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EXISTENCE AND CONCENTRATION OF POSITIVE GROUND STATE SOLUTIONS FOR A (P, Q)-KIRCHHOFF TYPE EQUATION WITH MOSER-TRUDINGER NONLINEARITY*

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Abstract In this work, we concern the existence and concentration of positive ground state solutions for the following (p,q)-Kirchhoff type problem

$$\begin{cases} -(1+a\int_{\mathbb{R}^N}|\nabla u|^p dx)\Delta_p u-(1+b\int_{\mathbb{R}^N}|\nabla u|^q dx)\Delta_q u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u) \\ = f(u) \quad \text{in}\, \mathbb{R}^N,\, u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon>0$ is a small parameter, $a,b>0, 1<p<q=N$, $\Delta_m u=\operatorname{div}(|\nabla u|^{m-2}\nabla u)$ with $m\in\{p,q\}$ is the m-Laplacian operator, the potential $V:\mathbb{R}^N\to\mathbb{R}$ is a positive continuous function and $f:\mathbb{R}\to\mathbb{R}$ is a continuous nonlinearity involving critical exponential growth and not satisfying usual Ambrosetti-Rabinowitz condition. The existence and concentration behavior of positive ground state solutions are established by variational methods combined with some sharp exponential type inequalities.

Keywords (p,q)-Kirchhoff equation, Nehari manifold, critical exponential growth.

MSC(2010) 35A01, 35A15, 35A23.

1. Introduction

In this paper, we investigate the existence and concentration behavior of positive solutions for the (p,q)-Kirchhoff type problem

$$\begin{cases} -(1+a\int_{\mathbb{R}^N}|\nabla u|^p dx)\Delta_p u-(1+b\int_{\mathbb{R}^N}|\nabla u|^q dx)\Delta_q u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u) \\ = f(u) \quad \text{in}\, \mathbb{R}^N,\, u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where ε is a positive small parameter, $a,b>0$ are fixed constants, $1<p<q=N$, $\Delta_m u=\operatorname{div}(|\nabla u|^{m-2}\nabla u)$ with $m\in\{p,q\}$ is the m-Laplacian operator. We let the potential $V:\mathbb{R}^N\to\mathbb{R}$ be a continuous function and satisfy

$$V_\infty := \liminf_{|x|\to\infty} V(x) > V_0 := \inf_{x\in\mathbb{R}^N} V(x) > 0, \tag{V}$$

where $V_\infty<\infty$. This type of hypothesis was first proposed by Rabinowitz [25]. The nonlinear term $f:\mathbb{R}\to\mathbb{R}$ is a continuous function. In order to find the positive solution for problem (1.1), we often assume that $f(t)=0$ for $t<0$. In addition, we need the following assumptions:

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(f_1) $\lim\limits_{t\to 0^+}\frac{f(t)}{t^{2p-1}}=0$;

(f_2) $f(t)$ satisfies critical exponential growth at $+\infty$; namely, there exists a positive constant α_0 such that

$$\lim_{t\to+\infty}\frac{f(t)}{e^{\alpha t^{\frac{N}{N-1}}}}=\begin{cases}0, & \forall\,\alpha>\alpha_0,\\ +\infty, & \forall\,\alpha<\alpha_0;\end{cases}$$

(f_3) $\lim\limits_{t\to+\infty}\frac{f(t)}{t^{2N-1}}=\infty$;

(f_4) For $t>0$, the function $\frac{f(t)}{t^{2N-1}}$ is strict increasing;

(f_5) There exists $M^*>0$ such that

$$F(t)\le M^*t^{2N}+M^*f(t),\ t\in\mathbb{R},$$

where $F(t)=\int_0^t f(s)ds$;

(f_6) $\lim\limits_{t\to+\infty} f(t)\exp(-\alpha_0 t^{\frac{N}{N-1}})t\ge\beta>\Big((\frac{\alpha_N}{\alpha_0})^{N-1}+b(\frac{\alpha_N}{\alpha_0})^{2N-2})\Big)/(\nu_N\mathcal{M})$, where

$$\mathcal{M}=\lim_{n\to\infty}N\ln n\int_0^1\exp\left[N\ln n\left(t^{\frac{N}{N-1}}-tN\ln n\right)\right]dt,$$

β is a positive constant, ν_N denotes the volume of the unit ball and the definition of α_N can be seen in Proposition 1.1.

In the case $a=b\neq 0, p=q=2$ and $N=3$, Equation (1.1) changes to a Kirchhoff equation of the form

$$-(1+a\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u+V(\varepsilon x)u=f(u)\quad\text{in }\mathbb{R}^3.\tag{1.2}$$

Equation (1.2) is related to the stationary analogue of the Kirchhoff equation

$$\frac{\partial^2 u}{\partial t^2}-\left(a^*+b^*\int_\Omega|\nabla u|^2dx\right)\Delta u=f(u)\quad\text{in }\Omega,$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $a^*,b^*>0$, and u satisfies some boundary condition, which was put forward by [19] as an generalization of the classical D'Alembert's wave equation for free vibration of elastic strings

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L|u_x|^2dx\right)\frac{\partial^2 u}{\partial x^2}.$$

Here, $u=u(x,t)$ is the transverse string displacement at the space coordinate x and the time t, L denotes the length of the string, h denotes the area of the cross section, E denotes the Young's modulus of the material, ρ denotes the mass density and p_0 is the initial axial tension. More and more attention has been paid to the study of nonlocal problem (1.2) only after the pioneer work of Lions [22], where the functional analysis approaches were first used and there are some interesting results in the knowledge, here we only refer to [13, 17] and the references therein.

As far as our knowledge is concerned, He and Zou [18] first obtained the existence, multiplicity and concentration of solutions for equation (1.2) with the nonlinearity $f\in C^1(\mathbb{R},\mathbb{R})$ satisfying usual Ambrosetti-Rabinowitz condition by using variational methods and Ljusternik-Schnirelmann theory. Wang et al. [28] regarded with the critical case. For related results, we refer to [16].

When $a = b = 0$, equation (1.1) changes to the (p, q) Laplacian equation

$$-\Delta_p u - \Delta_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

This equation stems from a stationary analogue of the reaction-diffusion system

$$u_t = \operatorname{div}(D(u)\nabla u) + f(x, u),$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This equation has rich physical background, for instance, biophysics, plasma physics and chemical reaction design. For more detailed applications, we refer to [10].

Recently, equation (1.3) and it's version have been concerned by several authors. In [14], He and Li obtained some regularity result for (1.3). They also established an existence result by combining concentration-compactness principle with mountain pass theorem in [15]. The existence of a ground-state positive solution for a (p, q)-Schrödinger equation involving critical growth was proved by Figueiredo [12]. By using refined variational methods, Morse theory and truncation technique, a multiplicity result for (p, q) Laplacian problem was obtained by Mugnai and Papageorgiou [23]. Ambrosio and Repovš [6] studied the the multiplicity and concentration of positive solutions for (1.3) under the condition (V). Ambrosio and Rădulescu [5] studied deeply multiplicity of concentrating solutions for(1.3) under some suitable assumptions by combining minimax theorems, penalization technique and Ljusternik-Schnirelmann category theory. For other interesting result, one can consult [7,8,26] and references therein. Since (p, q)-Laplacian is a nonlinear and nonhomogeneous operator, It will create challenges in addressing such problems.

Using penalization method and Ljusternik-Schnirelmann theory, the multiplicity and concentration behavior of positive solutions for equation (1.1) in [4], in which V satisfies (V) and f is subcritical growth and satisfies Ambrosetti-Rabinowitz condition was investigated by Ambrosio and Isernia. Particularly, in [31], Zhang et al. established the multiplicity and concentration of positive solutions to equation (1.1) by using Nehari manifold technique, perturbation methods and Ljusternik-Schnirelmann theory. It is worth noting that their methods are skill because they provide a new method to verify the compactness of the Palais-Smale sequence.

To the best of our knowledge, all of the works above mentioned about (p, q)-Laplacian problem involve the nonlinearity satisfying polynomial growth. But, there are only a few paper (see [9, 29]) dealing with the nonlinear term satisfying the exponential growth.

