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DYNAMICAL BEHAVIORS FOR A DISCRETE TWO-NEURON SYSTEM*

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Abstract In this paper, a discrete two-neuron system is investigated, wherein the comparison principle, invariant sets, existence, non-existence, and stability of steady-state solutions are established. These results are new, sharp, and valid for high-dimensional systems. Numerical simulations not only confirmed the obtained theoretical results but also inspired some new reflections. Furthermore, a new local stability theorem and its corollary are derived by applying the Courant-Weyl inequalities. In particular, the obtained theoretical results and numerical simulations will be beneficial for more general bistable discrete systems.

Keywords Discrete Nagumo equation, comparison principle, invariant interval, stability.

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1. Introduction

A spike fired from a neuron is typically followed by a brief refractory period, during which no further spikes can be fired. These facts also motivate the use of discrete-time modeling to capture dynamic transitions [1, 19, 20, 33]. In general, neurons do not fire on their own. They receive signals by the incoming spikes from other neurons [21, 23]. Thus, a single-equation model is unrealistic.

Indeed, the human brain contains approximately 10^{11} neurons, and a typical neuron connects with 10^4 other neurons. Neurons show a wide diversity in morphology and physiology, while the system architecture, the individual neural units, the details of the dynamics of specific neurons, as well as the interneuronal connections remain incompletely characterized. Given current limitations in understanding brain function, rigorous mathematical analysis is not feasible [21, 23, 36]. In order to understand the dynamics of neurons and neural networks, phenomenological models are developed. Recently, at least seven papers have been published in Nature, 640 (2025), 435–505 (also see Petkova and Schuhknecht [34] and the listed references). The authors constructed the most comprehensive mammalian brain structure and neural function connection database to

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date. Simultaneously, Petkova and Schuhknecht [34] provided a commentary article in the same issue of Nature.

Should we then necessarily build an extensive system of equations to model our brains? Clearly, this is neither a sound approach nor a realistic one. In fact, given current limitations in understanding brain function, rigorous mathematical analysis is not feasible [21,23,36]. Although neural connections provide clues for functional predictions, the validity of these predictions, in other words, depends critically on the very few "strong connections". It is indeed the "golden threads" within the neural network that carry the key pathways of function [6].

Based on the facts above, it follows that a nervous system comprises at least two neurons. Neurons integrate simultaneous inputs from excitatory postsynaptic potentials (EPSPs) and inhibitory postsynaptic potentials (IPSPs), which compete spatiotemporally to determine whether an action potential is triggered, where EPSP depolarizes the membrane potential (driving it toward the threshold, promoting excitation) and IPSP hyperpolarizes the membrane potential (pushing it away from the threshold, suppressing excitation) [23], for example, GHRH and SST in Ding et al. [8]. In this paper, we consider the simplest discrete equations system:

$$\begin{cases} u_1^{t+1} - u_1^t = \delta\left(u_2^t - u_1^t\right) + \alpha f\left(u_1^t\right), \\ u_2^{t+1} - u_2^t = \delta\left(u_1^t - u_2^t\right) + \alpha f\left(u_2^t\right), \end{cases} \tag{1.1}$$

where $\delta > 0$ is the coupling parameter, $\alpha > 0$ is a constant, and $f \in C^1[0,1]$ with the threshold $a \in (0,1)$, $f(0) = f(a) = f(1) = 0$, $f(x) < 0$ for $x \in (0,a)$, and $f(x) > 0$ for $x \in (a,1)$. A typical example is given by

$$f(x) = x(1-x)(x-a). \tag{1.2}$$

In system (1.1), $u_2^t - u_1^t$ and $u_1^t - u_2^t$ represent the competition between EPSP and IPSP, and δ is their competition intensity. In the following, we provide some reasons for choosing the above function f. Multistability in a dynamical system means the coexistence of multiple attractors separated in phase space for the same set of system parameters. In such a system, qualitative dynamical transitions can result from changes in the initial conditions, see [2]. For example, the conditions for multistability under which time-delayed recurrent loops of spiking neurons were derived by Foss et al. [11]. Marder et al. [29] and Turrigiano et al. [44] presented examples where bistable neurons support short-term memory mechanisms relying solely on intrinsic neuronal properties. A decision is taken when the decision variable crosses a threshold, see Ratcliff and Rouder [37,38], also see [13,21,23,35,36]. In fact, problem (1.1)-(1.2) can also be seen as a special case of the spatiotemporal discrete version for the Nagumo equation [5,21,27,30,32].

It is well known that Li-Yorke [25] asserts "Period three implies chaos" in certain discrete equations. However, this is not the case for a two-dimensional discrete system, for example, Guckenheomer, Oster and Ipaktchi [16] considered the two-dimensional Leslie model:

$$\begin{cases} x_{t+1} = \left(b_1 x_t + b_2 y_t\right)\exp\left(-a\left(x_t + y_t\right)\right), \\ y_{t+1} = s x_t, \end{cases} \tag{1.3}$$

where b_1, b_2, a and s are positive constants. System (1.3) possesses 3-cycles for certain choices of the parameters which appear numerically to be globally stable [16].

System (1.1) is a two-dimensional discrete system. Such systems have been extensively established by a number of authors, for example, see [9,10,22,26] and the listed references.

Similar bistable dynamics (continuous-time) have been studied in stage-structured two-species Gilpin-Ayala models, see [45], offering analogous stability and state transition insights for our discrete system. However, we will obtain some different results which are decided by the characteristics of the model.

