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SOLUTIONS TO NON-HOMOGENEOUS SCHRÖDINGER-POISSON SYSTEM INVOLVING A (P, Q)-LAPLACIAN OPERATOR*

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Abstract In this paper, we consider the following generalized Schrödinger-Poisson system

$$\begin{cases} -\Delta_p u - \Delta_q u + |u|^{p-2}u + \lambda\phi|u|^{s-2}u = |u|^{l-2}u + f(x) & \text{in } \mathbb{R}^3, \\ -\Delta_r \phi = |u|^s & \text{in } \mathbb{R}^3, \end{cases}$$

where $p, q, r \in (1, 3)$ with $p < q$, $\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$ and $m^* = \frac{3m}{3-m}$ with $m \in \{p, q, r\}$, stand for the m-Laplacian operator and Sobolev critical exponent respectively. $\max\{1, \frac{p(r^*-1)}{r^*}, \frac{q(r-1)}{r}\} < s < \frac{q^*(r^*-1)}{r^*}$, $q < l < q^*$, λ is a positive parameter, f satisfies certain integrability conditions. First, one solution with negative energy is obtained by Ekeland variational principle for any $\lambda > 0$ and $l \in (q, q^*)$. Second, according to the range of l, we use two methods to obtain one solution with positive energy. Precisely, for the case $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$, we employ a scaling technique to demonstrate the boundedness of Palais-Smale sequence, furthermore, one solution with positive energy is got by mountain pass theorem for any $\lambda > 0$; for the case $q < l < \frac{qr(s+1)}{r+q(r-1)}$, the cut-off technique is used to obtain a bounded Palais-Smale sequence for $\lambda > 0$ small enough, then one solution with positive energy is also obtained for $\lambda > 0$ small enough.

Keywords Schrödinger-Poisson system, (p, q)-Laplacian operator, Ekeland variational principle, cut-off technique, multiple solutions.

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1. Introduction and main results

In this paper, we study the following generalized Schrödinger-Poisson system

$$\begin{cases} -\Delta_p u - \Delta_q u + |u|^{p-2}u + \lambda\phi|u|^{s-2}u = |u|^{l-2}u + f(x) \text{ in } \mathbb{R}^3, \\ -\Delta_r \phi = |u|^s \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p, q, r \in (1, 3)$ with $p < q$, $\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$ and $m^* = \frac{3m}{3-m}$ with $m \in \{p, q, r\}$, stand for the m-Laplacian operator and Sobolev critical exponent respectively. λ is a positive parameter, s satisfies

$$\max\left\{1, \frac{p(r^*-1)}{r^*}, \frac{q(r-1)}{r}\right\} < s < \frac{q^*(r^*-1)}{r^*}.$$

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System (1.1) is formed by coupling two equations, one of which is a quasilinear nonhomogeneous equation involving a (p, q)-Laplacian operator, and the other is a Poisson equation of r-Laplacian. f is a positive radial function and satisfies the following assumptions

- (f_1) $f \in L^t(\mathbb{R}^3)$ for some $t \in [(q^*)', p']$, where $(q^*)'$ and p' are the conjugate indices of q^* and p respectively;
- (f_2) $(x, \nabla f) \in L^t(\mathbb{R}^3)$, where the gradient ∇f is in the weak sense.

As far as we know, the first equation of system (1.1) is called a double-phase problem, because the equation is driven by a differential operator with unbalanced growth due to the presence of the (p, q)-Laplacian operator. Interest in such problems stems from the study of stationary solutions of the following reaction-diffusion system

$$u_t = \operatorname{div}[A(\nabla u)\nabla u] + c(x, u) \quad \text{and} \quad A(\nabla u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}.$$

It is widely found in biophysics, plasma physics and chemical reaction design, see [13]. In such contents, the u is a state variable and describes the density or concentration of multicomponent substances, $\operatorname{div}[A(\nabla u)\nabla u]$ corresponds to the diffusion with a diffusion coefficient $A(\nabla u)$, and the term $c(x, u)$ is related to sources and loss processes.

In recent years, (p, q)-Laplacian equations of type

$$-\Delta_p u - \Delta_q u = g(x, u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

with $N \geqslant 3$, $1 < p < q < N$, have been widely studied. In [21], the existence of positive ground state solutions for equation (1.2) was studied with $m, n > 0$, $g(x, u) = -m|u|^{p-2}u - n|u|^{q-2}u + f(x, u)$ and f asymptotic to u^{q-1} at infinity uniformly in $x \in \mathbb{R}^N$. In [12], the authors obtained one nontrivial solution of equation (1.2) with $g(x, u) = -V(x)|u|^{p-2}u - W(x)|u|^{q-2}u + f(x, u)$, where potentials V and W are continuous, positive, coercive, and the nonlinear term f satisfies some hypotheses which do not include the Ambrosetti-Rabinowitz condition. Later, the problem studied in [12] was extended in [9], where the potential functions need not be positive and the nonlinear term f satisfies other conditions. Through a new decomposition of the space, the existence of infinitely many solutions was obtained in [9]. In [8], the authors obtained infinitely many weak solutions with negative energy of equation (1.2) with $1 < p < l < q < N$ and $g(x, u) = \lambda V(x)|u|^{l-2}u + W(x)|u|^{q^*-2}u$. For more recent studies on (p, q)-Laplacian equations, we can refer to [3, 6, 27] and the related literature therein.

According to (p, q)-Laplacian equations with nonlocal nonlinear terms, there are also some results. The existence of nodal solutions of a class of (p, q)-Laplacian Kirchhoff type problem was got by a minimization argument and a quantitative deformation lemma in [23]. A multiplicity result for a (p, q)-Schrödinger-Kirchhoff type equation was obtained in [4]. Existence of positive solutions for a Choquard equation involving (p, q)-Laplacian was considered in [29]. The ground state solution was obtained in [5] for a Choquard equation with a (p, q)-Laplacian operator and a general nonlinearity of Berestycki-Lions type. In [1], the authors concerned with Schrödinger-Poisson system with zero mass in $\mathbb{R}^2$ involving $(2, q)$-Laplacian. The existence of positive least energy solution was obtained in [1]. As far as we have known, there is still very little research on Schrödinger-Poisson system of type similar to system (1.1).

One highlight of system (1.1) is the appearance of the Poisson equation involving r-Laplacian. This kind of system was first proposed in reference [17]. Precisely, the following system was studied in [17],

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{s-2}u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta_r \phi = |u|^s & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $g(x,u) = |u|^{l-2}u$ with $p < l < p^*$. Using mountain pass theorem, the existence of nontrivial solutions to system (1.3) was obtained for appropriate $\lambda > 0$. System (1.3) with $r = s = 2$ was studied in [18, 19, 22]. Particularly, in [22], for $g(x,u) = |u|^{l-2}u + f(x)$, f is a positive radial function and satisfies (f_1) and (f_2), some multiplicity results were given by using Ekeland variational principle and mountain pass theorem. Recently, the following quasilinear Schrödinger-Poisson system was considered in [16],

$$\begin{cases} -\Delta_p u + \lambda\phi|u|^{s-2}u = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta_r \phi = |u|^s & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where g satisfies the Berestycki-Lions type conditions. One nontrivial solution of system (1.4) with $\lambda > 0$ small enough was got in [16].

For $p = r = s = 2$, system (1.3) reduces to the classical Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^{s-2}u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

It describes the interaction of a charged particle with an electromagnetic field, see [10, 11]. If $g(x,u) = |u|^{l-2}u$ and $l \in (2,6)$, the existence of solutions to the system (1.5) has been discussed in [7, 14, 15]. Furthermore, when $g(x,u) = |u|^{l-2}u + f(x)$ and f satisfies certain conditions, by applying Ekeland variational principle and mountain pass theorem, it was proved in [26] that system (1.5) admitted two radial solutions.

Inspired by the above literature, we seek multiple solutions of system (1.1). It is worth noting that the appearance of the p, q, r, s makes the problem more complex, and we need to consider the relationship among them. In order to apply the reduction method to handle system (1.1), it seems to be necessary to require s to meet the constraint

$$\max\left\{1, \frac{p(r^*-1)}{r^*}\right\} < s < \frac{q^*(r^*-1)}{r^*}.$$

Furthermore, in Section 4, we also need

$$s > \frac{q(r-1)}{r},$$

which guarantees

$$q < \frac{qr(s+1)}{r + q(r-1)} < q^*.$$

In order to deal with system (1.1) by the reduction method and variational methods, we have to overcome certain difficulties. The first obstacle is to guarantee the existence and uniqueness of solution to the second equation in system (1.1). Since Lax-Milgram theorem is no longer valid for the case $r \neq 2$, we adapt Minty-Browder theorem to overcome it. The second one is to obtain the properties of the solution to the second equation in system (1.1). Because it doesn't have an explicit expression, we apply the uniqueness of the solution and the strict convexity of the energy functional associated to the second equation in system (1.1) to get some important properties of the solution, such as the non-negativeness, homogeneity and weak convergence. In addition, the absence of the explicit expression of the solution also makes it extremely difficult to prove the C^1 property of the functional associated to system (1.1). Similar to [17], by some basic methods in nonlinear functional analysis, we can also get the Gâteaux differentiability of the functional and prove that it is also continuous.

Our main results are as follows.

Theorem 1.1. *If $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$, (f_1) and (f_2) hold, then there exists $\Lambda > 0$ such that system (1.1) with $|f|_t < \Lambda$ and $\lambda > 0$ admits two solutions.*

Theorem 1.2. *If $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$, (f_1) holds, then there exists $\lambda_0 > 0$ such that system (1.1) with $|f|_t < \Lambda$ and $\lambda \in (0, \lambda_0)$ admits two solutions.*

This article is organized as follows. In Section 2, we present some propositions, which are crucial to establish the variational structure of system (1.1). In Section 3, when $q < l < q^*$, by Ekeland variational principle [20], we get a solution of system (1.1) with negative energy in Theorem 3.1. In Section 4, we obtain a solution of system (1.1) with positive energy if s lies in a suitable range. We prove the conclusion in two cases: $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$ and $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$. Precisely, in Subsection 4.1, we employ a scaling technique, initially introduced in [24], to demonstrate the boundedness of a Palais-Smale sequence for the case $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$. Further, by mountain pass theorem [30], we establish the existence of a positive energy solution, as summarized in Theorem 4.1. In Subsection 4.2, we adapt the cut-off technique from [25] to get one positive energy solution for system (1.1) under the condition that $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$ and $\lambda > 0$ small. This result is presented in Theorem 4.2. Finally, Theorems 1.1 and 1.2 follow as direct consequences of Theorem 3.1, Theorems 4.1 and 4.2 repectively.

2. Variational framework

In this section, the variational structure of system (1.1) is established. The main working spaces are as follows. We define $W^{1,p}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to

$$\|u\|_{1,p} = \left(\int_{\mathbb{R}^3} (|\nabla u|^p + |u|^p) dx\right)^{\frac{1}{p}}$$

and $D^{1,q}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to

$$\|u\|_{D^{1,q}} = \left(\int_{\mathbb{R}^3} |\nabla u|^q dx\right)^{\frac{1}{q}}.$$

$L^s(\mathbb{R}^3)$, for $1 \leqslant s < \infty$, denotes the usual Lebesgue space with the norm

$$|u|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{\frac{1}{s}}.$$

For the sake of simplicity, we denote $X := W^{1,p}(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$, equipped with the norm

$$\|u\| = \|u\|_{1,p} + \|u\|_{D^{1,q}}.$$

It's easy to see that the continuous embeddings $X \hookrightarrow L^l(\mathbb{R}^3)$ for all $l \in [p, q^*]$ and $D^{1,r}(\mathbb{R}^3) \hookrightarrow L^{r^*}(\mathbb{R}^3)$.

The following proposition is important to establish the variational structure of system (1.1).