Motivated by the works above mentioned, the purpose of this paper is to investigate the existence and concentration of positive solutions for (1.1) where the nonlinearity has a critical exponential growth at $+\infty$ and not satisfies Ambrosetti-Rabinowitz condition by using variational methods combined with Moser-Trudinger inequality and additional analysis technique. According to our knowledge, our results are new. For the purpose of presenting our main result, we first define

$$\Lambda := \{x \in \mathbb{R}^N : V(x) = V_0\}.$$

Due to the condition V, we know that the set Λ is compact. Now, we state our main result as follows:

Theorem 1.1. *Suppose that conditions* (f_1)-(f_6) *and* (V) *hold. Then there exists* $\bar{\varepsilon} > 0$ *such that for any* $\varepsilon \in (0, \bar{\varepsilon})$*, equation* (1.1) *has a positive ground state solution. Moreover, if* u_ϵ *denotes one of these positive solutions with* $\eta_\epsilon \in \mathbb{R}^N$ *its global maximum point, then*

$$\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.$$

Before finishing this introduction, we introduce two important Moser-Trudinger inequalities, which will be used to prove our main results.

Proposition 1.1. (see [1]) *For all* $0 < \alpha$ *and* $u \in W^{1,N}(\mathbb{R}^N)$*, there holds*

$$\int_{\mathbb{R}^N} \left\{ \exp\left(\alpha |u|^{\frac{N}{(N-1)}} \right) - S_{N-2}(\alpha, u) \right\} dx < +\infty.$$

Furthermore, we have for all $\alpha \le \alpha_N$ *and* $\tau > 0$*,*

$$\sup_{\|u\|_{1,\tau} \le 1} \int_{\mathbb{R}^N} \left\{ \exp\left(\alpha |u|^{\frac{N}{(N-1)}} \right) - S_{N-2}(\alpha, u) \right\} dx < +\infty,$$

where $\|u\|_{1,\tau} = (\int_{\mathbb{R}^N} (|\nabla u|^N + \tau |u|^N) dx)^{\frac{1}{N}}$*. The inequality is sharp: For any* $\alpha > \alpha_N$*, the supremum is infinity. Here*

$$S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{\frac{kN}{N-1}}}{k!}$$

and $\alpha_N = N \omega_{N-1}^{\frac{1}{N-1}}$ *(*ω_{N-1} *is the surface area of the unit sphere in* $\mathbb{R}^N$*).*

Proposition 1.2. (see [24]) *If* $u \in W^{1,N}(\mathbb{R}^N)$*,* $0 < \alpha < \alpha_N$*,* $\|\nabla u\|_{L^N}^N \le 1$ *and* $\|u\|_{L^N} \le K$*,* $\forall\, n \in \mathbb{N}$*, for some positive constant* K*, then there exists a positive constant* $C = C(N, K, \alpha)$*, which depends only on* N, K *and* α *such that*

$$\int_{\mathbb{R}^N} \left\{ \exp\left(\alpha |u|^{\frac{N}{(N-1)}} \right) - S_{N-2}(\alpha, u) \right\} dx < C.$$

Here, the respective definition of $S_{N-2}(\alpha, u)$ *and* α_N *is made as above.*

For convenience, we use the following notations in the sequel:

- Let $C, C_i, i = 1, 2, \cdots$, denote different positive constants in different places;
- Let B_R be a open ball with radius R in $\mathbb{R}^N$;
- Let $|u|_r$ denote $|u|_{L^r(\mathbb{R}^N)}$.

This paper is organized as follows. In Section 2 we state some facts about involved Sobolev spaces, prove several important lemmas and establish the existence of one positive ground state solution for problem (1.1). In the last section we complete the proof of Theorem 1.1.

2. Variational framework and some lemmas

In this section, we recall some facts on the Sobolev spaces and we prove some important lemmas which will be used later. Let us define the Sobolev space

$$X_\epsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)(|u|^p + |u|^N) dx < +\infty \right\},$$

which is equipped with the norm

$$\|u\|_\epsilon = \|u\|_{p,V} + \|u\|_{N,V},$$

where $\|u\|_{p,V}^p = |\nabla u|_p^p + \int_{\mathbb{R}^N} V(\epsilon x)|u|^p dx$ and $\|u\|_{N,V}^N = |\nabla u|_N^N + \int_{\mathbb{R}^N} V(\epsilon x)|u|^N dx$.

Now, we introduce the following embedding result.

Lemma 2.1. (see [2]) *The space X_ϵ is continuously embedded into $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$. Therefore X_ε is continuously embedded in $L^s(\mathbb{R}^N)$ for any $[p,+\infty)$ and compactly embedded in $L^s_{loc}(\mathbb{R}^N)$ for any $s \in [1,+\infty)$.*

Proof. For completeness, we give a brief proof. Since the definition of X_ϵ, we have

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \le (\min\{1, V_0\})^{-\frac{1}{p}} \|u\|_{p,V} \le (\min\{1, V_0\})^{-\frac{1}{p}} \|u\|_\epsilon,$$

similarly, we have

$$\|u\|_{W^{1,N}(\mathbb{R}^N)} \le (\min\{1, V_0\})^{-\frac{1}{N}} \|u\|_\epsilon.$$

Thus, X_ϵ is continuously embedded into $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$. Next, by the Sobolev embedding theorem, we know that X_ε is continuously embedded in $L^s(\mathbb{R}^N)$ for any $[p,+\infty)$ and compactly embedded in $L^s_{loc}(\mathbb{R}^N)$ for any $s \in [1,+\infty)$. □

In the sequel, we consider problem (1.1). From (f_1) and (f_2), we know that problem (1.1) is variational and its weak solutions are the critical points of the functional given by

$$\mathcal{I}_\epsilon(u) = \frac{1}{p}\|u\|^p_{p,V} + \frac{a}{2p}|\nabla u|^{2p}_p + \frac{1}{N}\|u\|^N_{N,V} + \frac{b}{2N}|\nabla u|^{2N}_N - \int_{\mathbb{R}^N} F(u)dx.$$

In addition, $\mathcal{I}_\epsilon$ is differentiable in X_ε and its derivative is given for any $u, v \in X_\epsilon$ as

$$\begin{aligned}\langle \mathcal{I}'_\epsilon(u), v\rangle = &\big(1 + a\int_{\mathbb{R}^N} |\nabla u|^p dx\big) \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v dx \\ &+ \big(1 + b\int_{\mathbb{R}^N} |\nabla u|^N dx\big) \int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u \nabla v dx \\ &+ \int_{\mathbb{R}^N} V_\epsilon(x)\big(|u|^{p-2}u + |u|^{N-2}u|\big) v dx - \int_{\mathbb{R}^N} f(u) v dx.\end{aligned}$$

Now, we define the Nehari manifold related to $\mathcal{I}_\epsilon$ as

$$\mathcal{N}_\epsilon = \{u \in X_\epsilon \backslash \{0\} : \langle \mathcal{I}'_\epsilon(u), u\rangle = 0\}.$$

Next, we prove that the functional $\mathcal{I}_\epsilon$ satisfies the mountain pass geometry.

Lemma 2.2. *Assume that (f_1)-(f_3) hold. Then:*

(i) *There exist $\rho, \alpha > 0$ such that $\mathcal{I}_\epsilon(u) \ge \alpha$ for all $u \in X_\epsilon$ with $\|u\|_\epsilon = \rho$.*
(ii) *There exists $\varphi \in X_\epsilon$ such that $\mathcal{I}_\epsilon(t\varphi) \to -\infty$ as $t \to +\infty$.*

Proof. Using (f_1) and (f_2), for any $u \in X_\epsilon \backslash \{0\}$ and $\tau > 0$ small enough, there exist $C_\tau, \beta > 0$ and $q > 2p$ such that

$$F(t) \le \frac{\tau}{2p}|t|^{2p} + C_\tau \left[\exp\left(\beta |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\beta, u)\right] |t|^q.$$

By Lemma 2.1 and Proposition 1.1, we have

$$\begin{aligned}\mathcal{I}_\epsilon(u) \ge &\frac{1}{p}\|u\|^p_{p,V} + \frac{1}{N}\|u\|^N_{N,V} - \tau C\|u\|^{2p}_\epsilon \\ &-C_\tau \int_{\mathbb{R}^N} \left[\exp\left(\beta |u|^{\frac{N}{N-1}}\right) - S_{N-2}(\beta, u)\right] |u|^q \\ \ge &\frac{1}{p}\|u\|^p_{p,V} + \frac{1}{N}\|u\|^N_{N,V} - \tau C\|u\|^{2p}_\epsilon\end{aligned}$$

$$
\begin{aligned}
&-C_\tau \left(\int_{\mathbb{R}^N} \Phi_{N,1}\left(\beta r\|u\|_{1,V_0}^{\frac{N}{N-1}}\left(\frac{|u|}{\|u\|_{1,V_0}}\right)^{\frac{N}{N-1}}\right)dx\right)^{\frac{1}{r}}\left(\int_{\mathbb{R}^N}|u|^{r'q}dx\right)^{\frac{1}{r'}}\\
&\geq \frac{1}{p}\|u\|_{p,V}^p+\frac{1}{N}\|u\|_{N,V}^N-\tau C\|u\|_\epsilon^{2p}-C\|u\|_\epsilon^q,
\end{aligned}
$$

where $\Phi_{N,1}(t)=e^t-\sum\limits_{i=0}^{N-2}\frac{t^i}{i!}$, $r>1$ is sufficiently close to 1, $\|u\|_\epsilon\leq\sigma$ and $\beta r\sigma^{\frac{N}{N-1}}\leq\alpha_N$. So, part (i) is proved if we take $\|u\|_\epsilon=\rho>0$ small enough.

