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A NEW APPROACH FOR LINEAR SYSTEMS OF THE FORM $(A + \gamma UU^T)X = B$

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Abstract In this paper, we consider the numerical solution of linear systems of the form $(A + \gamma UU^T)x = b$. A new approach for this linear systems is proposed: by transforming this linear systems into the equivalent saddle point problem, we propose a new and effective preconditioner for this equivalent saddle point form and discuss the spectral properties of the corresponding preconditioned matrix. When dealing with the associated residual equations, compared with some existing preconditioners, the proposed precondnitioner can save computational workload, running time and computer memory in actual implementations. To illustrate the performance of the proposed preconditioner, some examples from different application cases are provided. Moreover, compared with some existing preconditioning strategies, the numerical results show that the proposed preconditioner is more competitive in a way.

Keywords Linear systems, saddle point problem, preconditioner, Krylov subspace method.

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1. Introduction

Consider the numerical solution of linear systems of the form

$$(A + \gamma UU^T)x = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is positive semidefinite (its symmetric part $\frac{1}{2}(A + A^T)$ is positive semidefinite), $U \in \mathbb{R}^{n\times m}$ with $m < n$, $\gamma > 0$, $b \in \mathbb{R}^n$ is given, see [4]. We assume that A and U satisfy $\mathrm{Ker}(A) \cap \mathrm{Ker}(U^T) = \{0\}$ such that the linear systems (1.1) has a unique solution for any vector b.

In scientific computing and engineering applications, we often need to deal with the linear systems (1.1), including the PDE-related saddle point problems [2, 5, 8], the sparse-dense least squares problems [16–18], the discrete operators on De Rham Complex [12,13], the KKT systems in constrained optimization [15], elliptic PDEs with non-local boundary conditions [11], and so on.

Whereas, for solving the linear systems (1.1), there exist only few numerical methods. For example, assume that A is solved efficiently, strategies on the base of the Sherman-Morrion-Woodbury (SMW) matrix identity can be adopted in [9]. Lu [12, 13] makes use of an auxiliary

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space preconditioning technique to address the solution of linear systems closely related to (1.1). In particular, in [4], Benzi and Faccio consider this case that $A + \gamma UU^T$ is nonsingular with matrix $A + A^T$ being positive semidefinite, and design the following alternating iteration method:

$$\begin{cases}(\alpha I + A)x^{k+\frac{1}{2}} = (\alpha I - \gamma UU^T)x^k + b, \\ (\alpha I + \gamma UU^T)x^{k+1} = (\alpha I - A)x^k + b,\end{cases} \quad k = 0, 1, \ldots . \tag{1.2}$$

They show that the alternating iteration method (1.2) is uncondition convergent for the positive definite $A + A^T$, incidentally, and propose the following preconditioner

$$P_\alpha = \frac{1}{2\alpha}(\alpha I + A)(\alpha I + \gamma UU^T) \tag{1.3}$$

and investigate its some derivative versions, too. Some numerical experiments are reported to illustrate the performance of the proposed preconditioning strategies.

By studying three main theoretical results in [4], i.e., Theorem 3.1, Theorem 5.1 and Theorem 5.5, the previous two are assumption that $A + A^T$ is positive definite; the last one is a hypothesis that A is symmetric positive definite (SPD). These assumptions are reasonable because these two cases often happen. For example, the former A is positive definite (its $A + A^T$ is SPD) from the Oseen problem in IFISS in [7]; the latter A is SPD from the Stokes problem by IFISS in [7] or the KKT systems in constrained optimization [14].

For the preconditioner P_α in (1.3), when it combines with the Krylov subspace methods (like GMRES), per step, one requires handling two linear residual equations with coefficient matrices $\alpha I + A$ and $\alpha I + \gamma UU^T$. For these two linear residual equations, we need to take appropriate measures according to the specific situation. For the linear residual equations with $\alpha I + A$, when $\alpha I + A$ is solved easily, exact solves can be advisable; when $\alpha I + A$ is solved difficulty, we obtain its inexact solves, its common means are to apply the precondition technique for it. For the linear residual equations with $\alpha I + \gamma UU^T$, the matrix $\alpha I + \gamma UU^T$ is SPD, and can be performed by the SMW formula or possibly by a suitable inner PCG iteration or maybe an (algebraic) multigrid method, one see [4] for more details. Exploiting the cheap and efficient preconditioner for solving the linear systems (1.1) is popular, and one of efficient approaches. Hence, designing the new, cheap and efficient preconditioner, together with the Krylov subspace methods, for solving the linear systems (1.1) is the main motivation for us. For this goal, we adopt a new approach for the linear systems (1.1). That is, by transforming the linear systems (1.1) into the equivalent saddle point problem, we propose a new, cheap and effective preconditioner for this equivalent saddle point form, and present some spectral properties of the corresponding preconditioned matrix. Although the order of system becomes large, when dealing with the associated residual equations, compared with some existing preconditioners involved in [4], the proposed precondnitioner can save computational workload, running time and computer memory in actual implementations. Further, to illustrate the performance of the proposed preconditioner, some examples from different application cases are provided. Moreover, compared with some existing preconditioning strategies, the numerical results show that the proposed preconditioner is more competitive in a way.

The remainder of the present paper is organized as follows. In Section 2, we present a new approach for solving the linear systems (1.1). In Section 3, some numerical experiments are provided. Finally, in Section 4, we use some conclusions to end up with this paper.

2. A new approach

In this section, we will adopt a new approach for solving the linear systems (1.1). That is, by expressing the linear systems (1.1) as the equivalent saddle point problem, we propose a new, cheap and effective preconditioner for this equivalent saddle point form.

It is easy to find that the linear systems (1.1) is equivalent to the following saddle point problem

$$\mathcal{A}\mathbf{x} :\equiv \begin{pmatrix} A & \beta U \\ -\beta U^T & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \equiv \mathbf{b} \tag{2.1}$$

with $\beta = \sqrt{\gamma}$. In the sequel, as the same as the assumption in [4], we always assume that A is positive definite or SPD.

