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# INTEGRABILITY, MATRIX EXACT SOLUTIONS AND DYNAMIC PROPERTIES FOR THE NONLOCAL MATRIX REVERSE SPACE-TIME MODIFIED KORTEWEG-DE VRIES EQUATION

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**Abstract** In this work, the nonlocal reverse space-time matrix version and its corresponding linear spectral problem are introduced from the well-known nonlocal reverse space-time modified Korteweg-de Vries (nmKdV) equation, and the integrability in the sense of infinitely many conservation laws is confirmed. The Darboux transformation for this nonlocal matrix integrable equation is constructed and proved, and several types of exact matrix solutions such as standard soliton solution, kink solution, singular periodic solution, rational solution, non-singular periodic solution, etc., are studied by taking different groups of seed solutions and spectral parameters. Furthermore, we investigate the dynamical properties of these matrix exact solutions.

**Keywords** Nonlocal matrix reverse space-time mKdV equation, infinite conservation laws, Darboux transformation, matrix exact solutions, dynamic property.

**MSC(2010)** 35C08, 35Q51, 37K10, 37K40.

## 1. Introduction

It is well known that Bender and Boettcher introduced the concept of parity-time ($\mathcal{PT}$)-symmetry when they studied the real energy eigenvalues associated with non-Hermitian Hamiltonian operators [4]. This concept is important, since it propose a method for the $\mathcal{PT}$-symmetric modification of traditional quantum mechanics and enables the characterization of dynamically stable and observable physical systems. Following this discovery, the integrable nonlocal nonlinear systems became an important research topic in the field of integrable system theory, and have attracted increasing attention and in-depth research in the last decades. People have extended the application of this conceptual framework to many fields such as quantum chromo dynamics, vortex dynamics, metamaterials, nonlinear optics, photonics, and so on [3,5,6,14,16,22–24,26], and constructed many nonlocal nonlinear models for describing and modeling the nonlocal phenomena in the related fields of physics. Konotop, Yang and Zezyulin studied, in detail, the physical mechanisms, nonlinear properties and experimental results generated by these nonlocal integrable systems [15]. Ma deduced many important nonlocal integrable models with various concrete examples of both real and complex equations, which can be generated by nonlocal group reductions [21]. In reference [17], Lou shown, mathematically, that multi-place nonlocal integrable nonlinear models can be systematically derived by means of the discrete symmetry reductions of coupled local nonlinear systems. As one of these important nonlocal nonlin-

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ear integrable equations, the nonlocal integrable reverse space-time or $\mathcal{PT}$-symmetry modified Korteweg-de Vries (nmKdV) equation,

$$u_t(x,t)+u_{xxx}(x,t)+6u(x,t)u(-x,-t)u_x(x,t)=0, \tag{1.1}$$

was recently obtained from AKNS scheme by Ablowitz-Musslimani type nonlocal reductions [1,2]. It is important that the nmKdV equation (1.1) can be derived from a physical application of atmospheric and oceanic dynamics, and a special approximate solution of this equation can be applied to theoretically capture the salient features of two correlated dipole blocking events in atmospheric dynamical systems [25]. Besides, the inverse scattering transformation, Riemann-Hilbert problem, Darboux transformation, long-time asymptotic behavior, gauge equivalence to a spin-like model for the nmKdV equation have also been discussed [9,18,20], and many different types of exact solutions including dark-soliton solution, W-type soliton solution, M-type soliton solution, kink solution, complexiton solution, breather solution, rogue-wave solution, singular solution and periodic solution, etc., have been investigated in references [8,10,11,27]. Recently Khare and Saxena provided several types of novel solutions for nmKdV and compared with the corresponding solutions of the relevant local version [12,13]. These works show that there exists significant difference between the nonlocal reverse space-time mKdV equation and the classical mKdV equation. On the other hand, nonlinear integrable systems with matrix or multi-component potentials and their integrable properties have also been a topic of ongoing interest, and some methods have been developed to derive and investigated these systems [19].

In this work, we generalize nmKdV equation (1.1) to nonlocal reverse space-time matrix mKdV (nMmKdV) equation with matrix (or noncommutative) potentials, i.e.

$$P_t(x,t)+P_{xxx}(x,t)+3(P_x(x,t)P(-x,-t)P(x,t)+P(x,t)P(-x,-t)P_x(x,t))=0, \tag{1.2}$$

and investigate the integrability in the senses of infinitely many conservation laws, Darboux transformation (DT), and studies several types of matrix exact solutions with the aid of DT, where $P(x,t)$ is $2\times 2$ matrix potential related to spatial variable $x$ and temporal variable $t$. In addition, the dynamical properties of exact solutions with different spectral parameters are also studied.

The rest of the paper is organized as follows: In Section 2, the infinitely many conservation laws are deduced to confirm the integrability of nMmKdV equation. In Section 3, the Darboux transformation of linear spectral problem is given and proven, and several types of exact solutions are acquired via the Darboux transformation by taking different seed solutions and different spectral parameters. In addition, the dynamical properties of exact solutions have been studied. The conclusions are drawn in Section 4.