On the other hand, from (f_3), for M large enough we obtain that $F(t)\geq M|t|^{2N}-C$. Thus, by choosing a positive function $\varphi\in C_0^\infty(\mathbb{R}^N)$, we have

$$
\begin{aligned}
\mathcal{I}_\epsilon(t\varphi)&\leq\frac{t^p}{p}\|\varphi\|_{p,V}^p+\frac{at^{2p}}{2p}|\nabla\varphi|_p^{2p}+\frac{t^N}{N}\|\varphi\|_{N,V}^N+\frac{bt^{2N}}{2N}|\nabla\varphi|_N^{2N}-Mt^{2N}|\varphi|_{2N}^{2N}+C|\mathrm{supp}\varphi|\\
&\to-\infty\ \ \mathrm{as}\,t\to\infty,
\end{aligned}
$$

where $|\mathrm{supp}\varphi|$ represents the measure of $\mathrm{supp}\varphi$. So part (ii) holds. □

Now, we state the definition of *Palais-Smale* sequence ((PS) sequence, for short). That is, a sequence $\{u_n\}$ is called (PS) sequence for $\mathcal{I}_\epsilon$ if $\{\mathcal{I}_\epsilon(u_n)\}$ is bounded and $\mathcal{I}_\epsilon'(u_n)\to0$. If $\mathcal{I}_\epsilon(u_n)\to c\in\mathbb{R}$ and $\mathcal{I}_\epsilon'(u_n)\to0$, then $\{u_n\}$ is called a $(PS)_c$ sequence. The functional $\mathcal{I}_\epsilon$ is said to satisfy the *Palais-Smale* condition ((PS) condition for short) (or $(PS)_c$ condition) if each *Palais-Smale* sequence (or $(PS)_c$ sequence) has a strong convergent subsequence.

By Lemma 2.2 and the mountain pass theorem without (PS) condition (cf. [30]), there exists a (PS) sequence $\{u_n\}\subset X_\epsilon$ such that $\mathcal{I}_\epsilon(u_n)\to c_\epsilon$ and $\mathcal{I}_\epsilon'(u_n)\to0$ in X_ϵ^{-1} at the minimax level

$$
c_\epsilon:=\inf_{g\in\Gamma}\sup_{t\in[0,1]}\mathcal{I}_\epsilon(g(t))>0,
$$

where $\Gamma:=\{g\in C([0,1],X_\epsilon):\ g(0)=0,\mathcal{I}_\epsilon(g(1))<0\}$.

Lemma 2.3. *Assume that (f_1)-(f_4) hold. Then for every $u\in X_\epsilon\backslash\{0\}$ there exists a unique $t(u)>0$ such that $t(u)u\in\mathcal{N}_\epsilon$. Furthermore, $\mathcal{I}_\epsilon(t(u)u)=\max\limits_{t\geq0}\mathcal{I}_\epsilon(tu)$.*

Proof. For every $u\in X_\epsilon\backslash\{0\}$ fixed, let $h(t)\doteq\mathcal{I}_\epsilon(tu)$ for $t\geq0$. By a similar argument as in the proof of (i) and (ii) in Lemma 2.2, we conclude that $h(0)=0$, $h(t)>0$ for $t>0$ small and $h(t)<0$ for t large. Hence, there exists a $t(u)>0$ such that $h'(t(u))=0$, that is, $t(u)u\in\mathcal{N}_\epsilon$. We only prove that $t(u)$ is the unique critical point of $h(t)$ in $(0,\infty)$. In contrary, assume that there exist $0<t_1<t_2<\infty$ such that $h'(t_1)=h'(t_2)=0$. Then

$$
t_1^p\|u\|_{p,V}^p+at_1^{2p}|\nabla u|_p^{2p}+t_1^N\|u\|_{N,V}^N+bt_1^{2N}|\nabla u|_N^{2N}=\int_{\mathbb{R}^N}f(t_1u)t_1u\,dx
$$

and

$$
t_2^p\|u\|_{p,V}^p+at_2^{2p}|\nabla u|_p^{2p}+t_2^N\|u\|_{N,V}^N+bt_2^{2N}|\nabla u|_N^{2N}=\int_{\mathbb{R}^N}f(t_2u)t_2u\,dx.
$$

From the above two formulas and (f_4), we get that

$$
\Big(\frac{1}{t_1^{2N-p}}-\frac{1}{t_2^{2N-p}}\Big)\|u\|_{p,V}^p+\Big(\frac{a}{t_1^{2N-2p}}-\frac{a}{t_2^{2N-2p}}\Big)\|\nabla u|_p^{2p}
$$

$$
\begin{aligned}
&+\Big(\frac{1}{t_1^{2N-N}}-\frac{1}{t_2^{2N-N}}\Big)\|u\|_{N,V}^N\\
=&\int_{\mathbb{R}^N}\Big(\frac{f(t_1u)}{t_1^{2N-1}}u-\frac{f(t_2u)}{t_2^{2N-1}}u\Big)dx\\
<&0.
\end{aligned}
$$

Thus, we have shown that $t(u)$ is unique. Moreover, $\mathcal{J}_\epsilon(t(u)u)=\max\limits_{t\geq 0}\mathcal{J}_\epsilon(tu)$. □

Next, we define the numbers

$$
c_\epsilon^*:=\inf_{u\in\mathcal{N}_\epsilon}\mathcal{I}_\epsilon(u),\quad c_\epsilon^{**}:=\inf_{u\in X_\epsilon\setminus\{0\}}\max_{t\geq 0}\mathcal{I}_\epsilon(tu).
$$

Lemma 2.4. *$c_\epsilon=c_\epsilon^*=c_\epsilon^{**}$, for any fixed $\epsilon>0$.*

Proof. The detailed proof of this kind of results is completely similar to the classical Schrödinger equations which has been proved in [25]. Here, we omit it. □

Lemma 2.5. *Assume that (f_4) holds. Then any $(PS)_c$ sequence $\{u_n\}$ of $\mathcal{I}_\epsilon$ is bounded in X_ϵ.*

Proof. Suppose that $\{u_n\}$ is a $(PS)_c$ sequence. So, we get

$$
o(1)\|u_n\|+c_\epsilon=\mathcal{I}_\epsilon(u_n)-\frac{1}{2N}\langle\mathcal{I}_\epsilon'(u_n),u_n\rangle.
$$

This together with (f_4), we obtain

$$
\begin{aligned}
o(1)\|u_n\|+c_\epsilon=&\left(\frac{1}{p}-\frac{1}{2N}\right)\|u_n\|_{p,V}^p+a\left(\frac{1}{p}-\frac{1}{2N}\right)|\nabla u_n|_p^{2p}+\frac{1}{2N}\|u\|_{N,V}^N\\
&+\frac{1}{2N}\int_{\mathbb{R}^N}\left(f(u_n)u_n-2NF(u_n)\right)dx\\
\geq&\left(\frac{1}{p}-\frac{1}{2N}\right)\|u_n\|_{p,V}^p+\frac{1}{2N}\|u\|_{N,V}^N.
\end{aligned}
$$

Hence, $\{u_n\}$ is bounded in X_ϵ and the conclusion follows. □

Remark 2.1. Notice that for any $u\in\mathcal{N}_\varepsilon$, from (f_1) and (f_2), we get for $\tau^*>0$ small enough and $q_1>N$

$$
\begin{aligned}
0=&\|u\|_{p,V}^p+a|\nabla u|_p^{2p}+\|u\|_{N,V}^N+b|\nabla u|_N^{2N}-\int_{\mathbb{R}^N}f(u)udx\\
\geq&\|u\|_{p,V}^p+\|u\|_{N,V}^N-\tau^*C\|u\|_\varepsilon^p-C_1\|u\|_\varepsilon^{q_1},
\end{aligned}
$$

which implies that

$$
\|u\|_\varepsilon>r^*>0 \tag{2.1}
$$

for some $r^*>0$.