For the saddle point problem (2.1), one easily considers the following two splittings

$$\begin{pmatrix} A & \beta U \\ -\beta U^T & I \end{pmatrix} = \begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I + I \end{pmatrix} - \begin{pmatrix} \alpha I & -\beta U \\ \beta U^T & \alpha I \end{pmatrix} \tag{2.2}$$

and

$$\begin{pmatrix} A & \beta U \\ -\beta U^T & I \end{pmatrix} = \begin{pmatrix} \alpha I & \beta U \\ -\beta U^T & \alpha I \end{pmatrix} - \begin{pmatrix} \alpha I - A & 0 \\ 0 & \alpha I - I \end{pmatrix}, \tag{2.3}$$

and establishes the alternating iteration similar to the alternating iteration method (1.2). Moreover, under the same condition, i.e., A is positive definite, the corresponding iteration method induced by the matrix splittings (2.2) and (2.3) is convergent. Further, the building preconditioner induced by the matrix splittings (2.2) and (2.3) is designed, i.e.,

$$P_{\alpha\beta} = \frac{1}{2\alpha} \begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I + I \end{pmatrix} \begin{pmatrix} \alpha I & \beta U \\ -\beta U^T & \alpha I \end{pmatrix}.$$

Like the spectrum of matrix $P_{\alpha}^{-1}A_{\gamma}$, where $A_{\gamma} = A + \gamma UU^T$, the spectrum of matrix $P_{\alpha\beta}^{-1}\mathcal{A}$ lies in the disk of center $(1, 0)$ and radius 1 in the complex plane for all $\alpha > 0$. In particular, when A is SPD, all the eigenvalues of $P_{\alpha\beta}^{-1}\mathcal{A}$ are real for $1 < \alpha < \lambda_{\min}$ or $\lambda_{\max} < \alpha < 1$, where $\lambda_{\min}$ and $\lambda_{\max}$ in order denote the smallest and largest eigenvalues of A, see Theorem 2.1 in [6].

To accelerate Krylov methods like GMRES with the preconditioners P_{α} or $P_{\alpha\beta}$, we solve the residual equation $P_{\alpha}z = r$ in the following two steps (note that the factor $\frac{1}{2\alpha}$ is omitted since it does not change the preconditioned system):

(1) solve $(A + \alpha I)v = r$;

(2) solve $(\gamma UU^T + \alpha I)z = v$,

and the residual equation $P_{\alpha\beta}z = r$ can be solved by the following two steps:

(1) solve $\begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I + I \end{pmatrix} v = r$;

(2) solve $\begin{pmatrix} \alpha I & \beta U \\ -\beta U^T & \alpha I \end{pmatrix} z = v.$

By investigating the solution procedures of these two residual equations, obviously, the preconditioner P_α with Krylov method overmatches the preconditioner $P_{\alpha\beta}$ with Krylov method. This implies that the building preconditioner $P_{\alpha\beta}$ induced by the matrix splittings (2.2) and (2.3) is not competitive, compared with the preconditioner P_α. This is because the order of the coefficient matrix of the residual equation $P_{\alpha\beta} z = r$ is not only increased, but also its structure is more complicated.

As is known, any matrix splitting not only can automatically lead to a splitting iteration method, but also can naturally induce a splitting preconditioner for the Krylov subspace methods. Hence, to design what we want, cheap and efficient preconditioner, here, we consider the following special block triangular splitting

$$\begin{pmatrix} A & \beta U \\ -\beta U^T & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -\beta U^T & I \end{pmatrix} - \begin{pmatrix} 0 & -\beta U \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

Based on (2.4), naturally, we can design the following iteration method for solving the saddle point problem (2.1).

Block triangular splitting (BTS) method: Given an initial guess $(x^0, y^0)^T$, for $k = 0, 1, 2, \ldots$, until $(x^k, y^k)^T$ converges, compute

$$\begin{pmatrix} A & 0 \\ -\beta U^T & I \end{pmatrix} \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & -\beta U \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{2.5}$$

By the simple computations, it is easy to obtain that the BTS method is convergent when $\rho(U^T A^{-1} U) < \frac{1}{\gamma}$. Not only that, a new preconditioner induced by the BTS method can be defined by

$$P_\beta = \begin{pmatrix} A & 0 \\ -\beta U^T & I \end{pmatrix}.$$

When applying this new preconditioner P_β to improve the convergence speed of GMRES, the residual equation $P_\beta z = r$, i.e.,

$$\begin{pmatrix} A & 0 \\ -\beta U^T & I \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

can be solved by the following two steps:

(1) solve $Az_1 = r_1$;

(2) solve $z_2 = r_2 + \beta U^T z_1$.

From solving the corresponding residual equations for three preconditioners P_α, $P_{\alpha\beta}$ and P_β, obviously, this new preconditioner P_β only involves the computation of the inverse of the matrix A. This implies that the new preconditioner P_β is superior to the preconditioners P_α

and $P_{\alpha\beta}$ from the view of computational workload, running time and computer memory. In the following, we turn to study some properties of the preconditioned matrix $P_\beta^{-1}\mathcal{A}$. By the simple computations,

$$P_\beta^{-1}\mathcal{A} = \begin{pmatrix} A & 0 \\ -\beta U^T & I \end{pmatrix}^{-1} \begin{pmatrix} A & \beta U \\ -\beta U^T & I \end{pmatrix}$$
$$= \begin{pmatrix} A^{-1} & 0 \\ \beta U^T A^{-1} & I \end{pmatrix} \begin{pmatrix} A & \beta U \\ -\beta U^T & I \end{pmatrix}$$
$$= \begin{pmatrix} I & \beta A^{-1}U \\ 0 & I + \gamma U^T A^{-1} U \end{pmatrix}, \tag{2.6}$$

from which we have the following result, see Lemma 2.1.

Lemma 2.1. *Let A satisfy that $H = \frac{1}{2}(A + A^T)$ is positive definite. Then the preconditioned matrix $P_\beta^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least n, the remaining eigenvalues are the eigenvalues of the matrix $I + \gamma U^T A^{-1} U$.*

To further discuss the distribution of eigenvalues of the matrix $I + \gamma U^T A^{-1} U$, Lemma 2.2 is recalled.

Lemma 2.2. [10] *Suppose that $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{n\times m}$ with $m \leq n$.Then NM has the same eigenvalues as MN, counting multiplicity, together with an additional $n - m$ eigenvalues equal to 0.*

Based on Lemma 2.2, matrix $U^T A^{-1} U$ and $A^{-1} U U^T$ have the same non-zero eigenvalues. By means of this, now we turn our attention to some properties of the matrix $A^{-1}UU^T$.