## 2. Lax integrability and infinitely many conservation laws of nMmKdV equation

In this section, we derive the infinitely many conservation laws for nMmKdV equation (1.2) which is the compatibility condition $M_t-N_x+[M,N]=0$ of the linear spectral problem

$$\partial_x\Psi=M\Psi=\lambda J\Psi+U\Psi, \tag{2.1a}$$

and the related temporal evolution equation

$$\partial_t\Psi=N\Psi=(-4\lambda^3J-4\lambda^2U+2\lambda V+W)\Psi, \tag{2.1b}$$

where

$$J=\begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}, \qquad U=\begin{pmatrix} 0 & P(x,t) \\ -P(-x,-t) & 0 \end{pmatrix},$$

$$V=\begin{pmatrix} -P(x,t)P(-x,-t) & -P_x(x,t) \\ -P_x(-x,-t) & P(-x,-t)P(x,t) \end{pmatrix},$$

$$W=\begin{pmatrix} P(x,t)P_x(-x,-t)-P_x(x,t)P(-x,-t) & -P_{xx}(x,t)-2P(x,t)P(-x,-t)P(x,t) \\ P_{xx}(-x,-t)+2P(-x,-t)P(x,t)P(-x,-t) & P(-x,-t)P_x(x,t)-P_x(-x,-t)P(x,t) \end{pmatrix},$$

$\lambda$ is complex spectral parameter, $I$ is $2\times 2$ unit matrix, $\Psi=\begin{pmatrix} \Psi_1(x,t,\lambda) \\ \Psi_2(x,t,\lambda) \end{pmatrix}_{4\times 2}$ is matrix spectral function. For convenience, let $P(-x,-t)\triangleq P^\dagger$ and $\Psi_2\Psi_1^{-1}\triangleq\Xi$, the equation (2.1a) can be written as the following matrix Riccati type equation

$$2\lambda\Xi+\Xi_x+\Xi P\Xi+P^\dagger=0. \tag{2.2}$$

We expand $\Xi$ into a Taylor series of spectral parameter $\lambda$ in the form $\Xi=P^{-1}\sum_{n=0}^{\infty}(2\lambda)^{-n-1}\Xi_n$. Comparing the coefficients of $(2\lambda)^{-n}$ $(n=0,1,2,...)$ gives

$$\begin{aligned}
&(2\lambda)^0:\Xi_0=-PP^\dagger, \qquad (2\lambda)^{-1}:\Xi_1=PP_x^\dagger, \qquad (2\lambda)^{-2}:\Xi_2=-(PP^\dagger)^2+PP_{xx}^\dagger,\\
&(2\lambda)^{-3}:\Xi_3=2(PP_x^\dagger PP^\dagger+PP^\dagger P_xP^\dagger)+PP^\dagger PP_x^\dagger-PP_{xxx}^\dagger,\\
&(2\lambda)^{-4}:\Xi_4=P(P_{xxx}^\dagger-2P_x^\dagger PP^\dagger-2P^\dagger P_xP^\dagger-P^\dagger PP_x^\dagger)_x+PP^\dagger PP_{xx}^\dagger\\
&\qquad\qquad -(PP_x^\dagger)^2+PP_{xx}^\dagger PP^\dagger-2(PP^\dagger)^3,\\
&......
\end{aligned}$$

From the linear spectral problem (2.1), one can achieve

$$\begin{aligned}
(\ln\Psi_1)_x&=\lambda+P\Psi_2\Psi_1^{-1},\\
(\ln\Psi_1)_t&=-4\lambda^3-4\lambda^2P\Psi_2\Psi_1^{-1}-2\lambda(P_x\Psi_2\Psi_1^{-1}+PP^\dagger)\\
&\quad+PP_x^\dagger-P_xP^\dagger-(P_{xx}-2PP^\dagger P)\Psi_2\Psi_1^{-1},
\end{aligned}$$

and then the compatibility $(\ln\Psi_1)_{xt}=(\ln\Psi_1)_{tx}$ can lead to the infinitely many conservation laws for the nMmKdV equation. The first three conservation equations are expressed as follows

$$\begin{aligned}
(PP^\dagger)_t&=\big(-6(PP^\dagger)^2-P_{xx}P^\dagger+4PP_{xx}^\dagger+2P_xP_x^\dagger\big)_x,\\
(PP_x^\dagger)_t&=\big(-8PP_x^\dagger PP^\dagger-8PP^\dagger P_xP^\dagger-6PP^\dagger PP_x^\dagger+4P_xP^\dagger PP^\dagger\\
&\quad+4PP_{xxx}^\dagger+2P_xP_{xx}^\dagger-P_{xx}P_x^\dagger\big)_x,\\
(PP_{xx}^\dagger-(PP^\dagger)^2)_t&=\big(-6PP^\dagger PP_{xx}^\dagger-3P_{xx}P^\dagger PP^\dagger+2P_xP_{xx}^\dagger-P_{xx}P_{xx}^\dagger\\
&\quad-2P_x(2P_x^\dagger PP^\dagger+2P^\dagger P_xP^\dagger+P^\dagger PP_x^\dagger)+4(PP_x^\dagger)^2+10(PP^\dagger)^3\\
&\quad-4P(P_{xxx}^\dagger-2P_x^\dagger PP^\dagger-2P^\dagger P_xP^\dagger-P^\dagger PP_x^\dagger)_x\big)_x.
\end{aligned}$$

**Remark 2.1.** If $P=u(x,t)$ and $P^\dagger=u(-x,-t)$, the last equations can be reduced to the infinitely many conversation laws of nonlocal mKdV equation (1.1).

## 3. Darboux transformation and matrix exact solutions for nMmKdV equation

In this section, we construct and prove the Darboux transformation (DT) for linear spectral problem (2.1), and study several types of matrix exact solutions for nMmKdV equation (1.2) with aid of DT.