As we shall see, it is useful to compare c_ε with the minimax level of the autonomous problem

$$
\begin{cases}
-(1+a\int_{\mathbb{R}^N}|\nabla u|^pdx)\Delta_pu-(1+b\int_{\mathbb{R}^N}|\nabla u|^qdx)\Delta_qu+\nu(|u|^{p-2}u+|u|^{q-2}u)\\
=f(u)\quad\text{in }\mathbb{R}^N,\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N),\quad u>0\text{ in }\mathbb{R}^N,
\end{cases}
\tag{2.2}
$$

where $\nu \in \mathbb{R}^+$.

Let $W_\nu = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ be equipped with the norm

$$\|u\|_\nu = \|u\|_{p,\nu} + \|u\|_{N,\nu},$$

where $\|u\|_{p,\nu} = (|\nabla u|_p^p + \nu |u|_p^p)^{\frac{1}{p}}$ and $\|u\|_{N,\nu} = (|\nabla u|_N^N + \nu |u|_N^N)^{\frac{1}{N}}$. The solutions of (2.2) are precisely the critical points of the functional defined by

$$\mathcal{I}_\nu(u) = \frac{1}{p}\|u\|_{p,\nu}^p + \frac{a}{2p}|\nabla u|_p^{2p} + \frac{1}{N}\|u\|_{N,\nu}^N + \frac{b}{2N}|\nabla u|_N^{2N} - \int_{\mathbb{R}^N} F(u)dx.$$

From (f_1) and (f_2), we can deduce that $\mathcal{I}_\nu(u) \in C^1(W_\nu, \mathbb{R})$ and its derivative is expressed for any $u, v \in W_\nu$ as

$$\begin{aligned}\langle \mathcal{I}_\nu'(u), v\rangle = &\big(1 + a\int_{\mathbb{R}^N} |\nabla u|^p dx\big)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v dx \\ &+ \big(1 + b\int_{\mathbb{R}^N} |\nabla u|^N dx\big)\int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u \nabla v dx \\ &+ \int_{\mathbb{R}^N} \nu\big(|u|^{p-2}u + |u|^{N-2}u|\big)v dx - \int_{\mathbb{R}^N} f(u) v dx.\end{aligned}$$

Define the Nehari manifold associated to $\mathcal{I}_\nu$ as

$$\mathcal{M}_\nu := \{u \in W_\nu \backslash \{0\} : \langle \mathcal{I}_\nu'(u), u\rangle = 0\}.$$

We denote m_ν as

$$d_\nu := \inf_{u \in \mathcal{M}_\nu} \mathcal{I}_\nu(u).$$

The properties of the number d_ν and the manifold $\mathcal{M}_\nu$ are similar to those of c_ϵ and $\mathcal{N}_\epsilon$.

To prove existence of ground state solutions to problem (2.2), we will give an important estimate for the minimax level. We consider the following sequence of nonnegative functions:

$$\widetilde{\omega}_n(x) = \omega_{N-1}^{-\frac{1}{N}} \begin{cases} (\ln n)^{\frac{N-1}{N}}, & \text{if } |x| \le \dfrac{1}{n}, \\ \dfrac{\ln |x|^{-1}}{(\ln n)^{\frac{1}{N}}}, & \text{if } \dfrac{1}{n} < |x| < 1, \\ 0, & \text{if } |x| \ge 1. \end{cases}$$

Write

$$\varpi_n(x) = \frac{\widetilde{\omega}_n}{\|\widetilde{\omega}_n\|_\nu}.$$

Thus, we have $\|\varpi_n\|_\nu = 1$.

Lemma 2.6. *Assume conditions* (f_2) *and* (f_6) *hold. Then there exists* n *such that*

$$\max\{\mathcal{I}_\nu(t\varpi_n) : t \ge 0\} < \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N}.$$

Proof. Assume by contradiction that for all n, this maximum is larger or equal to

$$\left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}+\frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N}.$$

Then there exists $t_n > 0$ such that

$$\mathcal{I}_\nu(t_n\varpi_n) \geq \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}+\frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N}. \tag{2.3}$$

From (H_1) and (2.3), we deduce that

$$t_n^N \geq \left[\frac{\alpha_N}{\alpha_0}\right]^{N-1}. \tag{2.4}$$

Also at $t = t_n$, we get

$$t_n^{N-1} + bt_n^{2N-1} - \int_{\mathbb{R}^N} f(t_n\varpi_n)\varpi_n dx = o(1),$$

which means that

$$t_n^N + bt_n^{2N} = \int_{\mathbb{R}^N} f(t_n\varpi_n)t_n\varpi_n dx + o(1). \tag{2.5}$$

It follows from (f_6) that for given $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$tf(t) \geq (\beta-\epsilon)\exp\left(\alpha_0|t|^{\frac{N}{N-1}}\right),\ t \geq R_\epsilon.$$

So from (2.5), we deduce that, for large n

$$t_n^N + bt_n^{2N} \geq (\beta-\epsilon)\nu_N \exp\left[\left(\frac{\alpha_0 t_n^{\frac{N}{N-1}}}{\alpha_N}-1\right)N\ln n\right] + o(1). \tag{2.6}$$

Owing to (2.4), the inequality above holds if, and only if

$$t_n^N \to \left[\frac{\alpha_N}{\alpha_0}\right]^{N-1},\ \text{as } n\to\infty. \tag{2.7}$$

Let

$$A_n^* = \{x \in B_1(0) : t_n\varpi_n(x) \geq R_\epsilon\},\quad B_n = B_1(0)\setminus A_n^*,$$

and break the integral in (2.5) into a sum of integrals over A_n^* and B_n. From common computation, we obtain

$$\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + b\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2} \geq (\beta-\epsilon)\lim_{n\to\infty}\int_{B_1(0)}\exp\left[\alpha_N\varpi_n^{\frac{N}{N-1}}\right]dx - (\beta-\epsilon)\nu_N. \tag{2.8}$$

The last integral in (2.8), write it ξ_n, is evaluated as follows:

$$\xi_n = \left\{\nu_N + N\nu_N\ln n\int_0^1 \exp\left(Nt^{\frac{N}{N-1}}\ln n - tN\ln n\right)dt\right\}.$$

So, by (2.8) we reach

$$\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + b\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2} \geq (\beta - \epsilon)\nu_N \mathcal{M},$$

which implies $\beta \leq \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + b\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2} / (\nu_N \mathcal{M})$. This causes a contradiction with (f_6). $\square$

Remark 2.2. From the above lemma, we know

$$d_{V_\infty} < \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N}$$

when $V_\infty < \infty$.