Neural dynamics is strongly dissipative. Energy derived from biochemical sources is used to drive neural activity with substantial energy loss in action-potential generation and propagation [36]. Assume that I and J are two intervals in $\mathbb{R}$. If $\left(u_1^0, u_2^0\right) \in I \times J$ implies that $\left(u_1^t, u_2^t\right) \in I \times J$ for all $t \geq 1$, then we say that $I \times J$ is an invariant set of (1.1). In the next section, we first give the sufficient conditions of the invariant sets for a more general function f. To resolve this, we establish a comparison principle and upper and lower solution method of (1.1). The invariant sets of (1.1) are additional products of the comparison principle. In particular, the conditions of the invariant sets are sharp, with an explanatory example in Section 2 to illustrate. Our results are valid for the following functions, for example, the logistic function

$$f(x) = x\left(1 - \frac{x}{K}\right),\ K > 0,$$

the weak Allee effect

$$f(x) = x\left(1 - \frac{x}{K}\right)\frac{x}{x+\theta},\ K, \theta > 0,$$

the strong Allee effect (1.2) [12, 31], or the Caspari-Watson function

$$f(x) = \frac{s_h x(1-x)(x - s_f/s_h)}{s_h x^2 - (s_f + s_h)x + 1},$$

where $s_h, s_f \in (0,1)$ with $s_f < s_h$ [3, 39, 42, 43, 46, 47].

When the function f is defined by (1.2), the global stability of the steady-state $(0,0)$ or $(1,1)$ is also established by using the comparison principle, see Theorem 2.2, where, we allow that one of neurons equals to the threshold a. When $u_1^0 \in [0,a)$ and $u_2^0 \in (a,1]$ or $u_2^0 \in [0,a)$ and $u_1^0 \in (a,1]$, unfortunately, the comparison principle is invalid. By using an alternative method [24], the invariant sets are also obtained, see Theorem 2.3 in Section 2. In Section 3, we firstly give a non-existence result of non-constant steady-state solutions in the invariant sets $[0,a]^2$ and $[a,1]^2$, see Theorem 3.1, which implies that the non-constant steady-state solutions exhibit a dichotomy. Furthermore, we also find that the corresponding $u_1 = 0$, a or 1 if $u_2 = 0$, a or 1 is a steady-state of (1.1). In this case, the non-constant steady-state solutions need to be $(u_1, u_2) \in (0,a) \times (a,1)$ or $(a,1) \times (0,a)$. To this end, we further establish another non-existence result, see Theorem 3.2. By using the implicit function theorem and the above invariant set principle, the exact existence results are obtained, where, one state for these non-constant steady-state solutions is less than the threshold a, while the other exceeds a [14, 17, 24, 27, 28, 40, 41].

The stability of those steady-state solutions is considered in Section 4. The upper bound of the diffusion coefficient δ and the regions of initial values are given. The numerical simulations are established in Section 5 when f defined by (1.2) with $a = 1/2$ and $\alpha = 1$. All numerical non-constant steady-state solutions and their stability are simulated, which verify the theoretical results and reveal new phenomena that motivate further investigations. To this end, a new local stability theorem and its corollary are derived by using the Courant-Weyl inequalities [7] in this section. In particular, the obtained theoretical results and numerical simulations will be beneficial for more general bistable discrete systems [5, 21, 27]. In the final section, some conclusions are given.

2. Comparison principle and invariant regions

First of all, we give a general comparison principle.

Theorem 2.1. *Let* $p, q \in \mathbb{R}$ *with* $p < q$, *and* $f \in C^1[p,q]$ *with* $f(p) = f(q) = 0$. *Denote* $m = \min_{x\in[p,q]} f'(x)$. *Assume that*

$$\begin{cases} u_1^{t+1} - u_1^t \le \delta\left(-u_1^t + u_2^t\right) + \alpha f\left(u_1^t\right), \\ u_2^{t+1} - u_2^t \le \delta\left(u_1^t - u_2^t\right) + \alpha f\left(u_2^t\right), \end{cases} \tag{2.1}$$

and

$$\begin{cases} v_1^{t+1} - v_1^t \ge \delta\left(-v_1^t + v_2^t\right) + \alpha f\left(v_1^t\right), \\ v_2^{t+1} - v_2^t \ge \delta\left(v_1^t - v_2^t\right) + \alpha f\left(v_2^t\right), \end{cases} \tag{2.2}$$

hold for $0 < \delta \le 1 + \alpha m$ *and* $u_i^t, v_i^t \in [p,q]$ *with* $u_i^0 \le v_i^0$ *for all* $i \in \{1,2\}$, *then* $u_i^t \le v_i^t$ *for all* $(i,t) \in \{1,2\} \times \mathbb{Z}^+$.

Proof. From (2.1), (2.2) and the mean value theorem, we have

$$\begin{aligned} v_1^1 - u_1^1 &\ge \delta\left(v_2^0 - u_2^0\right) + (1-\delta)\left(v_1^0 - u_1^0\right) + \alpha\left(f\left(v_1^0\right) - f\left(u_1^0\right)\right) \\ &= \delta\left(v_2^0 - u_2^0\right) + \left(1 - \delta + \alpha f'(\xi_1)\right)\left(v_1^0 - u_1^0\right), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} v_2^1 - u_2^1 &\ge \delta\left(v_1^0 - u_1^0\right) + (1-\delta)\left(v_2^0 - u_2^0\right) + \alpha\left(f\left(v_2^0\right) - f\left(u_2^0\right)\right) \\ &= \delta\left(v_1^0 - u_1^0\right) + \left(1 - \delta + \alpha f'(\xi_2)\right)\left(v_2^0 - u_2^0\right), \end{aligned} \tag{2.4}$$

where ξ_i is some point between u_i^0 and v_i^0 for $i \in \{1,2\}$. By using $0 < \delta \le 1 + \alpha m$, we have $1 - \delta + \alpha f'(\xi_i) \ge 0$ for all $\xi_i \in [p,q]$. From (2.3) and (2.4), we obtain that $u_i^1 \le v_i^1$ for $i \in \{1,2\}$. Similarly, we can also get that $u_i^2 \le v_i^2$, $u_i^3 \le v_i^3$, $\cdots$ for $i \in \{1,2\}$. The proof is complete. □

Corollary 2.1. *Let* $p, q \in \mathbb{R}$ *with* $p < q$, *and* $f \in C^1[p,q]$ *with* $f(p) = f(q) = 0$. *Then,* $[p,q]$ *is an invariant region of (1.1) when* $0 < \delta \le 1 + \alpha m$, *where* m *is defined in Theorem 2.1.*