**Theorem 3.1.** *Suppose* $\widetilde{\Psi} = (\lambda I - D)\Psi$, $\Psi^{[j]} = \begin{pmatrix} \Psi_1^{[j]}(x,t) \\ \Psi_2^{[j]}(x,t) \end{pmatrix}$ *is solution of linear equations* (2.1) *as* $\lambda = \lambda_j$ $(j = 1, 2)$, *then the linear system* (2.1) *has Darboux matrix in the form of* $\lambda I - D$, *where*

$$D = \lambda_2 I + (\lambda_1 - \lambda_2)T, \tag{3.1}$$

*here* $T = (T_{jl})$ *is a* $2 \times 2$ *block matrix defined as*

$$\begin{aligned} T_{11} &= (I - \Gamma_2\Gamma_1^{-1})^{-1}, \qquad T_{12} = (\Gamma_1^{-1} - \Gamma_2^{-1})^{-1}, \\ T_{21} &= (\Gamma_1 - \Gamma_2)^{-1}, \qquad T_{22} = (I - \Gamma_2^{-1}\Gamma_1)^{-1} \end{aligned} \tag{3.2}$$

*with* $\Gamma_j = \Psi_1^{[j]}(x,t)(\Psi_2^{[j]}(x,t))^{-1}$. *Furthermore,*

$$\widetilde{P} = P + 2(\lambda_1 - \lambda_2)T_{12} \tag{3.3}$$

*gives a new matrix exact solution for nMmKdV equation* (1.2).

**Proof.** We take

$$\Phi = \begin{pmatrix} \Psi_1^{[1]}(x,t,\lambda_1) & \Psi_1^{[2]}(x,t,\lambda_2) \\ \Psi_2^{[1]}(x,t,\lambda_1) & \Psi_2^{[2]}(x,t,\lambda_2) \end{pmatrix}_{4\times 4}$$

as nonsingular matrix solution for the linear equations (2.1) related to spectral parameters $\lambda_1$ and $\lambda_2$, which means $\Phi$ satisfies equations

$$\Phi_x = J\Phi\Lambda + U\Phi, \qquad \Phi_t = -4J\Phi\Lambda^3 - 4U\Phi\Lambda^2 + 2V\Phi\Lambda + W\Phi, \tag{3.4}$$

where $\Lambda = \begin{pmatrix} \lambda_1 I & \mathbf{0} \\ \mathbf{0} & \lambda_2 I \end{pmatrix}$. Taking $D = \Phi\Lambda\Phi^{-1}$, one can verify that $D$ satisfies equation (3.1), that is,

$$\begin{pmatrix} \Psi_1^{[1]} & \Psi_1^{[2]} \\ \Psi_2^{[1]} & \Psi_2^{[2]} \end{pmatrix} \begin{pmatrix} \lambda_1 I & \\ 0 & \lambda_2 I \end{pmatrix} = \begin{pmatrix} \lambda_2 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} \begin{pmatrix} \Psi_1^{[1]} & \Psi_1^{[2]} \\ \Psi_2^{[1]} & \Psi_2^{[2]} \end{pmatrix} + (\lambda_1 - \lambda_2) \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \Psi_1^{[1]} & \Psi_1^{[2]} \\ \Psi_2^{[1]} & \Psi_2^{[2]} \end{pmatrix}.$$

In fact, let's take the first submatrix of the last $2 \times 2$ block matrix as an example, i.e.,

$$\lambda_1\Psi_1^{[1]} = \lambda_2\Psi_1^{[1]} + (\lambda_1 - \lambda_2)(T_{11}\Psi_1^{[1]} + T_{12}\Psi_2^{[1]}),$$

which is equivalent to

$$(I - T_{11})\Psi_1^{[1]} = T_{12}\Psi_2^{[1]}.$$

Multiply both sides of the last matrix equation by $(\Psi_2^{[1]})^{-1}$ on the right and substituting (3.2) into the last equation gives

$$(I-\Gamma_2\Gamma_1^{-1})^{-1}+(I-\Gamma_1\Gamma_2^{-1})^{-1}=I$$

or

$$(\Gamma_1\Gamma_2^{-1}-I)^{-1}(\Gamma_1\Gamma_2^{-1}-I)=I,$$

which is an identity. Consequently the equations (3.4) can be rewritten as

$$\begin{aligned}
D_x &= \Phi_x\Lambda\Phi^{-1}-\Phi\Lambda\Phi^{-1}\Phi_x\Phi^{-1}=[\Phi_x\Phi^{-1},D]=[JD+U,D],\\
D_t &= \Phi_t\Lambda\Phi^{-1}-\Phi\Lambda\Phi^{-1}\Phi_t\Phi^{-1}=[\Phi_t\Phi^{-1},D]=[-4JD^3-4UD^2+2VD+W,D].
\end{aligned}$$

Furthermore, noting that linear equations (2.1) and $\widetilde{\Psi}=(\lambda I-D)\Psi$, we can get

$$\begin{aligned}
&(\lambda J+\widetilde{U})(\lambda I-D)\Psi=(\lambda I-D)(\lambda J+U)\Psi-D_x\Psi,\\
&(-4\lambda^3J-4\lambda^2\widetilde{U}+2\lambda\widetilde{V}+\widetilde{W})(\lambda I-D)\Psi=(\lambda I-D)(-4\lambda^3J-4\lambda^2U+2\lambda V+W)\Psi-D_t\Psi.
\end{aligned}$$