Lemma 2.7. *Let $\{u_n\} \subset \mathcal{M}_\nu$ be a sequence converging weakly to 0 satisfying $\int_{\mathbb{R}^N} |\nabla u|^N dx < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$. If there exists $R > 0$ such that $\liminf\limits_{n\to+\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p dx = 0$, then it follows that*

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \to 0 \text{ and } \int_{\mathbb{R}^N} F(u_n) dx \to 0.$$

Proof. By our hypothesis

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p dx = 0,$$

together with Lions' lemma [21], we obtain

$$u_n \to 0 \text{ in } L^\varrho(\mathbb{R}^N), \ \forall \varrho \in (p, +\infty). \tag{2.9}$$

Again applying (f_1)-(f_2), for any $\tau_1 > 0$ small and $\beta_1 > 1$ closed to 1, it follows that

$$|f(u_n)u_n| \leq \tau_1 |u_n|^{2p} + C_{\tau_1}\left[\exp\left(\beta_1 |u_n|^{\frac{N}{N-1}}\right) - S_{N-2}(\beta_1, u_n)\right]|u_n|. \tag{2.10}$$

Because $\{u_n\}$ is bounded in $\mathcal{M}_\nu$ satisfying

$$|\nabla u_n|_N^N < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1},$$

from Proposition 1.2, there exist $C_2 > 0$, $r > 1$ closed to 1, such that

$$\mathcal{H}_n := \left[\exp\left(\beta_1 \alpha_0 |u_n|^{\frac{N}{N-1}}\right) - S_{N-2}(\beta_1\alpha_0, u_n)\right]$$

belongs to $L^r(\mathbb{R}^N)$ and $|\mathcal{H}_n|_r \leq C_2$ for all $n \in \mathbb{N}$. Thus, there exists $H \in L^r(\mathbb{R}^N)$ such that $\mathcal{H}_n \rightharpoonup H$ weakly in $L^r(\mathbb{R}^N)$. By (2.10), we get

$$u_n \to 0 \ \text{ in } L^{r'}(\mathbb{R}^N) \text{ for } \frac{1}{r} + \frac{1}{r'} = 1$$

and, so,

$$\int_{\mathbb{R}^N} \mathcal{H}_n |u_n| dx \to 0,$$

which is equivalent to

$$\int_{\mathbb{R}^N} \left[\exp\left(\beta_1\alpha_0|u_n|^{\frac{N}{N-1}}\right) - S_{N-2}(\beta_1\alpha_0, u_n)\right] |u_n|dx \to 0.$$

Thus, this together with (2.10), and the boundedness of $\{u_n\}$, we deduce that

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \to 0.$$

Using similar proof of as above, we also obtain

$$\int_{\mathbb{R}^N} F(u_n)dx \to 0.$$

□

Proposition 2.1. *Problem* (2.2) *has a ground state solution for any given* $\nu > 0$.

Proof. Similar to the proof of Lemma 2.2, we know that $\mathcal{I}_\nu$ also satisfies mountain pass geometry, then there exists a bounded $(PS)_{c^*}$ sequence $\{u_n\}$ at the level

$$0 < c^* < \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N}.$$

Write

$$\delta = \liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p dx. \tag{2.11}$$

Now, we show that $\delta > 0$.

If $\delta = 0$, applying the Lions' Lemma [21] again, we get

$$u_n \to 0 \text{ in } L^\varrho(\mathbb{R}^N),\ \forall \varrho \in (p, +\infty). \tag{2.12}$$

Furthermore, from (f_5), we adopt similar method of the proof of Proposition 5.2 in [20], we obtain

$$\int_{\mathbb{R}^N} F(u_n)dx \to 0. \tag{2.13}$$

So, this results in

$$|\nabla u_n|_N^N < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Thus, from Lemma 2.7, we get

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \to 0.$$

Hence, as result of the above limit formula, we get $c^* = 0$. This is a contradiction. By (2.11), we can take $\{z_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_2(z_n)} |u_n|^p dx \geq \frac{\delta}{2}.$$

It is clear to note that the number of points in $\mathbb{Z}^N \cap B_2(z_n)$ is less than 4^N. Hence, there exists $y_n \in \mathbb{Z}^N \cap B_2(z_n)$, such that

$$\int_{B_2(y_n)} |u_n|^p dx \geq \sigma > 0.$$

Write $\tilde{u}_n = u_n(\cdot + y_n)$. Then we have $\|\tilde{u}_n\|_\nu = \|u_n\|_\nu$ and

$$\int_{B_2(0)} |\tilde{u}_n|^p dx = \int_{B_2(y_n)} |u_n|^p dx \geq \sigma > 0. \tag{2.14}$$

Going if necessary up to a subsequence, we get

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } \mathcal{M}_\nu, \ \tilde{u}_n \to \tilde{u} \text{ in } L^\gamma_{loc}(\mathbb{R}) \quad \text{for any } \gamma \in [1, \infty),$$

and, from (2.14) we get that $\tilde{u} \neq 0$. In addition, by the $\mathbb{Z}^N$ translation invariance of the problem, we have that $\{\tilde{u}_n\}$ is also $(\mathrm{PS})_{c^*}$ sequence of $\mathcal{I}_\nu$. Without loss of generality, suppose that $|\nabla \tilde{u}_n|_p \to A$ and $|\nabla \tilde{u}_n|_N \to B$. Thus for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned}(1 + aA^p)&\int_{\mathbb{R}^N} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla \varphi dx + (1 + bB^N) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^{N-2} \nabla \tilde{u} \nabla \varphi dx \\ &+ \nu \int_{\mathbb{R}^N} (|\tilde{u}|^{p-2}\tilde{u} + |\tilde{u}|^{N-2}\tilde{u})\varphi dx = \int_{\mathbb{R}^N} f(\tilde{u})\varphi.\end{aligned}$$

We claim that $A = |\nabla \tilde{u}|_p$ and $B = |\nabla \tilde{u}|_N$. If $A > |\nabla \tilde{u}|_p$ or $B > |\nabla \tilde{u}|_N$, we have

$$\langle \mathcal{I}'_\nu(\tilde{u}), \tilde{u} \rangle < 0.$$

By (f_4) and Sobolev imbedding, we obseve that $\langle \mathcal{I}'_\nu(t\tilde{u}), \tilde{u} \rangle > 0$ for t small enough. So, there exists $\sigma \in (0, 1)$ such that $\langle \mathcal{I}'_\nu(\sigma\tilde{u}), \sigma\tilde{u} \rangle = 0$, i.e., $\sigma\tilde{u} \in \mathcal{N}_\nu$. Noting that $c^* = d_\nu$. by (f_4), we can deduce that

$$tf(t) - 2NF(t) \text{ is strict increasing for } t > 0.$$

Hence, based on (f_4) and semicontinuous property of the norm and Fatou's lemma we obtain

$$\begin{aligned}
d_\nu \leq& \mathcal{I}_\nu(\sigma\tilde{u}) \\
=& \mathcal{I}_\nu(\sigma\tilde{u}) - \frac{1}{2N}\langle \mathcal{I}'_\nu(\sigma\tilde{u}), \sigma\tilde{u})\rangle \\
=& \sigma^p \left(\frac{1}{p} - \frac{1}{2N}\right) \|\tilde{u}\|^p_{p,\nu} + a\sigma^{2p} \left(\frac{1}{2p} - \frac{1}{2N}\right) |\nabla \tilde{u}|^{2p}_p \\
&+ \frac{\sigma^N}{2N}\|\tilde{u}\|^N_{N,\nu} + \frac{1}{2N}\int_{\mathbb{R}^N} (f(\sigma\tilde{u})\sigma\tilde{u} - 2NF(\sigma\tilde{u}))dx \\
<& \left(\frac{1}{p} - \frac{1}{2N}\right) \|\tilde{u}\|^p_{p,\nu} + a\left(\frac{1}{2p} - \frac{1}{2N}\right) |\nabla \tilde{u}|^{2p}_p \\
&+ \frac{1}{2N}\|\tilde{u}\|^N_{N,\nu} + \frac{1}{2N}\int_{\mathbb{R}^N} (f(\tilde{u})\tilde{u} - 2NF(\tilde{u}))dx \\
\leq& \lim_{n\to\infty} [\mathcal{I}_\nu(u_n) - \frac{1}{2N}\langle \mathcal{I}'_\nu(u_n), u_n\rangle] \\
=& d_\nu,
\end{aligned}$$

which brings about a contradiction. Therefore, we get $A = |\nabla \tilde{u}|_p$, $B = |\nabla \tilde{u}|_N$ and $\tilde{u}$ is a nontrivial positive ground state solution of problem (2.2). In fact, because $f(t) = 0$ for $t < 0$, so we can conclude that $\tilde{u} > 0$ by the regularity result in [14] and the Harnack inequality in [27]. □

Lemma 2.8. *Let c_ϵ be minimax value which is defined before Lemma 2.3, then there holds*

$$c_\epsilon < d_{V_\infty}.$$

Proof. From Lemma 2.2, we know that the functional $\mathcal{I}_\epsilon$ satisfies the mountain pass geometry. Then by a version of the mountain pass theorem without (PS) condition (see [30]), there exists a (PS) sequence $\{u_n\} \subset X_\epsilon$ such that $\mathcal{I}_\epsilon(u_n) \to c_\epsilon$ and $\mathcal{I}'_\epsilon(u_n) \to 0$. Now, without loss of generality, we may assume that $V_0 = V(0) = \inf_{x\in\mathbb{R}} V(x)$. We fix $\zeta \in \mathbb{R}$ satisfying $V_0 < \zeta < V_\infty$. Applying Proposition 2.1, denote by $w = w_\zeta$ a ground state solution of problem (2.2) $(\nu = \zeta)$. For any given $r > 0$, let $\eta_r \in C_0^\infty(\mathbb{R}^N)$ satisfying $\eta_r(x) = 1$ if $|x| < r$ and $\eta_r(x) = 0$ if $|x| \geq 2r$. We write $u_r(x) = \eta_r(x)w(x)$ and choose $t_r > 0$ such that $\tilde{u}_r = t_r u_r \in \mathcal{M}_\zeta$.