If (μ, x) with $\|x\|_2 = 1$ is a real eigenpair of the matrix $A^{-1}UU^T$, then we have

$$A^{-1}UU^T x = \mu x,$$

which is expressed as

$$UU^T x = \mu A x. \tag{2.7}$$

Premultiplying (2.7) by x^*, together with $\|x\|_2 = 1$, we have

$$\mu = \frac{x^* U U^T x}{x^* A x}. \tag{2.8}$$

Noting that when μ is real, x can be taken real and x^* becomes x^T. Hence, from (2.8), it is easy to find that μ satisfies

$$0 \leq \mu \leq \frac{\|U\|_2^2}{\lambda_{\min}(H)}.$$

Hence, we have

Theorem 2.1. *Let the conditions of Lemma 2.1 be satisfied. If (μ, x) with $\|x\|_2 = 1$ is a real eigenpair of the matrix $A^{-1}UU^T$, then all the real eigenvalues of the matrix $I + \gamma U^T A^{-1} U$ lie in*

$$[1, 1 + \gamma \|U\|_2^2 \lambda_{\min}^{-1}(H)].$$

In addition, the real and imaginary parts of all the complex eigenvalues of the matrix $I + \gamma U^T A^{-1} U$*, respectively, satisfy*

$$1 \le \Re(\lambda(I + \gamma U^T A^{-1} U)) \le 1 + \gamma \|U\|_2^2 \theta^{-1}$$

and

$$|\Im(\lambda(I + \gamma U^T A^{-1} U)| \le \gamma \|U\|_2^2 \theta^{-2} \tau,$$

with $\Re(\lambda(A)) \ge \theta$ *and* $|\Im(\lambda(A))| \le \tau$*, where* $\lambda(\cdot)$ *represents the eigenvalue of matrix,* $\Re(\cdot)$ *and* $\Im(\cdot)$ *in order represent the real and the imaginary parts of eigenvalue.*

Proof. Now, we only remain to show that the distribution of the complex eigenvalues holds.

Let $\nu = x^* U U^T x$ and $x^* A x = \varepsilon + i\upsilon$ with $\varepsilon > 0$. From (2.8), we have

$$\mu = \frac{x^* U U^T x}{\varepsilon + i\upsilon} = \frac{\nu}{\varepsilon + i\upsilon} = \frac{\nu(\varepsilon - i\upsilon)}{\varepsilon^2 + \upsilon^2}.$$

Denote

$$\Re(\mu) = \frac{\nu\varepsilon}{\varepsilon^2 + \upsilon^2} \text{ and } \Im(\mu) = \frac{-\nu\upsilon}{\varepsilon^2 + \upsilon^2}.$$

Together with $0 \le \nu \le \|U\|_2^2$, we easily obtain that

$$0 \le \Re(\mu) \le \|U\|_2^2 \varepsilon^{-1} = \|U\|_2^2 \theta^{-1}$$

and

$$|\Im(\mu)| \le \frac{\nu|\upsilon|}{\varepsilon^2 + \upsilon^2} \le \frac{\nu|\upsilon|}{\varepsilon^2} \le \|U\|_2^2 \theta^{-2} \tau.$$

This completes the proof. □

Based on the results in Lemma 2.1 and Theorem 2.1, it is easy to find that when $\gamma \to 0$, all the eigenvalues of $P_\beta^{-1}\mathcal{A}$ are very close to 1. This is to be expected because the more the spectrum of the preconditioned matrix (around 1) is aggregated, the faster the GMRES method is. Beyond this, there is another situation that $(\eta, [x, y])$ is a eigenpair of $P_\beta^{-1}\mathcal{A}$ with $y \ne 0 \in \mathrm{Ker}(U)$. In such case, *1 is the eigenvalue of* $P_\beta^{-1}\mathcal{A}$ *with algebraic multiplicities* $n + k$*, where* k *denotes the dimension of Ker*(U). In fact, based on this assumption and (2.6), we have

$$\begin{pmatrix} I & \beta A^{-1} U \\ 0 & I + \gamma U^T A^{-1} U \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \eta \begin{pmatrix} x \\ y \end{pmatrix},$$

from which we have

$$\begin{cases} x = \eta x, \\ y = \eta y. \end{cases}$$

Together with Lemma 2.1, this result is easy to obtain.

Now, we recall the preconditioner P_α for the linear systems (1.1). In theory, Theorem 5.1 in [4] describes the distribution of the eigenvalues of the preconditioned matrix $P_\alpha^{-1} A_\gamma$, see Theorem 2.2.

Theorem 2.2. (see Theorem 5.1 in [4]) *Let A be such that $A + A^T$ is positive definite. Assume that $\|A\|_2 = \|U\|_2 = 1$. Let P_α be given by (1.3). If (λ, x) is a real eigenpair of the preconditioned matrix $P_\alpha^{-1}A_\gamma$, with $\|x\|_2 = 1$, then $\lambda \in [\nu, 2)$ where*

$$\nu = \frac{\alpha\lambda_{\min}(A + A^T)}{(1+\alpha)(\alpha+\gamma)}.$$

If (ϱ, x) is an eigenpair of A with $x \in \mathrm{Ker}(U^T)$, then x is eigenvector of $P_\alpha^{-1}A_\gamma$ associated to the eigenvalue

$$\lambda = \frac{2\varrho}{\alpha+\varrho}$$

(independent of γ).

By investigating the distribution of the eigenvalues of the preconditioned matrices $P_\beta^{-1}\mathcal{A}$ and $P_\alpha^{-1}A_\gamma$ (i.e., see Lemma 2.1 and Theorem 2.2), it is not difficult to find that the eigenvalues of the former are more aggregation than that of the latter, even if the lower bound in Theorem 2.2 (Theorem 5.1 in [4]) is maximized by taking $\alpha = \sqrt{\gamma}$. From the view of the aggregation of eigenvalues, it also implies that the former with GMRES may be more efficient than the latter with GMRES.

When A is SPD, in [4], Benzi and Faccio also investigate the following preconditioner

$$P_\alpha^s = \frac{1}{2\alpha}L(\alpha I + \gamma UU^T)L^T, \tag{2.9}$$

where L is the Cholesky factor of $\alpha I + A$, and provide the following result:

Theorem 2.3. (see Theorem 5.5 in [4]) *Let A be SPD, $A_\gamma = A + \gamma UU^T$, with $\|A\|_2 = \|U\|_2 = 1$. Let P_α^s be given by (2.9). Then the eigenvalue λ of the preconditioned matrix $(P_\alpha^s)^{-1}A_\gamma$ is real and lies in the interval*

$$\frac{2\alpha\lambda_{\min}(A)}{(1+\alpha)(\gamma+\alpha)} < \lambda < \frac{2(1+\gamma)}{\lambda_{\min}(A)+\alpha}.$$

For the new preconditioner P_β, when A is SPD, all the eigenvalues of the preconditioned matrix $P_\beta^{-1}\mathcal{A}$ are real as well. Concretely, Corollary 2.1 can be provided.