Comparing the coefficients of $\lambda^0$, $\lambda^1$ and $\lambda^2$ in the last equations leads to

$$\widetilde{U}=U+[J,D],\qquad D_x=[JD+U,D],\qquad D_t=[-4JD^3-4UD^2+2VD+W,D],$$

which means $\lambda I-D$ is the Darboux matrix of the linear system (2.1), and (3.3) is the Darboux transformation for nMmKdV equation (1.2) [7]. □

**Remark 3.1.** The Darboux transformation (3.3) can be continued using pure algebraic algorithms to obtain a series of matrix exact solutions for nMmKdV equation. For example, using DT (3.3) two times, one can get 2-fold DT matrix solution for nMmKdV equation (1.2)

$$\widetilde{\widetilde{P}}=P+2(\lambda_1-\lambda_2)T_{12}+2(\lambda_3-\lambda_4)\widetilde{T}_{12}, \tag{3.5}$$

where $\widetilde{T}_{12}=(\widetilde{\Gamma}_1^{-1}-\widetilde{\Gamma}_2^{-1})^{-1}$ with $\widetilde{\Gamma}_j=\widetilde{\Psi}_1^{[j]}(x,t)(\widetilde{\Psi}_2^{[j]}(x,t))^{-1}$ and $\widetilde{\Psi}^{[j]}=\begin{pmatrix}\widetilde{\Psi}_1^{[j]}(x,t)\\ \widetilde{\Psi}_2^{[j]}(x,t)\end{pmatrix}=(\lambda I-D)\Psi^{[j]}$.

If we take matrix potentials $P(x,t)$ in the form of components

$$P(x,t)=\begin{pmatrix}p(x,t) & q(x,t)\\ q(x,t) & r(x,t)\end{pmatrix},$$

a coupled nonlocal reverse space-time mKdV equation with three components $p(x,t)$, $q(x,t)$ and $r(x,t)$ are obtained, i.e.,

$$\begin{cases}
p_t(x,t)+p_{xxx}(x,t)+6[p(x,t)p(-x,-t)p_x(x,t)+p(x,t)q(-x,-t)q_x(x,t)\\
+p_x(x,t)q(x,t)q(-x,-t)+q(x,t)q_x(x,t)r(-x,-t)]=0, & (3.6a)\\
q_t(x,t)+q_{xxx}(x,t)+6q(x,t)q_x(x,t)q(-x,-t)+3[p_x(x,t)p(-x,-t)q(x,t)\\
+p_x(x,t)q(-x,-t)r(x,t)+q_x(x,t)r(x,t)r(-x,-t)+p(x,t)p(-x,-t)q_x(x,t)\\
+p(x,t)q(-x,-t)r_x(x,t)+q(x,t)r(-x,-t)r_x(x,t)]=0, & (3.6b)\\
r_t(x,t)+r_{xxx}(x,t)+6[p(-x,-t)q(x,t)q_x(x,t)+q(x,t)q(-x,-t)r_x(x,t)\\
+q(-x,-t)q_x(x,t)r(x,t)+r(x,t)r(-x,-t)r_x(x,t)]=0. & (3.6c)
\end{cases}$$

Next we are dedicated to deriving several types of new matrix exact solutions for the nMmKdV equation.

**Case (I).** $P(x,t) = \mathbf{0}$. Solving the linear system (2.1) as $\lambda = \lambda_j \in \mathbb{C}$ gives

$$\Psi = \begin{pmatrix} \delta_{j1}^{(1)} e^{(\lambda_j x - 4\lambda_j^3 t)I} & 0 & 0 & 0 \\ 0 & \delta_{j2}^{(1)} e^{(\lambda_j x - 4\lambda_j^3 t)I} & 0 & 0 \\ 0 & 0 & \delta_{j1}^{(2)} e^{-(\lambda_j x - 4\lambda_j^3 t)I} & 0 \\ 0 & 0 & 0 & \delta_{j2}^{(2)} e^{-(\lambda_j x - 4\lambda_j^3 t)I} \end{pmatrix},$$

where $\delta_{j1}^{(k)}, \delta_{j2}^{(k)}$ $(k, j = 1, 2)$ are arbitrary complex constants, then the Darboux transformations (3.3) leads to a new 1-fold DT solution

$$\widetilde{p}(x,t) = \frac{2(\lambda_1 - \lambda_2)\delta_{11}^{(1)}\delta_{21}^{(1)}}{\delta_{21}^{(1)}\delta_{11}^{(2)} e^{-2\lambda_1(x-4\lambda_1^2 t)} - \delta_{11}^{(1)}\delta_{21}^{(2)} e^{-2\lambda_2(x-4\lambda_2^2 t)}}, \tag{3.7a}$$

$$\widetilde{r}(x,t) = \frac{2(\lambda_1 - \lambda_2)\delta_{12}^{(1)}\delta_{22}^{(1)}}{\delta_{22}^{(1)}\delta_{12}^{(2)} e^{-2\lambda_1(x-4\lambda_1^2 t)} - \delta_{12}^{(1)}\delta_{22}^{(2)} e^{-2\lambda_2(x-4\lambda_2^2 t)}}, \tag{3.7b}$$