Next, we show that there exists an $r_0 > 0$ such that $\tilde{u}_1 = \tilde{u}_{r_0}$ satisfies $\mathcal{I}_\zeta(\tilde{u}_1) < d_{V_\infty}$. Argue by contradiction that $\mathcal{I}_\zeta(t_r u_r) \geq d_{V_\infty}$ for all $r > 0$. Since $w \in \mathcal{M}_\zeta$ and $u_r \to w$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ as $r \to \infty$, we can deduce that $t_r \to 1$ and

$$d_{V_\infty} \leq \liminf_{r\to+\infty} \mathcal{I}_\zeta(t_r u_r) = \mathcal{I}_\zeta(w) = d_\zeta < d_{V_\infty}.$$

This results in a contradiction.

Because supp $\tilde{u}_1$ is compact, we can select $\epsilon_0 > 0$ such that $V(\epsilon x) \leq \zeta$ for any $x \in$ supp $\tilde{u}_1$. In addition, we have

$$\mathcal{I}_\epsilon(t\tilde{u}_1) \leq \mathcal{I}_\zeta(t\tilde{u}_1)$$

for any $\epsilon \in [0, \epsilon_0)$ and $t \geq 0$, and

$$\max_{t\geq 0} \mathcal{I}_\epsilon(t\tilde{u}_1) \leq \max_{t\geq 0} \mathcal{I}_\zeta(t\tilde{u}_1) = \mathcal{I}_\zeta(\tilde{u}_1) < d_{V_\infty}$$

for any $\epsilon \in [0, \epsilon_0)$. Therefore, from Lemma 2.4 and the definition of d_{V_∞}, we have $c_\epsilon < d_{V_\infty}$ for any $\epsilon \in [0, \epsilon_0)$. □

Lemma 2.9. *Assume that $\{u_n\}$ be a $(PS)_{c_\epsilon}$ sequence with $\epsilon \in [0, \epsilon_0)$ and denote u_ϵ be the weak limit of $\{u_n\}$, then $\{u_n\}$ converges strongly to u_ϵ in X_ϵ, that is, $\mathcal{I}_\epsilon$ satisfies $(PS)_{c_\epsilon}$ for $\epsilon \in [0, \epsilon_0)$.*

Proof. By Remark 2.2, we get that

$$c_\epsilon < \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N} \tag{2.15}$$

and there exist two positive constants C_3 and C_4 satisfying

$$C_3 < \|u_n\|_\epsilon < C_4 \tag{2.16}$$

in the sense of subsequence. In fact, since $\{u_n\}$ is a bounded $(PS)_{c_\epsilon}$ $(c_\epsilon > 0)$ sequence and $F(t) \geq 0, t \in \mathbb{R}$, we know that (2.16) holds in the sense of subsequence.

Next, our first purpose is to show that $u_\epsilon \neq 0$. For doing this, argue by the contradiction that $u_\epsilon = 0$.

Claim. There exist $\delta^*, R_1 > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_{R_1}(y_n)} |u_n|^p dx \geq \delta^*.$$

Otherwise, repeatedly utilizing Lions' Lemma [21], we obtain

$$u_n \to 0 \ \text{ in } L^{\varrho}(\mathbb{R}^N), \quad \forall \varrho \in (p, +\infty).$$

Based on the proof of Proposition 2.1, we get

$$\int_{\mathbb{R}^N} F(u_n)dx \to 0, \quad \text{as } n \to \infty.$$

So, together with the above formula and $\{u_n\}$ being a $(PS)_{c_\epsilon}$ sequence with

$$c_\epsilon < \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N},$$

we deduce that

$$\begin{aligned} c_\epsilon &= \lim_{n\to\infty} \mathcal{I}_\epsilon(u_n) \\ &= \lim_{n\to\infty} (\frac{1}{p}\|u_n\|_{p,V}^p + \frac{a}{2p}|\nabla u_n|_p^{2p} + \frac{1}{N}\|u_n\|_{N,V}^N + \frac{b}{2N}|\nabla u_n|_N^{2N}) \\ &< \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2}\right]\frac{1}{N}. \end{aligned} \tag{2.17}$$

Thus, according to the proof of Lemma 2.7, we still can deduce that

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \to 0, \quad \text{as } n \to \infty.$$

This combining with $\langle \mathcal{I}_\epsilon'(u_n), u_n\rangle \to 0$ implies that

$$\lim_{n\to\infty} \|u_n\|_\epsilon = 0,$$

so, we have $\lim\limits_{n\to\infty} \mathcal{I}_\epsilon(u_n) = 0$ contradicts with $c_\epsilon > 0$ in (2.17). Hence, this claim can be proved.

Now, we let $\{t_n\} \subset (0,\infty)$ be a sequence such that $\{t_n u_n\} \subset \mathcal{M}_{V_\infty}$. We claim that $\{t_n\}$ is bounded. In fact, setting $v_n = u_n(x + y_n)$, by the above claim, we may suppose that, passing to a subsequence, $v_n \rightharpoonup v$ in X_ϵ. Furthermore, utilizing the fact that $u_n \geq 0$ for all $n \in \mathbb{N}$, there exists $C_5 > 0$ and a subset $\mathcal{A} \subset \mathbb{R}^N$ with positive measure such that $v(x) > C_5$ for all $x \in \mathcal{A}$. Thus, we obtain

$$\frac{1}{t_n^{2N-p}}\|u_n\|_{p,V_\infty}^p + \frac{a}{t_n^{2N-2p}}|\nabla u_n|_p^{2p} + \frac{1}{t_n^N}\|u_n\|_{N,V_\infty}^N + b|\nabla u_n|_N^{2N} = \int_{\mathbb{R}^N} \frac{f(t_n u_n)t_n u_n}{t_n^{2N}}dx$$

and so,

$$\frac{1}{t_n^{2N-p}}\|u_n\|_{p,V_\infty}^p + \frac{a}{t_n^{2N-2p}}|\nabla u_n|_p^{2p} + \frac{1}{t_n^N}\|u_n\|_{N,V_\infty}^N + b|\nabla u_n|_N^{2N} = \int_{\mathbb{R}^N} \frac{f(t_n v_n)t_n v_n}{t_n^{2N}}dx,$$

from which

$$\frac{1}{t_n^{2N-p}}\|u_n\|_{p,V_\infty}^p + \frac{a}{t_n^{2N-2p}}|\nabla u_n|_p^{2p} + \frac{1}{t_n^N}\|u_n\|_{N,V_\infty}^N + b|\nabla u_n|_N^{2N} \geq \int_{\mathcal{A}} \frac{f(t_n v_n)t_n v_n}{t_n^{2N}}dx.$$

Since

$$\int_{\mathcal{A}} \frac{f(t_n v_n) t_n v_n}{t_n^4} dx \to +\infty$$

for all a.e. $x \in \mathcal{A}$. Thus, from Fatou's lemma, one concludes that

$$\liminf_{n\to+\infty} |\nabla u_n|_N^{2N} = +\infty,$$

which leads to a contradiction since $\{u_n\}$ is bounded in X_ϵ. Thus, without loss of generality, we may suppose that

$$\lim_{n\to\infty} t_n = t_0 > 0.$$

Next, we divide the remaining part of the proof into three steps.

Step 1. The number t_0 is less than or equal to 1.