Proof. Assume that u_i^t with $u_i^0 \in [p,q]$ for $i \in \{1,2\}$ is a solution of (1.1). Note that $v_i^t \equiv p$ and $w_i^t \equiv q$ are solutions of (1.1). In view of Theorem 2.1, we have

$$p = v_i^t \le u_i^t \le w_i^t = q$$

for any $(i,t) \in \{1,2\} \times \mathbb{Z}^+$. The proof is complete. □

Corollary 2.2. *Let* $p, q, r \in \mathbb{R}$ *with* $p < r < q$, *and* $f \in C^1[p,q]$ *with* $f(p) = f(r) = f(q) = 0$. *Denote* $\min_{x\in[p,r]} f'(x) = m_1$, $\min_{x\in[r,q]} f'(x) = m_2$ *and* $\min_{x\in[p,q]} f'(x) = m = \min\{m_1, m_2\}$. *Then,* $[p,r]$, $[r,q]$ *and* $[p,q]$ *are invariant regions of (1.1) when* $0 < \delta \le 1 + \alpha m_1$, $0 < \delta \le 1 + \alpha m_2$ *and* $0 < \delta \le 1 + \alpha m$, *respectively.*

The proof of Corollary 2.2 is similar to that of Corollary 2.1, and is thus omitted. We remark that in the following discussions, the term $f(u)$ in (1.1) can be a more general bistable or strong

Allee effect function. However, without loss of generality, we keep the simple form (1.2). Then, we have

$$\begin{aligned}&f'(x) = 2ax - a + 2x - 3x^2,\\&f'(0) = -a,\ f'(a) = a(1-a) \text{ and } f'(1) = -(1-a).\end{aligned}$$

In view of Corollary 2.2, we immediately obtain the following result. In the following, we always assume $f(x) = x(1-x)(x-a)$ unless specifically emphasized.

Corollary 2.3. *$[0,a]$, $[a,1]$ and $[0,1]$ are the invariant regions of (1.1) when $0 < \delta \le 1 - \alpha a$, $0 < \delta \le 1 - \alpha(1-a)$, and $0 < \delta \le 1 - \alpha \max\{a, 1-a\}$, respectively.*

Remark 2.1. When $1-\delta-\alpha \max\{a, 1-a\} < 0$, Corollary 2.3 is invalid. Indeed, let $1/2 < a < 1$, we have $\max\{a, 1-a\} = a$. Let $u_1^0 = 0$. Then, we have

$$u_2^1 = u_2^0 - \delta u_2^0 + \alpha u_2^0 \left(u_2^0 - a\right)\left(1 - u_2^0\right).$$

Notice that

$$g(x) = x\left(1 - \delta - \alpha a + \alpha(1+a)x - \alpha x^2\right),$$

we can choose $0 < u_2^0 \ll 1$ such that $u_2^1 = g\left(u_2^0\right) < 0$ when $g'(0) = 1 - \delta - \alpha a < 0$. So, the conditions of invariant regions are sharp.

By using Theorem 2.1, we can also obtain the following global stability result.

Theorem 2.2. *When $0 < \delta \le 1 - \alpha(1-a)$ holds, any solution $\left(u_1^t, u_2^t\right)$ of (1.1) satisfies $\lim_{t\to\infty} u_i^t = 1$ for $i \in \{1,2\}$, where $u_1^0, u_2^0 \in [a,1]$ with $u_1^0 \ne a$ or $u_2^0 \ne a$. Similarly, if $0 < \delta \le 1 - \alpha a$, any solution $\left(u_1^t, u_2^t\right)$ of (1.1) satisfies $\lim_{t\to\infty} u_i^t = 0$ for $i \in \{1,2\}$, where $u_1^0, u_2^0 \in [0,a]$ with $u_1^0 \ne a$ or $u_2^0 \ne a$.*

Proof. For $u_1^0, u_2^0 \in (a,1]$, let $v_i^0 = \min\left\{u_1^0, u_2^0\right\}$ for $i \in \{1,2\}$. Clearly, $\left(v_1^t, v_2^t\right)$ is a sub-solution of (1.1), and $\lim_{t\to\infty} v_i^t = 1$ for $i \in \{1,2\}$. For $u_1^0, u_2^0 \in [0,a)$, the case is similar.

If $u_1^0 = a$ and $u_2^0 \in (a,1]$, then,

$$u_1^1 - a = \delta\left(u_2^0 - a\right) > 0,$$

and

$$u_2^1 = a + \left(u_2^0 - a\right)\left[1 + \alpha u_2^0\left(1 - u_2^0\right) - \delta\right] > a,$$

by using $0 < \delta \le 1 - \alpha(1-a)$. In view of the above discussion, we have $\lim_{t\to\infty} u_i^t = 1$ for $i \in \{1,2\}$. When $u_1^0 = a$ and $u_2^0 \in [0,a)$, the case is similar. The proof is complete. □

In the following, we assume $0 < a \le 1/2$, and introduce the substitution $u_i^t = 1 - v_i^t$ into (1.1). This transforms the analysis for $0 < a < 1/2$ to a symmetric case, yielding:

$$\begin{cases} v_1^{t+1} - v_1^t = \delta\left(-v_1^t + v_2^t\right) + \alpha v_1^t\left(1 - v_1^t\right)\left(v_1^t - (1-a)\right),\\ v_2^{t+1} - v_2^t = \delta\left(v_1^t - v_2^t\right) + \alpha v_2^t\left(1 - v_2^t\right)\left(v_2^t - (1-a)\right),\end{cases}$$

which remains valid for $1/2 < a < 1$. Thus, we only consider the case $0 < a \le 1/2$ unless specifically emphasized. At this time, we have $\max\{a, 1-a\} = 1-a$.