and $\widetilde{q}(x,t) = 0$. The exact solution of this structure admits several different types of exact solutions for the nMmKdV equation.
(A). If taking $\lambda_2 \neq 0$ and $\delta_{2j}^{(1)}\delta_{1j}^{(2)} \cdot \delta_{1j}^{(1)}\delta_{2j}^{(2)} < 0$, (3.7) gives bright single soliton solution of nMmKdV equation which have been obtained in [27] for nmKdV equation (1.1), and the profile of $\widetilde{p}(x,t)$ was plotted in Fig.1(a).
(B). If $\lambda_2 = 0$ and $\delta_{2j}^{(1)}\delta_{1j}^{(2)} \cdot \delta_{1j}^{(1)}\delta_{2j}^{(2)} < 0$, the solution (3.7) can be rewritten as

$$\widetilde{p}(x,t) = \frac{-2\lambda_1}{1 + e^{-2\lambda_1(x-4\lambda_1^2 t)+\delta_1}}, \qquad \widetilde{r}(x,t) = \frac{-2\lambda_1}{1 + e^{-2\lambda_1(x-4\lambda_1^2 t)+\delta_2}}, \qquad \widetilde{q}(x,t) = 0,$$

where $\delta_j = \ln\big(-\frac{\delta_{2j}^{(1)}\delta_{1j}^{(2)}}{\delta_{1j}^{(1)}\delta_{2j}^{(2)}}\big)$. Let $\lambda_1 = \mu_1 + i\nu_1$, if $\nu_1 = 0$ and $\mu_1 \neq 0$, then the last solution gives single kink solution for nMmKdV which has the same structure with the kink solution obtained for nmKdV in [27]. The profile of kink solution for $\widetilde{p}$ was plotted in Fig.1(b). As we known, it is interesting that the standard mKdV equation does not admit kink solutions.
(C). If taking $\nu_1 \neq 0$, one can derive singular periodic solution for $\widetilde{p}$, which has infinite number of singular points $(\hat{x}_k, \hat{t}_k)$ on the $x - t$ plane, where

$$\hat{x}_k = \frac{2\mu_1^2 - 6\nu_1^2}{4\nu_1(\mu_1^2 + \nu_1^2)} k\pi + \frac{\mu_1^2 - 3\nu_1^2}{4\nu_1(\mu_1^2 + \nu_1^2)}\pi + \frac{3\mu_1^2 - \nu_1^2}{4\mu_1(\mu_1^2 + \nu_1^2)}\delta_1,$$
$$\hat{t}_k = \frac{2k+1}{16\nu_1(\mu_1^2 + \nu_1^2)}\pi + \frac{\delta_1}{16\mu_1(\mu_1^2 + \nu_1^2)}, \qquad k \in \mathbb{Z}.$$

These singular points lie on a straight line with gradient of $4\mu_1^2 - 12\nu_1^2$ (Fig.1(c)).

It is shown that the nonlocal equations have different types of exact solutions compared to the classical mKdV equation. The profiles and properties of $\widetilde{r}(x,t)$ are similar with the ones of $\widetilde{p}(x,t)$.

Furthermore, one can get the 2-fold DT solutions with the aid of (3.5)

$$\widetilde{\widetilde{p}}(x,t)=\frac{\delta_{11}^{(1)}}{\delta_{11}^{(2)}}\frac{c_1^{(1)}e^{2(2\lambda_2-\lambda_1)x+8(\lambda_1^3-2\lambda_2^3)t}+\delta_{11}^{(1)}\delta_{21}^{(2)}(c_2^{(1)}e^{2\lambda_1(x-4\lambda_1^2t)}+c_3^{(1)}e^{2\lambda_2(x-4\lambda_2^2t)})}{(\delta_{11}^{(1)}\delta_{21}^{(2)})^2-(\delta_{21}^{(1)}\delta_{11}^{(2)})^2e^{4(\lambda_1-\lambda_2)(4\lambda_1^2t+4\lambda_1\lambda_2+4\lambda_2^2t-x)}}, \tag{3.8a}$$

$$\widetilde{\widetilde{r}}(x,t)=\frac{\delta_{12}^{(1)}}{\delta_{12}^{(2)}}\frac{c_1^{(2)}e^{2(2\lambda_2-\lambda_1)x+8(\lambda_1^3-2\lambda_2^3)t}+\delta_{12}^{(1)}\delta_{22}^{(2)}(c_2^{(2)}e^{2\lambda_1(x-4\lambda_1^2t)}+c_3^{(2)}e^{2\lambda_2(x-4\lambda_2^2t)})}{(\delta_{12}^{(1)}\delta_{22}^{(2)})^2-(\delta_{22}^{(1)}\delta_{12}^{(2)})^2e^{4(\lambda_1-\lambda_2)(4\lambda_1^2t+4\lambda_1\lambda_2+4\lambda_2^2t-x)}}, \tag{3.8b}$$

and $\widetilde{\widetilde{q}}(x,t)=0$, where $c_1^{(j)}=(\delta_{1j}^{(2)}\delta_{2j}^{(1)})^2(2\lambda_1-2\lambda_2+\lambda_3+\lambda_3\lambda_4+\lambda_4+1)$, $c_2^{(j)}=\delta_{1j}^{(1)}\delta_{2j}^{(2)}(\lambda_3-1)(\lambda_4-1)$, $c_3^{(j)}=2\delta_{2j}^{(1)}\delta_{1j}^{(2)}(\lambda_1-\lambda_2-\lambda_3\lambda_4+1)$.