Indeed, argue by the contradiction that the above claim does not hold. Then there exist $\delta_1 > 0$ and a subsequence still denoted by $\{t_n\}$ such that $t_n \geq 1 + \delta_1$ for all $n \in \mathbb{N}$. Since $\langle \mathcal{I}'_\epsilon(u_n), u_n\rangle \to 0$ and $\{t_n u_n\} \subset \mathcal{M}_{V_\infty}$, we get

$$\|u_n\|_{p,V}^p + a|\nabla u_n|_p^{2p} + \|u_n\|_{N,V}^N + b|\nabla u_n|_N^{2N} = \int_{\mathbb{R}^N} f(u_n)u_n dx + o(1) \tag{2.18}$$

and

$$t_n^p\|u_n\|_{p,V_\infty}^p + at_n^{2p}|\nabla u_n|_p^{2p} + t_n^N\|u_n\|_{N,V_\infty}^N + bt_n^{2N}|\nabla u_n|_N^{2N} = \int_{\mathbb{R}^N} f(t_n u_n)t_n u_n dx. \tag{2.19}$$

Together with (2.18) and (2.19), we conclude that

$$\begin{aligned}
&o(1) + \left(\frac{1}{t_n^{2N-p}}\|u_n\|_{p,V_\infty}^p - \|u_n\|_{p,V}^p\right) + a|\nabla u_n|_p^{2p}\left(\frac{1}{t_n^{2N-2p}} - 1\right)\\
&+ \left(\frac{1}{t_n^N}\|u_n\|_{N,V_\infty}^N - \|u_n\|_{N,V}^N\right)\\
&= \int_{\mathbb{R}^N}\left(\frac{f(t_n u_n)}{t_n^{2N-1}u_n^{2N-1}} - \frac{f(u_n)}{u_n^{2N-1}}\right)u_n^{2N} dx.
\end{aligned}$$

By condition (V) and $t_n > 1$, given $\epsilon_1 > 0$, there exists $R_{\epsilon_1} > 0$ such that

$$V(\epsilon x) \geq V_\infty - \epsilon_1 > \frac{V_\infty}{t_n^2} - \epsilon_1 \tag{2.20}$$

for any $|x| \geq R_{\epsilon_1}$. Since $\|u_n\|_\epsilon < C_6$ and $u_n \to 0$ in $L^N(B_{R_{\epsilon_1}}(0))$, we deduce that

$$\int_{\mathbb{R}^N}\left(\frac{f(t_n u_n)}{t_n^{2N-1}u_n^{2N-1}} - \frac{f(u_n)}{u_n^{2N-1}}\right)u_n^{2N} dx \leq \epsilon_1 C_7 + o(1). \tag{2.21}$$

Applying the sequence $v_n = u_n(x + y_n)$ again, it follows from (f_4) and (2.21) that

$$0 < \int_{\mathbb{R}^N}\left(\frac{f((1+\delta_1)v_n)}{((1+\delta_1)v_n)^{2N-1}} - \frac{f(v_n)}{v_n^{2N-1}}\right)v_n^{2N} dx \leq \epsilon_1 C_8 + o(1)$$

for any $\epsilon_1 > 0$. Letting $n \to \infty$ in the last inequality and using the Fatou's lemma, this causes a contradiction.

Step 2. $t_0 = 1$.

In this case, since $d_{V_\infty} \le \mathcal{I}_{V_\infty}(t_n u_n)$, we get

$$c_\epsilon + o(1) = \mathcal{I}_\epsilon(u_n) \ge \mathcal{I}_\epsilon(u_n) + m_{V_\infty} - \mathcal{I}_{V_\infty}(t_n u_n).$$

Note that

$$\begin{aligned}\mathcal{I}_\epsilon(u_n) - \mathcal{I}_{V_\infty}(t_n u_n) = &\frac{1}{p}(1-t_n^p)|\nabla u_n|_p^p + \int_{\mathbb{R}^N}(V(\varepsilon x) - t_n^p V_\infty)|u_n|^p dx\\ &+\frac{a}{2p}(1-t_n^{2p})|\nabla u_n|_p^{2p} + \frac{1}{N}(1-t_n^N)|\nabla u_n|_N^N\\ &+\int_{\mathbb{R}^N}(V(\varepsilon x) - t_n^N V_\infty)|u_n|^N dx + \frac{b}{2N}(1-t_n^{2N})|\nabla u_n|_N^{2N}\\ &+\int_{\mathbb{R}^N}(F(t_n u_n) - F(u_n))dx.\end{aligned}$$

Using the boundedness of $\{u_n\}$ in X_ϵ, (2.17), Proposition 1.2 and condition (V), we obtain

$$c_\epsilon + o(1) \ge d_{V_\infty} - \epsilon_1,$$

which means that

$$c_\epsilon \ge d_{V_\infty}.$$

This leads to a contradiction.

Step 3. $t_0 < 1$.

In this case, we may suppose that $t_n < 1$ for all $n \in \mathbb{N}$. From $d_{V_\infty} \le \mathcal{I}_{V_\infty}(t_n u_n)$ and $\langle \mathcal{I}'_{V_\infty}(t_n u_n), t_n u_n\rangle = 0$, we have

$$\begin{aligned}d_{V_\infty} \le &\mathcal{I}_{V_\infty}(t_n u_n) - \frac{1}{2N}\langle \mathcal{I}'_{V_\infty}(t_n u_n), t_n u_n\rangle\\ \le &\left(\frac{1}{p} - \frac{1}{2N}\right) t_n^p \|u_n\|_{p,V_\infty}^p + a\left(\frac{1}{2p} - \frac{1}{2N}\right) t_n^{2p}|\nabla u_n|_p^{2p} + \frac{1}{2N} t_n^N \|u_n\|_{N,V_\infty}^N\\ &+\frac{1}{2N}\int_{\mathbb{R}^N}\left(f(t_n u_n)t_n u_n - 2NF(t_n u_n)\right)dx\\ \le &\mathcal{I}_\epsilon(u_n) - \frac{1}{2N}\langle \mathcal{I}'_\epsilon(u_n), u_n\rangle + \epsilon_1 C + o(1)\\ \le &c_\epsilon + \epsilon_1 C + o(1).\end{aligned}$$

Letting $\epsilon_1 \to 0$ and $n \to \infty$, we know $c_\epsilon \ge d_{V_\infty}$. This causes a contradiction. By Steps 1, 2 and 3, we infer that $u_\epsilon \ne 0$. Thus, from Fatou's Lemma and the characterization of c_ϵ, we deduce that

$$\begin{aligned}c_\epsilon \le &\mathcal{I}_\epsilon(u_\epsilon) - \frac{1}{2N}\langle \mathcal{I}'_\epsilon(u_\epsilon), u_\epsilon\rangle\\ \le &\liminf_{n\to\infty} \mathcal{I}_\epsilon(u_n) - \frac{1}{2N}\langle \mathcal{I}'_\epsilon(u_n), u_n\rangle\\ \le &\limsup_{n\to\infty} \mathcal{I}_\epsilon(u_n) - \frac{1}{2N}\langle \mathcal{I}'_\epsilon(u_n), u_n\rangle\\ \le &c_\epsilon.\end{aligned}$$

In fact, this means that

$$u_n \to u_\epsilon \text{ in } X_\epsilon,$$

and consequently, $\mathcal{I}_\epsilon$ satisfies the $(PS)_{c_\epsilon}$ condition. □

As a direct consequence of Lemmas 2.8-2.9, we get the following conclusion:

Corollary 2.1. *The minimax value c_ϵ is achieved if ϵ is small enough and problem* (1.1) *has a ground state solution if ϵ is small enough. Moreover $c_\epsilon \to d_{V_0}$, as $\epsilon \to 0$.*

Remark 2.3. Applying $f(t) = 0$ for $t \le 0$ and $\langle \mathcal{I}'_\epsilon(u_\epsilon), u_\epsilon^- \rangle = 0$, we get $\|u_\epsilon^-\|_\epsilon = 0$. Hence, we have $u_\epsilon \ge 0$ in $\mathbb{R}^N$. Using the regularity results (see Theorems 2,3) in [14], we conclude that $u_\epsilon \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^N)$. By means of the Harnack inequality in [27], we obtain that $u_\epsilon > 0$ in $\mathbb{R}^N$.

3. Concentration phenomena: Proof of the Theorem 1.1 is completed

In this section, our purpose is to study the concentration phenomenon from ground state solutions of problem (1.1). For the purpose of doing this, we will prove the following several useful lemmas.

