: From past to present*, Fract. Differ. Calc., 2021, 11(1), 147–159.

[7] O. A. Arqub, *Series solution of fuzzy differential equations under strongly generalized differentiability*, J. Adv. Res. Appl. Math., 2013, 5(1), 31–52.

[8] G. Barbero, L. R. Evangelista, R. S. Zola, E. K. Lenzi and A. M. Scarfone, *A brief review of fractional calculus as a tool for applications in physics: Adsorption phenomena and electrical impedance in complex fluids*, Fractal and Fractional, 2024, 8(7), 369.

[9] M. Benbachir and Z. Dahmani, *The ADM method for solving the modified kawahara equation with fractional spatial and temporal derivatives*, Journal of Interdisciplinary Mathematics, 2011, 14(5–6), 523–533.

[10] S. Bhatter, A. Mathur, D. Kumar, K. S. Nisar and J. Singh, *Fractional modified Kawahara equation with Mittag–Leffler law*, Chaos, Solitons & Fractals, 2020, 131, 109508.

[11] Y. H. Cheow, K. H. Ng, C. Phang and K. H. Ng, *The application of fractional calculus in economic growth modelling: An approach based on regression analysis*, Heliyon, 2024, 10(15).

[12] *Comparative analysis of the time-fractional Black–Scholes Option Pricing Equations (BSOPE) by the Laplace Residual Power Series Method (LRPSM).*

[13] S. A. Elsamani, A. A. Mohmmed, K. M. Kogan, S. Y. M. Salih and I. A. Ahmed, *Fractional calculus applications in engineering insights into four-dimensional chaotic systems*, Journal of Theory, Mathematics and Physics, 2024, 3(5), 75–84.

[14] E. M. Elsayed, R. Shah and K. Nonlaopon, *The analysis of the fractional-order Navier-Stokes equations by a novel approach*, Journal of Function Spaces, 2022, 8979447.

[15] T. Eriqat, A. El-Ajou, N. O. Moa'ath, Z. Al-Zhour and S. Momani, *A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations*, Chaos, Solitons & Fractals, 2020, 138, 109957.

[16] *Exact and approximate solutions for linear and nonlinear partial differential equations via Laplace residual power series method.*

[17] L. Jin, *Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation*, Mathematical and Computer Modelling, 2009, 49(3–4), 573–578.

[18] R. Kamocki, *A new representation formula for the Hilfer fractional derivative and its application*, Journal of Computational and Applied Mathematics, 2016, 308, 39–45.

[19] L. Kaur and R. K. Gupta, *Kawahara equation and modified Kawahara equation with time dependent coefficients: Symmetry analysis and generalized-expansion method*, Mathematical Methods in the Applied Sciences, 2013, 36(5), 584–600.

[20] S. Khirsariya, S. Rao and J. Chauhan, *Solution of fractional modified Kawahara equation: A semi-analytic approach*, Mathematics in Applied Sciences and Engineering, 2023, 4(4), 264–284.

[21] J. Lu, *Analytical approach to Kawahara equation using variational iteration method and homotopy perturbation method*, Topol. Methods Nonlinear Anal., 2008, 31(2), 287–293.

[22] X. Luo and M. Nadeem, *Laplace residual power series method for the numerical solution of time-fractional Newell–Whitehead–Segel model*, International Journal of Numerical Methods for Heat & Fluid Flow, 2023, 33(7), 2377–2391.

[23] E. Viera-Martin, J. F. Gómez-Aguilar, J. E. Solís-Pérez, J. A. Hernández-Pérez and R. F. Escobar-Jiménez, *Artificial neural networks: A practical review of applications involving fractional calculus*, The European Physical Journal Special Topics, 2022, 231(10), 2059–2095.

[24] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc., New York, 1993.

[25] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

[26] M. Al-Refai, *On weighted Atangana–Baleanu fractional operators*, Advances in Difference Equations, 2020, 2020(1), 3.

[27] M. Mossa Al-Sawalha, R. Shah, A. Khan, O. Y. Ababneh and T. Botmart, *Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives*, AIMS Mathematics, 2022, 7(10), 18334–18359.

[28] A. Shafee, Y. Alkhezi and R. Shah, *Efficient solution of fractional system partial differential equations using Laplace residual power series method*, Fractal and Fractional, 2023, 7(6), 429.

[29] *Using Laplace residual power series method in solving coupled fractional Neutron diffusion equations with delayed Neutrons system.*

[30] A. M. Wazwaz, *New solitary wave solutions to the modified Kawahara equation*, Physics Letters A, 2007, 360(4–5), 588–592.

[31] J. Zhang and X. Tian, *Laplace-residual power series method for solving fractional generalized long wave equations*, Ocean Engineering, 2024, 310, 118693.

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