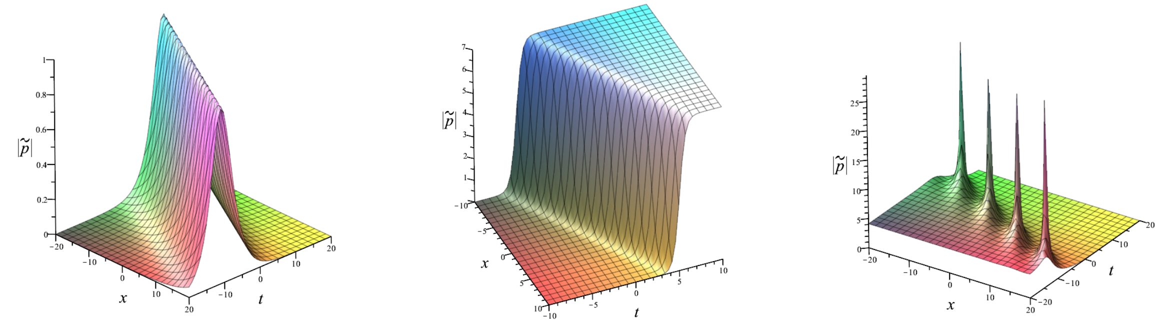


**Figure 1.** Profiles of $|\widetilde{p}|$ for 1-DT solutions as taking different groups of parameters: (a) $\lambda_1=-0.1+0.4i$, $\lambda_2=0.1+0.4i$, $\delta_{11}^{(1)}=\delta_{21}^{(1)}=5$, $\delta_{11}^{(2)}=-\delta_{21}^{(2)}=-1$; (b) $\lambda_1=0.7$, $\lambda_2=0$, $\delta_{11}^{(1)}=\delta_{21}^{(1)}=5$, $\delta_{11}^{(2)}=-\delta_{21}^{(2)}=-1$; (c) $\lambda_1=-0.1+0.4i$, $\lambda_2=0$, $\delta_{11}^{(1)}=\delta_{21}^{(1)}=5$, $\delta_{11}^{(2)}=-\delta_{21}^{(2)}=-1$.

**Case (II).** One can find that the nMmKdV (1.2) admits the following matrix seed solution

$$P(x,t)=e^{mx+nt}\begin{pmatrix}\beta & \alpha\\ \alpha & -\beta\end{pmatrix} \tag{3.9}$$

with $n=-m^3-6m(\alpha^2+\beta^2)$, $\alpha$, $\beta$, $m\in\mathbb{C}$ but $\alpha\neq0$ and $\alpha^2+\beta^2\neq0$. Solving the linear system (2.1) as $\lambda=\lambda_1\in\mathbb{C}$ and $\lambda_2=-\lambda_1$ gives base matrix solution

$$\Psi(x,t)=\begin{pmatrix}\Psi_1^{[1]}(x,t,\lambda_1) & \Psi_2^{[1]}(-x,-t,-\lambda_1)\\ \Psi_2^{[1]}(x,t,\lambda_1) & \Psi_1^{[1]}(-x,-t,-\lambda_1)\end{pmatrix}, \tag{3.10}$$

where

$$\Psi_1^{[1]}(x,t,\lambda_1)=(\rho_1-\gamma^{-1})e^{\frac{1}{2}(mx+nt)}\begin{pmatrix}\beta & \alpha\\ \alpha & -\beta\end{pmatrix}, \qquad \Psi_2^{[1]}(x,t,\lambda_1)=\rho_1\gamma e^{-\frac{1}{2}(mx+nt)}I,$$

and $\rho_1 = x-(4\lambda_1^2+2m\lambda_1+2(\alpha^2+\beta^2)+m^2)t$, $\lambda_1 = \frac{m}{2}-\gamma$, $\gamma = i\sqrt{\alpha^2+\beta^2}$. By using of Darboux transformation (3.3), one can get the new matrix-type exact solution of rational components

$$\widetilde{P}(x,t) = (1+\frac{4\lambda_1 A}{B})e^{mx+nt}\begin{pmatrix}\beta & \alpha \\ \alpha & -\beta\end{pmatrix}, \tag{3.11}$$

where

$$\begin{aligned}A =& \frac{\gamma}{4}x^2 - (\frac{1}{4}+(2\lambda_1^2+\alpha^2+\beta^2+\frac{m^2}{2})\gamma t)x \\ &+ (\gamma(2\lambda_1^2+m\lambda_1+\alpha^2+\beta^2+\frac{m^2}{2})t+\frac{1}{2})(2\lambda_1^2-m\lambda_1+\alpha^2+\beta^2+\frac{m^2}{2})t, \\ B =& (\gamma^4-\gamma^2)x^2-2\gamma^2(\gamma^2-1)(4\lambda_1^2+2(\alpha^2+\beta^2)+m^2)tx \\ &+ 4\gamma^2(\gamma^2-1)^2(2\lambda_1^2-m\lambda_1+\alpha^2+\beta^2+\frac{m^2}{2})(2\lambda_1^2+m\lambda_1+\alpha^2+\beta^2+\frac{m^2}{2})t^2+4m\gamma\lambda_1+1.\end{aligned}$$

We find that, on the one hand, the last solution shows a pair of colliding singular traveling waves moving to the opposite direction as $\lambda_1$ being pure imaginary number and $\nu_1 = -\sqrt{\alpha^2+\beta^2} < 0$ (Fig.2(a)). The solution satisfies

$$\lim_{x\to\pm\infty}|\widetilde{p}| = \lim_{x\to\pm\infty}|\widetilde{r}| = 1+\frac{|\beta|}{1+\alpha^2+\beta^2}, \qquad \lim_{x\to\pm\infty}|\widetilde{q}| = 1+\frac{|\alpha|}{1+\alpha^2+\beta^2},$$

and

$$\lim_{t\to\pm\infty}|\widetilde{p}| = \lim_{t\to\pm\infty}|\widetilde{r}| = 1+\frac{|\beta|}{(1+\alpha^2+\beta^2)^2}, \qquad \lim_{t\to\pm\infty}|\widetilde{q}| = 1+\frac{|\alpha|}{(1+\alpha^2+\beta^2)^2}.$$

On the other hand, the matrix exact solution (3.11) can give rogue wave solution for the nMmKdV equation as $m \neq 0$ (Fig.2(b)).