Lemma 3.1. *Let $\epsilon_n \to 0$ and $\{u_n\}$ be the sequence of solutions got in Corollary 2.13. Then, there is a sequence $\{y_n\}$ in $\mathbb{R}^N$ satisfying that $v_n = u_n(x + y_n)$ has a convergent subsequence in X_ϵ. Furthermore, up to a subsequence, $\tilde{y}_n := \epsilon_n y_n \to y \in \Lambda$.*

Proof. By Corollary 2.1, there is a sequence $\{u_n\}$ of solutions which is bounded such that $c_{\epsilon_n} = \mathcal{I}_{\epsilon_n}(u_n) \to d_{V_0}$ and

$$0 < d_{V_0} = \lim_{n\to\infty} c_{\epsilon_n} < \left[\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + \frac{b}{2}\left(\frac{\alpha_N}{\alpha_0}\right)^{2N-2} \right] \frac{1}{N}.$$

Applying the same method of proof of previous section of Lemma 2.9, we can deduce that there exist $r, \delta_2 > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p > \delta_2. \tag{3.1}$$

Setting $v_n(x) = u_n(x + y_n)$, passing to a subsequence, we may assume that $v_n \rightharpoonup v \ne 0$ in X_ϵ. Let $t_n > 0$ satisfying $\hat{v}_n = t_n v_n \in \mathcal{M}_{V_0}$. Thus, we get

$$d_{V_0} \le \mathcal{I}_{V_0}(\hat{v}_n) = \mathcal{I}_{V_0}(t_n u_n) \le \mathcal{I}_\epsilon(t_n u_n) \le \mathcal{I}_\epsilon(u_n) \to d_{V_0}$$

and

$$\mathcal{I}_{V_0}(\hat{v}_n) \to d_{V_0}.$$

Thus, the sequence $\{\hat{v}_n\}$ is a minimizing sequence, and utilizing the Ekeland variational principle [11], we may also suppose it is a bounded (PS) sequence at d_{V_0}. Thus, for some subsequence, $\hat{v}_n \rightharpoonup \tilde{v}$ in X_ϵ with $\tilde{v} \ne 0$ and $\mathcal{I}'_{V_0}(\tilde{v}) = 0$. Again applying the proof of Lemma 2.9, we conlude that $\hat{v}_n \to \tilde{v}$ in X_ϵ. Because $\{t_n\}$ is bounded, we can assume that for some subsequence $t_n \to t_0 > 0$, and $v_n \to v$ in X_ϵ.

Next, we will claim that $\{\tilde{y}_n\} = \{\epsilon_n y_n\}$ has a subsequence such that $\tilde{y}_n \to y \in \Lambda$. First, we show that $\{\tilde{y}_n\}$ is bounded. Indeed, argue by contradiction that there exists a subsequence, which is still represented by $\{\tilde{y}_n\}$, satisfying $|\tilde{y}_n| \to \infty$. By $\hat{v}_n \to \tilde{v}$ in X_ϵ and $V_0 < V_\infty$, we obtain

$$\begin{aligned}
d_{V_0} &= \frac{1}{p}\|\tilde{v}\|_{p,V_0}^p + \frac{a}{2p}|\nabla\tilde{v}|_p^{2p} + \frac{1}{N}\|\tilde{v}\|_{N,V_0}^N + \frac{b}{2N}|\nabla\tilde{v}|_N^{2N} - \int_{\mathbb{R}^N} F(\tilde{v})dx \\
&< \frac{1}{p}\|\tilde{v}\|_{p,V_\infty}^p + \frac{a}{2p}|\nabla\tilde{v}|_p^{2p} + \frac{1}{N}\|\tilde{v}\|_{N,V_\infty}^N + \frac{b}{2N}|\nabla\tilde{v}|_N^{2N} - \int_{\mathbb{R}^N} F(\tilde{v})dx \\
&< \liminf_{n\to\infty}\Big(\frac{1}{p}|\nabla\hat{v}_n|_p^p + \frac{a}{2p}|\nabla\hat{v}_n|_p^{2p} + \frac{1}{N}|\nabla\hat{v}_n|_N^N + \frac{b}{2N}|\nabla\hat{v}_n|_N^{2N} \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon_n x + \tilde{y}_n)|\hat{v}_n|^p dx + \frac{1}{N}\int_{\mathbb{R}^N} V(\varepsilon_n x + \tilde{y}_n)|\hat{v}_n|^N dx - \int_{\mathbb{R}^N} F(\hat{v}_n)dx\Big) \\
&= \liminf_{n\to\infty}\Big(\frac{t_n^p}{p}|\nabla u_n|_p^p + \frac{a t_n^{2p}}{2p}|\nabla u_n|_p^{2p} + \frac{t_n^N}{N}|\nabla u_n|_N^N + \frac{b t_n^{2N}}{2N}|\nabla u_n|_N^{2N} \\
&\quad + \frac{t_n^p}{p}\int_{\mathbb{R}^N} V(\varepsilon_n x)|u_n|^p dx + \frac{t_n^N}{N}\int_{\mathbb{R}^N} V(\varepsilon_n x)|u_n|^N dx - \int_{\mathbb{R}^N} F(t_n u_n)dx\Big) \\
&= \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(t_n u_n) \\
&\le \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(u_n) \\
&= d_{V_0}.
\end{aligned}$$

This yields a contradiction. Hence, $\{\tilde{y}_n\}$ is bounded and by choosing a subsequence, $\tilde{y}_n \to y$ in $\mathbb{R}^N$. Then, necessarily $y \in \Lambda$ otherwise we would get a contradiction by the same arguments made as above. □

Let $\epsilon_n \to 0$ as $n \to \infty$ and u_n be the ground state solution of problem (1.1). By Corollary 2.1 we know $\mathcal{I}_\epsilon(u_n) \to m_{V_0}$. Then there exists a sequence $y_n \in \mathbb{R}^N$, such that $v_n = u_n(x + y_n)$ is a solution of

$$\begin{cases} -(1 + a\int_{\mathbb{R}^N} |\nabla v_n|^p dx)\Delta_p v_n - (1 + b\int_{\mathbb{R}^N} |\nabla v_n|^q dx)\Delta_q v_n + V_n(x)(|v_n|^{p-2}v_n \\ +|v_n|^{q-2}v_n) = f(v_n) \quad \text{in } \mathbb{R}^N, \end{cases} \tag{3.2}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n y_n)$. Furthermore, $\{v_n\}$ has a convergent subsequence in X_ϵ and $\tilde{y}_n \to y \in \Lambda$, up to a subsequence, where $\tilde{y}_n = \epsilon_n y_n$. Hence, there exists $\mathcal{H} \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ such that

$$|v_n(x)| \le \mathcal{H}(x) \ \text{ a.e. in } \mathbb{R}^{\mathbb{N}},\ \forall n \in \mathbb{N}. \tag{3.3}$$

Lemma 3.2. *Suppose that conditions (f_1)-(f_6) and (V) hold. Then there exists $C > 0$ such that $\|v_n\|_\infty \le C$ for all $n \in \mathbb{N}$. In addition,*

$$\lim_{n\to\infty} v_n(x) = 0 \text{ uniformly for } n \in \mathbb{N}.$$

Proof. Here, we only give a described proof. By (3.2) and (3.3) and combining with the proofs of [6, Lemma 15] and [3, Lemma 15], this conclusion holds. □

Lemma 3.3. *There exists $\delta_* > 0$ such that $|v_n|_\infty \ge \delta_*$ for all $n \in \mathbb{N}$.*

Proof. Recall that,

$$\delta_2 \le \int_{B_r(y_n)} |u_n|^p dx,$$

then

$$\delta_2 \le \int_{B_r(0)} |v_n|^p dx \le |B_r||v_n|_\infty^p.$$

The above formula implies that

$$|v_n|_\infty \ge \delta_*.$$

Thus, our conclusion holds. □

Concentration around maxima. Let $\tilde{p}_n$ be the global maximum of point of v_n. By Lemma 3.1, we notice that $\tilde{p}_n \in B_R(0)$ for some $R > 0$. Thus, the global maximum of point of u_{ϵ_n} given by $z_{\epsilon_n} = \tilde{p}_n + y_n$ satisfies $\epsilon_n z_{\epsilon_n} = \epsilon_n \tilde{p}_n + \tilde{y}_n$. Because $\{\tilde{p}_n\}$ is bounded, this means that $\epsilon_n z_{\epsilon_n} \to y$, and so $\lim\limits_{n\to\infty} V(\epsilon_n z_{\epsilon_n}) = V_0$.

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Competing interests

The authors declares that they have no competing interests.

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