Corollary 2.1. *Let A be SPD. Then the preconditioned matrix $P_\beta^{-1}\mathcal{A}$ has an eigenvalue 1 with multiplicity at least n, the remaining eigenvalues are $1 + \gamma\sigma_i^2$ $(1 \le i \le m)$, where σ_i is the positive singular value of the matrix $U^TA^{-\frac{1}{2}}$.*

Proof. Let

$$S = \begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix}.$$

Then the preconditioned matrix $P_\beta^{-1}\mathcal{A}$ similar to the matrix $\bar{S}$ with

$$\bar{S} = SP_\beta^{-1}\mathcal{A}S^{-1} = \begin{pmatrix} I & \beta A^{-\frac{1}{2}}UA^{-\frac{1}{2}} \\ 0 & I + \gamma U^TA^{-1}U \end{pmatrix}.$$

Let

$$U^TA^{-\frac{1}{2}} = P^T\hat{\Sigma}Q = P^T[\Sigma\ 0]Q$$

be the singular value decomposition of the matrix $U^T A^{-\frac{1}{2}}$, where $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are orthogonal matrices and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}^{m\times m}$ is a diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m > 0$ being the nonzero singular values of $U^T A^{-\frac{1}{2}}$. Then $\bar{S}$ can be rewritten as

$$\bar{S} = \begin{pmatrix} I & \beta(Q[\Sigma\ 0]^T P^T)A^{-\frac{1}{2}} \\ 0 & I + \gamma P^T [\Sigma\ 0] Q Q^T [\Sigma\ 0]^T P \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 & \beta Q \Sigma P^T A^{-\frac{1}{2}} \\ 0 & I & 0 \\ 0 & 0 & I + \gamma P^T \Sigma^2 P \end{pmatrix}.$$

Define

$$F = \begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}.$$

Since F is an orthogonal matrix, it holds that

$$F\bar{S}F^T = \hat{S} = \begin{pmatrix} I & 0 & \beta Q \Sigma P^T A^{-\frac{1}{2}} \\ 0 & I & 0 \\ 0 & 0 & I + \gamma \Sigma^2 \end{pmatrix}.$$

Noting that $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) \in \mathbb{R}^{m\times m}$ is a diagonal matrix, it is easy to check that the matrix $\hat{S}$ has an eigenvalue 1 with multiplicity n, the remaining eigenvalues are of the form $1 + \gamma\sigma_i^2$ $(1 \leq i \leq m)$. □

Compared Theorem 2.3 with Corollary 2.1, obviously, we can draw the same conclusion as before that for the symmetric positive definite A, the eigenvalues of the preconditioned matrix $P_\beta^{-1}\mathcal{A}$ are more aggregation than that of the preconditioned matrix $(P_\alpha^s)^{-1}A_\gamma$. In addition, when we deal with the corresponding residual equations of both, for P_α^s, it needs to solve three subsystems even if the involved matrix L is the Cholesky factor of $\alpha I + A$. For the new preconditioner P_β, it only needs to solve two subsystems, and only involve the computations of the inverse of the matrix A. From the point of this, the new preconditioner P_β is more efficient than the preconditioner P_α^s as well.

In addition, in [4], numerical results show that P_α overmatches P_α^s by means of the iteration counts and timings.

Finally, we mention the following two preconditioners:

$$P_S = \begin{pmatrix} A & 0 \\ -\beta U^T & S \end{pmatrix}, P_{-S} = \begin{pmatrix} A & 0 \\ -\beta U^T & -S \end{pmatrix},$$

where $S = I + \gamma U^T A^{-1} U$. By the simple computations, we have

$$P_S^{-1}\mathcal{A} = \begin{pmatrix} I & \beta A^{-1}U \\ & I \end{pmatrix}, P_{-S}^{-1}\mathcal{A} = \begin{pmatrix} I & \beta A^{-1}U \\ & -I \end{pmatrix}.$$

This shows that $P_S^{-1}\mathcal{A}$ has one eigenvalue: 1, of multiplicity $n+m$; $P_{-S}^{-1}\mathcal{A}$ has precisely two eigenvalue: 1, of multiplicity n, and -1, of multiplicity m. In theory, for these two precondidtioners P_S and P_{-S}, any Krylov subspace method with optimality and Galerkin property terminates in at most two steps if roundoff errors are ignored. Whereas, in practice, P_S and P_{-S} are unpopular because their corresponding residual equations with P_S and P_{-S} are more complicated, compared with the preconditioner P_β.

3. Numerical experiments

In this section, we consider two aspects: one is to investigate the distribution of the eigenvalues of the preconditioned matrices $P_\beta^{-1}\mathcal{A}$ and $P_\alpha^{-1}A_\gamma$ from three application areas to further compare our theoretical results (Lemma 2.1 and Theorem 2.1) with Theorem 2.2 (Theorem 5.1 [4]); the other is to compare the performance of GMRES method with the preconditioners P_β and P_α on a selection of test problems by means of the iteration counts, CPU times and relative residual errors. All the computations are done with MATLAB R2016b on a Lenovo PC (Intel(R) Core(TM)i7-10700 CPU @ 2.90GHz, 16.00 GB of RAM).

3.1. Spectral distribution

As is known, the spectral properties of the preconditioned matrix normally give important insight in the convergence behavior of the preconditioned Krylov subspace methods. Based on this, in this subsection, we investigate the distribution of the eigenvalues of the preconditioned matrices $P_\beta^{-1}\mathcal{A}$ and $P_\alpha^{-1}A_\gamma$ from three application areas. Specifically, see Examples 3.1, 3.2 and 3.3. In our computations, to investigate Theorem 2.2 (Theorem 5.1 [4]), A and U in Examples 3.1, 3.2 and 3.3 have been normalized so that $\|A\|_2 = \|U\|_2 = 1$, and we take $\alpha = \sqrt{\gamma}$ to obtain the maximum lower bound of the real eigenvalues for $P_\alpha^{-1}A_\gamma$.