Theorem 2.3. *Assume that* $0 < \delta \leq 1 - \alpha(1-a)$ *with* $\delta/\alpha = \kappa \leq \widehat{u}_1(a - \widehat{u}_1)$, *then,* $[\widetilde{u}_1, \widehat{u}_1] \times [\widehat{u}_2, \widetilde{u}_2]$ *is an invariant region of (1.1), where*

$$\widetilde{u}_1 = \frac{a - \sqrt{a^2 - 4\kappa}}{2}, \widetilde{u}_2 = \frac{a + 1 + \sqrt{(a-1)^2 - 4\kappa}}{2},$$
$$\widehat{u}_1 = \frac{a + 1 - \sqrt{a^2 - a + 1}}{3} \text{ and } \widehat{u}_2 = \frac{a + 1 + \sqrt{a^2 - a + 1}}{3}.$$

Proof. For $0 < a \leq 1/2$, we have

$$0 < \sqrt{a^2 - a + 1} < 1.$$

The algebraic equation

$$f'(u) = 2au - a + 2u - 3u^2 = 0,$$

has two roots

$$\widehat{u}_1 = \frac{a + 1 - \sqrt{a^2 - a + 1}}{3} \text{ and } \widehat{u}_2 = \frac{a + 1 + \sqrt{a^2 - a + 1}}{3}. \tag{2.5}$$

Denote $\kappa \triangleq \delta/\alpha$, and let

$$\begin{aligned} u_1^1 - u_1^0 &= \delta\left(-u_1^0 + u_2^0\right) - \alpha u_1^0\left(a - u_1^0\right)\left(1 - u_1^0\right) \\ &\leq \alpha\left(\kappa - u_1^0\left(a - u_1^0\right)\right)\left(1 - u_1^0\right) \\ &\leq 0, \end{aligned} \tag{2.6}$$

and

$$u_2^1 - u_2^0 \geq -\alpha u_2^0\left[\kappa - \left(1 - u_2^0\right)\left(u_2^0 - a\right)\right] \geq 0. \tag{2.7}$$

When $0 < \delta \leq 1 - \alpha(1-a)$,

$$\kappa - u_1(a - u_1) = 0,$$

has a root

$$\widetilde{u}_1 = \frac{a - \sqrt{a^2 - 4\kappa}}{2} \tag{2.8}$$

which implies that $0 < \widetilde{u}_1 \leq \widehat{u}_1 < a$. Similarly,

$$\kappa - (1 - u_2)(u_2 - a) = 0$$

has a root

$$\widetilde{u}_2 = \frac{1 + a + \sqrt{(1-a)^2 - 4\kappa}}{2}, \tag{2.9}$$

which implies that $a < \widehat{u}_2 \leq \widetilde{u}_2 < 1$.

On the other hand, we easily prove that

$$\widehat{u}_1(a - \widehat{u}_1) \leq (1 - \widehat{u}_2)(\widehat{u}_2 - a),$$

because

$$\begin{aligned} \widehat{u}_1(a - \widehat{u}_1) - (1 - \widehat{u}_2)(\widehat{u}_2 - a) &= \frac{1}{3}\left(2a + \frac{1 - 2a}{3}\sqrt{a^2 - a + 1} - 1\right) \\ &= \frac{1}{9}(1 - 2a)\left(\sqrt{a^2 - a + 1} - 3\right) \\ &\leq 0. \end{aligned}$$

In summary, when $\left(u_1^0, u_2^0\right) \in [\widetilde{u}_1, \widehat{u}_1] \times [\widehat{u}_2, \widetilde{u}_2]$ with $0 < \delta \leq 1 - \alpha(1-a)$ and $\kappa \leq \widehat{u}_1(a - \widehat{u}_1)$, (2.6) and (2.7) hold. By using (2.6), (2.7) and the induction method, the proof is complete. □

3. Existence of non-constant steady-state solutions

Now, we consider the existence of non-constant steady-state solutions of (1.1) or

$$\begin{cases} \kappa\left(-u_1+u_2\right)+f\left(u_1\right)=0, \\ \kappa\left(u_1-u_2\right)+f\left(u_2\right)=0, \end{cases} \tag{3.1}$$

where $\kappa=\delta/\alpha$.

First of all, to analyze the non-existence of the solutions, let

$$F\left(\kappa, a; u_1, u_2\right)=\begin{pmatrix} \kappa\left(-u_1+u_2\right)+f\left(u_1\right) \\ \kappa\left(u_1-u_2\right)+f\left(u_2\right) \end{pmatrix}.$$

Assume that $(u_1, u_2) \in [0,1]\times[0,1]$ is a solution of (3.1) for some (κ, a), take the inner product of the system with (u_1, u_2), we have

$$0 \leq \kappa\left(u_1-u_2\right)^2=u_1 f\left(u_1\right)+u_2 f\left(u_2\right)<0 \text{ for } u_1, u_2 \in (0,1) \tag{3.2}$$

which implies that u_1 and u_2 cannot both belong to the interval $(0, a)$. Let $u_i=1-v_i$ for $i \in \{1,2\}$. Similarly, we can prove that v_1 and v_2 cannot both belong to the interval $(0, 1-a)$. That is, u_1 and u_2 cannot both belong to the interval $(a, 1)$.

When $u_1=0$, a, or 1, clearly, u_2 must also equal to 0, a, or 1 from (3.1), respectively. This leads to the following result.

Theorem 3.1. *Assume that $u_1 \neq u_2$ and $u_1, u_2 \in (0, a)$ or $u_1, u_2 \in (a, 1)$, then (u_1, u_2) is not a non-constant solution of (3.1) for any $\kappa>0$ and $0<a<1$.*

Remark 3.1. Theorem 3.1 implies that the non-constant steady-state solutions of (3.1) need to be $(u_1, u_2) \in (0, a)\times(a, 1)$ or $(u_1, u_2) \in (a, 1)\times(0, a)$. Thus, Theorem 3.1 can also be seen another version of the special case of Lemma 3.2 in Stehlík [40].

In the following, we establish a non-existence result for $(u_1, u_2) \in (0, a)\times(a, 1)$. By symmetry, the result can be extended to the case of $(u_1, u_2) \in (a, 1)\times(0, a)$.