Proposition 2.1. *Let* $\frac{p(r^*-1)}{r^*} < s < \frac{q^*(r^*-1)}{r^*}$. *For any given* $u \in X$, *there exists a unique* $\phi_u \in D^{1,r}(\mathbb{R}^3)$ *solving*

$$-\Delta_r \phi = |u|^s. \tag{2.1}$$

Proof. To illustrate the existence of a unique $\phi_u \in D^{1,r}(\mathbb{R}^3)$, for any $u \in X$, we define the linear functional

$$\mathcal{L}(v) = \int_{\mathbb{R}^3} |u|^s v dx \text{ for all } v \in D^{1,r}(\mathbb{R}^3).$$

It is easy to see that $\mathcal{L}$ is continuous in $D^{1,r}(\mathbb{R}^3)$. In fact, for $u \in X$ and $v \in D^{1,r}(\mathbb{R}^3)$, by Hölder and Sobolev inequalities, we have

$$\left|\int_{\mathbb{R}^3} |u|^s v dx\right| \leqslant \left(\int_{\mathbb{R}^3} |u|^{\frac{r^* s}{r^*-1}} dx\right)^{\frac{r^*-1}{r^*}} \left(\int_{\mathbb{R}^3} |v|^{r^*} dx\right)^{\frac{1}{r^*}} \leqslant C\|u\|^s \|v\|_{D^{1,r}}.$$

For $r = 2$, we can easily come to the conclusion by Lax-Milgram theorem. We just need to deal with the case for $r \neq 2$. By Minty-Browder theorem, we only show that $-\Delta_r : D^{1,r}(\mathbb{R}^3) \to (D^{1,r}(\mathbb{R}^3))^*$ is a continuous, coercive and injective monotone operator. We first prove the continuity of it. Indeed, let $v_n \to v$ in $D^{1,r}(\mathbb{R}^3)$. Then $\nabla v_n \to \nabla v$ in $L^r(\mathbb{R}^3)$. For any $\phi \in D^{1,r}(\mathbb{R}^3)$, we have

$$\begin{aligned} |\langle -\Delta_r v_n - (-\Delta_r v), \phi\rangle| &= \left|\int_{\mathbb{R}^3} (|\nabla v_n|^{r-2}\nabla v_n - |\nabla v|^{r-2}\nabla v) \cdot \nabla \phi dx\right| \\ &\leqslant \left(\int_{\mathbb{R}^3} ||\nabla v_n|^{r-2}\nabla v_n - |\nabla v|^{r-2}\nabla v|^{\frac{r}{r-1}} dx\right)^{\frac{r-1}{r}} \|\phi\|_{D^{1,r}} \\ &= o(1)\|\phi\|_{D^{1,r}}. \end{aligned}$$

So, $-\Delta_r$ is continuous in $D^{1,r}(\mathbb{R}^3)$. From

$$\frac{\langle -\Delta_r v, v\rangle}{\|v\|_{D^{1,r}}} \to +\infty \quad \text{as} \quad \|v\|_{D^{1,r}} \to +\infty,$$

it is easy to see that $-\Delta_r$ is coercive. Finally, in order to prove the injective monotonicity, we need the following Simon inequality: There exist $d_1, d_2 > 0$ such that for all $x, y \in \mathbb{R}^3$,

$$\begin{aligned} &(|x|^{r-2}x - |y|^{r-2}y) \cdot (x - y) \geqslant d_1|x - y|^r, \quad \text{for } \ 2 \leqslant r < 3, \\ &\big(|x| + |y|\big)^{2-r}(|x|^{r-2}x - |y|^{r-2}y) \cdot (x - y) \geqslant d_2|x - y|^2, \quad \text{for } \ 1 < r < 2. \end{aligned}$$

It follows that

$$d_1 \int_{\mathbb{R}^3} |\nabla v_1 - \nabla v_2|^r dx \leqslant \int_{\mathbb{R}^3} (|\nabla v_1|^{r-2}\nabla v_1 - |\nabla v_2|^{r-2}\nabla v_2) \cdot (\nabla v_1 - \nabla v_2) dx \quad \text{for } \ 2 \leqslant r < 3,$$

and for $1 < r < 2$,

$$d_2^{\frac{r}{2}} \int_{\mathbb{R}^3} |\nabla v_1 - \nabla v_2|^r dx$$

$$\leqslant \left(\int_{\mathbb{R}^3}(|\nabla v_1|^{r-2}\nabla v_1-|\nabla v_2|^{r-2}\nabla v_2)\cdot(\nabla v_1-\nabla v_2)dx\right)^{\frac{r}{2}}\left(\int_{\mathbb{R}^3}(|\nabla v_1|+|\nabla v_2|)^r dx\right)^{\frac{2-r}{2}}.$$

Thus,

$$\langle -\Delta_r v_1-(-\Delta_r v_2), v_1-v_2\rangle \geqslant 0.$$

In addition, if $v_1\neq v_2$, then

$$\langle -\Delta_r v_1-(-\Delta_r v_2), v_1-v_2\rangle > 0.$$

Hence, $-\Delta_r$ is an injective monotone operator. The conclusion follows from Minty-Browder theorem. □

Similar to reference [17], from the uniqueness of solution to equation (2.1), we can conclude that ϕ_u has the following properties. It is worth noting that the conclusion regarding weak continuity of $u\to\phi_u$ has not been presented in the previous studies.

Proposition 2.2. *For any $u\in X$, the solution $\phi_u\in D^{1,r}(\mathbb{R}^3)$ has the following properties.*

(i) $\int_{\mathbb{R}^3}\left(\frac{1}{r}|\nabla\phi_u|^r-|u|^s\phi_u\right)dx=\min_{\phi\in D^{1,r}(\mathbb{R}^3)}\int_{\mathbb{R}^3}\left(\frac{1}{r}|\nabla\phi|^r-|u|^s\phi\right)dx$, *and* $\phi_u(x)\geqslant 0$, *a.e.* $x\in\mathbb{R}^3$.

(ii) *For any* $t>0$, $\phi_{tu}=t^{\frac{s}{r-1}}\phi_u$, *and* $\phi_{u_t}(x)=t^{\frac{ks-r}{r-1}}\phi_u(tx)$, *where* $u_t(x)=t^k u(tx)$. *Moreover, for any* $y\in\mathbb{R}^3$, $\phi_{u(\cdot+y)}=\phi_u(\cdot+y)$.

(iii) $\|\phi_u\|_{D^{1,r}}\leqslant C\|u\|^{\frac{s}{r-1}}$ *for any* $u\in X$.

(iv) *If* $u_n\rightharpoonup u$ *in* X, *then* $\phi_{u_n}\rightharpoonup\phi_u$ *in* $D^{1,r}(\mathbb{R}^3)$ *and*

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{s-2}u_n\varphi dx\to\int_{\mathbb{R}^3}\phi_u|u|^{s-2}u\varphi dx \quad \text{for any} \quad \varphi\in X.$$

(v) *If* $u_n\to u$ *in* X, *then* $\phi_{u_n}\to\phi_u$ *in* $D^{1,r}(\mathbb{R}^3)$.

Proof. (i) First, for any given $u\in X$, we define a functional $I: D^{1,r}(\mathbb{R}^3)\to\mathbb{R}$ by

$$I(\phi)=\int_{\mathbb{R}^3}\left(\frac{1}{r}|\nabla\phi|^r-|u|^s\phi\right)dx.$$

Obviously, $I\in C^1(D^{1,r}(\mathbb{R}^3),\mathbb{R})$ and

$$\langle I'(\phi),v\rangle=\int_{\mathbb{R}^3}|\nabla\phi|^{r-2}\nabla\phi\cdot\nabla v dx-\int_{\mathbb{R}^3}|u|^s v dx \quad \text{for all} \quad v\in D^{1,r}(\mathbb{R}^3).$$

By

$$\langle I'(w)-I'(v),w-v\rangle=\int_{\mathbb{R}^3}(|\nabla w|^{r-2}\nabla w-|\nabla v|^{r-2}\nabla v)\cdot(\nabla w-\nabla v)dx\geqslant 0,$$

we conclude that I is strictly convex. Combined with the continuity and coercive of I, we state that I achieves its unique global minimum point at ϕ_u. Finally, it is easy to get $\phi_u\geqslant 0$. Indeed, from

$$\frac{1}{r}\int_{\mathbb{R}^3}|\nabla|\phi_u||^r dx-\int_{\mathbb{R}^3}|u|^s|\phi_u|dx\leqslant\frac{1}{r}\int_{\mathbb{R}^3}|\nabla\phi_u|^r dx-\int_{\mathbb{R}^3}|u|^s\phi_u dx.$$

I also achieves its global minimum point at $|\phi_u|$. By the uniqueness, one has $\phi_u=|\phi_u|\geqslant 0$.

(ii) By the uniqueness of ϕ_u, it follows that, for any $t>0$, $\phi_{tu}=t^{\frac{s}{r-1}}\phi_u$. Indeed, by a simple calculation, we have

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla\phi_{tu}|^{r-2}\nabla\phi_{tu}\cdot\nabla\varphi dx&=\int_{\mathbb{R}^3}|tu|^s\varphi dx\\&=t^s\int_{\mathbb{R}^3}|u|^s\varphi dx\\&=t^s\int_{\mathbb{R}^3}|\nabla\phi_u|^{r-2}\nabla\phi_u\cdot\nabla\varphi dx\\&=\int_{\mathbb{R}^3}|\nabla(t^{\frac{s}{r-1}}\phi_u)|^{r-2}\nabla(t^{\frac{s}{r-1}}\phi_u)\cdot\nabla\varphi dx.\end{aligned}\tag{2.2}$$

From (2.2), the conclusion is obviously true.

Next, we will show that, for any $t>0$,

$$\phi_{u_t}(x)=t^{\frac{ks-r}{r-1}}\phi_u(tx),\ \text{where}\ \ u_t(\cdot)=t^ku(t\cdot).$$

On the one hand, for every $\varphi\in X$, one has

$$\int_{\mathbb{R}^3}|\nabla\phi_u(x)|^{r-2}\nabla\phi_u(x)\cdot\nabla\varphi(x)dx=\int_{\mathbb{R}^3}|u(x)|^s\varphi(x)dx.$$

Using the variable change of integral, we deduce that

$$\int_{\mathbb{R}^3}|(\nabla\phi_u)(tx)|^{r-2}(\nabla\phi_u)(tx)\cdot(\nabla\varphi)(tx)dx=\int_{\mathbb{R}^3}|u(tx)|^s\varphi(tx)dx.$$

So,

$$t^{-r}\int_{\mathbb{R}^3}|\nabla(\phi_u(tx))|^{r-2}\nabla(\phi_u(tx))\cdot\nabla(\varphi(tx))dx=\int_{\mathbb{R}^3}|u(tx)|^s\varphi(tx)dx.\tag{2.3}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla\phi_{u_t}(x)|^{r-2}\nabla\phi_{u_t}(x)\cdot\nabla(\varphi(tx))dx&=\int_{\mathbb{R}^3}|u_t(x)|^s\varphi(tx)dx\\&=t^{ks}\int_{\mathbb{R}^3}|u(tx)|^s\varphi(tx)dx.\end{aligned}\tag{2.4}$$

Recalling (2.3) and (2.4), it's easy to get

$$t^{ks-r}\int_{\mathbb{R}^3}|\nabla(\phi_u(tx))|^{r-2}\nabla(\phi_u(tx))\cdot\nabla(\varphi(tx))dx=\int_{\mathbb{R}^3}|\nabla\phi_{u_t}(x)|^{r-2}\nabla\phi_{u_t}(x)\cdot\nabla(\varphi(tx))dx.$$

So,

$$\phi_{u_t}(x)=t^{\frac{ks-r}{r-1}}\phi_u(tx).$$

Similarly, we also get that

$$\phi_{u(\cdot+y)}(\cdot)=\phi_u(\cdot+y).$$

In fact, for every $\varphi\in X$ and $y\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}|\nabla\phi_u(x)|^{r-2}\nabla\phi_u(x)\cdot\nabla\varphi(x-y)dx=\int_{\mathbb{R}^3}|u(x)|^s\varphi(x-y)dx.$$

By the translation invariance of Lebesgue integral on $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} |\nabla\phi_u(x+y)|^{r-2}\nabla\phi_u(x+y)\cdot\nabla\varphi(x)dx = \int_{\mathbb{R}^3} |u(x+y)|^s\varphi(x)dx. \tag{2.5}$$

The uniqueness of solution to equation (2.1) and (2.5) leads to that $\phi_{u(\cdot+y)}(\cdot) = \phi_u(\cdot+y)$.