Example 3.1. Consider the linear systems (1.1), in which

$$A = \begin{pmatrix} I\otimes T + T\otimes I & 0 \\ 0 & I\otimes T + T\otimes I \end{pmatrix} \in \mathbb{R}^{2p^2\times 2p^2}, U = \begin{pmatrix} I\otimes F \\ F\otimes I \end{pmatrix} \in \mathbb{R}^{2p^2\times p^2},$$

and

$$T = \text{tridiag}(-1,2,-1) \in \mathbb{R}^{p\times p}, \quad F = \text{tridiag}(0,1,-1) \in \mathbb{R}^{p\times p},$$

where $\otimes$ stands for the Kronecker product symbol. Example 3.1 is from the constrained quadratic programming problems of the form

$$\min f(x) = c^T x + \frac{1}{2}x^T A x$$

and

$$\text{s.t. } Ux \geq d, x \geq 0,$$

where A is SPD and U is full coloum rank, see [1] for more details. Obviously, $n = 2p^2$ and $m = p^2$.

For the convenience of calculations, we take $p = 32$, and consider three cases: $\gamma = 0.001$, $\gamma = 0.005$ and $\gamma = 0.01$. Based on this, Table 1 lists the minimum and maximum real parts of

Table 1. Results of eigenvalues of $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for Example 3.1 with $\gamma = 0.001$, $\gamma = 0.005$ and $\gamma = 0.01$.

γ	$P_\alpha^{-1}A_\gamma$		$P_\beta^{-1}\mathcal{A}$	
	$\max \Re(P_\alpha^{-1}A_\gamma)$	$\min \Re(P_\alpha^{-1}A_\gamma)$	$\min \Re(P_\beta^{-1}\mathcal{A})$	$\max \Re(P_\beta^{-1}\mathcal{A})$
0.001	1.9387	0.1340	1	1.0010
0.005	1.8679	0.0623	1	1.0050
0.01	1.8182	0.0446	1	1.0100

the preconditioned matrices $P_\beta^{-1}\mathcal{A}$ and $P_\alpha^{-1}A_\gamma$. Figure 1 plots the distribution of the eigenvalues of the preconditioned matrices $P_\beta^{-1}\mathcal{A}$ and $P_\alpha^{-1}A_\gamma$.

Example 3.2. We make use of the driver stokes_testproblem of IFISS [7] package to discretize a stationary Stokes problem with the default stabilization parameter (1/4) and Q1-P0 mixed finite elements on the uniform grid of square elements (leaky-lid driven cavity), in such case, we get that matrix 'Ast' is symmetric positive definite and the matrix 'Bst' produced by this software is rank deficient. In our computations, we take $A = \text{Ast}$ and $U = \text{Bst}^T$.

For Example 3.2, we take 32×32 uniform grids and consider three cases for the choice of γ: $\gamma = 0.1$, $\gamma = 1$ and $\gamma = 10$. Based on this, Table 2 lists the minimum and maximum real parts of the preconditioned matrices $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$. Figure 2 plots the distribution of the eigenvalues of the preconditioned matrices $P_\beta^{-1}\mathcal{A}$ and $P_\alpha^{-1}A_\gamma$.

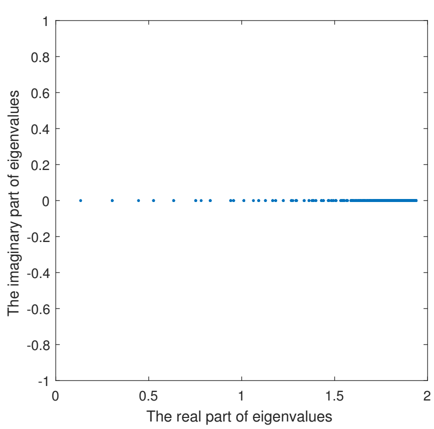
Table 2. Results of eigenvalues of $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for Example 3.2 with $\gamma = 0.1$, $\gamma = 1$ and $\gamma = 10$.

γ	$P_\alpha^{-1}A_\gamma$		$P_\beta^{-1}\mathcal{A}$	
	$\max \Re(P_\alpha^{-1}A_\gamma)$	$\min \Re(P_\alpha^{-1}A_\gamma)$	$\min \Re(P_\beta^{-1}\mathcal{A})$	$\max \Re(P_\beta^{-1}\mathcal{A})$
0.1	1.8531	0.0081	1	1.0004
1	1.5990	0.0144	1	1.0039
10	1.1155	0.0110	1	1.0390

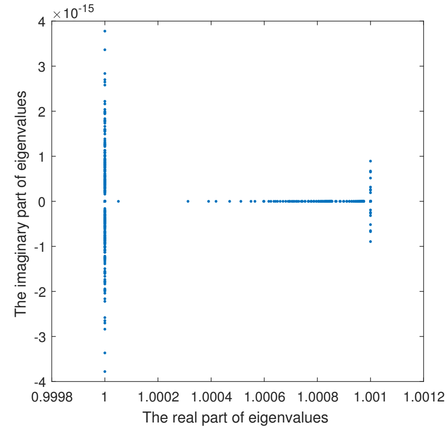
Example 3.3. Similar to Example 3.2, we make use of the driver navier_testproblem of IFISS [7] package with default viscosity parameter (1/100) and Q2-Q1 mixed finite elements on the uniform grid of square elements for the stationary Oseen problem (leaky-lid driven cavity), in this way, we get that matrix 'Ast' is positive definite and the matrix 'Bst' produced by this software is rank deficient. In the same way, we take $A = \text{Ast}$ and $U = \text{Bst}^T$.

In Table 3, we list the minimum and maximum real parts of the preconditioned matrices $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for 32×32 uniform grids with the different γ. Figure 3 plots the distribution of the eigenvalues of the preconditioned matrices $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$.

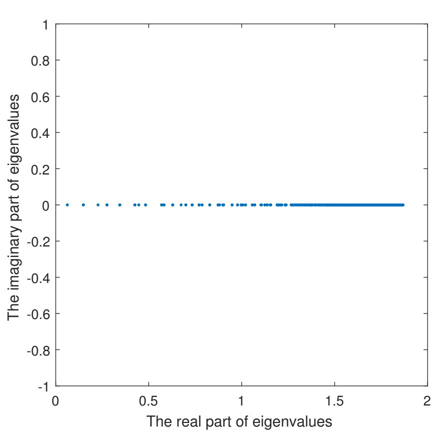
From all the numerical results of Examples 3.1, 3.2 and 3.3, together with Figures 1, 2 and 3, we come to the same conclusion as before, i.e., the eigenvalues of $P_\beta^{-1}\mathcal{A}$ are more aggregation than that of $P_\alpha^{-1}A_\gamma$.