Theorem 3.2. *If*

$$\kappa>\frac{1}{6}\left(a^2-a+1\right),$$

then, (3.1) has no non-constant solution (u_1, u_2) with $(u_1, u_2) \in (0, a)\times(a, 1)$ or $(u_1, u_2) \in (a, 1)\times(0, a)$.

Proof. Assume that $(u_1, u_2) \in (0, a)\times(a, 1)$ is a solution of (3.1). The difference between the two equations of (3.1) is

$$2\kappa\left(u_2-u_1\right)+f\left(u_1\right)-f\left(u_2\right)=0,$$

or

$$\kappa=\frac{f\left(u_2\right)-f\left(u_1\right)}{2\left(u_2-u_1\right)}=\frac{f'(\zeta)}{2} \leq \frac{1}{6}\left(a^2-a+1\right).$$

This is a contradiction. The proof is complete. □

Remark 3.2. Theorem 3.2 is new. In particular,

$$\kappa_{\max} = \frac{1}{6}\left(a^2 - a + 1\right) \le \frac{1}{8},$$

which is a best estimate. For example, let $f(u) = u(1-u)(u-0.5)$. In such a discrete Nagumo networks, the non-trivial symmetric steady states satisfy the condition $u_1 = 1 - u_2$. By substituting this into the equilibrium equation $\kappa(u_2 - u_1) = f(u_2)$, we obtain:

$$\kappa(2u_2 - 1) = \frac{1}{2}u_2(1-u_2)(2u_2-1),$$

or

$$\kappa = \frac{1}{2}u_2(1-u_2) \le \frac{1}{8} = 0.125.$$

Next, to obtain the non-constant solutions of (3.1), we need the following implicit function theorem which can be seen in Grandmont [15].

Lemma 3.1. (Implicit function theorem) *Let W be an open set in $\mathbb{R}^m \times \mathbb{R}^p$, and F a C^r map from W into $\mathbb{R}^p$, that is $(x,y) \to F(x,y)$ where x and y are vectors of $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. Let (x_0,y_0) in W be such that $F(x_0,y_0) = c$ and suppose that the Jacobian matrix of the map $F(x_0,\cdot)$ is invertible at $y = y_0$. Then there are open sets U and V in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, with x_0 in U, y_0 in V and $U \times V$ contained in W, and a unique C^r map $G: U \to V$, such that $F(x, G(x)) = c$ for all x in U, and moreover, $F(x,y) \ne c$ if x is in U, y in V and $y \ne G(x)$.*

Assume that (u_1^*, u_2^*) is a solution of (3.1) for some (κ_0, a_0). Then, the Jacobian matrix:

$$J(u_1^*, u_2^*) = \begin{pmatrix} -\kappa + f'(u_1^*) & \kappa \\ \kappa & -\kappa + f'(u_2^*) \end{pmatrix} = \begin{pmatrix} -\kappa + \sigma & \kappa \\ \kappa & -\kappa + \eta \end{pmatrix},$$

has the eigenvalues

$$\lambda_{1,2}^J(u_1^*, u_2^*) = \frac{\sigma + \eta - 2\kappa \pm \sqrt{(\sigma - \eta)^2 + 4\kappa^2}}{2}, \tag{3.3}$$

where $\sigma = f'(u_1^*)$ and $\eta = f'(u_2^*)$. In view of Lemma 3.1, there exists a unique implicit function when the Jacobian matrix $J(u_1^*, u_2^*)$ is reversible. When $(u_1^*, u_2^*) = (0,1)$, we have

$$\lambda_1^J(0,1) = \frac{\sqrt{(1-2a)^2 + 4\kappa^2} - (2\kappa + 1)}{2} < 0,$$

and

$$\lambda_2^J(0,1) = -\frac{1 + 2\kappa + \sqrt{(1-2a)^2 + 4\kappa^2}}{2} < 0,$$

for any $a \in (0,1)$ and $\kappa > 0$. For $a_0 \in (0,1)$ and $\kappa_0 = 0$, we have $F(0, a_0; 0, 1) = 0$ which implies that there exists a unique implicit function for sufficiently small $\kappa > 0$. For $(u_1^*, u_2^*) = (1,0)$, the analysis is similar.

By using the above analysis and Theorem 2.3, the following result is clear.

Theorem 3.3. *For $\delta/\alpha = \kappa < \widehat{u}_1(a - \widehat{u}_1)$, (3.1) has a unique non-constant solution $(u_1^*(\kappa,a), u_2^*(\kappa,a)) \in [0, \widetilde{u}_1) \cup (\widetilde{u}_2, 1]$, where $\widehat{u}_1$, $\widetilde{u}_1$ and $\widetilde{u}_2$ are defined in Theorem 2.3.*

Remark 3.3. Theorem 3.3 provides an upper bound of κ, improving prior results in [17,18,40,41], for example, Lemma 2 in [18].

4. The stability of steady-state solutions

Assume that (u_1^*, u_2^*) is a solution of problem (3.1), we consider the following linearized form

$$\begin{cases} v_1^{t+1} = v_1^t + \delta\left(-v_1^t + v_2^t\right) + \alpha f'\left(u_1^*\right) v_1^t, \\ v_2^{t+1} = v_2^t + \delta\left(v_1^t - v_2^t\right) + \alpha f'\left(u_2^*\right) v_2^t, \end{cases} \tag{4.1}$$

or

$$\begin{pmatrix} v_1^{t+1} \\ v_2^{t+1} \end{pmatrix} = \begin{pmatrix} 1-\delta+\alpha f'\left(u_1^*\right) & \delta \\ \delta & 1-\delta+\alpha f'\left(u_2^*\right) \end{pmatrix} \begin{pmatrix} v_1^t \\ v_2^t \end{pmatrix}, \tag{4.2}$$

where

$$f'\left(u_i^*\right) = 2au_i^* - a + 2u_i^* - 3\left(u_i^*\right)^2 \text{ for } i \in \{1,2\}.$$

Denote

$$C = \begin{pmatrix} 1-\delta+\alpha f'\left(u_1^*\right) & \delta \\ \delta & 1-\delta+\alpha f'\left(u_2^*\right) \end{pmatrix}$$

Case (i). $u_1^* = u_2^* = 0$. We have

$$\lambda_1^C(0,0) = 1 - 2\delta - \alpha a$$

and

$$\lambda_2^C(0,0) = 1 - \alpha a.$$

It is easy to know that

$$1 > \lambda_2^C(0,0) = 1 - \alpha a > \lambda_1^C(0,0) = 1 - 2\delta - \alpha a > -1$$

or

$$2\delta + \alpha a < 2, \tag{4.3}$$

which implies that $(0,0)$ is local asymptotically stable.