(iii) By

$$\|\phi_u\|^r_{D^{1,r}} = \int_{\mathbb{R}^3} |u|^s\phi_u dx \leqslant C\|u\|^s\|\phi_u\|_{D^{1,r}},$$

we get the conclusion.

(iv) Since $u_n \rightharpoonup u$ in X, then $\{u_n\}$ is bounded in X. So, up to a subsequence, by Sobolev embedding theorem and local compact embedding theorem, we deduce that

$$\begin{cases} u_n \rightharpoonup u & \text{in } X, \\ u_n \to u & \text{in } L^t_{loc}(\mathbb{R}^3),\ p \leqslant t < q^*, \\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Using of (iii), it is easy to see that $\{\phi_{u_n}\}$ is also bounded in $D^{1,r}(\mathbb{R}^3)$. Going if necessary to a subsequence, there exists $\phi \in D^{1,r}(\mathbb{R}^3)$ such that

$$\phi_{u_n} \rightharpoonup \phi \ \text{ in } D^{1,r}(\mathbb{R}^3).$$

Furthermore, we can also assume that

$$\begin{cases} \phi_{u_n} \rightharpoonup \phi & \text{in } L^{r^*}(\mathbb{R}^3), \\ \phi_{u_n} \to \phi & \text{in } L^t_{loc}(\mathbb{R}^3),\ 1 \leqslant t < r^*, \\ \phi_{u_n}(x) \to \phi(x) & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

On the one hand, from Proposition 2.1, one has, for any $\varphi \in D^{1,r}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n}\cdot\nabla\varphi dx = \int_{\mathbb{R}^3} |u_n|^s\varphi dx \tag{2.6}$$

and

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^{r-2}\nabla\phi_u\cdot\nabla\varphi dx = \int_{\mathbb{R}^3} |u|^s\varphi dx. \tag{2.7}$$

Set $\varphi = (\phi_{u_n} - \phi)\psi_R$ in (2.6), where $\psi_R \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leqslant \psi_R(x) \leqslant 1$ for all $x \in \mathbb{R}^N, \psi_R(x) = 1$ for all $x \in B_R(0)$, $\psi_R(x) = 0$ for all $x \in B^c_{2R}(0)$ and $|\nabla\psi_R| \leqslant \frac{2}{R}$. So,

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n}\cdot\nabla(\phi_{u_n}-\phi)\psi_R dx \\ &+ \int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n}\cdot\nabla\psi_R(\phi_{u_n}-\phi)dx \\ &- \int_{\mathbb{R}^3} |u_n|^s(\phi_{u_n}-\phi)\psi_R dx. \end{aligned} \tag{2.8}$$

On the other hand, by using the definition of weak convergence in $D^{1,r}(\mathbb{R}^3)$ and local compact embedding theorem, respectively, we get that

$$\int_{\mathbb{R}^3} |\nabla\phi|^{r-2}\nabla\phi \cdot \nabla(\phi_{u_n} - \phi)\psi_R dx \to 0 \ \text{ as } n \to \infty,$$

and

$$\int_{\mathbb{R}^3} |\nabla\phi|^{r-2}\nabla\phi \cdot \nabla\psi_R(\phi_{u_n} - \phi) dx \to 0 \ \text{ as } n \to \infty, \tag{2.9}$$

namely,

$$\begin{aligned} o(1) = &\int_{\mathbb{R}^3} |\nabla\phi|^{r-2}\nabla\phi \cdot \nabla(\phi_{u_n} - \phi)\psi_R dx \\ &+ \int_{\mathbb{R}^3} |\nabla\phi|^{r-2}\nabla\phi \cdot \nabla\psi_R(\phi_{u_n} - \phi) dx. \end{aligned} \tag{2.10}$$

By (2.8) and (2.10), we deduce that

$$\begin{aligned} o(1) = &\int_{\mathbb{R}^3} (|\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n} - |\nabla\phi|^{r-2}\nabla\phi) \cdot \nabla(\phi_{u_n} - \phi)\psi_R dx \\ &+ \int_{\mathbb{R}^3} (|\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n} - |\nabla\phi|^{r-2}\nabla\phi) \cdot \nabla\psi_R(\phi_{u_n} - \phi) dx \\ &- \int_{\mathbb{R}^3} |u_n|^s(\phi_{u_n} - \phi)\psi_R dx. \end{aligned} \tag{2.11}$$

Since $\phi_{u_n} \to \phi$ in $L^t_{loc}(\mathbb{R}^3)$, $t \in [1, r^*)$, by Hölder inequality and the definition of ψ_R, we can get that

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n} \cdot \nabla\psi_R(\phi_{u_n} - \phi) dx \to 0 \ \text{ as } n \to \infty, \\ &\int_{\mathbb{R}^3} |u_n|^s(\phi_{u_n} - \phi)\psi_R dx \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Combine (2.9) with (2.11), we get that

$$o(1) = \int_{\mathbb{R}^3} (|\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n} - |\nabla\phi|^{r-2}\nabla\phi) \cdot \nabla(\phi_{u_n} - \phi)\psi_R dx.$$

Then Simon inequality leads to that

$$\int_{\mathbb{R}^3} |\nabla\phi_{u_n} - \nabla\phi|^r \psi_R dx \to 0 \text{ as } n \to \infty.$$

Up to a subsequence, we have

$$\nabla\phi_{u_n}(x) \to \nabla\phi(x) \text{ a.e. } x \in B_R(0) \ \text{ as } n \to \infty.$$

The arbitrariness of R implies that, going to a subsequence,

$$\nabla\phi_{u_n}(x) \to \nabla\phi(x) \text{ a.e. } x \in \mathbb{R}^3 \ \text{ as } n \to \infty.$$

The boundedness of $\{|\nabla\phi_{u_n}|\}$ in $L^r(\mathbb{R}^3)$ ensures that $\{|\nabla\phi_{u_n}|^{r-1}\}$ is also bounded in $L^{\frac{r}{r-1}}(\mathbb{R}^3)$. Thus, it follows from [31] that

$$|\nabla\phi_{u_n}|^{r-2}D_i\phi_{u_n} \rightharpoonup |\nabla\phi|^{r-2}D_i\phi \ \text{ in } L^{\frac{r}{r-1}}(\mathbb{R}^3), \quad i=1,2,3.$$

That is, for every $\varphi \in X$,

$$\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^{r-2}D_i\phi_{u_n}D_i\varphi dx \to \int_{\mathbb{R}^3} |\nabla\phi|^{r-2}D_i\phi D_i\varphi dx, \quad i=1,2,3.$$

Then

$$\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^{r-2}\nabla\phi_{u_n}\cdot\nabla\varphi dx \to \int_{\mathbb{R}^3} |\nabla\phi|^{r-2}\nabla\phi\cdot\nabla\varphi dx.$$

Since $\varphi \in L^{r^*}(\mathbb{R}^3)$ and $|u_n|^s \rightharpoonup |u|^s$ in $L^{\frac{r^*}{r^*-1}}(\mathbb{R}^3)$, by [31], we have

$$\int_{\mathbb{R}^3} |u_n|^s\varphi dx \to \int_{\mathbb{R}^3} |u|^s\varphi dx \ \text{ as } n\to\infty.$$

Therefore, by taking limits as $n\to\infty$ on both sides of (2.6), we can obtain that

$$\int_{\mathbb{R}^3} |\nabla\phi|^{r-2}\nabla\phi\cdot\nabla\varphi dx = \int_{\mathbb{R}^3} |u|^s\varphi dx. \tag{2.12}$$

The uniqueness of solution for equation (2.1) with given u, (2.7) and (2.12) result that $\phi=\phi_u$. Then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,r}(\mathbb{R}^3)$. By Sobolev inequality, we also have $\phi_{u_n} \rightharpoonup \phi_u$ in $L^{r^*}(\mathbb{R}^3)$. Furthermore, it's easy to know that

$$\phi_{u_n}(x) \to \phi_u(x) \text{ a.e. } x\in\mathbb{R}^3 \ \text{ as } n\to\infty. \tag{2.13}$$

By Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} |\phi_{u_n}|u_n|^{s-2}u_n|^{\frac{r^*s}{r^*(s-1)+1}}dx\\ &\leqslant \left(\int_{\mathbb{R}^3}|\phi_{u_n}|^{r^*}dx\right)^{\frac{s}{r^*(s-1)+1}}\left(\int_{\mathbb{R}^3}|u_n|^{\frac{r^*s}{r^*-1}}dx\right)^{\frac{(r^*-1)(s-1)}{r^*(s-1)+1}}.\end{aligned}$$

So, $\{\phi_{u_n}|u_n|^{s-2}u_n\}$ is bounded in $L^{\frac{r^*s}{r^*(s-1)+1}}(\mathbb{R}^3)$. Combined with (2.13), it follows from [31] that

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{s-2}u_n\varphi dx \to \int_{\mathbb{R}^3} \phi_u|u|^{s-2}u\varphi dx \quad \text{for all } \varphi\in X \text{ as } n\ \to\infty.$$

(v) Let $u_n \to u$ in X. Then $|u_n|^s \to |u|^s$ in $L^{\frac{r^*}{r^*-1}}(\mathbb{R}^3)$, since $\frac{p(r^*-1)}{r^*} < s < \frac{q^*(r^*-1)}{r^*}$. In addition, using (iii) and (iv), we know $\{\phi_{u_n}\}$ is bounded in $L^{r^*}(\mathbb{R}^3)$ and

$$\phi_{u_n}(x) \to \phi_u(x) \ \text{ a.e. } x\in\mathbb{R}^3.$$

Then, we have

$$\int_{\mathbb{R}^3} (\phi_{u_n}-\phi_u)|u|^s dx \to 0 \ \text{ as } n\to\infty.$$

Furthermore, by Hölder inequality, we obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(|\nabla\phi_{u_n}|^r-|\nabla\phi_u|^r)dx\right|&=\left|\int_{\mathbb{R}^3}(\phi_{u_n}(|u_n|^s-|u|^s)+(\phi_{u_n}-\phi_u)|u|^s)dx\right|\\&\leqslant|\phi_{u_n}|_{r^*}||u_n|^s-|u|^s|_{\frac{r^*}{r^*-1}}+\left|\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)|u|^sdx\right|\to 0.\end{aligned}$$

Then, we conclude that $\phi_{u_n}\to\phi_u$ in $D^{1,r}(\mathbb{R}^3)$. □

By Proposition 2.2 and Sobolev inequality, the functional

$$J_\lambda(u)=\frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p+|u|^p)dx+\frac{1}{q}\int_{\mathbb{R}^3}|\nabla u|^qdx+\frac{\lambda(r-1)}{rs}\int_{\mathbb{R}^3}\phi_u|u|^sdx-\frac{1}{l}\int_{\mathbb{R}^3}|u|^ldx-\int_{\mathbb{R}^3}f(x)udx$$

is well-defined in X. Now, we will prove that $J_\lambda\in C^1(X,\mathbb{R})$.