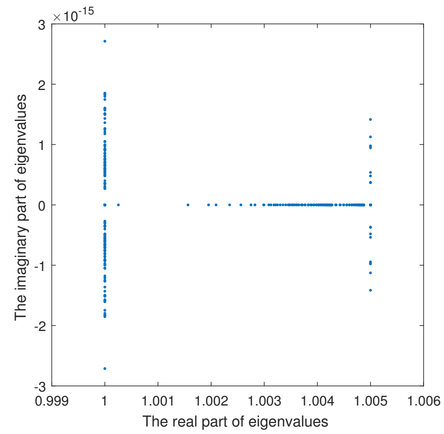
(a) $P_\alpha^{-1}A_\gamma$ with $\gamma = 0.001$



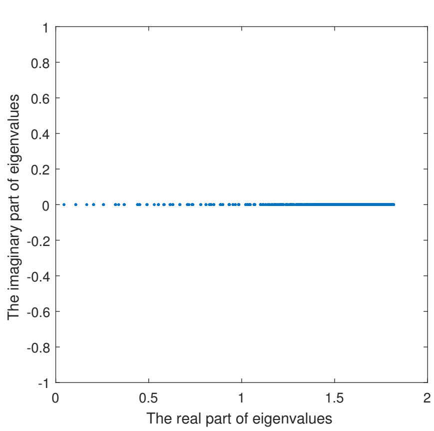
(b) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 0.001$



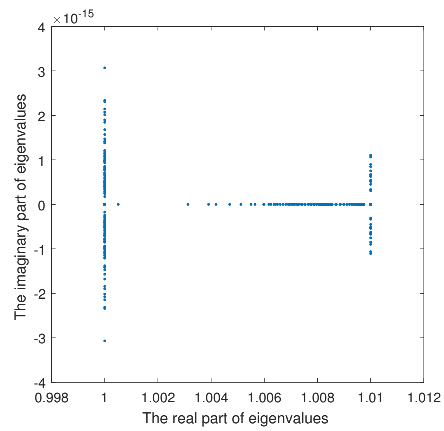
(c) $P_\alpha^{-1}A_\gamma$ with $\gamma = 0.005$



(d) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 0.005$

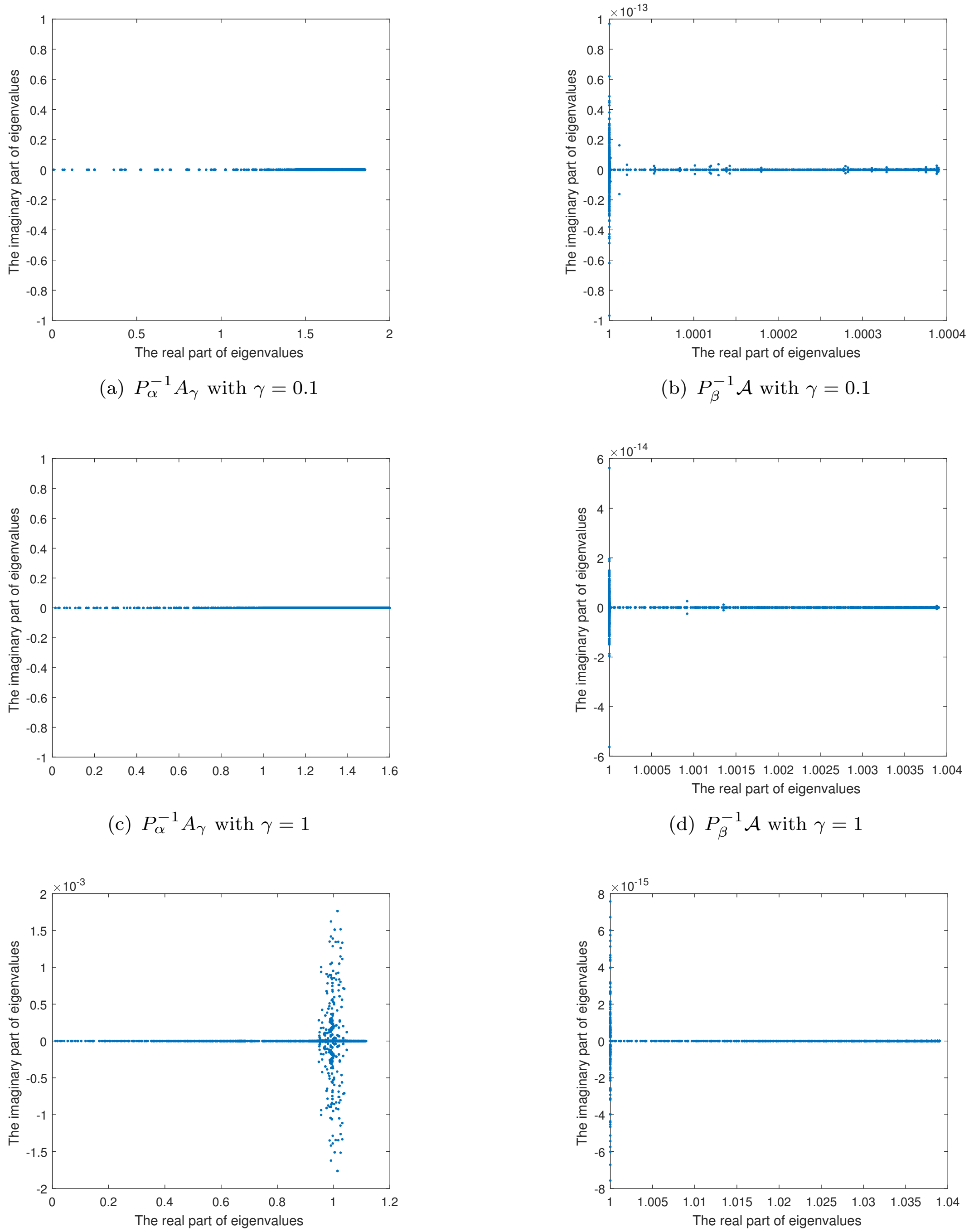


(e) $P_\alpha^{-1}A_\gamma$ with $\gamma = 0.01$



(f) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 0.01$

Figure 1. Spectra of $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for Example 3.1 with $\gamma = 0.001$, $\gamma = 0.005$ and $\gamma = 0.01$.