Case (ii). $u_1^* = u_2^* = 1$. We have

$$\lambda_1^C(1,1) = 1 - 2\delta - \alpha(1-a)$$

and

$$\lambda_2^C(1,1) = 1 - \alpha(1-a).$$

Similarly, we get

$$1 > \lambda_2^C(1,1) = 1 - \alpha(1-a) > \lambda_1^C(1,1) = 1 - 2\delta - \alpha(1-a) > -1$$

or

$$2\delta + \alpha(1-a) < 2, \tag{4.4}$$

which implies that $(1,1)$ is local asymptotically stable.

Case (iii). $u_1^* = u_2^* = a$. We have

$$\lambda_1^C(a,a) = 1 + \alpha a(1-a) - 2\delta$$

and

$$\lambda_2^C(a,a) = 1 + \alpha a(1-a) > 1, \tag{4.5}$$

which implies that (a,a) is not local asymptotically stable.

Theorem 4.1. *Assume that*

$$\delta < 1 - \frac{\alpha(1-a)}{2},$$

then the corresponding stationary solutions $(0,0)$ *and* $(1,1)$ *of (1.1) are locally asymptotically stable, while the fixed point* (a,a) *is unstable.*

Remark 4.1. Theorem 4.1 provides an upper bound of δ.

Theorem 4.2. *Suppose the hypotheses of Theorem 2.3 hold. For any initial condition* $\left(u_1^0, u_2^0\right) \in [0, \widetilde{u}_1) \times (\widetilde{u}_2, 1]$, *then,* $\lim_{t\to\infty} u_i^t = u_i^*(\kappa)$ *for* $i \in \{1,2\}$, *where* $(u_1^*(\kappa), u_2^*(\kappa)) \in [0, \widetilde{u}_1) \times (\widetilde{u}_2, 1]$ *are the non-constant steady-state solutions of (1.1).*

The proof of Theorem 4.2 is similar with Theorem 2.5 in Keener [24], also see Lemma 3.1 in Chow and Shen [5], thus, it is omitted.

5. Numerical simulations and further results

For the convenience of simulation, we denote

$$\begin{aligned}
R_{00} &= \{(\delta,\alpha,a) : \delta + \alpha a \le 1\},\\
R_{00}^* &= \left\{(\delta,\alpha,a) : \delta + \frac{\alpha a}{2} < 1\right\},\\
R_{11} &= \{(\delta,\alpha,a) : \delta + \alpha(1-a) \le 1\},\\
R_{11}^* &= \left\{(\delta,\alpha,a) : \delta + \frac{\alpha(1-a)}{2} < 1\right\},\\
R &= R_{00} \cap R_{11} = \{(\delta,\alpha,a) : \delta + \alpha \max\{a, 1-a\} \le 1\}
\end{aligned}$$

and

$$R^* = R_{00}^* \cap R_{11}^* = \left\{(\delta,\alpha,a) : \delta + \frac{\alpha \max\{a,1-a\}}{2} < 1\right\}.$$

When $\alpha = 1$ and $a = 1/2$, we have

$$\begin{aligned}
&R_{00} = R_{11} = R = \left\{\left(\delta, 1, \frac{1}{2}\right) : \delta \le \frac{1}{2}\right\},\\
&R_{00}^* = R_{11}^* = R^* = \left\{\left(\delta, 1, \frac{1}{2}\right) : \delta < \frac{3}{4}\right\},\\
&\widehat{u}_1 = \frac{1}{2} - \frac{\sqrt{3}}{6} \text{ and } \widehat{u}_2 = \frac{1}{2} + \frac{\sqrt{3}}{6},\\
&\widehat{u}_1(a - \widehat{u}_1) = (1-\widehat{u}_2)(\widehat{u}_2 - a) = \frac{\sqrt{3}-1}{12},\\
&\widetilde{u}_1 = \frac{1}{4} - \sqrt{\frac{1}{16} - \kappa},
\end{aligned}$$

and

$$\widetilde{u}_2 = \frac{3}{4} + \sqrt{\frac{1}{16} - \kappa}.$$

Additionally, we let

$$\kappa < \frac{\sqrt{3}-1}{12} \approx 0.061 < \frac{1}{16},$$

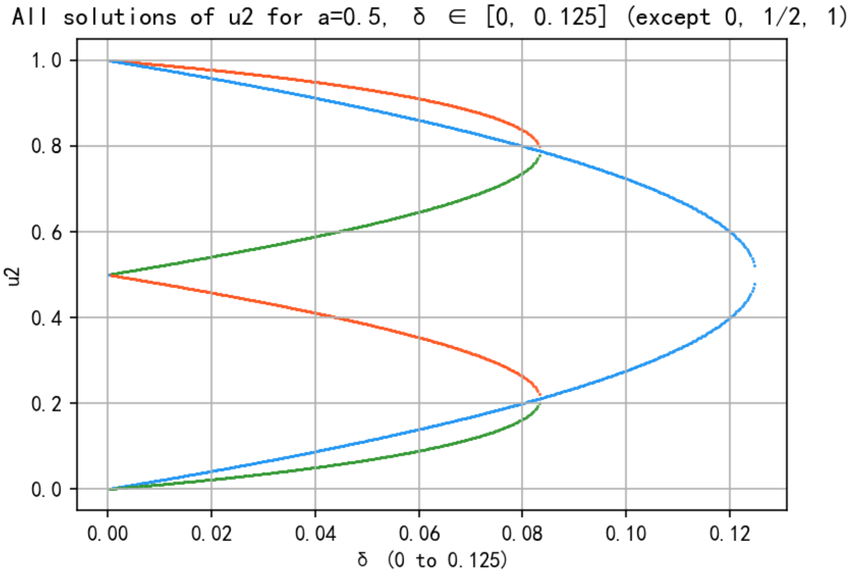

Figure 1. u_2 for $\alpha = 1,\ a = 0.5$ and $\delta \in [0, 0.125]$.

and get that

$$\frac{1}{16} - \frac{\sqrt{3}-1}{12} \approx 1.4958 \times 10^{-3}.$$

In Figure 1, we can see all non-constant solutions of (3.1) for $\alpha = 1$ and $a = 1/2$.