Proposition 2.3. *$J_\lambda\in C^1(X,\mathbb{R})$, and for any $u,v\in X$,*

$$\begin{aligned}\langle J_\lambda'(u),v\rangle=&\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\cdot\nabla v+|u|^{p-2}uv)dx+\int_{\mathbb{R}^3}|\nabla u|^{q-2}\nabla u\cdot\nabla vdx\\&+\lambda\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx-\int_{\mathbb{R}^3}|u|^{l-2}uvdx-\int_{\mathbb{R}^3}f(x)vdx.\end{aligned}$$

Proof. By the definition of Gâteaux differentiable, we just need to prove that

$$\lim_{t\to 0}\frac{J_\lambda(u+tv)-J_\lambda(u)}{t}=\langle J_G'(u),v\rangle,\quad u,v\in X,$$

where

$$\begin{aligned}\langle J_G'(u),v\rangle=&\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\cdot\nabla v+|u|^{p-2}uv)dx+\int_{\mathbb{R}^3}|\nabla u|^{q-2}\nabla u\cdot\nabla vdx\\&+\lambda\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx-\int_{\mathbb{R}^3}|u|^{l-2}uvdx-\int_{\mathbb{R}^3}f(x)vdx.\end{aligned}$$

Set

$$J_\lambda(u+tv)-J_\lambda(u)-t\langle J_G'(u),v\rangle=\mathcal{A}+\mathcal{B}+\mathcal{C}+\lambda\mathcal{D},$$

where

$$\begin{aligned}\mathcal{A}&=\frac{1}{p}\|u+tv\|_{1,p}^p-\frac{1}{p}\|u\|_{1,p}^p-t\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\cdot\nabla v+|u|^{p-2}uv)dx,\\\mathcal{B}&=\frac{1}{q}\|u+tv\|_{D^{1,q}}^q-\frac{1}{q}\|u\|_{D^{1,q}}^q-t\int_{\mathbb{R}^3}|\nabla u|^{q-2}\nabla u\cdot\nabla vdx,\\\mathcal{C}&=-\frac{1}{l}\int_{\mathbb{R}^3}|u+tv|^ldx+\frac{1}{l}\int_{\mathbb{R}^3}|u|^ldx+t\int_{\mathbb{R}^3}|u|^{l-2}uvdx,\\\mathcal{D}&=\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_{u+tv}|u+tv|^sdx-\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_u|u|^sdx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx.\end{aligned}$$

By a standard process, it can reach that

$$\mathcal{A}=o(t),\quad\mathcal{B}=o(t),\quad\mathcal{C}=o(t),\quad t\to 0.$$

We only need to prove that

$$\mathcal{D}=o(t), \quad t\to 0.$$

According to Proposition 2.2(i), we can get that

$$\mathcal{M}(u,\phi_u)=\min_{\phi\in D^{1,r}(\mathbb{R}^3)}\mathcal{M}(u,\phi),$$

where

$$\mathcal{M}(u,\phi)=\frac{1}{r}\int_{\mathbb{R}^3}|\nabla\phi|^r dx-\int_{\mathbb{R}^3}|u|^s\phi dx.$$

So,

$$\mathcal{M}(u+tv,\phi_u)\geqslant\mathcal{M}(u+tv,\phi_{u+tv}).$$

Combine the definition of $\mathcal{M}$ with the equation

$$-\Delta_r\phi_{u+tv}=|u+tv|^s,$$

one has

$$\mathcal{M}(u+tv,\phi_{u+tv})=\left(\frac{1}{r}-1\right)\int_{\mathbb{R}^3}|u+tv|^s\phi_{u+tv}dx.$$

Then, we deduce that

$$\begin{aligned}\mathcal{D}=&\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_{u+tv}|u+tv|^s dx-\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ =&-\frac{1}{s}\mathcal{M}(u+tv,\phi_{u+tv})-\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ \geqslant&-\frac{1}{s}\mathcal{M}(u+tv,\phi_u)-\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ =&-\frac{1}{s}\left(\frac{1}{r}\int_{\mathbb{R}^3}|\nabla\phi_u|^r dx-\int_{\mathbb{R}^3}|u+tv|^s\phi_u dx\right)-\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx\\ &-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ =&\frac{1}{s}\int_{\mathbb{R}^3}|u+tv|^s\phi_u dx-\frac{1}{s}\int_{\mathbb{R}^3}|u|^s\phi_u dx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ =&\frac{1}{s}\int_{\mathbb{R}^3}\phi_u\big(|u+tv|^s-|u|^s-st|u|^{s-2}uv\big)dx.\end{aligned}\tag{2.14}$$

Next, we claim that

$$\int_{\mathbb{R}^3}\phi_u\big(|u+tv|^s-|u|^s-st|u|^{s-2}uv\big)dx=o(t), \quad t\to 0.\tag{2.15}$$

For any $s>1$, we have that for a.e. $x\in\mathbb{R}^3$,

$$\lim_{t\to 0}\frac{\phi_u(|u(x)+tv(x)|^s-|u(x)|^s-st|u(x)|^{s-2}u(x)v(x))}{t}=0.\tag{2.16}$$

Then, by the mean value theorem, there exists $\theta\in\mathbb{R}$, $|\theta|\leqslant 1$ such that

$$\begin{aligned}&\frac{1}{t}|\phi_u(x)(|u(x)+tv(x)|^s-|u(x)|^s-st|u(x)|^{s-2}u(x)v(x))|\\ \leqslant&s\phi_u(x)(||u(x)+\theta tv(x)|^{s-2}(u(x)+\theta tv(x))v(x)|+||u(x)|^{s-2}u(x)v(x)|)\\ \leqslant&C\phi_u(x)(|u(x)|^{s-1}|v(x)|+|v(x)|^s).\end{aligned}\tag{2.17}$$

So, it follows from Hölder and Sobolev inequalities, Proposition 2.2 (iii) that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\phi_u(|u|^{s-1}|v|+|v|^s)dx\right| &\leqslant |\phi_u|_{r^*}\left(\int_{\mathbb{R}^3}(|u|^{s-1}|v|+|v|^s)^{\frac{r^*}{r^*-1}}dx\right)^{\frac{r^*-1}{r^*}}\\ &\leqslant C\|\phi_u\|_{D^{1,r}}\left(\int_{\mathbb{R}^3}((|u|^{s-1}|v|)^{\frac{r^*}{r^*-1}}+|v|^{\frac{r^*s}{r^*-1}})dx\right)^{\frac{r^*-1}{r^*}}\\ &\leqslant C\|u\|^{\frac{s}{r-1}}\left(\int_{\mathbb{R}^3}((|u|^{s-1}|v|)^{\frac{r^*}{r^*-1}}+|v|^{\frac{r^*s}{r^*-1}})dx\right)^{\frac{r^*-1}{r^*}}.\end{aligned}$$

Furthermore, by Young inequality and $\max\{1,\frac{p(r^*-1)}{r^*}\}<s<\frac{q^*(r^*-1)}{r^*}$, we have $\phi_u(|u|^{s-1}|v|+|v|^s)\in L^1(\mathbb{R}^3)$. Combining (2.16) and (2.17), by Lebesgue dominated convergence theorem, we get (2.15).

By (2.14), one has

$$\liminf_{t\to 0}\frac{\mathcal{D}}{t}\geqslant 0.$$

Using the same method, we can get

$$\limsup_{t\to 0}\frac{\mathcal{D}}{t}\leqslant 0.$$

Indeed, using

$$\mathcal{M}(u,\phi_u)\leqslant\mathcal{M}(u,\phi_{u+tv}),$$

we deduce that

$$\begin{aligned}\mathcal{D}&=\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_{u+tv}|u+tv|^sdx-\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_u|u|^sdx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ &=\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_{u+tv}|u+tv|^sdx+\frac{1}{s}\mathcal{M}(u,\phi_u)-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ &\leqslant\frac{r-1}{rs}\int_{\mathbb{R}^3}\phi_{u+tv}|u+tv|^sdx+\frac{1}{s}\mathcal{M}(u,\phi_{u+tv})-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ &=\frac{1}{s}\int_{\mathbb{R}^3}|u+tv|^s\phi_{u+tv}dx-\frac{1}{s}\int_{\mathbb{R}^3}|u|^s\phi_{u+tv}dx-t\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\\ &=\frac{1}{s}\int_{\mathbb{R}^3}\big(\phi_{u+tv}\big(|u+tv|^s-|u|^s-st|u|^{s-2}uv\big)+st(\phi_{u+tv}-\phi_u)|u|^{s-2}uv\big)\,dx\\ &\leqslant\frac{1}{s}\big(\int_{\mathbb{R}^3}|\phi_{u+tv}|^{r^*}dx\big)^{\frac{1}{r^*}}\big(\int_{\mathbb{R}^3}||u+tv|^s-|u|^s-st|u|^{s-2}uv|^{\frac{r^*}{r^*-1}}dx\big)^{\frac{r^*-1}{r^*}}\\ &\quad+t\big(\int_{\mathbb{R}^3}|\phi_{u+tv}-\phi_u|^{r^*}dx\big)^{\frac{1}{r^*}}\big(\int_{\mathbb{R}^3}||u|^{s-2}uv|^{\frac{r^*}{r^*-1}}dx\big)^{\frac{r^*-1}{r^*}}.\end{aligned}\tag{2.18}$$

Similar to the proof of (2.15), we have that

$$\lim_{t\to 0}\frac{1}{t}\left(\int_{\mathbb{R}^3}||u+tv|^s-|u|^s-st|u|^{s-2}uv|^{\frac{r^*}{r^*-1}}dx\right)^{\frac{r^*-1}{r^*}}=0.\tag{2.19}$$

By Proposition 2.2 (v) and Sobolev embedding theorem, we can get that

$$\lim_{t\to 0}\int_{\mathbb{R}^3}|\phi_{u+tv}-\phi_u|^{r^*}dx=0.\tag{2.20}$$

By (2.19) and (2.20), we deduce from (2.18) that $\limsup\limits_{t\to 0}\frac{\mathcal{D}}{t}\leqslant 0$. So,

$$\mathcal{D}=o(t),\quad t\to 0.$$

Finally, we show that the continuity of $J'_G: X\to X^*$. Now, we define

$$\langle \Phi'_G(u),v\rangle=\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uvdx\ \text{ for all } u,v\in X.$$

So, we only need to prove that Φ'_G is continuous on X. Let $u_n\to u$ in X. By Proposition 2.2(v), $\phi_{u_n}\to\phi_u$ in $L^{r^*}(\mathbb{R}^3)$, combining $|u_n|^{s-2}u_n\to|u|^{s-2}u$ in $L^{\frac{r^*s}{(s-1)(r^*-1)}}(\mathbb{R}^3)$, then, for any $v\in X$,

$$\begin{aligned}
&|\langle\Phi'_G(u_n)-\Phi'_G(u),v\rangle|\\
\leqslant&\int_{\mathbb{R}^3}|\phi_{u_n}|u_n|^{s-2}u_n-\phi_u|u|^{s-2}u||v|dx\\
\leqslant&\int_{\mathbb{R}^3}|\phi_{u_n}||u_n|^{s-2}u_n-|u|^{s-2}u||v|dx+\int_{\mathbb{R}^3}|\phi_{u_n}-\phi_u||u|^{s-1}|v|dx\\
\leqslant&|\phi_{u_n}|_{r^*}||u_n|^{s-2}u_n-|u|^{s-2}u|_{\frac{r^*s}{(s-1)(r^*-1)}}|v|_{\frac{r^*s}{r^*-1}}+|\phi_{u_n}-\phi_u|_{r^*}|u|^{s-1}_{\frac{r^*s}{r^*-1}}|v|_{\frac{r^*s}{r^*-1}}\\
\leqslant&C\left(|\phi_{u_n}|_{r^*}||u_n|^{s-2}u_n-|u|^{s-2}u|_{\frac{r^*s}{(s-1)(r^*-1)}}+|\phi_{u_n}-\phi_u|_{r^*}|u|^{s-1}_{\frac{r^*s}{r^*-1}}\right)\|v\|\\
=&o(1)\|v\|.
\end{aligned}$$

So, the proof is complete. □

In order to find a solution to system (1.1), we need the following proposition which shows the relationship between the solution of system (1.1) and the critical point of functional J_λ.