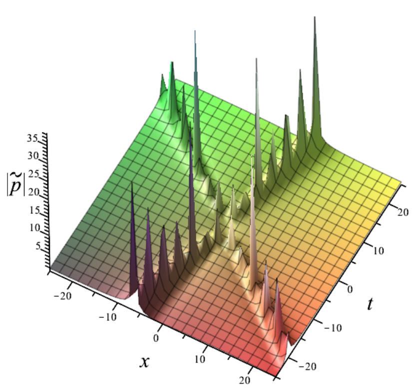


Fig.2(a)

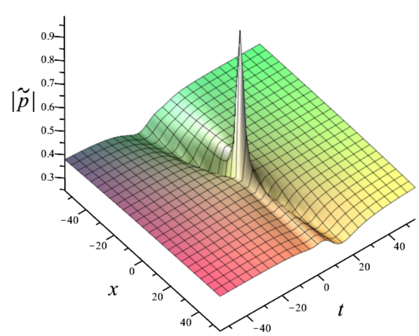


Fig.2(b)

**Figure 2.** Profiles of $|\widetilde{p}|$ as taking two different groups of parameters: (a) $m=0$, $\lambda_1=-0.5i$, $\alpha=0.3$, $\beta=0.4$; (b) $m=i$, $\lambda_1=1.4+0.5i$, $\alpha=\beta=-0.99i$.

**Case (III).** Based on matrix seed solution (3.9), the linear system (2.1) also admits solution (3.10), where

$$\begin{aligned}
\Psi_1^{[1]}(x,t,\lambda_1) &= \frac{e^{\xi_+}}{\beta(m-2\lambda_1+2z_+)}\begin{pmatrix} (\alpha^2+2\beta^2)a_+ - \alpha^2 b_+ & \alpha\beta(a_+-b_+) \\ \alpha\beta(a_+-b_+) & \alpha^2 a_+ - (\alpha^2+2\beta^2)b_+ \end{pmatrix} \\
&\quad + \frac{2e^{-\eta_+}}{m-2\lambda_1-2z_+}\begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix}, \\
\Psi_2^{[1]}(x,t,\lambda_1) &= e^{-\xi_+}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{\eta_+}\begin{pmatrix} a_+ & \frac{\alpha(a_+-b_+)}{2\beta} \\ \frac{\alpha(a_+-b_+)}{2\beta} & b_+ \end{pmatrix}, \\
\xi_+ &= (z_+ + \frac{m}{2})x + (y_+z_+ + \frac{n}{2})t, \\
\eta_+ &= (z_+ - \frac{m}{2})x + (y_+z_+ - \frac{n}{2})t,
\end{aligned} \tag{3.12}$$

and $y_+ = -4\lambda_1^2 - 2m\lambda_1 - m^2 - 2(\alpha^2+\beta^2)$, $z_+ = \sqrt{(\lambda_1 - m/2)^2 - (\alpha^2+\beta^2)}$, $a_+$ and $b_+$ are arbitrary complex constants. Without loss of generality, taking $a_+ = b_+$, with the aid of DT (3.3), one can obtain a new exact solutions for nMmKdV in matrix form

$$\widetilde{P}(x,t) = e^{mx+nt}\begin{pmatrix} \beta & \alpha \\ \alpha & -\beta \end{pmatrix} + \frac{4\lambda_1}{(A_+-A_-)^2+(B_+-B_-)^2}\begin{pmatrix} A_+-A_- & B_+-B_- \\ B_+-B_- & A_--A_+ \end{pmatrix} \tag{3.13}$$

where

$$\begin{aligned}
A_+ = -D_+ &= \frac{\beta\rho_+\zeta_+(1+b_+e^{\xi_++\eta_+})(\rho_+ + b_+\zeta_+e^{\xi_++\eta_+})e^{\eta_+-\xi_+}}{2\beta^2b_+^2\zeta_+^2e^{2(\xi_++\eta_+)} + 2\rho_+(2\beta^2b_+\zeta_+e^{\xi_++\eta_+} + (\alpha^2+\beta^2)\rho_+)}, \\
B_+ = C_+ &= \frac{\alpha\rho_+^2\zeta_+(b_+e^{2\eta_+} + e^{\eta_+-\xi_+})}{2\beta^2b_+^2\zeta_+^2e^{2(\xi_++\eta_+)} + 2\rho_+(2\beta^2b_+\zeta_+e^{\xi_++\eta_+} + (\alpha^2+\beta^2)\rho_+)}, \\
A_- = -D_- &= \frac{2\beta(b_-\zeta_-e^{\xi_-+\eta_-} + \rho_-)e^{\xi_--\eta_-}}{\rho_-\zeta_-(1+b_-e^{\xi_-+\eta_-})}, \\
B_- = C_- &= \frac{2\alpha e^{-\eta_-}}{\eta_-(b_-e^{\zeta_-} + e^{-\xi_-})},
\end{aligned}$$