In view of Theorem 3.1, the non-constant solutions of (3.1) lie in the cross interval $(0, a) \times (a, 1)$ or its symmetric counterpart $(a, 1) \times (0, a)$. Assume that $(u_1(\kappa), u_2(\kappa)) \in (0, a) \times (a, 1)$ or $(u_2(\kappa), u_1(\kappa)) \in (0, a) \times (a, 1)$ is a solution of (3.1) for sufficiently small κ. From (3.1), we have

$$\begin{cases} -u_1 + u_2 + \kappa\left(-\dfrac{du_1}{d\kappa} + \dfrac{du_2}{d\kappa}\right) + f'(u_1)\dfrac{du_1}{d\kappa} = 0, \\ u_1 - u_2 + \kappa\left(\dfrac{du_1}{d\kappa} - \dfrac{du_2}{d\kappa}\right) + f'(u_2)\dfrac{du_2}{d\kappa} = 0, \end{cases}$$

that is

$$f'(u_1)\frac{du_1}{d\kappa} = -f'(u_2)\frac{du_2}{d\kappa}, \tag{5.1}$$

and

$$\left(\left(f'(u_2) - \kappa\right) - \frac{\kappa f'(u_2)}{f'(u_1)}\right)\frac{du_2}{d\kappa} = u_2 - u_1. \tag{5.2}$$

When $u_1 = 0$ and $u_2 = a$, substitute the values into (5.2), we have

$$(1 - a - \kappa)\frac{du_2}{d\kappa} = 1,$$

which implies that $du_2/d\kappa > 0$ for sufficiently small κ. Similarly, we also have $du_2/d\kappa < 0$ for $u_1 = 1$ and $u_2 = a$, $u_1 = 0$ and $u_2 = 1$, with sufficiently small κ. Thus, in Figure 1, the curves of the same color represent a solution set. we also observe the maximum value of δ is $\delta \approx 0.125$, however, $R = \{(\delta, 1, 1/2) : \delta \leq 1/2\}$ and $R^* = \{(\delta, 1, 1/2) : \delta < 3/4\}$.

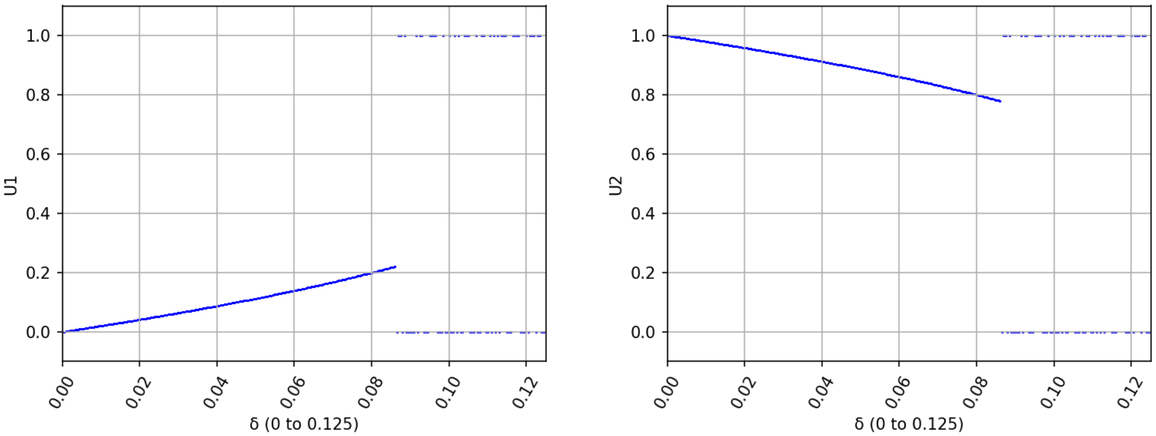

Figure 2. Stable solutions for $\alpha = 1$, $a = 1/2$ and $\left(u_1^0, u_2^0\right) = (0, 1)$.

In the following, we simulate the non-constant solutions of (3.1) for $\alpha = 1$, using the initial condition $\left(u_1^0, u_2^0\right) = (0, 1)$ over the time domain $t \in [0, 2000]$.

From Figure 2, numerical simulations show that the steady-state solutions are stable for $0 < \delta < 0.086$. However, in view of Theorem 4.2, the stability condition is stricter: $\delta < \left(\sqrt{3} - 1\right)/12 \approx 0.061$. This confirms Theorem 4.2 as a sufficient condition. In particular, we also note that the number 0.082 happens to be the maximum range of the steady-state solutions reduced by $(0, a)$ or $(1, a)$. On the other hand, from Figure 2, $\max\left\{u_1^*(\kappa)\right\} \approx 0.211\,32 = \widehat{u}_1$.

To achieve further results, we need the following Courant-Weyl inequalities which can be seen in Cvetković, Doob and Sachs [7].