Proposition 2.4. *Let $(u,\phi)\in X\times D^{1,r}(\mathbb{R}^3)$. Then (u,ϕ) is a solution of system* (1.1) *if and only if u is a critical point of J_λ and $\phi=\phi_u$.*

Proof. For convenience, let's firstly define the functional $\mathcal{F}: X\times D^{1,r}(\mathbb{R}^3)\to\mathbb{R}$ by

$$\begin{aligned}
\mathcal{F}(u,\phi)=&\frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p+|u|^p)dx+\frac{1}{q}\int_{\mathbb{R}^3}|\nabla u|^q dx+\frac{\lambda}{s}\int_{\mathbb{R}^3}\phi|u|^s dx\\
&-\frac{\lambda}{rs}\int_{\mathbb{R}^3}|\nabla\phi|^r dx-\frac{1}{l}\int_{\mathbb{R}^3}|u|^l dx-\int_{\mathbb{R}^3}f(x)udx.
\end{aligned}$$

Clearly, by simple calculations, for any $(v,w)\in X\times D^{1,r}(\mathbb{R}^3)$, one has

$$\begin{aligned}
&\langle\partial_u\mathcal{F}(u,\phi),v\rangle\\
=&\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\cdot\nabla v+|u|^{p-2}uv)dx+\int_{\mathbb{R}^3}|\nabla u|^{q-2}\nabla u\cdot\nabla vdx\\
&+\lambda\int_{\mathbb{R}^3}\phi|u|^{s-2}uvdx-\int_{\mathbb{R}^3}|u|^{l-2}uvdx-\int_{\mathbb{R}^3}f(x)vdx,\\
=&\langle J'_\lambda(u),v\rangle,\\
&\langle\partial_\phi\mathcal{F}(u,\phi),w\rangle\\
=&\frac{\lambda}{s}\int_{\mathbb{R}^3}|u|^s wdx-\frac{\lambda}{s}\int_{\mathbb{R}^3}|\nabla\phi|^{r-2}\nabla\phi\cdot\nabla wdx.
\end{aligned}$$

The result of this proposition is obtained from the following equivalence

$$(u,\phi) \text{ is a solution of (1.1)} \Leftrightarrow \partial_u\mathcal{F}(u,\phi)=0 \text{ and } \partial_\phi\mathcal{F}(u,\phi)=0 \Leftrightarrow\ J'_\lambda(u)=0 \text{ and } \phi=\phi_u.$$

□

In the forthcoming sections, to obtain solutions of system (1.1), we need to seek critical points of the functional J_λ in the space X. However, due to the translation invariance of J_λ, the problem lacks compactness, which complicates the application of standard variational methods. To overcome this obstacle, we restrict J_λ to the radially symmetric subspace of X, denoted by X_r. It is a natural constraint for J_λ by Palais' principle of symmetric criticality [28] and the existence and uniqueness of solution for equation (2.1).

3. A solution with negative energy

In this section, we seek a negative energy solution to system (1.1) with (f_1).

Lemma 3.1. *Assume that f satisfies (f_1) and $q<l<q^*$. Then, there exist $\rho>0$ small enough, $\Lambda>0$ and $\alpha>0$ such that $J_\lambda(u)\geqslant\alpha$ for all $u\in X_r$ with $\|u\|=\rho$ and $|f|_t<\Lambda$.*

Proof. First, let's review a fundamental inequality which is important for the proof. There exist $C_1, C_2>0$ such that for all $a,b>0$ and $1<m<n$,

$$a^m+b^n\geqslant\begin{cases}C_1(a+b)^n, & a+b\leqslant 1,\\ C_2(a+b)^m, & a+b>1.\end{cases}\tag{3.1}$$

For any $u\in X_r$ and $\|u\|\leqslant 1$, it is easy to see that

$$\begin{aligned}J_\lambda(u)=&\frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p+|u|^p)dx+\frac{1}{q}\int_{\mathbb{R}^3}|\nabla u|^q dx+\frac{\lambda(r-1)}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx\\&-\frac{1}{l}\int_{\mathbb{R}^3}|u|^l dx-\int_{\mathbb{R}^3}f(x)u dx\\ \geqslant&\frac{C_1}{q}\|u\|^q-\frac{S_l^l}{l}\|u\|^l-S_{t'}|f|_t\|u\|\\ =&\|u\|\left(\frac{C_1}{q}\|u\|^{q-1}-\frac{S_l^l}{l}\|u\|^{l-1}-S_{t'}|f|_t\right),\end{aligned}$$

where S_ξ stands for the Sobolev embedding constant of $X\hookrightarrow L^\xi(\mathbb{R}^3)$ for $p<\xi<q^*$. Let

$$g(t)=\frac{C_1}{q}t^{q-1}-\frac{S_l^l}{l}t^{l-1}.$$

After a simple calculation, since $q<l$, there exists a unique $\rho>0$ such that g attains its maximum value

$$g(\rho)=\max_{t\in[0,1]}g(t)>0.$$

By choosing

$$\Lambda=\frac{g(\rho)}{S_{t'}}\quad\text{and}\quad\alpha=\rho(g(\rho)-S_{t'}|f|_t),$$

the proof is complete. □

Lemma 3.2. *Assume that f satisfies (f_1) and $q < l < q^*$. Let $\{u_n\} \subset X_r$ be a bounded sequence with $J'_\lambda(u_n) \to 0$. Then, up to a subsequence, $u_n \to u$ in X_r for some $u \in X_r$.*

Proof. Since $\{u_n\} \subset X_r$ is bounded, we can deduce that there exists $u \in X_r$ such that, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } X_r, \\ u_n \to u & \text{in } L^t(\mathbb{R}^3),\ p < t < q^*, \\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

We will claim that $u_n \to u$ in X_r, namely,

$$\|u_n - u\| \to 0,\ n \to \infty. \tag{3.2}$$

Indeed, by Hölder inequality, one has

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} (|u_n|^{l-2}u_n - |u|^{l-2}u)(u_n - u)dx \right| \\ \leqslant &\int_{\mathbb{R}^3} |u_n|^{l-1}|u_n - u|dx + \int_{\mathbb{R}^3} |u|^{l-1}|u_n - u|dx \\ \leqslant &|u_n|_l^{l-1}|u_n - u|_l + |u|_l^{l-1}|u_n - u|_l \\ = &o(1). \end{aligned} \tag{3.3}$$

Combining Proposition 2.2 (iii) and Hölder inequality, one has that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} (\phi_{u_n}|u_n|^{s-2}u_n - \phi_u|u|^{s-2}u)(u_n - u)dx \right| \\ \leqslant &|\phi_{u_n}|_{r^*}||u_n|^{s-2}u_n(u_n - u)|_{(r^*)'} + |\phi_u|_{r^*}||u|^{s-2}u(u_n - u)|_{(r^*)'} \\ \leqslant &C\|\phi_{u_n}\|_{D^{1,r}}|u_n|^{s-1}_{\frac{r^*s}{r^*-1}}|u_n - u|_{\frac{r^*s}{r^*-1}} + C\|\phi_u\|_{D^{1,r}}|u|^{s-1}_{\frac{r^*s}{r^*-1}}|u_n - u|_{\frac{r^*s}{r^*-1}} \\ \leqslant &C\|u_n\|^{\frac{s}{r-1}+(s-1)}|u_n - u|_{\frac{r^*s}{r^*-1}} + C\|u\|^{\frac{s}{r-1}+(s-1)}|u_n - u|_{\frac{r^*s}{r^*-1}} \\ = &o(1). \end{aligned} \tag{3.4}$$

By $J'_\lambda(u_n) \to 0$ and $\{u_n\} \subset X_r$ is bounded, it results that

$$\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u \rangle \to 0 \ \text{ as } n \to \infty. \tag{3.5}$$

It follows from (3.3)-(3.5) that

$$\begin{aligned} &\int_{\mathbb{R}^3} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big) \cdot (\nabla u_n - \nabla u)dx \\ &+ \int_{\mathbb{R}^3} \big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)dx \\ &+ \int_{\mathbb{R}^3} \big(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\big) \cdot (\nabla u_n - \nabla u)dx = o(1). \end{aligned} \tag{3.6}$$

Similar to the proof of Proposition 2.1, for the case $2 \leqslant p < 3$, we get

$$\int_{\mathbb{R}^3} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big) \cdot (\nabla u_n - \nabla u)dx \geqslant C\int_{\mathbb{R}^3} |\nabla u_n - \nabla u|^p dx, \tag{3.7}$$

and

$$\int_{\mathbb{R}^3}\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)dx \geqslant C\int_{\mathbb{R}^3}|u_n-u|^p dx. \tag{3.8}$$

For the case $1<p<2$, from the boundedness of $\{u_n\}$, Hölder inequality, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla(u_n-u)|^p dx\\
\leqslant& C\int_{\mathbb{R}^3}\left[\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u)\right]^{\frac{p}{2}}\left(|\nabla u_n|+|\nabla u|\right)^{\frac{p(2-p)}{2}}dx\\
\leqslant& C\left(\int_{\mathbb{R}^3}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u)dx\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^3}\left(|\nabla u_n|^p+|\nabla u|^p\right)dx\right)^{\frac{2-p}{2}}\\
\leqslant& C\left(\int_{\mathbb{R}^3}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n-u)dx\right)^{\frac{p}{2}}.
\end{aligned} \tag{3.9}$$

In the same way, we obtain

$$\int_{\mathbb{R}^3}|u_n-u|^p dx\leqslant C\left(\int_{\mathbb{R}^3}\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)(u_n-u)dx\right)^{\frac{p}{2}}. \tag{3.10}$$

Therefore, it follows from (3.6)-(3.10) that $u_n\to u$ in $W^{1,p}(\mathbb{R}^3)$. Similar to (3.7) and (3.9), we can also get that $u_n\to u$ in $D^{1,q}(\mathbb{R}^3)$. Thus, (3.2) holds. □

Theorem 3.1. *Assume that (f_1) holds and $q<l<q^*$. Then, for any $\lambda>0$, $|f|_t<\Lambda$, J_λ has a critical point u_1 with $J_\lambda(u_1)<0$ where Λ was given in Lemma 3.1.*

Proof. First, we will find a function $w\in X_r$ such that $\int_{\mathbb{R}^3}f(x)w(x)dx>0$. In fact, it is obvious that $|f|^{t-2}f\in L^{t'}(\mathbb{R}^3)$ from $f\in L^t(\mathbb{R}^3)$. Because $C_0^\infty(\mathbb{R}^3)$ is dense in $L^{t'}(\mathbb{R}^3)$ and f is radial. Then, there exists a radial sequence $\{f_n\}\subset C_0^\infty(\mathbb{R}^3)$ such that $f_n\to|f|^{t-2}f$ in $L^{t'}(\mathbb{R}^3)$. Furthermore,

$$\int_{\mathbb{R}^3}ff_n dx\to\int_{\mathbb{R}^3}|f|^t dx \text{ as } n\to\infty.$$

Let's pick n_0 such that $\int_{\mathbb{R}^3}ff_{n_0}>0$. Selecting $w(x)=f_{n_0}(x)$, we will get $\int_{\mathbb{R}^3}f(x)w(x)dx>0$.

So, one has, for any $t>0$ small enough,

$$\begin{aligned}
J_\lambda(tw)=&\frac{t^p}{p}\int_{\mathbb{R}^3}(|\nabla w|^p+|w|^p)dx+\frac{t^q}{q}\int_{\mathbb{R}^3}|\nabla w|^q dx+\frac{\lambda(r-1)t^{\frac{sr}{r-1}}}{rs}\int_{\mathbb{R}^3}\phi_w|w|^s dx\\
&-\frac{t^l}{l}\int_{\mathbb{R}^3}|w|^l dx-t\int_{\mathbb{R}^3}f(x)w dx\\
<&0.
\end{aligned}$$

Thus, it is easy to prove that

$$c=\inf_{u\in\overline{B}_\rho}J_\lambda(u)<0,$$

where $\overline{B}_\rho=\{u\in X_r:\|u\|\leqslant\rho\}$ and ρ is in Lemma 3.1.