(a) $P_\alpha^{-1}A_\gamma$ with $\gamma = 0.1$ (b) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 0.1$

(c) $P_\alpha^{-1}A_\gamma$ with $\gamma = 1$ (d) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 1$

(e) $P_\alpha^{-1}A_\gamma$ with $\gamma = 10$ (f) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 10$

Figure 2. Spectra of $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for Example 3.2 with $\gamma = 0.1$, $\gamma = 1$ and $\gamma = 10$.

3.2. Test results of GMRES

As is known, an effective and cheap preconditioner is popular, and a key to improve the convergence of an iteration method [3]. Based on this, in this subsection, our goal is to present

Table 3. Results of eigenvalues of $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for Example 3.3 with $\gamma = 0.1$, $\gamma = 1$ and $\gamma = 10$.

γ	$P_\alpha^{-1}A_\gamma$		$P_\beta^{-1}\mathcal{A}$	
	$\max \Re(P_\alpha^{-1}A_\gamma)$	$\min \Re(P_\alpha^{-1}A_\gamma)$	$\min \Re(P_\beta^{-1}\mathcal{A})$	$\max \Re(P_\beta^{-1}\mathcal{A})$
0.1	1.9205	0.0117	1	1.0015
1	1.7686	0.0234	1	1.0147
10	1.4146	0.0110	1	1.1475

some numerical experiments to illustrate the performance of the new preconditioner P_β with GMRES(20) for solving the linear systems (2.1). In the meantime, we compare P_β together with GMRES for solving (2.1) and P_α together with GMRES for solving (1.1). These numerical comparisons are to show the advantage of the preconditioner P_β. In our computations, we adjust the right-hand side b such that the exact solution of the linear systems (1.1) is $\mathbf{1} = (1, 1, \ldots, 1)^T$. The iterations start with a zero vector and are stopped when the iteration counts are larger than 500 or the respective relative residual errors satisfy

$$\frac{\|b - A_\gamma x^{(k)}\|_2}{\|b\|_2} \le 10^{-6} \text{ and } \frac{\|\mathbf{b} - \mathcal{A}\mathbf{x}\|}{\|\mathbf{b}\|_2} \le 10^{-6}.$$

The iteration parameter α_* used in the preconditioner P_α is the experimental optimal one, which minimizes the numbers of iteration steps. If the experimental optimal iteration parameters form an interval, then they are further optimized according to the least relative residual error. We present the numerical results in the tables. In the tables, "IT", "CPU" and "ERR" in order stand for the iteration counts, the elapsed CPU time (in second) and the relative residual errors.

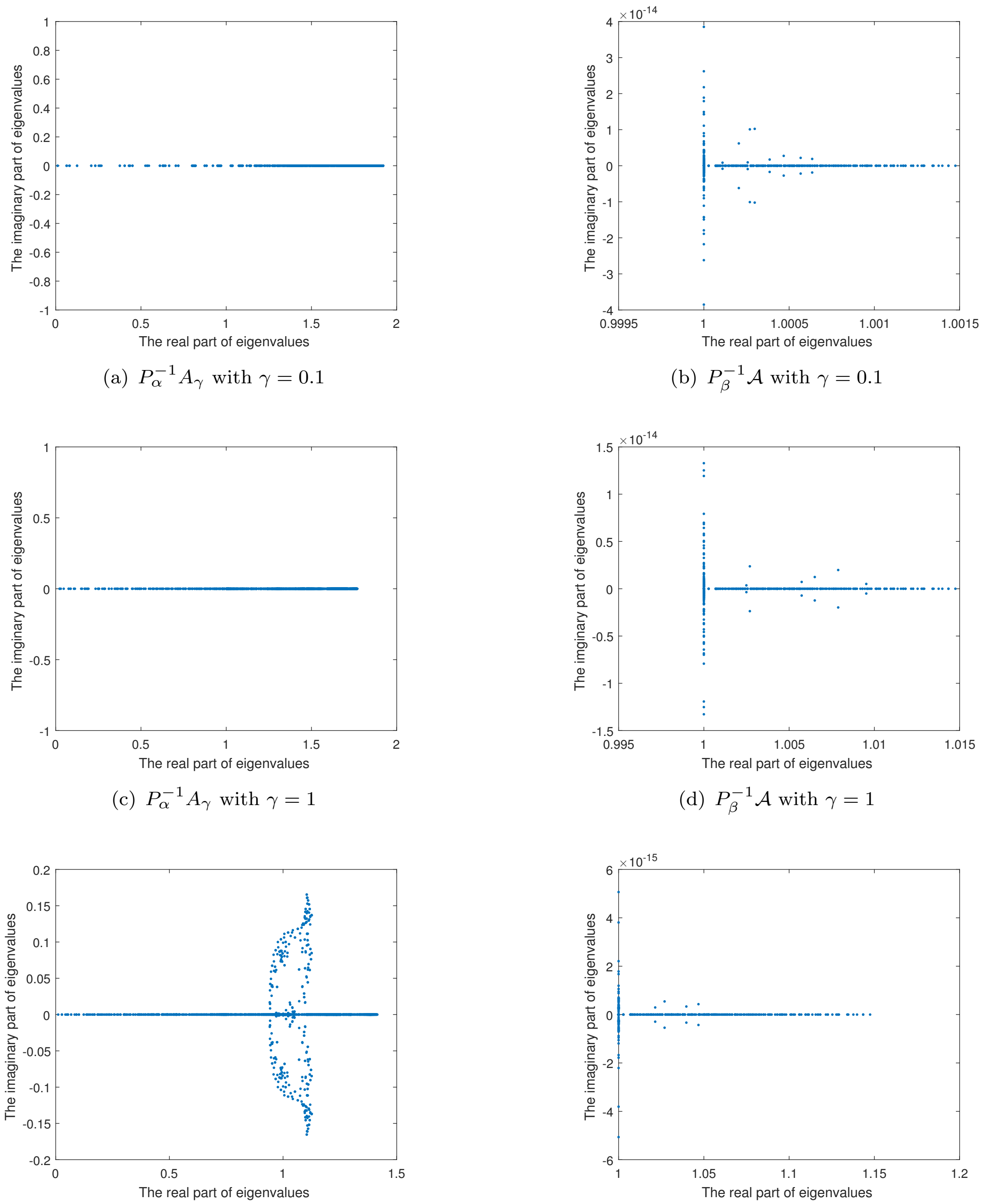
In our computations, for the preconditioner P_α, when $\alpha I + A$ is SPD, the linear residual equations with $\alpha I + A$ is solved by the Choleskey decomposition, and when $\alpha I + A$ is positive definite, the linear residual equations with $\alpha I + A$ is solved by the LU decomposition; for the linear residual equations with $\alpha I + \gamma UU^T$, the matrix $\alpha I + \gamma UU^T$ is SPD, it is performed by the PCG iteration. For the preconditioner P_β, when dealing with the linear residual equations with A, it is completely similar to the preconditioner P_α with $\alpha I + A$.