Lemma 5.1. *Let* $\lambda_1(X), \cdots, \lambda_m(X)$ $(\lambda_1(X) \geq \lambda_2(X) \geq \cdots, \lambda_m(X))$ *be the eigenvalues of a real symmetric matrix* X. *If* A *and* B *are real symmetric matrices of order* m, *and* $C = A + B$, *then*

$$\lambda_{i+j+1}(C) \leq \lambda_{i+1}(A) + \lambda_{j+1}(B),$$
$$\lambda_{m-i-j}(C) \geq \lambda_{m-i}(A) + \lambda_{m-j}(B),$$

where $0 \leq i, j,\ i + j + 1 \leq m$. *In particular,*

$$\lambda_1(C) \leq \lambda_1(A) + \lambda_1(B).$$

Note that

$$\begin{pmatrix} 1-\delta & \delta \\ \delta & 1-\delta \end{pmatrix}$$

has the eigenvalues 1 and $1 - 2\delta$. By Lemma 5.1, we can conclude that $f'\left(u_1^*\right) < 0$ and $f'\left(u_2^*\right) < 0$, which implies that $u_1^* \in [0, \widehat{u}_1)$ and $u_2^* \in (\widehat{u}_2, 1]$ or $u_2^* \in [0, \widehat{u}_1)$ and $u_1^* \in (\widehat{u}_2, 1]$, where $\widehat{u}_1$ and $\widehat{u}_2$ are defined by (2.5). Let

$$1 - 2\delta + \alpha f'(u) > -1.$$

By the monotonicity of $f'(u)$, we have the following result.

Theorem 5.1. *Assume that* $\kappa < \widehat{u}_1 (a - \widehat{u}_1)$ *and*

$$\delta < 1 - \frac{\alpha (1-a)}{2},$$

then, the stationary solutions $(u_1^* (\kappa), u_2^* (\kappa))$ *of (1.1) is locally asymptotically stable, where* $\widehat{u}_1$ *is defined by (2.5).*

Proof. From Theorem 2.3, we have

$$\widehat{u}_1 (a - \widehat{u}_1) \leq (1 - \widehat{u}_2)(\widehat{u}_2 - a)$$

for $0 < a \leq 1/2$, which implies that $u_1^* (\kappa) < \widehat{u}_1 < a < \widehat{u}_2 < u_2^* (\kappa)$ and that

$$-a < f' (u_1^* (\kappa)) < 0 \text{ and } 1 - a < f' (u_2^* (\kappa)) < 0.$$

The proof is complete. □

In view of Theorem 5.1, we can immediately obtain the following result.

Corollary 5.1. *Assume that all conditions of Theorem 5.1 hold. Then, there is a unique* $(u_1^* (\kappa, a), u_2^* (\kappa, a)) \in [0, \widehat{u}_1) \times (\widehat{u}_2, 1]$*, where* $\widehat{u}_1$ *and* $\widehat{u}_2$ *are defined by (2.5).*

Remark 5.1. Theorem 5.1 and Corollary 5.1 are novel results.

6. Conclusions

In this paper, the comparison principle, invariant intervals, existence, non-existence, and stability of steady-state solutions of two-neuron system (1.1) have been investigated. The following conclusions are obtained.

(i) A comparison principle (or upper and lower solution method) is first obtained for a more general function f of (1.1), see Theorem 2.1. The invariant intervals of (1.1) are additional products of the comparison principle, refer to Corollaries 2.1 and 2.2. When f is defined by (1.2), the conditions of the invariant intervals are sharp. An explanatory example is also given, see Corollary 2.3 and Remark 2.1. These results guarantee the invariance of both neuronal states within the intervals $[0, a]$, $[a, 1]$, and $[0, 1]$.

(ii) By using the comparison principle, a global stability result is obtained when f is defined by (1.2), as stated in Theorem 2.2 in Section 2. Theorem 2.2 implies that the states converge to 1 when one state is greater than the threshold a and the other is greater than or equal to a, otherwise, they converge to 0.

(iii) Noting that the duality of the system, we only consider the case $0 < a \leq 1/2$ when f is defined by (1.2). In this case, we focus on the existence, non-existence, and stability of steady-state solutions induced by the states $(0, 1)$ or $(1, 0)$. Thus, the conditions of the invariant intervals are investigated in Section 2, see Theorem 2.3, which provides a dichotomy of two neuronal states.

(iv) Non-existence results are important as they provide necessary conditions for existence. We first establish a non-existence result for non-constant steady-state solutions within the invariant intervals $[0, a]$ and $[a, 1]$, see Theorem 3.1. On the other hand, we find that if $u_2 = 0$, a, 1, then $u_1 = 0$, a, 1 corresponds to a steady-state of (1.1). Non-constant steady-state solutions must satisfy $(u_1, u_2) \in (0, a) \times (a, 1)$ or $(a, 1) \times (0, a)$, see Theorem 3.2. Theorems 3.1 and 3.2 provide the non-existence conditions for non-constant steady-states of the two-neuron system.

(v) By using the implicit function theorem [15], the existence of non-constant steady-state solutions is considered. They can also be obtained using methods from [17, 40, 41]. However, the existence of non-constant steady-state solutions induced by $(0, 1)$ is established in this section, see Theorem 3.3 in Section 3. Correspondingly, their stability is also considered, see Theorems 4.1 and 4.2 in Section 4. Theorem 3.3 provides a dichotomy for the existence result of non-constant steady-states for the two-neuron system, and Theorem 4.2 corresponds to the stability analysis of these states as in Theorem 3.3.

(vi) To verify our theoretical findings, numerical simulations of the stationary solutions and their stability are conducted in Section 5 for $\alpha = 1$ and $a = 1/2$. These simulations reveal new insights, leading to the addition of a supplementary theorem and a corollary in Section 5, see Theorem 5.1 and Corollary 5.1.

(vii) From Figure 1, we find that the maximum value of δ is $\delta \approx 0.125$. However, $R = \{(\delta, 1, 1/2) : \delta \leq 1/2\}$ and $R^* = \{(\delta, 1, 1/2) : \delta < 3/4\}$. Theorem 3.2 is supported by illustrative numerical simulations.

(viii) All mathematical results and numerical simulations consistently support the dynamical behaviors of 2^2 states for the two-neuron system [4, 13, 14, 34, 36].

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Data availability. No data was used for the research described in the article.

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