The next step involves utilizing Ekeland variational principle to demonstrate the existence of local minimizers of J_λ. First, by Ekeland variational principle [20], the sequence $\{u_n\}\subset\overline{B}_\rho$ is obtained which satisfies

$$c\leqslant J_\lambda(u_n)\leqslant c+\frac{1}{n} \tag{3.11}$$

and

$$J_\lambda(v) \geqslant J_\lambda(u_n) - \frac{1}{n}\|v - u_n\| \quad \text{for all} \quad v \in \overline{B}_\rho. \tag{3.12}$$

Next, we assert that $\|u_n\| < \rho$ for n large enough. Indeed, if $\|u_n\| = \rho$, applying Lemma 3.1, we can get that $J_\lambda(u_n) \geqslant \alpha > 0$ which is contradict with (3.11) for n large enough. Finally, we will show that $J'_\lambda(u_n) \to 0$. For any $v \in X_r$ and $\|v\| = 1$, we can choose sufficiently small $\epsilon > 0$ such that $\|u_n + tv\| < \rho$ for all $|t| < \epsilon$. Furthermore, we get that

$$\frac{J_\lambda(u_n + tv) - J_\lambda(u_n)}{t} \geqslant -\frac{1}{n}\|v\| = -\frac{1}{n},$$

where we have used (3.12). Letting $t \to 0$, we deduce that $\langle J'_\lambda(u_n), v\rangle \geqslant -\frac{1}{n}$. Replacing v with $-v$ for the formula above, we have $\langle J'_\lambda(u_n), v\rangle \leqslant \frac{1}{n}$. Thus, $|\langle J'_\lambda(u_n), v\rangle| \leqslant \frac{1}{n}$ for every $v \in X_r$ with $\|v\| = 1$. Therefore,

$$J'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Then, $\{u_n\}$ is a bounded $(\mathrm{PS})_c$ sequence of J_λ. By Lemma 3.2, there exists $u_1 \in X_r$ such that $J_\lambda(u_1) = c < 0$ and $J'_\lambda(u_1) = 0$. The proof is complete. □

4. A solution with positive energy

In this section, we will prove that system (1.1) has a solution with positive energy. It is worth noting that here we need

$$\max\left\{1, \frac{p(r^*-1)}{r^*}, \frac{q(r-1)}{r}\right\} < s < \frac{q^*(r^*-1)}{r^*}.$$

It ensures that

$$q < \frac{qr(s+1)}{r+q(r-1)} < q^*.$$

We handle system (1.1) in the following two cases.

4.1. The case $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$

In this subsection, we consider the case $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$ and prove the following theorem.

Theorem 4.1. *Assume that (f_1), (f_2) hold and $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$. Then, for any $\lambda > 0$, $|f|_t < \Lambda$, J_λ has a critical point u_2 with $J_\lambda(u_2) > 0$, where Λ is given in Lemma 3.1.*

Lemma 4.1. *Let $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$ and $\lambda > 0$. Then, we have the following:*

(i) *there exist $\rho, \Lambda, \alpha > 0$ such that $J_\lambda(u) \geqslant \alpha$ for $u \in X$ with $\|u\| = \rho$ and $|f|_t < \Lambda$;*

(ii) *there exists $v \in X$ with $\|v\| > \rho$ such that $J_\lambda(v) < 0$.*

Proof. (i) The proof here is the same as that of Lemma 3.1. We omit it.

(ii) Set

$$\eta := \frac{r + q(r-1)}{rs + q(1-r)}.$$

For any $u \in X\backslash\{0\}$ and $t > 0$, we denote $u_t(x) = t^{\eta} u(tx)$. By Proposition 2.2 (ii), a direct computation gives that

$$\begin{aligned} J_\lambda(u_t) =& \frac{t^{\eta_1-3}}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx + \frac{t^{\eta_2-3}}{p}\int_{\mathbb{R}^3}|u|^p dx + \frac{t^{\eta_3-3}}{q}\int_{\mathbb{R}^3}|\nabla u|^q dx + \frac{\lambda(r-1)t^{\eta_3-3}}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx \\ &- \frac{t^{\eta_4-3}}{l}\int_{\mathbb{R}^3}|u|^l dx - t^{\eta_5-3}\int_{\mathbb{R}^3} f(\frac{x}{t})u dx, \end{aligned}$$

where

$$\begin{aligned} &\eta_1 := \frac{pr(s+1)}{rs+q(1-r)}, \quad \eta_2 := \frac{p[r+q(r-1)]}{rs+q(1-r)}, \; \eta_3 := \frac{qr(s+1)}{rs+q(1-r)}, \\ &\eta_4 := \frac{l[r+q(r-1)]}{rs+q(1-r)}, \quad \eta_5 := \eta. \end{aligned}$$

We claim that

$$\eta_4 > \eta_3 > \eta_1 > \eta_2, \quad \eta_3 - 3 > 0.$$

Indeed, by $\max\{1, \frac{p(r^*-1)}{r^*}, \frac{q(r-1)}{r}\} < s < \frac{q^*(r^*-1)}{r^*}$, we have

$$rs + q(1-r) > 0, \; s < \frac{q(4r-3)}{r(3-q)}.$$

Combining $\frac{qr(s+1)}{r+q(r-1)} < l < q^*$ and $q > p$, we get that $\eta_4 > \eta_3 > \eta_1 > \eta_2$ and $\eta_3 - 3 > 0$. Consequently, for some sufficiently large $t_0 > 0$, the choice $v = u_{t_0}$ yields $J_\lambda(v) < 0$. $\square$

Now, we deduce that the functional J_λ satisfies mountain pass theorem [2]. The mountain pass level of J_λ can be defined as

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X_r) : \gamma(0) = 0, \; J_\lambda(\gamma(1)) < 0\}.$$

To prove the following Lemma, by (f_1) and (f_2), we define an auxiliary functional $P_\lambda : X_r \to \mathbb{R}$ as follows

$$\begin{aligned} P_\lambda(u) =& \frac{\beta_1}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx + \frac{\beta_2}{p}\int_{\mathbb{R}^3}|u|^p dx + \frac{\beta_3}{q}\int_{\mathbb{R}^3}|\nabla u|^q dx + \frac{\lambda(r-1)\beta_3}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx \\ &- \frac{\beta_4}{l}\int_{\mathbb{R}^3}|u|^l dx - \beta_5\int_{\mathbb{R}^3} f(x)u dx + \int_{\mathbb{R}^3}(x, \nabla f(x))u dx, \end{aligned}$$

where $\beta_i = \eta_i - 3$, for $i = 1, 2, 3, 4, 5$.

Lemma 4.2. *Under the assumption of theorem 4.1, there exists a bounded sequence* $\{u_n\} \subset X_r$ *satisfying*

$$J_\lambda(u_n) \to c, \quad J'_\lambda(u_n) \to 0, \quad P_\lambda(u_n) \to 0. \tag{4.1}$$

Proof. The following method adopts the idea of Jeanjean in [24].

Here, we introduce the action of group $\mathbb{R}$ on X_r defined as follows. For any $t \in \mathbb{R}$, $u \in X_r$, we can define $t * u$ by

$$t * u(x) = e^{\eta t}u(e^t x), \; \text{a.e.} \; x \in \mathbb{R}^N,$$

where η is given in Lemma 4.1. So, it is easy to see that

$$\begin{aligned} J_\lambda(t*u) = &\frac{e^{\beta_1 t}}{p}\int_{\mathbb{R}^3}|\nabla u|^p dx + \frac{e^{\beta_2 t}}{p}\int_{\mathbb{R}^3}|u|^p dx + \frac{e^{\beta_3 t}}{q}\int_{\mathbb{R}^3}|\nabla u|^q dx \\ &+ \frac{\lambda(r-1)e^{\beta_3 t}}{rs}\int_{\mathbb{R}^3}\phi_u|u|^s dx - \frac{e^{\beta_4 t}}{l}\int_{\mathbb{R}^3}|u|^l dx - e^{\beta_5 t}\int_{\mathbb{R}^3} f(\frac{x}{e^t})u dx. \end{aligned}$$

Now, we define $\tilde{J}_\lambda : \mathbb{R}\times X_r \to \mathbb{R}$ by

$$\tilde{J}_\lambda(t,u) = J_\lambda(t*u) \text{ for all } (t,u)\in\mathbb{R}\times X_r.$$

By a simple calculation, for any $u, v \in X_r$, it holds that

$$\tilde{J}_\lambda(0,u) = J_\lambda(u), \quad \partial_t\tilde{J}_\lambda(0,u) = P_\lambda(u), \quad \langle\partial_u\tilde{J}_\lambda(0,u), v\rangle = \langle J'_\lambda(u), v\rangle.$$

Let $D = (\partial_t, \partial_u)$ for all $(t,u)\in\mathbb{R}\times X_r$. Then we have

$$\langle D\tilde{J}_\lambda(t,u),(\kappa,v)\rangle = \langle\partial_t\tilde{J}_\lambda(t,u),\kappa\rangle + \langle\partial_u\tilde{J}_\lambda(t,u),v\rangle \text{ for all } (\kappa,v)\in\mathbb{R}\times D^{1,p}(\mathbb{R}^N). \tag{4.2}$$

We set the standard product norm $\|(t,u)\|_{\mathbb{R}\times X_r} = (|t|^p + \|u\|^p)^{\frac{1}{p}}$ for all $(t,u)\in\mathbb{R}\times X_r$. The dual norm of $\|\cdot\|_{\mathbb{R}\times X_r}$ is defined by

$$\|g\|_{(\mathbb{R}\times X_r)^*} = \sup\{|g(\theta,u)| : \|(\theta,u)\|_{\mathbb{R}\times X_r}\leqslant 1\} \text{ for each } g\in(\mathbb{R}\times X_r)^*.$$

The mountain pass level for $\tilde{J}_\lambda$ can also be defined by

$$\tilde{c} = \inf_{\tilde{\gamma}\in\tilde{\Gamma}}\max_{t\in[0,1]}\tilde{J}_\lambda(\tilde{\gamma}(t)),$$

where

$$\tilde{\Gamma} = \{\tilde{\gamma}\in C([0,1],\mathbb{R}\times X_r) : \tilde{\gamma}(0) = (0,0), \tilde{J}_\lambda(\tilde{\gamma}(1)) < 0\}.$$

From the above definitions, it follows that $c = \tilde{c}$. Indeed, on the one hand, for any $\gamma\in\Gamma$, set $\tilde{\gamma}(t) = (0,\gamma(t))$ for all $t\in[0,1]$, which is in $\tilde{\Gamma}$. Thus, it leads to that $c\geqslant\tilde{c}$. On the other hand, for any $\tilde{\gamma}\in\tilde{\Gamma}$, that is, $\tilde{\gamma}(t) = (\theta(t),\eta(t))$ for all $t\in[0,1]$. Let

$$\gamma(t) : \gamma(t)(\cdot) = \eta(t)(e^{-\theta(t)}\cdot) \text{ for all } t\in[0,1].$$

Then $\gamma\in\Gamma$ and $J_\lambda(\gamma(t)) = \tilde{J}_\lambda(\tilde{\gamma}(t))$ for all $t\in[0,1]$. Thus, we also have $\tilde{c}\geqslant c$. By the general minimax principle, there exist sequences $\{\gamma_n\}\subset\Gamma$ and $\{(t_n,v_n)\}\subset\mathbb{R}\times X_r$ such that

$$|\tilde{J}_\lambda(t_n,v_n) - c| \leqslant \frac{1}{n}, \tag{4.3}$$

$$\text{dist}_{\mathbb{R}\times X_r}((t_n,v_n),\{0\}\times\gamma_n([0,1])) \leqslant \frac{2}{\sqrt{n}}, \tag{4.4}$$

$$\|D\tilde{J}_\lambda(t_n,v_n)\|_{(\mathbb{R}\times X_r)^*} \leqslant \frac{2}{\sqrt{n}}, \tag{4.5}$$

where $\text{dist}_{\mathbb{R}\times X_r}((t,u),A) = \inf_{(\kappa,v)\in A}(|t-\kappa|^p + \|u-v\|^p)^{\frac{1}{p}}$ for all $(t,u)\in\mathbb{R}\times X_r$. It follows from (4.3)-(4.5) that

$$t_n \to 0, \tag{4.6}$$

$$J_\lambda(t_n * v_n) \to c, \tag{4.7}$$
$$J'_\lambda(t_n * v_n) \to 0 \ \text{ in } X_r^*, \tag{4.8}$$
$$P_\lambda(t_n * v_n) \to 0. \tag{4.9}$$

Indeed, clearly (4.3) and (4.4) imply (4.7) and (4.6) respectively. For (4.9), we recall (4.2), $t_n \to 0$ and (4.5). Finally, we just have to prove (4.8). By a simple calculation, one has

$$\langle J'_\lambda(t_n * v_n), \varphi\rangle = \langle \partial_u \tilde{J}_\lambda(t_n, v_n), (-t_n) * \varphi\rangle \ \text{ for all } \ \varphi \in X_r.$$

Then

$$\begin{aligned}\|J'_\lambda(t_n * v_n)\|_{X_r^*} &= \sup_{\|\varphi\|\leqslant 1} |\langle J'_\lambda(t_n * v_n), \varphi\rangle| \\ &= \sup_{\|\varphi\|\leqslant 1} |\langle \partial_u \tilde{J}_\lambda(t_n, v_n), (-t_n) * \varphi\rangle| \\ &\leqslant \sup_{\|\varphi\|\leqslant 1} \|\partial_u \tilde{J}_\lambda(t_n, v_n)\|_{X_r^*} \|(-t_n) * \varphi\|.\end{aligned}$$

Thus, it is clearly that (4.6) and (4.5) imply (4.8).