and $\xi_- = -(z_- + \frac{m}{2})x - (y_-z_- + \frac{n}{2})t$, $\eta_- = -(z_- - \frac{m}{2})x - (y_-z_- - \frac{n}{2})t$, $\rho_\pm = m \mp 2\lambda_1 + 2z_\pm$, $\zeta_\pm = m \mp 2\lambda_n - 2z_\pm$ with $y_- = -4\lambda_1^2 + 2m\lambda_1 - m^2 - 2(\alpha^2+\beta^2)$, $z_- = \sqrt{(\lambda_1 + m/2)^2 - (\alpha^2+\beta^2)}$, and $a_-$, $b_-$ are complex constants. For simplicity, we set $a_\pm = b_\pm = 0$, then

$$\begin{aligned}
\widetilde{p}(x,t) = \widetilde{r}(x,t) &= \frac{2be^{\tau}}{(m-2\lambda_1-2z_+)e^{2\tau} + 2z_+ - m - 2\lambda_1}, \\
\widetilde{q}(x,t) &= \frac{2ae^{\tau}}{(m-2\lambda_1-2z_+)e^{2\tau} + 2z_+ - m - 2\lambda_1},
\end{aligned} \tag{3.14}$$

where $\tau = -mx + m(6\alpha^2 + 6\beta^2 + m^2)t$. On the one hand, if $m \in \mathbb{R}$, the last solution (3.14) obviously gives soliton solution with velocity of $6\alpha^2 + 6\beta^2 + m^2$. However, if $m$ is a pure imaginary number, which means $\tau$ is pure imaginary number, one can get the periodic solution for nMmKdV

$$\widetilde{p}(x,t)=\widetilde{r}(x,t)=\frac{\beta}{i(2z_{+}-m)\sin(i\tau)-2\lambda_1\cos(i\tau)},$$
$$\widetilde{q}(x,t)=\frac{\alpha}{i(2z_{+}-m)\sin(i\tau)-2\lambda_1\cos(i\tau)}.$$

The profile for periodic solution $\widetilde{p}(x,t)$ is plotted in Fig.3(a).

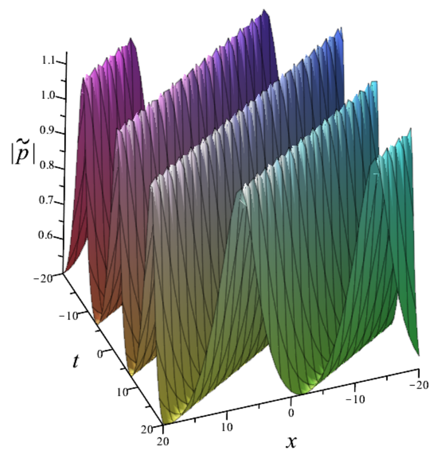


Fig.3(a)

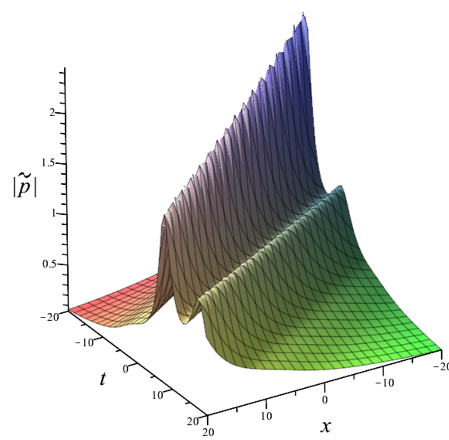


Fig.3(b)

**Figure 3.** Profiles of $|\widetilde{p}|$ as taking two different groups of parameters: (a) $a_{\pm}=b_{\pm}=0$, $\alpha=0.1$, $\beta=0.5$, $m=0.15i$, $\lambda_1=0.25i$; (b) $a_{\pm}=b_{\pm}=0$, $\alpha=0.1$, $\beta=0.5$, $m=0.07+0.18i$, $\lambda_1=0.25i$.

On the other hand, if spectral parameter $m\in\mathbb{C}$ but not pure imaginary number, the solution (3.14) is non-stationary traveling wave solution (Fig.3(b)). The dynamics of the solutions $\widetilde{r}$ and $\widetilde{q}$ have the similar properties.

**Remark 3.2.** By using of pure algebraic method, we can obtain higher-fold Darboux transformation solutions. The various types of exact solutions of nMmKdV obtained above can be reduced to exact solutions of nonlocal mKdV equation (1.1), respectively.

## 4. Conclusions

In this work, we introduced the nonlocal matrix reverse space-time mKdV equation, constructed its Darboux transformation, and confirmed its integrability by deducing its matrix-type infinitely many conservation laws. We studied some types of matrix exact solutions including soliton solution, kink solution, singular periodic solution, rational solution, non-singular periodic solution, investigated the evolution behaviors of these exact solutions as taking different spectral parameters. It was shown that nMmKdV have different types of exact solutions compared to the classical mKdV equation, and the configuration of exact solutions exhibit a significant dependence on spectral parameters. In addition, it should be noted that the nMmKdV equation (1.2), its integrability and the matrix exact solutions can be reduced to the counterparts of the nmKdV equation (1.1), respectively.

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