For Example 3.1, we test three cases: $\gamma = 1$, $\gamma = 10$ and $\gamma = 50$. In such case, Table 4 lists some numerical results when GMRES(20) together with the preconditioners P_β and P_α is adopted to solve the respective linear systems for the different mesh.

For all considered dimensions and γ, the preconditioners P_β and P_α together with GMRES(20) can rapidly achieve a well pleasing approximation to the solution of the respective linear systems induced by Example 3.1. Further, we find that when γ is fixed, for the iteration counts required for convergence, with the problem size n increasing, they increase quite slowly, even no increase (like P_β for $\gamma = 1$). Not surprisingly, for the CPU times, they increase with the dimensions of the problem.

From the numerical results in Table 4, obviously, the new preconditioner P_β requires the less iteration counts and the CPU times. This implies that the new preconditioner P_β is more effective than the preconditioner P_α by means of the iteration counts and the elapsed CPU times. That is to say, the new preconditioner P_β is more competitive, compared with the preconditioner P_α.

Table 5 and Table 6 in order list some numerical results for Example 3.2 and Example 3.3.



(a) $P_\alpha^{-1}A_\gamma$ with $\gamma = 0.1$ (b) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 0.1$

(c) $P_\alpha^{-1}A_\gamma$ with $\gamma = 1$ (d) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 1$

(e) $P_\alpha^{-1}A_\gamma$ with $\gamma = 10$ (f) $P_\beta^{-1}\mathcal{A}$ with $\gamma = 10$

Figure 3. Spectra of $P_\alpha^{-1}A_\gamma$ and $P_\beta^{-1}\mathcal{A}$ for Example 3.3 with $\gamma = 0.1$, $\gamma = 1$ and $\gamma = 10$.

From the numerical results of Tables 5 and 6, it is easy to see that the new preconditioner P_β overmatches the preconditioner P_α from the iteration counts and CPU times when both together with GMRES(20) are considered as a solver for solving the respective linear systems. In particular, for our considered dimensions and γ, for the new preconditioner P_β, it always converges in less than 10 iteration counts.

Table 4. Numerical results of P_β and P_α for Example 3.1 with $\gamma = 1$, $\gamma = 10$ and $\gamma = 50$.

γ	Mesh	P_α				P_β		
		α_*	IT	CPU	ERR	IT	CPU	ERR
1	32×32	0.3	19	0.08	8.0e-7	8	0.01	1.1e-7
	64×64	0.2	27	0.45	9.3e-7	8	0.03	1.7e-7
	128×128	0.07	33	4.61	9.8e-7	8	0.14	1.9e-7
10	32×32	0.6	19	0.09	7.4e-7	12	0.01	9.2e-7
	64×64	0.3	27	0.44	6.2e-7	13	0.04	8.1e-7
	128×128	0.2	44	7.01	7.7e-7	14	0.21	4.0e-7
50	32×32	0.7	19	0.09	6.1e-7	14	0.02	7.2e-7
	64×64	0.3	29	0.51	9.5e-7	16	0.04	2.2e-7
	128×128	0.2	52	8.01	8.3e-7	17	0.25	2.5e-7

Table 5. Numerical results of P_β and P_α for Example 3.2 with $\gamma = 1$, $\gamma = 10$ and $\gamma = 50$.

γ	Mesh	P_α				P_β		
		α_*	IT	CPU	ERR	IT	CPU	ERR
1	32×32	0.03	7	0.05	3.6e-7	3	0.01	1.4e-9
	64×64	0.009	7	0.16	2.2e-7	2	0.03	9.3e-8
	128×128	0.002	7	0.63	2.9e-7	2	0.06	5.8e-9
10	32×32	0.1	10	0.06	8.1e-7	4	0.01	3.0e-9
	64×64	0.03	11	0.23	2.9e-7	3	0.03	2.2e-8
	128×128	0.007	11	1.01	2.7e-7	2	0.08	5.8e-7
50	32×32	0.2	14	0.07	7.0e-7	5	0.02	2.9e-8
	64×64	0.06	15	0.34	3.6e-7	4	0.04	8.2e-9
	128×128	0.01	15	1.78	9.9e-7	3	0.08	4.5e-8

4. Conclusion

In this paper, we propose a new approach for solving linear systems of the form $(A+\gamma UU^T)x = b$. By transforming this linear systems into the equivalent saddle point problem, a new and effective preconditioner for this equivalent saddle point form is proposed. Meanwhile, the spectral properties of the corresponding preconditioned matrix are presented. From three aspects: Spectral distribution, residual equations and numerical results, we compare the proposed new preconditioner P_β with P_α in [4]. The comparative results show that the proposed new preconditioner P_β is superior to P_α under certain conditions.

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Table 6. Numerical results of P_β and P_α for Example 3.3 with $\gamma = 1$, $\gamma = 10$ and $\gamma = 50$.

γ	Mesh	P_α				P_β		
		α_*	IT	CPU	ERR	IT	CPU	ERR
1	32×32	0.03	7	0.18	2.8e-7	3	0.01	3.0e-8
	64×64	0.008	7	0.59	3.5e-7	2	0.02	7.3e-7
	128×128	0.002	7	2.41	2.3e-7	2	0.06	5.0e-8
10	32×32	0.1	10	0.25	7.4e-7	4	0.02	5.0e-7
	64×64	0.02	11	1.07	3.8e-7	3	0.02	6.3e-7
	128×128	0.007	10	3.88	8.3e-7	3	0.08	1.2e-8
50	32×32	0.2	14	0.38	9.0e-7	7	0.02	2.1e-7
	64×64	0.06	14	1.21	9.8e-7	5	0.03	5.4e-8
	128×128	0.01	15	7.47	1.0e-6	4	0.1	5.8e-9

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