Next, let $u_n = t_n * v_n$, $\{u_n\} \subset X_r$ is the sequence that satisfies (4.1), we will prove that $\{u_n\}$ is bounded in X_r. by (4.1), for n large enough, one has

$$\begin{aligned} c+1 \geqslant\ & J_\lambda(u_n) - \frac{1}{\beta_4} P_\lambda(u_n) \\ =\ & \frac{\beta_4-\beta_1}{p\beta_4}\int_{\mathbb{R}^3} |\nabla u_n|^p dx + \frac{\beta_4-\beta_2}{p\beta_4}\int_{\mathbb{R}^3} |u_n|^p dx + \frac{\beta_4-\beta_3}{q\beta_4}\int_{\mathbb{R}^3} |\nabla u_n|^q dx \\ & + \frac{\lambda(r-1)(\beta_4-\beta_3)}{rs\beta_4}\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s dx + \frac{\beta_5-\beta_4}{\beta_4}\int_{\mathbb{R}^3} f(x)u_n dx \\ & - \frac{1}{\beta_4}\int_{\mathbb{R}^3} (x, \nabla f(x))u_n dx. \end{aligned}$$

Furthermore,

$$\begin{aligned} & c+1 + \frac{\beta_4-\beta_5}{\beta_4}\int_{\mathbb{R}^3} f(x)u_n dx + \frac{1}{\beta_4}\int_{\mathbb{R}^3} (x, \nabla f(x))u_n dx \\ \geqslant & \frac{\beta_4-\beta_1}{p\beta_4}\int_{\mathbb{R}^3} |\nabla u_n|^p dx + \frac{\beta_4-\beta_2}{p\beta_4}\int_{\mathbb{R}^3} |u_n|^p dx + \frac{\beta_4-\beta_3}{q\beta_4}\int_{\mathbb{R}^3} |\nabla u_n|^q dx. \end{aligned} \tag{4.10}$$

From (f_1) and (f_2), Hölder and Sobolev inequalities, we have

$$\left|\int_{\mathbb{R}^3} f(x)u_n dx\right| + \left|\int_{\mathbb{R}^3} (x, \nabla f(x))u_n dx\right| \leqslant C\|u_n\|. \tag{4.11}$$

So, by (4.10) and (4.11), it follows that $\{u_n\}$ is bounded in X_r. □

Proof of Theorem 4.1. The result is immediately obtained by Lemmas 4.1, 4.2 and 3.2. □

4.2. The case $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$

In this subsection, we consider the case $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$. the main result is as follows.

Theorem 4.2. *Assume that* (f_1) *holds and* $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$. *Then, there exists* $\lambda_0 > 0$ *such that, for all* $\lambda \in (0, \lambda_0)$, $|f|_t < \Lambda$, J_λ *has a critical point* u_2 *with* $J_\lambda(u_2) > 0$, *where* Λ *is given in Lemma 3.1.*

Now, we need to do some technical processing, refer to [25]. To do this, we need to define a smooth function $\eta \in C^\infty([0, +\infty), [0, 1])$ such that

$$\eta(t) = \begin{cases} 1, & \text{for } t \in [0, 1/2], \\ \in [0,1], & \text{for } \ t \in (1/2, 1), \\ 0, & \text{for } t \in [1, +\infty), \end{cases}$$

and $|\eta'|_\infty \leqslant 4$. So, we can define a penalized functional $J_{\lambda,M} : X_r \to \mathbb{R}$ as

$$\begin{aligned} J_{\lambda,M}(u) = & \frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)dx + \frac{1}{q}\int_{\mathbb{R}^3}|\nabla u|^q dx + \frac{\lambda(r-1)}{rs}L_M(u)\int_{\mathbb{R}^3}\phi_u|u|^s dx \\ & - \frac{1}{l}\int_{\mathbb{R}^3}|u|^l dx - \int_{\mathbb{R}^3} f(x)u dx, \end{aligned} \tag{4.12}$$

where $M > 0$ and

$$L_M(u) = \eta\left(\frac{\|u\|_{1,p}^p}{M^p} + \frac{\|u\|_{D^{1,q}}^q}{M^q}\right).$$

It is easy to see that $J_{\lambda,M}$ belongs to C^1 with

$$\begin{aligned} \langle J'_{\lambda,M}(u), v\rangle = & (1 + H_{\lambda,M}(u))\int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)dx \\ & + (1 + G_{\lambda,M}(u))\int_{\mathbb{R}^3}|\nabla u|^{q-2}\nabla u \cdot \nabla v dx \\ & + \lambda L_M(u)\int_{\mathbb{R}^3}\phi_u|u|^{s-2}uv dx \\ & - \int_{\mathbb{R}^3}|u|^{l-2}uv dx - \int_{\mathbb{R}^3} f(x)v dx \ \text{ for all } \ v \in X_r, \end{aligned} \tag{4.13}$$

where

$$H_{\lambda,M}(u) = \frac{\lambda(r-1)p}{rsM^p}\eta'\left(\frac{\|u\|_{1,p}^p}{M^p} + \frac{\|u\|_{D^{1,q}}^q}{M^q}\right)\int_{\mathbb{R}^3}\phi_u|u|^s dx,$$

$$G_{\lambda,M}(u) = \frac{\lambda(r-1)q}{rsM^q}\eta'\left(\frac{\|u\|_{1,p}^p}{M^p} + \frac{\|u\|_{D^{1,q}}^q}{M^q}\right)\int_{\mathbb{R}^3}\phi_u|u|^s dx.$$

It is obviously that if u is a critical point of $J_{\lambda,M}$ and $\|u\| \leqslant \frac{M}{4}$, then u is a critical point of J_λ. To do this, we first prove that for any $M > 0$, the penalized functional $J_{\lambda,M}$ enjoys the mountain pass structure.

Lemma 4.3. *Assume that* (f_1) *holds and* $p < q < l < \frac{qr(s+1)}{r+q(r-1)}$. *For any* $M > 0$,

(i) *there exist* $\rho, \Lambda, \alpha > 0$ *such that* $J_{\lambda,M}(u) \geqslant \alpha$ *for* $u \in X_r$ *with* $\|u\| = \rho$ *and* $|f|_t < \Lambda$;

(ii) *there exists* $v \in X_r$ *with* $\|v\| > \rho$ *such that* $J_{\lambda,M}(v) < 0$.

Proof. (i) The proof here is the same as the proof of Lemma 3.1. We omit it.

(ii) Similar to that in Theorem 3.1, we choose $\varphi \in X_r$ such that $\int_{\mathbb{R}^3} f(x)\varphi(x)dx > 0$ and $\|\varphi\|_{1,p} = 1$ or $\|\varphi\|_{D^{1,q}} = 1$. For any given $M > 0$ and $t \geqslant M$, it's easy to know that $L_M(t\varphi) = 0$. So,

$$J_{\lambda,M}(t\varphi) = \frac{t^p}{p}\|\varphi\|_{1,p}^p + \frac{t^q}{q}\|\varphi\|_{D^{1,q}}^q - \frac{t^l}{l}\int_{\mathbb{R}^3} |\varphi|^l dx - t\int_{\mathbb{R}^3} f(x)\varphi dx.$$

Since $p < q < l$, we can choose $t_M > M$ large enough, so that $\|t\varphi\| > \rho$ and $J_{\lambda,M}(t\varphi) < 0$. □

By Lemma 4.3, for any $M > 0$, we have stated that the penalized functional $J_{\lambda,M}$ has a mountain pass geometry. Then, we can define the following mountain pass level of $J_{\lambda,M}$,

$$c_{\lambda,M} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,M}(\gamma(t)),$$

where

$$\Gamma_M = \{\gamma \in C([0,1], X_r) : \gamma(0) = 0,\ J_{\lambda,M}(\gamma(1)) < 0\}.$$

Hence, there exists $\{u_n\} \subset X_r$ such that

$$J_{\lambda,M}(u_n) \to c_{\lambda,M}, \quad J'_{\lambda,M}(u_n) \to 0 \text{ in } X_r^*. \tag{4.14}$$

Lemma 4.4. *For* $M > 0$ *large enough, there exists* $\lambda_M > 0$ *such that, for all* $\lambda \in (0, \lambda_M)$, *the sequence* $\{u_n\}$ *given by* (4.14) *satisfies*

$$\limsup_{n\to\infty} \|u_n\| \leqslant \frac{M}{4}.$$

Proof. For any given $M > 0$, if $\|u_n\| \to \infty$, as $n \to \infty$, there exists $N > 0$ such that for every $n > N$

$$\|u_n\| > 1 + \left(\frac{M^p + M^q}{C_2}\right)^{\frac{1}{p}},$$

by (3.1), one has

$$L_M(u_n) = 0, \text{ for } n \text{ large enough}.$$

So, for all $n \in \mathbb{N}$ large enough, we deduce that

$$J_{\lambda,M}(u_n) = \frac{1}{p}\|u_n\|_{1,p}^p + \frac{1}{q}\|u_n\|_{D^{1,q}}^q - \frac{1}{l}\int_{\mathbb{R}^3} |u_n|^l dx - \int_{\mathbb{R}^3} f(x)u_n dx,$$

and

$$\langle J'_{\lambda,M}(u_n), u_n\rangle = \|u_n\|_{1,p}^p + \|u_n\|_{D^{1,q}}^q - \int_{\mathbb{R}^3} |u_n|^l dx - \int_{\mathbb{R}^3} f(x)u_n dx.$$

By (3.1) again, for $n \in \mathbb{N}$ large enough, we have

$$
\begin{aligned}
c_{\lambda,M} + 1 + \|u_n\| &\geqslant J_{\lambda,M}(u_n) - \frac{1}{l}\langle J'_{\lambda,M}(u_n), u_n\rangle \\
&= \left(\frac{1}{p} - \frac{1}{l}\right)\|u_n\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{l}\right)\|u_n\|_{D^{1,q}}^q - \frac{l-1}{l}\int_{\mathbb{R}^3} f(x)u_n dx \qquad (4.15)\\
&\geqslant C\|u_n\|^p - \frac{l-1}{l}\int_{\mathbb{R}^3} f(x)u_n dx.
\end{aligned}
$$

It is easy to see that (4.15) contradicts with the fact that $\|u_n\| \to \infty$, $n \to \infty$. Therefore, $\{u_n\}$ is bounded in X_r which may be dependent on M.

Now, by contradiction, we assume that $\limsup\limits_{n\to\infty} \|u_n\| > \frac{M}{4}$. Up to a subsequence and still denoted by $\{u_n\}$, we have $\lim\limits_{n\to\infty} \|u_n\| > \frac{M}{4}$. In fact, by using (4.12)-(4.14), for $n \in \mathbb{N}$ large enough, we have

$$
\begin{aligned}
c_{\lambda,M} + 1 + \|u_n\| \geqslant & J_{\lambda,M}(u_n) - \frac{1}{l}\langle J'_{\lambda,M}(u_n), u_n\rangle \\
= & \left(\frac{1}{p} - \frac{1}{l}\right)\|u_n\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{l}\right)\|u_n\|_{D^{1,q}}^q \\
& + \left(\frac{\lambda(r-1)}{rs} - \frac{\lambda}{l}\right) L_M(u_n)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s dx - \frac{H_{\lambda,M}(u_n)}{l}\|u_n\|_{1,p}^p \\
& - \frac{G_{\lambda,M}(u_n)}{l}\|u_n\|_{D^{1,q}}^q - \frac{l-1}{l}\int_{\mathbb{R}^3} f(x)u_n dx.
\end{aligned}
$$

Then, for $n \in \mathbb{N}$ large enough,

$$
\begin{aligned}
& \left(\frac{1}{p} - \frac{1}{l}\right)\|u_n\|_{1,p}^p + \left(\frac{1}{q} - \frac{1}{l}\right)\|u_n\|_{D^{1,q}}^q \\
\leqslant & c_{\lambda,M} + 1 + \|u_n\| + \left(\frac{\lambda}{l} - \frac{\lambda(r-1)}{rs}\right) L_M(u_n)\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s dx \qquad (4.16)\\
& + \frac{H_{\lambda,M}(u_n)}{l}\|u_n\|_{1,p}^p + \frac{G_{\lambda,M}(u_n)}{l}\|u_n\|_{D^{1,q}}^q + \frac{l-1}{l}\int_{\mathbb{R}^3} f(x)|u_n| dx.
\end{aligned}
$$

Next, for any $n \in \mathbb{N}$, by Proposition 2.2 (iii), one has

$$
\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s dx = \|\phi_{u_n}\|_{D^{1,r}}^r \leqslant C^r\|u_n\|^{\frac{rs}{r-1}}. \qquad (4.17)
$$

Using the fact that $\eta\left(\frac{\|u_n\|_{1,p}^p}{M^p} + \frac{\|u_n\|_{D^{1,q}}^q}{M^q}\right) = 0$ and $\eta'\left(\frac{\|u_n\|_{1,p}^p}{M^p} + \frac{\|u_n\|_{D^{1,q}}^q}{M^q}\right) = 0$ if $\|u_n\|_{1,p} \geqslant M$ or $\|u_n\|_{D^{1,q}} \geqslant M$, combined with the definitions of $H_{\lambda,M}$, $G_{\lambda,M}$ and (4.17), then

$$
\begin{aligned}
|H_{\lambda,M}(u_n)| &= \frac{\lambda(r-1)p}{rsM^p}\left|\eta'\left(\frac{\|u_n\|_{1,p}^p}{M^p} + \frac{\|u_n\|_{D^{1,q}}^q}{M^q}\right)\right|\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^s dx \\
&\leqslant \frac{2^{\frac{rs}{r-1}+2}\lambda(r-1)p}{rs} C^r M^{\frac{rs}{r-1}-p},
\end{aligned}
$$

$$
\begin{aligned}
|G_{\lambda,M}(u_n)| &= \frac{\lambda(r-1)q}{rsM^q}\left|\eta'\left(\frac{\|u_n\|_{1,p}^p}{M^p}+\frac{\|u_n\|_{D^{1,q}}^q}{M^q}\right)\right|\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^s dx\\
&\leqslant \frac{2^{\frac{rs}{r-1}+2}\lambda(r-1)q}{rs}C^r M^{\frac{rs}{r-1}-q}.
\end{aligned}
$$

Further, by a simple calculation, we have

$$
\begin{aligned}
&\left(\frac{\lambda}{l}-\frac{\lambda(r-1)}{rs}\right)L_M(u_n)\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^s dx \leqslant \lambda\left|\frac{1}{l}-\frac{r-1}{rs}\right|C^r M^{\frac{rs}{r-1}},\\
&\frac{H_{\lambda,M}(u_n)}{l}\|u_n\|_{1,p}^p \leqslant \frac{2^{\frac{rs}{r-1}+2}\lambda(r-1)p}{rs}C^r M^{\frac{rs}{r-1}},\\
&\frac{G_{\lambda,M}(u_n)}{l}\|u_n\|_{D^{1,q}}^q \leqslant \frac{2^{\frac{rs}{r-1}+2}\lambda(r-1)q}{rs}C^r M^{\frac{rs}{r-1}},\\
&\frac{l-1}{l}\int_{\mathbb{R}^3}f(x)|u_n|dx \leqslant C\Lambda\|u_n\|.
\end{aligned}
\tag{4.18}
$$

Let φ be the function chosen in the proof of Lemma 4.3. We deduce that

$$
J_{\lambda,M}(M\varphi)=\frac{M^p}{p}\|\varphi\|_{1,p}^p+\frac{M^q}{q}\|\varphi\|_{D^{1,q}}^q-\frac{M^l}{l}\int_{\mathbb{R}^3}|\varphi|^l dx - M\int_{\mathbb{R}^3}f(x)\varphi dx.
$$

Then, there exists $M_1>0$ such that $J_{\lambda,M}(M\varphi)<0$ for all $M\geqslant M_1$. Further, by the definition of $c_{\lambda,M}$ and (4.17), one has

$$
\begin{aligned}
c_{\lambda,M} &\leqslant \max_{t\in[0,1]} J_{\lambda,M}(tM\varphi)\\
&\leqslant \max_{t\in[0,1]}\left\{\frac{(tM)^p}{p}\|\varphi\|_{1,p}^p+\frac{(tM)^q}{q}\|\varphi\|_{D^{1,q}}^q-\frac{(tM)^l}{l}\int_{\mathbb{R}^3}|\varphi|^l dx\right\}\\
&\quad+\max_{t\in[0,1]}\frac{\lambda(r-1)}{rs}(tM)^{\frac{rs}{r-1}}L_M(tM\varphi)\int_{\mathbb{R}^3}\phi_\varphi|\varphi|^s dx\\
&\leqslant C_3+\lambda C_4 C^r M^{\frac{rs}{r-1}}.
\end{aligned}
\tag{4.19}
$$

Now recall (4.16), (4.18) and (4.19), we deduce that

$$
\begin{aligned}
&\left(\frac{1}{p}-\frac{1}{l}\right)\|u_n\|_{1,p}^p+\left(\frac{1}{q}-\frac{1}{l}\right)\|u_n\|_{D^{1,q}}^q\\
\leqslant &C_3+\lambda C_4 C^r M^{\frac{rs}{r-1}}+1+(1+C\Lambda)\|u_n\|+\lambda\left|\frac{1}{l}-\frac{r-1}{rs}\right|C^r M^{\frac{rs}{r-1}}\\
&+\frac{2^{\frac{rs}{r-1}+3}\lambda(r-1)q}{rs}C^r M^{\frac{rs}{r-1}}\\
\leqslant &C_3+1+(1+C\Lambda)\|u_n\|+\lambda C^r M^{\frac{rs}{r-1}}\left(C_4+\left|\frac{1}{l}-\frac{r-1}{rs}\right|+\frac{2^{\frac{rs}{r-1}+3}(r-1)q}{rs}\right).
\end{aligned}
\tag{4.20}
$$

In order to obtain λ_M in the conclusion, since $\lim_{n\to\infty}\|u_n\|>\frac{M}{4}$, we can assume that for every $n\in\mathbb{N}$,

$$
\|u_n\|_{1,p}+\|u_n\|_{D^{1,q}}\geqslant 1.
$$

Thus, recalling the inequality (3.1) again, from (4.20), we deduce that

$$\left(\frac{1}{q}-\frac{1}{l}\right)\|u_n\|^p \leqslant \frac{C_3+1}{C_2}+\frac{1+C\Lambda}{C_2}\|u_n\|+\frac{\lambda C^r M^{\frac{rs}{r-1}}}{C_2}\left(C_4+\left|\frac{1}{l}-\frac{r-1}{rs}\right|+\frac{2^{\frac{rs}{r-1}+3}(r-1)q}{rs}\right).$$

Furthermore, for $\epsilon>0$ small enough, by using Young inequality, we obtain that

$$\begin{aligned}\left(\frac{1}{q}-\frac{1}{l}-\frac{\epsilon^p}{p}\right)\|u_n\|^p \leqslant &\frac{C_3+1}{C_2}+\frac{C_\epsilon}{p'}\left(\frac{1+C\Lambda}{C_2}\right)^{p'}\\ &+\frac{\lambda C^r M^{\frac{rs}{r-1}}}{C_2}\left(C_4+\left|\frac{1}{l}-\frac{r-1}{rs}\right|+\frac{2^{\frac{rs}{r-1}+3}(r-1)q}{rs}\right).\end{aligned} \tag{4.21}$$

Let $n\to\infty$ on both sides of (4.21), it reaches that

$$\begin{aligned}\left(\frac{1}{q}-\frac{1}{l}-\frac{\epsilon^p}{p}\right)\frac{M^p}{4^p} \leqslant &\frac{C_3+1}{C_2}+\frac{C_\epsilon}{p'}\left(\frac{1+C\Lambda}{C_2}\right)^{p'}\\ &+\frac{\lambda C^r M^{\frac{rs}{r-1}}}{C_2}\left(C_4+\left|\frac{1}{l}-\frac{r-1}{rs}\right|+\frac{2^{\frac{rs}{r-1}+3}(r-1)q}{rs}\right).\end{aligned}$$

For any M large enough such that

$$\left(\frac{1}{q}-\frac{1}{l}-\frac{\epsilon^p}{p}\right)\frac{M^p}{4^p} > \frac{C_3+1}{C_2}+\frac{C_\epsilon}{p'}\left(\frac{1+C\Lambda}{C_2}\right)^{p'}+1,$$

by choosing $\lambda_M>0$ such that

$$\lambda_M\frac{C^r M^{\frac{rs}{r-1}}}{C_2}\left(C_4+\left|\frac{1}{l}-\frac{r-1}{rs}\right|+\frac{2^{\frac{rs}{r-1}+3}(r-1)q}{rs}\right)=1,$$

we can get a contradiction for every $\lambda\in(0,\lambda_M)$. ☐

Proof of Theorem 4.4. By Lemmas 4.3, 4.4 and mountain pass theorem, for $M>0$ large enough, we can choose $\lambda_M>0$ such that for any $\lambda\in(0,\lambda_M)$,

$$\limsup_{n\to\infty}\|u_n\|\leqslant\frac{M}{4}.$$

Combining this with the definition of L_M, we get that for $n\in\mathbb{N}$ large enough,

$$J_{\lambda,M}(u_n)=J_\lambda(u_n) \text{ and } J'_{\lambda,M}(u_n)=J'_\lambda(u_n).$$

Therefore, we have $J_\lambda(u_n)\to c_{\lambda,M}>0$ and $J'_\lambda(u_n)\to 0$ as $n\to\infty$. By Lemma 3.2, there exists $u_2\in X_r\setminus\{0\}$ such that $J_\lambda(u_2)=c_{\lambda,M},\ J'_\lambda(u_2)=0$. Thus, u_2 is also a critical point of J_λ, which is different from u_1. The proof is complete. ☐

5. Proof of main results

Proof of Theorem 1.1. Theorem 1.1 can be got by Theorem 3.1 and Theorem 4.1. ☐

Proof of Theorem 1.2. Theorem 1.1 can be got by Theorem 3.1 and Theorem 4.2. ☐

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